ON NON-EQUILIBRIUM STOCHASTIC DYNAMICS FOR INTERACTING PARTICLE SYSTEMS IN CONTINUUM

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Abstract

We propose a general scheme for the construction of Markov stochastic dynamics on configuration spaces in the continuum. An application to the Glauber-type dynamics with competitions is considered.

Keywords: configuration space; Glauber dynamics, non-equilibrium Markov process.

1 Introduction

Interacting particle systems (IPS) is a large and growing area of probability theory and infinite dimensional analysis which is devoted to the study of certain models that arises in statistical physics, biology, economics etc. Most of results in the theory of IPS are related to the study of the so-called lattice systems and their Markov stochastic evolutions. In such systems the spatial structure of the considered model is presented by a lattice (or an infinite graph). Considered processes are usually specified by transition rates and associated Markov generators. The existence problem for the corresponding Markov process on the lattice configuration space can be solved positively under quite general assumptions about the transition rates, see e.g. [22].

Comparing with the lattice case, the situation with Markov stochastic dynamics for IPS in continuum is essentially different. In particular, it is true for an important class of birth-and-death processes in continuum (or so-called spatial birth-and-death processes). To this class belong Glauber type dynamics in continuum which are under active considerations, see [3], [29]. Another class of interesting stochastic processes is formed by Kawasaki type dynamics in continuum [12] and gradient diffusions [1], [17]. Most of results we have up to now for these processes are related to the equilibrium case (via the Dirichlet forms approach) [15], [16] or to the processes in bounded domains (see, e.g., [8], [25]). The situation with the non-equilibrium case is much pure. In particular, non-equilibrium spatial birth-and-death processes were constructed recently by Garcia-Kurtz for a special class of transition rates using techniques of stochastic differential equations [6] and a graphical construction was applied in [5]. Note that in both mentioned papers the death rate was considered to be a constant and the latter has an essential technical role. A continuous version of the lattice contact model was analyzed in [19].

In contrast to the lattice case, constructions of the stochastic dynamics in continuum show essential difference between the Markov processes and Markov functions concepts. The latter notion (due to E. Dynkin) concerns the case of the processes with given initial distributions contrary to the more usual initial points framework. This weaker notion of the Markov function is not so essential in lattice models because corresponding Markov processes can be constructed (typically) under very general assumptions. A principal role of dynamics with given classes of initial distributions was clarified at first for (deterministic) Hamiltonian dynamics in continuum, see e.g. [4].

In the present paper the role of the Markov functions approach is clarified for an infinite particle stochastic dynamics in continuum. This approach is based on the study of the corresponding (dual) Kolmogorov equation on measures. Such equation can be transported to an equation for corresponding correlation functions. Typically, this correlation functions equation does not admit a direct perturbation theory approach. In fact, the main technical observation made in the paper is related to the consideration of its dual time evolution on the so-called quasi-observables. This approach appeared for the first time in the literature on stochastic IPS in continuum in our paper [14] in the particular case of a Glauber-type dynamics. The idea to move dynamics on a proper quasi-observables space (as well as the notion of quasi-observables itself) follows naturally from the harmonic analysis on configuration space concepts [11]. We apply a perturbation techniques to this dynamics in proper weighted L^1-spaces of functions on finite configurations and produce time evolutions of correlation functions as a dual object. The choice of corresponding weights gives precise description of the class of admissible initial distributions for our processes.

One should emphasize also another principal moment of the paper. Namely, even if we have constructed a time evolution of the correlation functions, we need to show that they correspond to a time evolution of measures. In fact, this point is hidden in several works in statistical physics concerning BBGKY-hierarchy etc. A rigorous mathematical analysis of this problem is based on a proper concept of positive definiteness of the correlation functions which was developed in [2], [11].

The power of the described general scheme we illustrate by the application to a particular model of Glauber-type stochastic dynamics in continuum. In this process birth of points are independent and uniformly distributed in the space. Without a death part, the density of the system will grow to the infinity with the time. To prevent such unbounded growth we can introduce a self-regulation in the model. The latter can be done in several ways and one of them is an introducing of the competition between points via a proper death rate. This competition (via density dependent mortality in terminology of the spatial ecology), we choose in such a way that a Gibbs measure on the configuration space became be a symmetrizing measure for the considered generator. Note that such type of stochastic dynamics with competitions may also realized as a proper framework for individual based models of complex socio-economic systems. Considered Glauber dynamics have unbounded death rates. Therefore, all known results in this field may not be applied for the construction of the corresponding Markov process.

In the present work, we have constructed a family of Markov functions for the Glauber dynamics with a competition corresponding to a class of initial distributions explicitly defined in the paper and depending on the interaction potential. This place needs an additional explanation: for the continuous IPS it would be too much to expect the existence of the stochastic dynamics for arbitrary initial distribution. The latter is wrong even for systems without interactions, see e.g. [16]. The class of admissible initial distributions shows "how far" from the a priori reversible state can be chosen an initial distribution to be able to prevent an explosion in the dynamics. Note that a right scale of the deviations from the equilibrium state is an important technical problem for several models of continuous infinite particle dynamics (stochastic or deterministic ones).

2 Foundations

We consider Euclidian space $\mathbb{R}^d$. By $\mathcal{B}(\mathbb{R}^d)$ we denote the family of all Borel sets in $\mathbb{R}^d$. $\mathcal{B}_b(\mathbb{R}^d)$ denotes the system of all sets in $\mathcal{B}(\mathbb{R}^d)$ which are bounded.

The space of *n-point configuration* is

$$\Gamma_0^{(n)} = \Gamma_{0,\mathbb{R}^d}^{(n)} := \left\{ \eta \subset \mathbb{R}^d \middle| \, |\eta| = n \right\}, \quad n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\},$$

where $|A|$ denotes the cardinality of the set A.

The space $\Gamma_\Lambda^{(n)} = \Gamma_{0,\Lambda}^{(n)}$ for $\Lambda \in \mathcal{B}_b(\mathbb{R}^d)$ is defined analogously to the space $\Gamma_0^{(n)}$. As a set $\Gamma_0^{(n)}$ is equivalent to the symmetrization of

$$\widetilde{(\mathbb{R}^d)^n} = \left\{ (x_1, \dots, x_n) \in (\mathbb{R}^d)^n \, | \, x_k \neq x_l \ \text{ if } \ k \neq l \right\},$$

i.e. to the $\widetilde{(\mathbb{R}^d)^n}/S_n$, where S_n is the permutation group of $\{1, \dots, n\}$. Hence, one can introduce the corresponding topology and Borel σ-algebra, which we denote by $\mathcal{O}(\Gamma_0^{(n)})$ and $\mathcal{B}(\Gamma_0^{(n)})$, respectively.

The space of *finite configurations*

$$\Gamma_0 := \bigsqcup_{n \in \mathbb{N}_0} \Gamma_0^{(n)}$$

is equipped with the topology $\mathcal{O}(\Gamma_0)$ of disjoint union. Let $\mathcal{B}(\Gamma_0)$ denotes the corresponding Borel σ-algebra.

A *set* $B \in \mathcal{B}(\Gamma_0)$ is called *bounded* if there exists $\Lambda \in \mathcal{B}_b(\mathbb{R}^d)$ and $N \in \mathbb{N}$ such that $B \subset \bigsqcup_{n=0}^N \Gamma_\Lambda^{(n)}$.

The *configuration space*

$$\Gamma := \left\{ \gamma \subset \mathbb{R}^d \middle| \, |\gamma \cap \Lambda| < \infty, \text{ for all } \Lambda \in \mathcal{B}_b(\mathbb{R}^d) \right\}$$

is equipped with the *vague* topology $\mathcal{O}(\Gamma)$. It is Polish space (see e.g. [13]). $\mathcal{B}(\Gamma)$ denotes the corresponding Borel σ-algebra. The *filtration* on Γ with a base set $\Lambda \in \mathcal{B}_b(\mathbb{R}^d)$ is given by

$$\mathcal{B}_\Lambda(\Gamma) := \sigma\left(N_{\Lambda'} \middle| \Lambda' \in \mathcal{B}_b(\mathbb{R}^d), \ \Lambda' \subset \Lambda \right),$$

where $N_\Lambda : \Gamma_0 \to \mathbb{N}_0$ is such that $N_\Lambda(\eta) := |\eta \cap \Lambda|$. For short we write $\eta_\Lambda := \eta \cap \Lambda$.

For every $\Lambda \in \mathcal{B}_b(\mathbb{R}^d)$ the *projection* $p_\Lambda : \Gamma \to \Gamma_\Lambda := \bigsqcup_{n\geq 0} \Gamma_\Lambda^{(n)}$ is defined as

$$p_\Lambda(\gamma) := \gamma_\Lambda$$

One can show that Γ is the projective limit of the spaces $\{\Gamma_\Lambda\}_{\Lambda\in\mathcal{B}_b(\mathbb{R}^d)}$ w.r.t. this projections.

In the sequel we will use the following classes of function on Γ_0:

• $L^0(\Gamma_0)$ - the set of all *measurable functions* on Γ_0;

• $L^0_{\rm ls}(\Gamma_0)$ - the set of measurable *functions* with *local support*, i.e. $G \in L^0_{\rm ls}(\Gamma_0)$ if there exists $\Lambda \in \mathcal{B}_b(\mathbb{R}^d)$ such that $G \upharpoonright_{\Gamma_0\setminus\Gamma_\Lambda} = 0$;

• $L^0_{\rm bs}(\Gamma_0)$ - the set of measurable *functions* with *bounded support*, i.e. $G \in L^0_{\rm bs}(\Gamma_0)$ if there exists $\Lambda \in \mathcal{B}_b(\mathbb{R}^d)$ and $N \in \mathbb{N}$ such that $G \upharpoonright_{\Gamma_0\setminus \bigsqcup_{n=0}^N \Gamma_\Lambda^{(n)}} = 0$;

• $B(\Gamma_0)$ - the set of *bounded* measurable *functions*

• $B_{\rm bs}(\Gamma_0)$ - the set of *bounded functions* with *bounded support*;

• $B^\Lambda_{\rm bs}(\Gamma_0)$, $\Lambda \in \mathcal{B}_b(\mathbb{R}^d)$ - the set of function from $B_{\rm bs}(\Gamma_0)$, whose support is a subset of Λ;

• $CB^\Lambda_{\rm bs}(\Gamma_0)$ - the set of *continuous functions* from $B^\Lambda_{\rm bs}(\Gamma_0)$.

On Γ we consider the set of a *cylinder functions* $\mathcal{F}L^0(\Gamma)$, i.e. the set of all measurable function $G \in L^0(\Gamma)$ which are measurable w.r.t. $\mathcal{B}_\Lambda(\Gamma)$ for some $\Lambda \in \mathcal{B}_b(\mathbb{R}^d)$. These functions are characterized by the following relation:

$$F(\gamma) = F \upharpoonright_{\Gamma_\Lambda} (\gamma_\Lambda).$$

Those cylinder functions which are measurable w.r.t. $\mathcal{B}_\Lambda(\Gamma)$ for fixed $\Lambda \in \mathcal{B}_b(\mathbb{R}^d)$ we will denote by $\mathcal{F}L^0(\Gamma,\ \mathcal{B}_\Lambda(\Gamma))$.

Next we would like to describe some facts from Harmonic analysis on configuration space based on [11].

The following mapping between functions on Γ_0, and functions on Γ, plays a key role in our further considerations:

$$KG(\gamma) := \sum_{\xi\Subset\gamma} G(\xi), \quad G \in L^0_{\rm ls}(\Gamma_0) \quad \gamma \in \Gamma,$$

see e.g. [20, 21]. The summation in the latter expression is taken over all finite subconfigurations of γ, which is denoted by symbol $\xi \Subset \gamma$.

K-transform is linear, positivity preserving, and invertible, with

$$K^{-1}F(\eta) := \sum_{\xi\subset\eta} (-1)^{|\eta\setminus\xi|} F(\xi), \quad F \in \mathcal{F}L^0(\Gamma) \quad \eta \in \Gamma_0. \tag{1}$$

It is easy to see that for any $\Lambda \in \mathcal{B}_b(\mathbb{R}^d)$ and arbitrary $F \in \mathcal{F}L^0(\Gamma,\ \mathcal{B}_\Lambda(\Gamma))$

$$K^{-1}F(\eta) = 1\!\!1_{\Gamma_\Lambda}(\eta) K^{-1}F(\eta), \quad \forall \eta \in \Gamma_0. \tag{2}$$

The map K, as well as map K^{-1}, can be extended to more wide classes of functions. For details and further properties of map K see, e.g. [11].

One can introduce a *convolution*

$$\begin{aligned} \star : L^0(\Gamma_0) \times L^0(\Gamma_0) &\rightarrow L^0(\Gamma_0) \\ (G_1, G_2) &\mapsto (G_1 \star G_2)(\eta) \end{aligned} \tag{3}$$

$$:= \sum_{(\xi_1,\xi_2,\xi_3)\in \mathcal{P}^3_\emptyset(\eta)} G_1(\xi_1 \cup \xi_2)\ G_2(\xi_2 \cup \xi_3),$$

where $\mathcal{P}^3_\emptyset(\eta)$ denotes the set of all partitions (ξ_1, ξ_2, ξ_3) of η in 3 parts, i.e., all triples (ξ_1, ξ_2, ξ_3) with $\xi_i \subset \eta$, $\xi_i \cap \xi_j = \emptyset$ if $i \neq j$, and $\xi_1 \cup \xi_2 \cup \xi_3 = \eta$.

It has the property that for $G_1, G_2 \in L^0_{\mathrm{ls}}(\Gamma_0)$

$$K(G_1 \star G_2) = KG_1 \cdot KG_2.$$

Due to this convolution we can interpret K-transform as Fourier transform in configuration space analysis, see also [2].

Let $\mathcal{M}^1_{\mathrm{fm}}(\Gamma)$ be the set of all probability measures μ which have *finite local moments* of all orders, i.e.

$$\int_\Gamma |\gamma_\Lambda|^n \mu(d\gamma) < +\infty$$

for all $\Lambda \in \mathcal{B}_b(\mathbb{R}^d)$ and $n \in \mathbb{N}_0$.

A measure ρ on Γ_0 is called *locally finite* if $\rho(A) < \infty$ for all bounded sets A from $\mathcal{B}(\Gamma_0)$. The set of such measures is denoted by $\mathcal{M}_{\mathrm{lf}}(\Gamma_0)$.

A measure $\rho \in \mathcal{M}_{\mathrm{lf}}(\Gamma_0)$ is called *positive definite* if

$$\int_{\Gamma_0} (G \star \overline{G})(\eta)\rho(d\eta) \geq 0, \quad \forall G \in B_{\mathrm{bs}}(\Gamma_0),$$

where $\overline{G}$ is a complex conjugate of G.

A measure ρ is called *normalized* iff $\rho(\{\emptyset\}) = 1$.

One can define a transform $K^* : \mathcal{M}^1_{\mathrm{fm}}(\Gamma) \rightarrow \mathcal{M}_{\mathrm{lf}}(\Gamma_0)$, which is *dual* to the K-transform, i.e., for every $\mu \in \mathcal{M}^1_{\mathrm{fm}}(\Gamma)$, $G \in \mathcal{B}_{\mathrm{bs}}(\Gamma_0)$ we have

$$\int_\Gamma KG(\gamma)\mu(d\gamma) = \int_{\Gamma_0} G(\eta)\,(K^*\mu)(d\eta).$$

The measure $\rho_\mu := K^*\mu$ is called the *correlation measure* of μ. As shown in [11] for $\mu \in \mathcal{M}^1_{\mathrm{fm}}(\Gamma)$ and any $G \in L^1(\Gamma_0, \rho_\mu)$ the series

$$KG(\gamma) := \sum_{\eta \Subset \gamma} G(\eta), \tag{4}$$

is μ-a.s. absolutely convergent. Furthermore, $KG \in L^1(\Gamma, \mu)$ and

$$\int_{\Gamma_0} G(\eta)\, \rho_\mu(d\eta) = \int_\Gamma (KG)(\gamma)\, \mu(d\gamma). \tag{5}$$

Fix a non-atomic and locally finite measure σ on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$. For any $n \in \mathbb{N}$ the product measure $\sigma^{\otimes n}$ can be considered by restriction as a measure on $\widetilde{(\mathbb{R}^d)^n}$ and hence on $\Gamma_0^{(n)}$. The measure on $\Gamma_0^{(n)}$ we denote by $\sigma^{(n)}$.

The *Lebesgue-Poisson measure* $\lambda_{z\sigma}$ on Γ_0 is defined as

$$\lambda_{z\sigma} := \sum_{n=0}^{\infty} \frac{z^n}{n!} \sigma^{(n)}.$$

Here $z > 0$ is the so-called activity parameter. The restriction of $\lambda_{z\sigma}$ to Γ_Λ will be also denoted by $\lambda_{z\sigma}$. We write λ_z instead of $\lambda_{z\sigma}$, if measure σ is considered to be fixed.

The *Poisson measure* $\pi_{z\sigma}$ on $(\Gamma, \mathcal{B}(\Gamma))$ is given as the projective limit of the family of measures $\{\pi^\Lambda_{z\sigma}\}_{\Lambda \in \mathcal{B}_b(\mathbb{R}^d)}$, where $\pi^\Lambda_{z\sigma}$ is the measure on Γ_Λ defined by $\pi^\Lambda_{z\sigma} := e^{-z\sigma(\Lambda)} \lambda_{z\sigma}$.

A measure $\mu \in \mathcal{M}^1_{\mathrm{fm}}(\Gamma)$ is called *locally absolutely continuous w.r.t.* $\pi_{z\sigma}$ iff $\mu_\Lambda := \mu \circ p_\Lambda^{-1}$ is absolutely continuous with respect to $\pi^\Lambda_{z\sigma} = \pi_{z\sigma} \circ p_\Lambda^{-1}$ for all $\Lambda \in \mathcal{B}_b(\mathbb{R}^d)$. In this case, $\rho_\mu := K^*\mu$ is absolutely continuous w.r.t $\lambda_{z\sigma}$. Let $k_\mu : \Gamma_0 \to \mathbb{R}_+$ be the corresponding Radon-Nikodym derivative, i.e.

$$k_\mu(\eta) := \frac{d\rho_\mu}{d\lambda_{z\sigma}}(\eta), \quad \eta \in \Gamma_0.$$

Remark 2.1 *The functions*

$$k_\mu^{(n)} : (\mathbb{R}^d)^n \longrightarrow \mathbb{R}_+ \tag{6}$$

$$k_\mu^{(n)}(x_1, \ldots, x_n) := \begin{cases} k_\mu(\{x_1, \ldots, x_n\}), & \text{if } (x_1, \ldots, x_n) \in \widetilde{(\mathbb{R}^d)^n} \\ 0, & \text{otherwise} \end{cases}$$

are well known correlation functions in statistical physics, see e.g [27], [28].

Next, we recall theorem about characterization of correlation measures (or correlation functions).

Theorem 2.1 *(cf. [2], [11]) Let* $\rho \in \mathcal{M}_{\mathrm{lf}}(\Gamma_0)$ *be given. Assume that* ρ *is positive definite, normalized and that for each bounded open* $\Lambda \subset \mathbb{R}^d$*, for every* $C > 0$ *there exists* $D_{\Lambda, C} > 0$ *such that*

$$\rho(\Gamma_\Lambda^n) \leq D_{\Lambda, C} C^n, \quad n \in \mathbb{N}_0.$$

Then, there exists a unique measure $\mu \in \mathcal{M}^1_{\mathrm{fm}}(\Gamma)$ *with* $\rho = K^\star \mu$.

Remark 2.2 *A sufficient condition for the bound in the theorem has form: for each bounded open* $\Lambda \subset \mathbb{R}^d$ *there exists* $\varepsilon_\Lambda > 0$ *and* $C_\Lambda > 0$ *such that*

$$\rho(\Gamma_\Lambda^{(n)}) \leq (n!)^{-\varepsilon_\Lambda} (C_\Lambda)^n. \tag{7}$$

For the technical purposes we also recall the following result:

Lemma 2.1 *Let* $n \in \mathbb{N}$*,* $n \geq 2$*, and* $z > 0$ *be given. Then*

$$\int_{\Gamma_0} \ldots \int_{\Gamma_0} G(\eta_1 \cup \ldots \cup \eta_n) H(\eta_1, \ldots, \eta_n) d\lambda_{z\sigma}(\eta_1) \ldots d\lambda_{z\sigma}(\eta_n) =$$

$$= \int_{\Gamma_0} G(\eta) \sum_{(\eta_1, \ldots, \eta_n) \in \mathcal{P}_n(\eta)} H(\eta_1, \ldots, \eta_n) d\lambda_{z\sigma}(\eta)$$

for all measurable functions $G : \Gamma_0 \mapsto \mathbb{R}$ *and* $H : \Gamma_0 \times \ldots \times \Gamma_0 \mapsto \mathbb{R}$ *with respect to which both sides of the equality make sense. Here* $\mathcal{P}_n(\eta)$ *denotes the set of all ordered partitions of* η *in* n *parts, which may be empty.*

This lemma is known in the literature as *Minlos lemma* (cf., [18], [23]) and it will be crucial for calculations in many places proposed in the next sections.

3 General approach to the construction of non-equilibrium dynamics for interacting particle systems (IPS)

In this section we investigate the existence problem for non-equilibrium Markov processes of IPS in continuum. The mechanism of an evolution of IPS on Γ, which we would like to study, is formally described by the heuristically given generator L, defined on some proper domain of functions on Γ. The problem of construction of the corresponding process in Γ, in mathematically rigorous sense, is related to the problem of construction of a semigroup associated with L on a functional space over Γ. The latter problem in its turn concerns the possibility to find solution to the Kolmogorov equation, which corresponds to the generator of this semigroup. Formally (only in the sense of action of operator), it has the form

$$\frac{dF_t}{dt} = LF_t,$$

$$F_t|_{t=0} = F_0.$$

In the most interesting cases the construction of the corresponding semigroup on functional spaces on Γ seems to be very difficult question. This difficulty is mostly related with the complex structure of infinite dimensional space Γ.

In this section we propose an alternative way for the construction of the corresponding dynamic which uses deeply the harmonic analysis technique described in the previous section.

Let

$$\hat{L} := K^{-1}LK$$

be the formal K-transform image of L or *symbol* of the operator L.

This object we consider as the starting point on the way to the mathematically rigorous description of the model.

Let $\varrho : \Gamma_0 \to \mathbb{R}_+$ be an arbitrary and fixed positive function, such that

$$\varrho(\eta) \le C^{|\eta|},$$

for some $C > 0$. We consider

$$\hat{L} : D(\hat{L}) \subset \mathcal{L}(\varrho) \to \mathcal{L}(\varrho)$$

in the Banach space

$$\mathcal{L} = \mathcal{L}(\varrho) := L^1(\Gamma_0, \varrho\, d\lambda_1),$$

where λ_1 is Lebesgue-Poisson measure with parameters $z = 1$ and σ - Lebesgue measure on $\mathbb{R}^d$.

One should emphasize, that the Banach space $\mathcal{L}$ has a Fock space structure:

$$\bigoplus_{n=0}^{\infty} L^1\left(\Gamma_0^{(n)},\, \varrho^{(n)}\sigma^{(n)}\right),$$

where $\varrho^{(n)}$ is the n-th component of function ϱ on $\Gamma_0^{(n)}$.

The following condition on operator $\hat{L}$ plays the crucial role in our technique:

Assumption 3.1 $(\hat{L},\, D(\hat{L}))$ *is a generator of a C_0-semigroup in $\mathcal{L}(\varrho)$, which will be denoted by $\hat{U}_{t,\ t\ge 0}$.*

Remark 3.1 *The semigroup $\hat{U}_t,\ _{t\geq 0}$ gives the solution $G_t = \hat{U}_t G_0$ to the following evolutional equation for the operator $\hat{L}$ in the Banach space $\mathcal{L}(\varrho)$:*

$$\frac{dG_t}{dt} = \hat{L}G_t,$$

$$G_t|_{t=0} = G_0.$$

The functional evolution on $\mathcal{L}(\rho)$ constructed via semigroup $\hat{U}_t,\ _{t\geq 0}$ gives possibility to construct the corresponding evolution of locally finite measures on Γ_0. In oder to do this we consider the dual space

$$\mathcal{K}(\varrho) := \left\{ k : \Gamma_0 \to \mathbb{R} \mid\ k \cdot \varrho^{-1} \in L^\infty(\Gamma_0, \lambda_1) \right\}.$$

to the Banach space $\mathcal{L}(\varrho)$. The duality is given by the scalar product in $L^2(\Gamma,\, \lambda_1)$, i.e.

$$<< G,\, k >> := < G,\, k >_{L^2(\Gamma,\, \lambda_1)} = \int_{\Gamma_0} G \cdot k \, d\lambda_1, \quad G \in \mathcal{L}(\varrho) \tag{8}$$

It is clear that $\mathcal{K}(\varrho)$ is the Banach space with norm

$$||k|| := ||k\varrho^{-1}||_{L^\infty(\Gamma_0, \lambda_1)}.$$

Note also, that $k \cdot \varrho^{-1} \in L^\infty(\Gamma_0, \lambda_1)$ means that function k satisfies bound

$$|k(\eta)| \leq const\, \varrho(\eta), \quad \lambda_1 \ - \ a.e.$$

The evolution on $\mathcal{K}(\varrho)$, which corresponds to $\hat{U}_t,\ _{t\geq 0}$, is constructed in the following way:

$$<< G,\, k_t >> := << \hat{U}_t G,\, k >> .$$

We denote

$$\hat{U}_t^\star k := k_t.$$

Remark 3.2 *$\hat{U}_t^\star,\ _{t\geq 0}$ is a semigroup on a Banach space $\mathcal{K}(\varrho)$. But, it is not necessarily C_0-semigroup. The continuity in 0 of a L^∞-semigroups implies the boundness of the corresponding generators, which is not necessarily the case in our situation.*

Let $k \in \mathcal{K}(\varrho)$ be a correlation function of some measure $\mu \in \mathcal{M}^1(\Gamma)$, where $\mathcal{M}^1(\Gamma)$ denotes the class of all probability measures on Γ. Let

$$k_t := \hat{U}_t^\star k,\ t \geq 0$$

be the corresponding evolution of function k in time. In order to say that there exists the corresponding evolution of probability measures on Γ we assume

Assumption 3.2 *For any $t \geq 0$, $k_t \in \mathcal{K}(\rho)$ is positive definite, normalized function.*

We set

$$\mathcal{M}^1_\rho(\Gamma) := \left\{\mu \in \mathcal{M}^1(\Gamma) \mid k_\mu \leq const \cdot \rho, \quad \lambda_1 - a.e\right\}.$$

Under Assumption 3.2, due to Theorem 2.1 about characterization of correlation measures, one can easily construct a time evolution of measures on $\mathcal{M}^1_\rho$:

$$\begin{aligned} k \quad &\rightarrow \quad \mu, \\ k_t = \hat{U}^\star_t k \quad &\rightarrow \quad \mu_t, \;\; t \geq 0, \\ U^\star_t \mu := \mu_t &\in \mathcal{M}^1_\rho. \end{aligned}$$

Remark 3.3 *It is not difficult to see that $U^\star_t, {}_{t\geq 0}$ is a semigroup. But, of course, not necessarily C_0-semigroup.*

Remark 3.4 *Suppose that operator L is a generator of a semigroup on some functional space on Γ. Suppose also that it is possible to define adjoint operator $L^\star$ to the operator L on $\mathcal{M}^1(\Gamma)$. Then, the constructed above semigroup $U^\star_t, {}_{t\geq 0}$ determines the solution $\mu_t := U^\star_t \mu_0$ to the dual Kolmogorov equation for the operator $L^\star$:*

$$\begin{aligned} \frac{\partial \mu_t}{\partial t} &= L^\star \mu_t, \\ \mu_t|_{t=0} &= \mu_0 \in \mathcal{M}^1_\rho. \end{aligned}$$

Theorem 3.1 *Suppose that Assumptions 3.1 - 3.2 are satisfied. Then, for any $\mu \in \mathcal{M}^1_\rho$, there exists a Markov process $(X^\mu_t)_{t\geq 0}$ on configuration space Γ with initial distribution μ associated with generator L.*

Proof. Let $n \in \mathbb{N}$, $A_1, ..., A_n \in \mathcal{B}(\Gamma)$ and moments of time $0 \leq t_1 \leq ... \leq t_n$ be arbitrary and fixed. Then there exists a process, defined on some probability space $(\Omega, \mathcal{F}, P)$, the finite-dimensional distribution of which is given by the following formula:

$$P\left(X^\mu_{t_1} \in A_1, \ldots, X^\mu_{t_n} \in A_n\right) = \int_\Gamma \Big((1\!\!1_{A_n} U^\star_{t_n - t_{n-1}} (\ldots U^\star_{t_2 - t_1} (1\!\!1_{A_1} U^\star_{t_1} \mu) \Big) (d\gamma),$$

where for $A \in \mathcal{B}(\Gamma)$ and $t \geq 0$ the measure $1\!\!1_A U^\star_t \mu$ on Γ is defined by

$$1\!\!1_A U^\star_t \mu(S) := \int_S 1\!\!1_A(\gamma) U^\star_t \mu(d\gamma), \quad S \in \mathcal{B}(\Gamma).$$

Moreover, $1\!\!1_A U^\star_t \mu \in \mathcal{M}^1_\rho$ since indicator of each $A \in \mathcal{B}(\Gamma)$ is bounded by 1. Eventually, we have constructed the non-equilibrium Markov process. ■

4 Application to Glauber dynamics with competition

The approach proposed in the previous section was successfully applied to a special class of Glauber dynamics on Γ with birth rate equal to constant (see [14]). Below we study the model with the death rate equal to some unbounded function, that makes impossible to apply approaches developed by [5] and [6]. In applications this death rate may be considered as reflection of the competition between particles of the system.

4.1 Potential and Gibbs measures on configuration spaces

A pair potential is a Borel, even function $\phi : \mathbb{R}^d \mapsto \mathbb{R} \cup \{+\infty\}$. Below we list some standard conditions on ϕ, known from statistical physics:

(S) (Stability) There exists $B > 0$ such that, for any $\eta \in \Gamma_0$

$$E(\eta) := \sum_{\{x,\, y\} \subset \eta} \phi(x - y) \geq -B|\eta|.$$

Notice that stability condition implies that the potential ϕ is semi-bounded from below.

(I) (Integrability) For any $\beta > 0$,

$$C(\beta) := \int_{\mathbb{R}^d} |1 - \exp\left[-\beta\phi(x)\right]| dx < \infty.$$

(SI) (Strong Integrability) For any $\beta > 0$,

$$C_{st}(\beta) := \int_{\mathbb{R}^d} |1 - \exp\left[\beta\phi(x)\right]| dx < \infty.$$

(P) (Positivity) $\phi(x) \geq 0$ for all $x \in \mathbb{R}^d$.

For $\gamma \in \Gamma$ and $x \in \mathbb{R}^d \setminus \gamma$ we define a relative energy of interaction as follows:

$$E(x,\, \gamma) := \begin{cases} \sum_{y\in\gamma} \phi(x-y), & \text{if } \sum_{y\in\gamma} |\phi(x-y)| < \infty, \\ +\infty, & \text{otherwise.} \end{cases}$$

The energy of configuration $\eta \in \Gamma_0$ or Hamiltonian $E^{\phi} : \Gamma_0 \to \mathbb{R} \cup \{+\infty\}$ which corresponds to potential ϕ is defined by

$$E^{\phi}(\eta) = \sum_{\{x,y\} \subset \eta} \phi(x - y),\ \eta \in \Gamma_0,\ |\eta| \geq 2.$$

The Hamiltonian $E^{\phi}_{\Lambda} : \Gamma_{\Lambda} \to \mathbb{R}$ for $\Lambda \in \mathcal{B}_b(\mathbb{R}^d)$ which corresponds to potential ϕ is defined by

$$E^{\phi}_{\Lambda}(\eta) = \sum_{\{x,y\}\subset\eta} \phi(x-y),\ \eta \in \Gamma_{\Lambda},\ |\eta| \geq 2.$$

For fixed ϕ we will write for short $E = E^{\phi}$ and $E_{\Lambda} = E^{\phi}_{\Lambda}$.

For given $\bar{\gamma} \in \Gamma$ we define the interaction energy between $\eta \in \Gamma_{\Lambda}$ and $\bar{\gamma}_{\Lambda^c} = \bar{\gamma} \cap \Lambda^c$, $\Lambda^c = \mathbb{R}^d \backslash \Lambda$:

$$W_{\Lambda}(\eta|\bar{\gamma}) = \sum_{x\in\eta,\ y\in\bar{\gamma}_{\Lambda^c}} \phi(x-y).$$

The interaction energy is said to be well-defined if for any $\Lambda \in \mathcal{B}_b(\mathbb{R}^d)$, $\eta \in \Gamma_{\Lambda}$ and $\bar{\gamma} \in \Gamma$ it is finite or $+\infty$.

For $\beta > 0$ we define

$$E_{\Lambda}(\eta|\bar{\gamma}) = E_{\Lambda}(\eta) + W_{\Lambda}(\eta|\bar{\gamma})$$

and

$$Z_{\Lambda}(\bar{\gamma}) := \int_{\Gamma_{\Lambda}} \exp\{-\beta E_{\Lambda}(\eta|\ \bar{\gamma})\} \lambda_z(d\eta)$$

the so-called partition function.

Let $\Lambda \in \mathcal{B}_b(\mathbb{R}^d)$, $\beta > 0$ be arbitrary, and let $\bar{\gamma} \in \Gamma$. The finite volume Gibbs measure on the space Γ_{Λ} with boundary configuration $\bar{\gamma}$ is defined by

$$\mathrm{P}_{\Lambda,\bar{\gamma}}(d\eta) = \frac{\exp\{-\beta E_{\Lambda}(\eta|\ \bar{\gamma})\}}{Z_{\Lambda}(\bar{\gamma})} \lambda_z(d\eta).$$

Let $\{\pi_{\Lambda}\}$ denote the specification associated with z and the Hamiltonian E (see [24]) which is defined by

$$\pi_{\Lambda,\bar{\gamma}}(A) = \int_{A'} P_{\Lambda,\bar{\gamma}}(d\eta)$$

where $A' = \{\eta \in \Gamma_{\Lambda} :\ \eta \cup (\bar{\gamma}_{\Lambda^c}) \in A\}$, $A \in \mathcal{B}(\Gamma)$ and $\bar{\gamma} \in \Gamma$.

A probability measure μ on Γ is called a Gibbs measure for E and z if

$$\mu(\pi_{\Lambda,\bar{\gamma}}(A)) = \mu(A)$$

for every $A \in \mathcal{B}(\Gamma)$ and every $\Lambda \in \mathcal{B}_b(\mathbb{R}^d)$.

This relation is the well-known (DLR)-equation (Dobrushin-Lanford-Ruelle equation), see [7] for more details.

The set of all Gibbs measures which corresponds to the potential ϕ, activity parameter $z > 0$, and inverse temperature $\beta > 0$ will be denoted by $\mathcal{G}(\phi,\ z,\ \beta)$. For fixed potential ϕ we will write $\mathcal{G}(z,\ \beta)$ instead of $\mathcal{G}(\phi,\ z,\ \beta)$.

4.2 Glauber type dynamics. Generator and the corresponding symbol on the space of finite configurations

Accordingly to the general scheme the mechanism of an evolution of configurations in Γ should be specified by some formally given generator. The action of such generator in the case of Glauber type dynamics has the following form

$$(LF)(\gamma) := (L_{b,d})F(\gamma) =$$
$$= \sum_{x\in\gamma} d(x, \gamma \setminus x) D_x^- F(\gamma) + \int_{\mathbb{R}^d} b(x,\gamma) D_x^+ F(\gamma) dx,$$

where $D_x^- F(\gamma) = F(\gamma \setminus x) - F(\gamma)$ and $D_x^+ F(\gamma) = F(\gamma \cup x) - F(\gamma)$.

It is known that the Gibbs measure $\mu \in \mathcal{G}(z, \beta)$ is reversible with respect to the Markov process associated with the L (i.e. the operator L is symmetrical in $L^2(\Gamma, \mu)$) iff the following condition on coefficients b and d (birth and death rates) is fulfilled:

$$b(x, \gamma) = ze^{-\beta E(x,\gamma)} d(x, \gamma). \tag{9}$$

In the sequel we will be interesting only in models with birth and death rates of the form:

- Glauber dynamics (G^+):

$$b(x, \gamma) = ze^{-\beta E(x,\gamma)}, \quad d(x, \gamma) = 1.$$

Such model was investigated by many authors, see e.g. [14], [15], [18]. As it was mentioned before, in the first paper authors used a particular realization of the general approach proposed here. Under conditions **(I)** and **(P)**, there was constructed non-equilibrium Glauber type dynamics on configuration space Γ.

In the present paper we consider another example of Glauber type dynamics:

- Glauber dynamics (G^-):

$$b(x, \gamma) = z, \quad d(x, \gamma) = e^{\beta E(x,\gamma)}.$$

The generator which corresponds to (G^-) we denote by the same symbol L.

Remark 4.1 *In the case, when $E(x, \gamma)$ is given via a potential with any positive part, the death rate of the operator L will be unbounded. In the considered model, the death rate reflects a competition between points in the configuration. In the spatial ecology models such a case is related with a density dependent mortality notion.*

For the technical reasons we will be also interested in the model with the birth and death rates localized in some volume $\Lambda \in \mathcal{B}_b(\mathbb{R}^d)$:

$$b_\Lambda(x,\,\gamma) = z 1\!\!1_\Lambda(x), \quad d_\Lambda(x,\,\gamma) = 1\!\!1_\Lambda(x) e^{\beta E(x,\,\gamma_\Lambda)}.$$

Corresponding operator we denote by L_Λ.

4.3 Symbol of the operator L

Let us consider operator L on functions $\mathcal{F}L^0(\Gamma,\ \mathcal{B}_\Lambda(\Gamma))$. One can easily check that this operator has the Markov property (satisfies maximum Principe for the generators of Markov semigroups). Therefore, one may think about this operator as about Markov pre-generator.

Proposition 4.1 *The image of L under the K-transform (or symbol of L) on functions $G \in B_{\mathrm{bs}}(\Gamma_0)$ is given by*

$$(\widehat{L}G)(\eta) := (K^{-1}LKG)(\eta) = -G(\eta) \sum_{x\in\eta} \prod_{y\in\eta\setminus x} e^{\beta\phi(x-y)} -$$

$$- \sum_{\xi\subset\eta,\,\xi\neq\eta} G(\xi) \sum_{x\in\xi} \prod_{y\in\xi\setminus x} e^{\beta\phi(x-y)} \prod_{y\in\eta\setminus\xi} (e^{\beta\phi(x-y)} - 1) + \int_{\mathbb{R}^d} G(\eta\cup x) z dx.$$

Proof. Accordingly to the definition of the operator $\widehat{L}$ we have

$$(\widehat{L}G)(\eta) = K^{-1} \left(\sum_{x\in\cdot} e^{\beta E(x,\,\cdot\setminus x)} D_x^- KG(\cdot) + z \int_{\mathbb{R}^d} D_x^+ KG(\cdot) dx \right)(\eta) =$$

$$\sum_{\xi\subset\eta} (-1)^{|\eta\setminus\xi|} \sum_{x\in\xi} e^{\beta E(x,\,\xi\setminus x)} D_x^- KG(\xi) + z \sum_{\xi\subset\eta} (-1)^{|\eta\setminus\xi|} \int_{\mathbb{R}^d} D_x^+ KG(\xi) dx. \quad (10)$$

At the beginning we transform the first expression in the sum (10)

$$I_1 G(\eta) := \sum_{\xi\subset\eta} (-1)^{|\eta\setminus\xi|} \sum_{x\in\xi} e^{\beta E(x,\,\xi\setminus x)} D_x^- KG(\xi) =$$

$$\sum_{\xi\subset\eta} (-1)^{|\eta\setminus\xi|} \sum_{x\in\xi} e^{\beta E(x,\,\xi\setminus x)} \left(\sum_{\rho\subset\xi\setminus x} G(\rho) - \sum_{\rho\subset\xi} G(\rho) \right) =$$

$$= - \sum_{\xi\subset\eta} (-1)^{|\eta\setminus\xi|} \sum_{x\in\xi} e^{\beta E(x,\,\xi\setminus x)} \sum_{\rho\subset\xi\setminus x} G(\rho\cup x).$$

Using the definitions of the K-transform and its inverse mapping we obtain

$$I_1G(\eta) = -\sum_{\xi\subset\eta}(-1)^{|\eta\setminus\xi|}\sum_{x\in\xi}e^{\beta E(x,\,\xi\setminus x)}K(G(\cdot\cup x))(\xi\setminus x) =$$

$$= -\sum_{x\in\eta}\sum_{\xi\subset\eta\setminus x}(-1)^{|\eta\setminus(\xi\cup x)|}e^{\beta E(x,\,\xi)}K(G(\cdot\cup x))(\xi) =$$

$$= -\sum_{x\in\eta}\sum_{\xi\subset\eta\setminus x}(-1)^{|(\eta\setminus x)\setminus\xi|}K\left(\prod_{y\in\cdot\setminus x}\left(e^{\beta\phi(x-y)}-1\right)\right)(\xi)\,K(G(\cdot\cup x))(\xi) =$$

$$= -\sum_{x\in\eta}K^{-1}\left(K\left(\prod_{y\in\cdot\setminus x}\left(e^{\beta\phi(x-y)}-1\right)\right)K(G(\cdot\cup x))\right)(\eta\setminus x).$$

A direct application of the definition of the convolution yields

$$I_1G(\eta) = -\sum_{x\in\eta}\left[e_\lambda\left(e^{\beta\phi(x-\cdot)}-1\right)\star G(\cdot\cup x)\right](\eta\setminus x) =$$

$$= -\sum_{x\in\eta}\sum_{(\xi_1,\xi_2,\xi_3)\in\mathcal{P}_3(\eta\setminus x)}e_\lambda\left(e^{\beta\phi(x-\cdot)}-1\right)(\xi_1\cup\xi_2)\,G(\xi_2\cup\xi_3\cup x) =$$

$$= -\sum_{x\in\eta}\sum_{\xi_1\subset\eta\setminus x}\sum_{(\xi_2,\xi_3)\in\mathcal{P}_2(\eta\setminus(\xi_1\cup x))}e_\lambda\left(e^{\beta\phi(x-\cdot)}-1\right)(\xi_1\cup\xi_2)\,G(\xi_2\cup\xi_3\cup x).$$

Picking out from the last expression item which corresponds to $\xi_1=\emptyset$ we get

$$I_1G(\eta) = -\sum_{x\in\eta}\sum_{(\xi_2,\xi_3)\in\mathcal{P}_2(\eta\setminus x)}G(\eta)e_\lambda\left(e^{\beta\phi(x-\cdot)}-1\right)(\xi_2)-$$

$$-\sum_{x\in\eta}\sum_{\substack{\xi_1\subset\eta\setminus x,\\ \xi_1\neq\emptyset}}\sum_{\substack{(\xi_2,\xi_3)\in\\ \mathcal{P}_2(\eta\setminus(\xi_1\cup x))}}e_\lambda\left(e^{\beta\phi(x-\cdot)}-1\right)(\xi_1\cup\xi_2)\,G(\xi_2\cup\xi_3\cup x).\quad(11)$$

Changing summation in (11) we have

$$I_1G(\eta) = -G(\eta)\sum_{x\in\eta}\sum_{\xi\subset\eta\setminus x}e_\lambda\left(e^{\beta\phi(x-\cdot)}-1\right)(\xi)-$$

$$-\sum_{x\in\eta}\sum_{\substack{(\xi_2,\xi_3)\in\mathcal{P}_2(\eta\setminus x),\\ \xi_1\cup\xi_2\neq\eta\setminus x}}e_\lambda\left(e^{\beta\phi(x-\cdot)}-1\right)((\eta\setminus x)\setminus(\xi_2\cup\xi_3)\cup\xi_2)\,G(\xi_2\cup\xi_3\cup x).$$

Using the fact that

$$Ke_\lambda\left(e^{\beta\phi(x-\cdot)}-1\right)(\eta)=e_\lambda\left(e^{\beta\phi(x-\cdot)}\right)(\eta),\quad \eta\in\Gamma_0 \tag{12}$$

we obtain

$$I_1G(\eta)=-G(\eta)\sum_{x\in\eta}\prod_{y\in\eta\setminus x}e^{\beta\phi(x-y)}-$$

$$-\sum_{x\in\eta}\sum_{\xi\subset\eta\setminus x,\,\xi\neq\eta\setminus x}\sum_{\rho\subset\xi}e_\lambda\left(e^{\beta\phi(x-\cdot)}-1\right)(((\eta\setminus x)\setminus\xi)\cup\rho)\,G(\xi\cup x).$$

Finally, changing summation in the last term and using (12) we get

$$I_1G(\eta)=-G(\eta)\sum_{x\in\eta}\prod_{y\in\eta\setminus x}e^{\beta\phi(x-y)}-$$

$$-\sum_{\xi\subset\eta,\,\xi\neq\eta}G(\xi)\sum_{x\in\xi}\prod_{y\in\xi\setminus x}e^{\beta\phi(x-y)}\prod_{y\in\eta\setminus\xi}(e^{\beta\phi(x-y)}-1).$$

Now, we transform the second item of (10):

$$I_2G(\eta):=z\sum_{\xi\subset\eta}(-1)^{|\eta\setminus\xi|}\int_{\mathbb{R}^d}D_x^+KG(\xi)dx=$$

$$=z\sum_{\xi\subset\eta}(-1)^{|\eta\setminus\xi|}\int_{\mathbb{R}^d}\left(\sum_{\rho\subset\xi\cup x}G(\rho)-\sum_{\rho\subset\xi}G(\rho)\right)dx.$$

A direct use of the definitions of the K-transform and its inverse yields

$$I_2G(\eta)=z\sum_{\xi\subset\eta}(-1)^{|\eta\setminus\xi|}\int_{\mathbb{R}^d}\sum_{\rho\subset\xi}G(\rho\cup x)dx=$$

$$z\int_{\mathbb{R}^d}\sum_{\xi\subset\eta}(-1)^{|\eta\setminus\xi|}K(G(\cdot\cup x))(\xi)dx=z\int_{\mathbb{R}^d}G(\eta\cup x)dx.\qquad\blacksquare$$

4.4 Verification of Assumption 3.1

In the following subsections, the Lebesgue-Poisson measure λ_1, defined for the general approach, will be denoted for simplicity by λ. We assume also that potential ϕ satisfies conditions **(S)** and **(SI)**.

For arbitrary and fixed $C>0$, we consider operator $\widehat{L}$ in the Banach space

$$\mathcal{L}_C:=L^1(\Gamma_0,C^{|\eta|}\lambda(d\eta)). \tag{13}$$

Symbol $||\cdot||$ stands for the norm of the space (13) and symbol $\overset{s}{\to}$ denotes the strong convergence of operators in $\mathcal{L}_C$.

Remark 4.2 *Accordingly to the general scheme* $\varrho(\eta) := C^{|\eta|}$, $\eta \in \Gamma_0$.

For any $\Lambda \in \mathcal{B}_b(\mathbb{R}^d)$ we set

$$\mathcal{L}_C^\Lambda := \left\{ G \in \mathcal{L}_C \mid G\restriction_{\Gamma_0 \setminus \Gamma_\Lambda} = 0 \right\}. \tag{14}$$

It is not difficult to show that $\mathcal{L}_C^\Lambda$ is a closed linear subset in $(\mathcal{L}_C\,, ||\cdot||)$. Therefore, $(\mathcal{L}_C^\Lambda\,, ||\cdot||)$ is the subspace of $(\mathcal{L}_C, ||\cdot||)$.

For any $\omega > 0$ we introduce a set $\mathcal{H}(\omega, 0)$ of all densely defined closed operators T on $\mathcal{L}_C$, the resolvent set $\rho(T)$ of which contains sector

$$Sect\left(\frac{\pi}{2} + \omega\right) := \left\{ \zeta \in \mathbb{C} \mid |arg\,\zeta| < \frac{\pi}{2} + \omega \right\}, \quad \omega > 0$$

and for any $\varepsilon > 0$

$$||(T - \zeta 1\!\!1)^{-1}|| \le \frac{M_\varepsilon}{|\zeta|}, \quad |arg\,\zeta| \le \frac{\pi}{2} + \omega - \varepsilon,$$

where M_ε does not depend on ζ.

Let $\mathcal{H}(\omega, \theta)$, $\theta \in \mathbb{R}$ denotes the set of all operators of the form $T = T_0 + \theta$ with $T_0 \in \mathcal{H}(\omega, 0)$.

Remark 4.3 *It is well-known (see e.g., [10]), that any* $T \in \mathcal{H}(\omega, \theta)$ *is a generator of a holomorphic semigroup* $U(t)$ *in the sector* $|arg\,t| < \omega$. *The function* $U(t)$ *is not necessary uniformly bounded, but it is quasi-bounded, i.e.*

$$||U(t)|| \le const\,|e^{\theta t}|$$

in any sector of the form $|arg\,t| \le \omega - \varepsilon$.

Proposition 4.2 *For any* $C > 0$, *the operator*

$$(L_0 G)(\eta) = (L_{0,\beta} G)(\eta) := -G(\eta) \sum_{x \in \eta} \prod_{y \in \eta \setminus x} e^{\beta \phi(x-y)},$$

$$D(L_0) = \left\{ G \in \mathcal{L}_C \left| \sum_{x \in \eta} \prod_{y \in \eta \setminus x} e^{\beta \phi(x-y)} G(\eta) \in \mathcal{L}_C \right. \right\}$$

is a generator of contraction semigroup on $\mathcal{L}_C$. *Moreover,* $L_0 \in \mathcal{H}(\omega, o)$ *for all* $\omega \in (0, \frac{\pi}{2})$.

Proof. It is not difficult to show that operator L_0 is densely defined and closed. Let $0 < \omega < \frac{\pi}{2}$ be arbitrary and fixed.

Since potential V satisfies **(S)**, for all $\eta \in \Gamma_0 : \, |\eta| > 1$ we have

$$\sum_{x\in\eta} \prod_{y\in\eta\setminus x} e^{\beta\phi(x-y)} \geq |\eta| \, e^{\frac{1}{|\eta|} \, 2\beta \sum_{\{x,\,y\}\in\eta} \phi(x-y)} \geq |\eta| e^{-2B\beta}. \tag{15}$$

This inequality implies that for all $\zeta \in Sect\left(\frac{\pi}{2}+\omega\right)$

$$\left| \sum_{x\in\eta} \prod_{y\in\eta\setminus x} e^{\beta\phi(x-y)} + \zeta \right| > 0, \quad \eta \in \Gamma_0.$$

Therefore, for any $\zeta \in Sect\left(\frac{\pi}{2}+\omega\right)$ the inverse operator $(L_0 - \zeta 1\!\!1)^{-1}$, the action of which is given by

$$[(L_0 - \zeta 1\!\!1)^{-1} G](\eta) = -\frac{1}{\sum_{x\in\eta} \prod_{y\in\eta\setminus x} e^{\beta\phi(x-y)} + \zeta} \, G(\eta), \tag{16}$$

is well-defined on the whole space $\mathcal{L}_C$. Moreover, it is bounded operator in this space and

$$||(L_0 - \zeta 1\!\!1)^{-1}|| \leq \begin{cases} \frac{1}{|\zeta|}, & \text{if } Re\,\zeta \geq 0, \\ \\ \frac{M}{|\zeta|}, & \text{if } Re\,\zeta < 0, \end{cases} \tag{17}$$

where constant M does not depend on ζ. Indeed, the case $Re\,\zeta \geq 0$ is a direct consequence of (16) and inequality

$$\sum_{x\in\eta} \prod_{y\in\eta\setminus x} e^{\beta\phi(x-y)} + Re\,\zeta \geq |\eta| e^{-2B\beta} + Re\,\zeta \geq Re\;\zeta \geq 0.$$

We prove now bound (17) for the case $Re\,\zeta < 0$. Using (16), we have

$$||(L_0 - \zeta 1\!\!1)^{-1} G|| = \left|\left| \frac{1}{|\sum_{x\in\cdot} \prod_{y\in\cdot\setminus x} e^{\beta\phi(x-y)} + \zeta|} \, G(\cdot) \right|\right| =$$

$$= \frac{1}{|\zeta|} \left|\left| \frac{|\zeta|}{|\sum_{x\in\cdot} \prod_{y\in\cdot\setminus x} e^{\beta\phi(x-y)} + \zeta|} G(\cdot) \right|\right|.$$

Since $\zeta \in Sect\left(\frac{\pi}{2}+\omega\right)$,

$$|Im\,\zeta| \geq |\zeta| \left| \sin\left(\frac{\pi}{2}+\omega\right) \right| = |\zeta| \cos\omega.$$

Hence,

$$\frac{|\zeta|}{|\sum_{x\in\eta} \prod_{y\in\eta\setminus x} e^{\beta\phi(x-y)} + \zeta|} \leq \frac{|\zeta|}{|Im\,\zeta|} \leq \frac{1}{\cos\omega} =: M$$

and (17) is fulfilled.

The rest statement of the lemma follows now directly from the theorem of Hille-Iosida (see e.g., [10]). ■

An ***additional parameter*** of the model: since intensity z of Lebesgue-Poisson measure in the definition of the Banach space $\mathcal{L}_C$ is equal to 1, let z which was involved in the structure of the birth rate of L plays now the role of additional parameter $\varkappa := z$.

We set now

$$(L_1 G)(\eta) := (L_{1,\,\beta} G)(\eta) =$$
$$= - \sum_{\xi\subset\eta,\,\xi\neq\eta} G(\xi) \sum_{x\in\xi} \prod_{y\in\xi\setminus x} e^{\beta\phi(x-y)} \prod_{y\in\eta\setminus\xi} (e^{\beta\phi(x-y)} - 1),\ D(L_1) := D(L_0)$$

and

$$(L_{2,\,\varkappa} G)(\eta) = \varkappa \int_{\mathbb{R}^d} G(\eta\cup x)dx, \quad D(L_2) := D(L_0).$$

The well-definiteness of these operators will be clear from the lemma below. We will sometimes use notation $\widehat{L_{\varkappa,\,\beta}}$ instead of $\widehat{L}$ to emphasize the dependence on $\varkappa$ and β.

Lemma 4.1 *For any $\delta > 0$ there exist $\varkappa_0 > 0$ and $\beta_0 > 0$ such that for all $\varkappa \le \varkappa_0$ and $\beta \le \beta_0$*

$$||(L_{1,\,\beta} + L_{2,\,\varkappa})G|| \le a||L_0 G|| + b||G||, \quad G \in D(L_0) \tag{18}$$

with $a = a(\varkappa,\,\beta) < \delta$, $b = b(\varkappa,\,\beta) < \delta$.

Proof. It is not difficult to show that

$$||L_1 G| \le \int_{\Gamma_0} \sum_{\xi\subset\eta,\,\xi\neq\eta} |G(\xi)| \sum_{x\in\xi} e^{\beta E(x,\,\xi\setminus x)} \left[\prod_{y\in\eta\setminus\xi} |1 - e^{\beta\phi(x-y)}| \right] C^{|\eta|}\lambda(d\eta) =$$

$$= \int_{\Gamma_0} \sum_{\xi\subset\eta} |G(\xi)| \sum_{x\in\xi} e^{\beta E(x,\,\xi\setminus x)} \left[\prod_{y\in\eta\setminus\xi} |1 - e^{\beta\phi(x-y)}| \right] C^{|\eta|}\lambda(d\eta) - \tag{19}$$
$$- \int_{\Gamma_0} |G(\eta)| \sum_{x\in\eta} e^{\beta E(x,\,\eta\setminus x)} C^{|\eta|}\lambda(d\eta).$$

The application of Minlos lemma to (19) give us

$$||L_1 G|| \le$$
$$\int_{\Gamma_0} |G(\eta)| \sum_{x\in\eta} e^{\beta E(x,\,\eta\setminus x)} C^{|\eta|}\lambda(d\eta) \int_{\Gamma_0} \prod_{y\in\,\xi} |1 - e^{\beta\phi(y)}| C^{|\xi|}\lambda(d\xi) - ||L_0 G|| =$$

$$= (e^{C_{st}(\beta)C} - 1)||L_0 G||.$$

The norm $||L_2 G||$ we will estimate using Minlos lemma and then (15). Namely,

$$||L_2 G|| \le \varkappa \int_{\Gamma_0} \int_{\mathbb{R}^d} |G(\eta \cup x)|\, dx\, C^{|\eta|} \lambda(d\eta) \le$$

$$\le \varkappa \int_{\Gamma_0} |\eta||G(\eta)|\, C^{|\eta|-1} \lambda(d\eta) \le$$

$$\varkappa e^{2B\beta} \int_{\Gamma_0} \sum_{x\in\eta} e^{\beta E(x,\, \eta\setminus x)} |G(\eta)|\, C^{|\eta|-1} \lambda(d\eta) \le = \varkappa e^{2B\beta} C^{-1} ||L_0 G||.$$

Therefore,

$$||(L_1 + L_2)G|| \le (e^{C_{st}(\beta)C} + \varkappa e^{2B\beta} C^{-1} - 1)||L_0 G||.$$

And hence the assertion of the lemma is fulfilled with coefficients

$$a := e^{C_{st}(\beta)C} + \varkappa e^{2B\beta} C^{-1} - 1, \quad b := 0$$

which can be taken less then δ for appropriate choice of $\varkappa$ and β. ■

Theorem 4.1 *For any $C > 0$, and for all $\varkappa$, $\beta > 0$ which satisfy*

$$2e^{C_{st}(\beta)C} + 2\varkappa e^{2B\beta} C^{-1} < 3 \tag{20}$$

the operator $\widehat{L_{\varkappa,\beta}}$ is a generator of a holomorphic semigroup in $\mathcal{L}_C$.

Proof. Follows from the theorem about perturbation of holomorphic semigroup (see e.g., [10]). For the reader's convenience, below we give its formulation:

for any $T \in \mathcal{H}(\omega,\, \theta)$ and for any $\varepsilon > 0$ there exists positive constants ϵ, δ such that if operator A satisfies

$$||Au|| \le a||Tu|| + b||u||, \quad u \in D(T) \subset D(A),$$

with $a < \delta$, $b < \delta$, then $T + A \in \mathcal{H}(\omega - \varepsilon,\, \epsilon)$. In particular, if $\theta = 0$ and $b = 0$, then $T + A \in \mathcal{H}(\omega - \varepsilon,\, 0)$. ■

Remark 4.4 *Applying the proof of the theorem about perturbation of the generator of a holomorphic semigroup (see, e.g. [10]) to our case and taking into account the fact that $L_0 \in \mathcal{H}(\omega, 0)$, for any $\omega \in (0, \frac{\pi}{2})$, one can conclude that δ in this theorem can be chosen to be $\frac{1}{2}$.*

For our further purposes we have to show that holomorphic semigroup constructed in Theorem 4.1 can be approximated by the semigroups localized in bounded volumes.

Let $\Lambda \in \mathcal{B}_b(\mathbb{R}^d)$ be arbitrary and fixed. Then all results proved in this subsection hold true for the operator

$$\widehat{L_\Lambda}G(\eta) := -\sum_{x\in\eta_\Lambda}\prod_{y\in\eta_\Lambda\setminus x} e^{\beta\phi(x-y)}G(\eta)-$$

$$-\sum_{\xi\subset\eta,\,\xi\neq\eta} G(\xi)\sum_{x\in\xi_\Lambda}\prod_{y\in\xi_\Lambda\setminus x} e^{\beta\phi(x-y)}\prod_{y\in\eta\setminus\xi}(e^{\beta\phi(x-y)1\!\!1_\Lambda(y)}-1)+\varkappa\int_\Lambda G(\eta\cup x)dx$$

acting in the functional space $\mathcal{L}_C^\Lambda$ with domain

$$D(\widehat{L_\Lambda}) := \{G\in\mathcal{L}_C \mid \sum_{x\in\eta_\Lambda}\prod_{y\in\eta_\Lambda\setminus x} e^{\beta\phi(x-y)}G(\eta)\in\mathcal{L}_C^\Lambda\}.$$

Namely, the main result can be formulated as follows

Theorem 4.2 *For any $\Lambda \in \mathcal{B}_b(\mathbb{R}^d)$, and any triple of constants C, $\varkappa > 0$, and $\beta > 0$ which satisfy*

$$2e^{C_{st}(\beta)C}+2\varkappa e^{2B\beta}C^{-1}<3$$

the operator $\widehat{L_\Lambda}$ is a generator of a holomorphic semigroup in $\mathcal{L}_C^\Lambda$.

Remark 4.5 *The arguments, analogous to those which were proposed in the proof of Lemma 4.1, imply the fulfillment of (18) for the operators*

$$\widehat{L_{0,\Lambda}}G(\eta) := -\sum_{x\in\eta_\Lambda}\prod_{y\in\eta_\Lambda\setminus x} e^{\beta\phi(x-y)}G(\eta),$$

$$\widehat{L_{1,\Lambda}}G(\eta) := -\sum_{\xi\subset\eta,\,\xi\neq\eta} G(\xi)\sum_{x\in\xi_\Lambda}\prod_{y\in\xi_\Lambda\setminus x} e^{\beta\phi(x-y)}\prod_{y\in\eta\setminus\xi}(e^{\beta\phi(x-y)1\!\!1_\Lambda(y)}-1),$$

and

$$(L_{2,\varkappa,\Lambda}G)(\eta) = \varkappa\int_\Lambda G(\eta\cup x)dx,$$

with

$$D(\widehat{L_{0,\Lambda}}) = D(\widehat{L_{1,\Lambda}}) = D(\widehat{L_{2,\Lambda}}) := \{G\in\mathcal{L}_C \mid \sum_{x\in\eta_\Lambda} e^{\beta E(x,\eta_\Lambda\setminus x)}G(\eta)\in\mathcal{L}_C^\Lambda\}.$$

Moreover, bound (18) in this case will be uniform with respect to the $\Lambda \in \mathcal{B}_b(\mathbb{R}^d)$, i.e. coefficients $a > 0$ and $b > 0$ in (18) can be chosen Λ independent.

Fix any triple of positive constants C, $\varkappa$ and β which satisfies (20) and any $\Lambda \in \mathcal{B}_b(\mathbb{R}^d)$.

Remark 4.6 *Let $\widehat{U_t^\Lambda}(C,\varkappa,\beta)$ be holomorphic semigroup generated by operator $\left(\widehat{L_\Lambda},\, D(\widehat{L_\Lambda})\right)$ on $\mathcal{L}_C^\Lambda$. Then $\widehat{U_t^\Lambda}(C,\varkappa,\beta)P_\Lambda,\ {}_{t\geq 0}$, where*

$$P_\Lambda G(\eta) := 1\!\!1_{\Gamma_\Lambda}(\eta)G(\eta), \quad G \in \mathcal{L}_C$$

is a semigroup on $\mathcal{L}_C$ generated by the operator $\widehat{L_\Lambda}P_\Lambda$ with domain

$$D(\widehat{L_\Lambda}P_\Lambda) := \left\{ G \in \mathcal{L}_C \mid \sum_{x\in\eta_\Lambda} \prod_{y\in\eta_\Lambda\setminus x} e^{\beta\phi(x-y)} 1\!\!1_{\Gamma_\Lambda}(\eta)G(\eta) \in \mathcal{L}_C \right\}.$$

Remark 4.7 *The theorem about perturbation of the generator of a holomorphic semigroup, mentioned before in this subsection (see also [10]), implies that for any $\Lambda \in \mathcal{B}_b(\mathbb{R}^d)$ and $\varepsilon > 0$ there exists $\epsilon > 0$ and constant $M > 0$ which is not depend on Λ such that for any $\zeta > \epsilon$ the following bound holds*

$$||(\widehat{L_\Lambda}P_\Lambda - \zeta)^{-1}|| \leq \frac{M_\varepsilon}{|\zeta-\epsilon|}, \quad |arg\,(\zeta-\epsilon)| \leq \frac{\pi}{2} + \omega - \varepsilon.$$

Let $\{\Lambda_n\}_{n\geq 1}$ be a sequence of bounded Borel sets such that $\Lambda_n \subset \Lambda_{n+1}$, for all $n \in \mathbb{N}$, and $\bigcup_{n\geq 1}\Lambda_n = \mathbb{R}^d$. Below, we formulate the following approximation theorem.

Theorem 4.3 *Let $\widehat{U_t}(C,\varkappa,\beta)$ and $\left\{\widehat{U_t^{\Lambda_n}}(C,\varkappa,\beta),\, n\geq 1\right\}$ be holomorphic semigroups generated by $\widehat{L_\varkappa}$ and $\left\{\widehat{L_{\Lambda_n,\varkappa}},\, n \geq 1\right\}$ in the spaces $\mathcal{L}_C$ and $\mathcal{L}_C^{\Lambda_n}$, respectively. Then,*

$$\widehat{U_t^{\Lambda_n}}(C,\varkappa,\beta)P_{\Lambda_n} \xrightarrow{s} \widehat{U_t}(C,\varkappa,\beta), \quad n\to\infty$$

uniformly on any finite interval of $t \geq 0$.

Proof. Using approximation theorem for quasi-bounded semigroups (see e.g. [10]), it is enough to show that

$$(\widehat{L_{\Lambda_n,\varkappa}}P_{\Lambda_n} - \zeta)^{-1} \xrightarrow{s} (\widehat{L_\varkappa} - \zeta)^{-1}$$

for some $\zeta \in \mathbb{C}$ such that $\mathrm{Re}\,\zeta > \theta$.

Let $\zeta \in \mathbb{C}$, $\mathrm{Re}\,\zeta > \theta$ be arbitrary and fixed. For any $G \in \mathcal{L}_C$ it holds

$$||(\widehat{L_{\Lambda_n,\varkappa}}P_{\Lambda_n} - \zeta)^{-1}G - (\widehat{L_\varkappa} - \zeta)^{-1}G|| =$$

$$= ||(\widehat{L_{\Lambda_n,\varkappa}P_{\Lambda_n}} - \zeta)^{-1}\left[\widehat{L_\varkappa} - \widehat{L_{\Lambda_n,\varkappa}P_{\Lambda_n}}\right](\widehat{L_\varkappa} - \zeta)^{-1}G||. \tag{21}$$

For any $G \in D(\widehat{L_\varkappa}) = D(L_0)$

$$\left[\widehat{L_\varkappa} - \widehat{L_{\Lambda_n,\varkappa}P_{\Lambda_n}}\right]G(\eta) = -\sum_{x\in\eta}\prod_{y\in\eta\setminus x} e^{\beta\phi(x-y)}\left[1 - 1\!\!1_{\Gamma_{\Lambda_n}}(\eta)\right]G(\eta) -$$

$$-\sum_{\xi\subset\eta,\,\xi\neq\eta} G(\xi)\sum_{x\in\xi}\prod_{y\in\xi\setminus x} e^{\beta\phi(x-y)}\prod_{y\in\eta\setminus\xi}(e^{\beta\phi(x-y)}-1)\left[1 - 1\!\!1_{\Gamma_{\Lambda_n}}(\xi)1\!\!1_{\Gamma_{\Lambda_n}}(\eta\setminus\xi)\right] +$$

$$+\varkappa\int_{\Lambda_n}\left[1 - 1\!\!1_{\Lambda_n}(\eta\cup x)\right]G(\eta\cup x)dx + \varkappa\int_{\Lambda_n^c} G(\eta\cup x)dx,$$

where $\Lambda_n^c = \mathbb{R}^d \setminus \Lambda_n$.

Using simple inequality

$$|1 - 1\!\!1_{\Gamma_{\Lambda_n}}(\xi)1\!\!1_{\Gamma_{\Lambda_n}}(\eta)| \leq |1 - 1\!\!1_{\Gamma_{\Lambda_n}}(\xi)| + |1 - 1\!\!1_{\Gamma_{\Lambda_n}}(\eta)|, \quad \xi,\, \eta \in \Gamma_0,$$

and estimates analogous to those which were proposed in Lemma 4.1 we obtain

$$\left|\left|\left[\widehat{L_\varkappa} - \widehat{L_{\Lambda_n,\varkappa}P_{\Lambda_n}}\right]G(\eta)\right|\right| \leq$$

$$\leq \left(e^{C_{st}(\beta)C} + \varkappa e^{2B\beta}C^{-1}\right)\left|\left|\left[1 - 1\!\!1_{\Gamma_{\Lambda_n}}(\cdot)\right]\sum_{x\in\cdot}\prod_{y\in\cdot\setminus x} e^{\beta\phi(x-y)}G(\cdot)\right|\right| +$$

$$+\varkappa e^{2B\beta}C^{-1}\,||\,|\cdot_{\Lambda_n^c}|G(\cdot)||+$$

$$+\left|\left|\sum_{x\in\cdot}\prod_{y\in\cdot\setminus x} e^{\beta\phi(x-y)}G(\cdot)\right|\right|\int_{\Gamma_{\Lambda_n}}|1 - 1\!\!1_{\Gamma_{\Lambda_n}}(\eta)|K(\eta)C^{|\eta|}\lambda(d\eta),$$

where

$$K(\eta) := \prod_{x\in\eta}\left|1 - e^{\beta\phi(x)}\right|, \quad \eta \in \Gamma_0.$$

All of the summands in the right-hand side of the last inequality definitely tends to zero, when $n \to \infty$.

Using Remark 4.7 and equality (21) we easily conclude that difference in (21) also tends to zero when $n \to \infty$. ∎

4.5 Verification of Assumption 3.2

Fix any triple of positive constants C, $\varkappa$ and β which satisfies (20). Let $\widehat{U_t}(C,\varkappa,\beta)$ be holomorphic semigroup generated by $\widehat{L_{\varkappa,\beta}}$ and let

$$\mathcal{K}_C := \Big\{ k : \Gamma_0 \to \mathbb{R} \mid k(\cdot)\, C^{-|\cdot|} \in L^\infty(\Gamma_0, \lambda) \Big\} \tag{22}$$

be the dual space to the space $\mathcal{L}_C$ w.r.t. the following duality

$$\langle\langle G,\, k\rangle\rangle := \langle G,\, k\rangle_{L^2(\Gamma_0,\,\lambda)}\,. \tag{23}$$

It is also called as the space of "so-called correlation functions". Analogously to the general scheme, $\mathcal{K}_C$ is a Banach space.

Note, also that $k(\cdot)\, C^{-|\cdot|} \in L^\infty(\Gamma_0, \lambda)$ means that function k satisfies the following bound

$$k(\eta) \le const\, C^{|\eta|}, \tag{24}$$

which is known as *classical Ruelle bound*, see e.g. [27].

Accordinglly to the general scheme, let $\widehat{U_t^\star}(C,\varkappa,\beta)$ be semigroup on $\mathcal{K}_C$ determined by $\widehat{U_t}(C,\varkappa,\beta)$ via duality (23).

Next, we solve the following problem: suppose that $k_0 \in \mathcal{K}_C$ is a correlation function which means, that there exists a probability measure $\mu_0 \in \mathcal{M}^1_{fm}(\Gamma)$, locally absolutely continuous with respect to Poisson measure, whose correlation function is exactly k_0. We would like to investigate now whether the evolution of k_0 in time given by the semigroup $\widehat{U_t^\star}(C,\varkappa,\beta)$ preserves the property described above. Namely, whether $\widehat{U_t^\star}(C,\varkappa,\beta)k_0$, for any moment of time $t > 0$, be a correlation function or not?

In order to answer this question, one can apply, for example, the theorem about characterization of correlation functions, proposed in [11]. The conditions of this theorem, which must to be checked for our particular model are the following:

$$for \quad any \quad t \ge 0: \quad \Big\langle\Big\langle G \star G,\, \widehat{U_t^\star}(C,\varkappa,\beta)k_0 \Big\rangle\Big\rangle \ge 0, \qquad \forall\, G \in B_{\mathrm{bs}}(\Gamma_0).$$

Further explanations will be devoted to the verification of the latter condition.

Let $\mu \in \mathcal{G}(\beta,\, z)$ and $\{\pi_{\Lambda,\,\emptyset}\}_{\Lambda\in\mathcal{B}_b(\mathbb{R}^d)}$ denotes the specification with empty boundary conditions corresponding to the Gibbs measure μ. We define

$$\mathcal{E}(F,\, G) := \int_\Gamma \sum_{x\in\gamma} D_x^+ F(\gamma) D_x^+ G(\gamma) \pi_\Lambda(d\gamma,\, \emptyset), \quad F,\, G \in K\, CB^\Lambda_{\mathrm{bs}}(\Gamma_0),$$

where $K\, CB^\Lambda_{\mathrm{bs}}(\Gamma_0)$ is K-image of $CB^\Lambda_{\mathrm{bs}}(\Gamma_0)$.

Now we would like to list some facts the proofs of which are completely analogous to those proposed in [15], [17]

Lemma 4.2 *The set* $KCB_{\mathrm{bs}}^{\Lambda}(\Gamma_0)$ *is dense in* $L^2(\Gamma,\, \pi_{\Lambda,\emptyset})$ *for any* $\Lambda \in \mathcal{B}_b(\mathbb{R}^d)$.

Lemma 4.3 *Let* $\Lambda \in \mathcal{B}_b(\mathbb{R}^d)$ *be arbitrary and fixed. Then* $(\mathcal{E},\, KCB_{\mathrm{bs}}^{\Lambda}(\Gamma_0))$ *is a well-defined bilinear form on* $L^2(\Gamma,\, \pi_{\Lambda,\emptyset})$.

Lemma 4.4 *Let* $\Lambda \in \mathcal{B}_b(\mathbb{R}^d)$ *be arbitrary and fixed. Suppose that conditions* ***(S)*** *and* ***(SI)*** *are satisfied. Then* $(L_\Lambda, KCB_{\mathrm{bs}}^{\Lambda}(\Gamma_0))$ *is an operator associated with bilinear form* $(\mathcal{E},\, KCB_{\mathrm{bs}}^{\Lambda}(\Gamma_0))$ *in* $L^2(\Gamma,\, \pi_{\Lambda,\emptyset})$*, i.e.*

$$\mathcal{E}(F,\, G) = \int_\Gamma L_\Lambda F(\gamma) G(\gamma) \pi_{\Lambda,\emptyset}(d\gamma), \quad F,\, G \in KCB_{\mathrm{bs}}^{\Lambda}(\Gamma_0).$$

Lemma 4.5 *Let* $\Lambda \in \mathcal{B}_b(\mathbb{R}^d)$ *be arbitrary and fixed. Suppose that conditions* ***(S)*** *and* ***(SI)*** *are satisfied and* $\mu \in \mathcal{G}(z,\, \beta)$*. Then there exists a self-adjoint positive Friedrichs' extension* $(\widetilde{L_\Lambda},\, D(\widetilde{L_\Lambda}))$ *of the operator* $(L_\Lambda,\, KCB_{\mathrm{bs}}^{\Lambda}(\Gamma_0))$ *in* $L^2(\Gamma,\, \pi_{\Lambda,\emptyset})$*. Moreover,* $(\widetilde{L_\Lambda},\, D(\widetilde{L_\Lambda}))$ *is a generator of a contraction semigroup which preserves 1 in* $L^2(\Gamma,\, \pi_{\Lambda,\emptyset})$*, associated with some Markov process.*

Remark 4.8 *It is well known (see e.g. [26]) that under condition of Lemma 4.5 the semigroup generated by* $(\widetilde{L_\Lambda},\, D(\widetilde{L_\Lambda}))$ *can be extended to* $L^1(\Gamma,\, \pi_{\Lambda,\emptyset})$*. For any* $\Lambda \in \mathcal{B}_b(\mathbb{R}^d)$*, the extension of this semigroup in* $L^1(\Gamma,\, \pi_{\Lambda,\emptyset})$ *we will denote by* $(\widetilde{U_t^\Lambda})_{t\geq 0}$*. For the generator of this semigroup we will use notation* $(\widetilde{L_\Lambda},\, D_1(\widetilde{L_\Lambda}))$*, where* $D_1(\widetilde{L_\Lambda}) \supset D(\widetilde{L_\Lambda})$ *is a domain of* $\widetilde{L_\Lambda}$ *in* $L^1(\Gamma,\, \pi_{\Lambda,\emptyset})$.

Now, we introduce one of the crucial lemma about the evolution of the "so-called correlation functions".

Lemma 4.6 *Let positive constants* C, $\varkappa$ *and* β *which satisfy (20) be arbitrary and fixed. The semigroup* $\widehat{U_t^\star}(C, \varkappa, \beta)$ *on* $\mathcal{K}_C$ *preserves positive semi-definiteness, i.e. for any* $t \geq 0$

$$\left\langle\left\langle G \star G,\, \widehat{U_t^\star}(C, \varkappa, \beta) k \right\rangle\right\rangle \geq 0, \qquad \forall\, G \in B_{\mathrm{bs}}(\Gamma_0)$$

iff

$$\langle\langle G \star G,\, k \rangle\rangle \geq 0, \tag{25}$$

for any $G \in B_{\mathrm{bs}}(\Gamma_0)$.

Remark 4.9 *Let* $\mathcal{M}_C$ *stands for the set of all probability measures on* Γ*, locally absolutely continuous with respect to Poisson measure, with locally finite moments, whose correlation functions satisfy bound (24). As it was pointed out at the beginning of this section, the condition (25) on function* $k \in \mathcal{K}_C$*, insures an existence of a unique measure* $\mu^\varkappa \in \mathcal{M}_C$ *whose correlation function is* k*, see [11].*

Proof of Lemma 4.6. Under assumptions of the lemma we have to show that for any $t \geq 0$

$$\left\langle\left\langle \widehat{U_t}(C, \varkappa, \beta)(G \star G),\, k \right\rangle\right\rangle \geq 0, \qquad \forall\, G \in B_{\mathrm{bs}}(\Gamma_0). \tag{26}$$

But $G \star G \in B_{\mathrm{bs}}(\Gamma_0)$ for any $G \in B_{\mathrm{bs}}(\Gamma_0)$. Therefore, due to Theorem 4.3 it is enough to show that for any $t \geq 0$ and any $G \in B_{\mathrm{bs}}(\Gamma_0)$ there exists $\Lambda' \in \mathcal{B}_b(\mathbb{R}^d)$ such that for all $\Lambda \in \mathcal{B}_b(\mathbb{R}^d)$, $\Lambda \supset \Lambda'$

$$\left\langle\left\langle \widehat{U_t^\Lambda}(C, \varkappa, \beta) P_\Lambda(G \star G),\, k \right\rangle\right\rangle \geq 0. \tag{27}$$

Let $\Lambda \in \mathcal{B}_b(\mathbb{R}^d)$ be arbitrary and fixed. We set

$$U_t^\Lambda := K \widehat{U_t^\Lambda}(C, \varkappa, \beta) K^{-1}, \quad t \geq 0.$$

$(U_t^\Lambda)_{t\geq 0}$ is a semigroup on

$$(\mathcal{L}_1^\Lambda := K \mathcal{L}_C^\Lambda,\, ||\cdot||_1 := ||K^{-1} \cdot ||_{\mathcal{L}_C})$$

which is the Banach space. Moreover, it is not difficult to show that a generator of this semigroup coincides with $(L_\Lambda,\, K\, D(\widehat{L_\Lambda}))$.

Proposition 4.3 *For any $F \in \mathcal{L}_1^\Lambda \subset L^1(\Gamma,\, \pi_{\Lambda,\emptyset})$,*

$$U_t^\Lambda F = \widetilde{U_t^\Lambda} F, \quad t \geq 0 \quad in \quad L^1(\Gamma,\, \pi_{\Lambda,\emptyset}),$$

where $(\widetilde{U_t^\Lambda})_{t\geq 0}$ is defined in Remark 4.8.

Proof. The fact that $(L_\Lambda,\, K\, D(\widehat{L_\Lambda}))$ is a generator of $(U_t^\Lambda)_{t\geq 0}$ in $(\mathcal{L}_1^\Lambda,\, ||\cdot||_1)$ means the following (see e.g. [9])

$$\left|\left| U_t^\Lambda F - \left(1\!\!1 - \frac{t}{n} L_\Lambda \right)^{-n} F \right|\right|_1 \to 0, \quad n \to \infty, \quad \text{for all } \; F \in \mathcal{L}_1^\Lambda.$$

Because $||\cdot||_1 \geq ||\cdot||$, the latter fact implies

$$\left|\left| U_t^\Lambda F - \left(1\!\!1 - \frac{t}{n} L_\Lambda \right)^{-n} F \right|\right| \to 0, \quad n \to \infty, \quad \text{for all } \; F \in \mathcal{L}_1^\Lambda. \tag{28}$$

Analogously, the fact that $(\widetilde{L_\Lambda},\, D_1(\widetilde{L_\Lambda}))$ is a generator of $(\widetilde{U_t^\Lambda})_{t\geq 0}$ gives us

$$\left|\left| \widetilde{U_t^\Lambda} F - \left(1\!\!1 - \frac{t}{n} \widetilde{L_\Lambda} \right)^{-n} F \right|\right| \to 0, \quad n \to \infty, \quad \text{for all } \; F \in \mathcal{L}_1^\Lambda. \tag{29}$$

As was shown before, there exists $\epsilon > 0$ such that for any real $\zeta > \epsilon$ and any $F \in \mathcal{L}_1^\Lambda$

$$\left(\widetilde{L_\Lambda} - \zeta 1\!\!1\right)^{-1} F - (L_\Lambda - \zeta 1\!\!1)^{-1} F = \left(\widetilde{L_\Lambda} - \zeta 1\!\!1\right)^{-1} \left[L_\Lambda - \widetilde{L_\Lambda}\right] (L_\Lambda - \zeta 1\!\!1)^{-1} F.$$

The function $F_\zeta := (L_\Lambda - \zeta 1\!\!1)^{-1} F \in K\, D(\widehat{L_\Lambda})$. Hence, $\left[L_\Lambda - \widetilde{L_\Lambda}\right] F_\zeta = 0$. The latter fact means that

$$\left\| \widetilde{U_t^\Lambda} F - \left(1\!\!1 - \frac{t}{n}\widetilde{L_\Lambda}\right)^{-n} F \right\| =$$

$$= \left\| \widetilde{U_t^\Lambda} F - \left(1\!\!1 - \frac{t}{n} L_\Lambda\right)^{-n} F \right\| \to 0, \quad n \to \infty, \quad \text{for all } F \in \mathcal{L}_1^\Lambda. \tag{30}$$

The convergence (28) and (29) imply the assertion of the proposition. ■

Corollary 4.1 *Lemma 4.5 implies that for any moment of time* $t \geq 0$

$$U_t^\Lambda F \geq 0, \quad for\ \ all\ \ F \geq 0\ \ in\ \ L^1(\Gamma,\, \pi_{\Lambda,\emptyset}). \tag{31}$$

Let $t \geq 0$ and $G \in B_{\mathrm{bs}}(\Gamma_0)$ be arbitrary and fixed. Suppose that $N' \in \mathbb{N}$ and $\Lambda' \in \mathcal{B}_b(\mathbb{R}^d)$ are such that

$$G \star G \restriction_{\Gamma_0 \setminus \sqcup_{n=0}^{N'} \Gamma_{\Lambda'}^{(n)}} = 0.$$

Then, $K(G \star G) = |KG|^2 \in \mathcal{L}_1^\Lambda$ for all $\Lambda \in \mathcal{B}_b(\mathbb{R}^d)$, $\Lambda \supset \Lambda'$. Moreover, $P_\Lambda |KG|^2 = |KG|^2$.

Hence, the left-hand side of (27) for any $\Lambda \in \mathcal{B}_b(\mathbb{R}^d)$, $\Lambda \supset \Lambda'$ is equal to the following expression

$$\left\langle\!\left\langle \widehat{U_t^\Lambda}(C, \varkappa, \beta) P_\Lambda (G \star G),\, k \right\rangle\!\right\rangle = \int_\Gamma K \widehat{U_t^\Lambda}(G \star G)(\gamma) \mu^\varkappa(d\gamma) =$$

$$= \int_\Gamma U_t^\Lambda K(G \star G)(\gamma) \mu^\varkappa(d\gamma) = \int_{\Gamma_\Lambda} U_t^\Lambda |KG|^2(\gamma) \mu_\Lambda^\varkappa(d\gamma),$$

where $\mu_\Lambda^\varkappa$ is a projection of $\mu^\varkappa$ on Γ_Λ (see Remark 4.9). Let us mention that measure $\mu^\varkappa$ is locally absolutely continuous with respect to Poisson measure π. Therefore,

$$\left\langle\!\left\langle \widehat{U_t^\Lambda}(C, \varkappa, \beta) P_\Lambda (G \star G),\, k \right\rangle\!\right\rangle = \int_{\Gamma_\Lambda} U_t^\Lambda |KG|^2(\gamma) \frac{d\mu_\Lambda^\varkappa}{d\pi_\Lambda}(\gamma) \pi_\Lambda(d\eta).$$

Corollary 4.1 implies that there exist set $S \subset \Gamma$, $\pi_{\Lambda,\emptyset}(S) = 0$ such that for all $\gamma \in \Gamma \setminus S$:

$$U_t^\Lambda |KG|^2(\gamma) \geq 0.$$

But $\pi_{\Lambda,\emptyset}$ is absolutely continuous with respect to π_Λ. Furthermore, the corresponding Radon-Nikodym derivative is positive almost surely with respect to π_Λ. Hence, $\pi_\Lambda(S_\Lambda) = 0$, where S_Λ is a projection of the set S to Γ_Λ, and

$$\left\langle\!\left\langle \widehat{U_t^\Lambda}(C,\varkappa,\beta) P_\Lambda(G \star G),\, k \right\rangle\!\right\rangle = \int_{\Gamma_\Lambda \setminus S_\Lambda} U_t^\Lambda |KG|^2(\gamma) \frac{d\mu_\Lambda^\varkappa}{d\pi_\Lambda}(\gamma) \pi_\Lambda(d\eta) \geq 0.$$

The latter proof the assertion of Lemma 4.6. ■

The result obtained in Lemma 4.6 and fact about characterization of correlation functions in [11] imply the following corollary

Corollary 4.2 *Let positive constants C, $\varkappa$ and β which satisfy (20) be arbitrary and fixed. Let $k \in \mathcal{K}_C$ be such that $\langle\langle G \star G,\, k\rangle\rangle \geq 0$, for any $G \in B_{\mathrm{bs}}(\Gamma_0)$. Then for any $t \geq 0$ there exists unique measure $\mu_t \in \mathcal{M}_C$ whose correlation function is $\widehat{U_t^\star}(C,\varkappa,\beta)k$.*

In Corollary 4.2, the evolution of the measure μ in time we denote by $U_t^\star(C,\varkappa,\beta)\mu := \mu_t$. Accordingly to the general scheme $(U_t^\star(C,\varkappa,\beta))_{t\geq 0}$ is a semigroup on $\mathcal{M}_{C,\beta}$. This leads us directly to the construction of a non-equilibrium Markov process (or rather Markov function) on Γ.

Theorem 4.4 *Suppose that conditions* **(S)** *and* ***(SI)*** *are satisfied. For any triple of positive constants C, $\varkappa$ and β which satisfies (20) and any $\mu \in \mathcal{M}_C$ there exists Markov process $X_t^\mu \in \Gamma$ with initial distribution μ associated with generator $L_\varkappa$.*

Proof. The proof is a direct consequence of Theorem 3.1. ■

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