CORRELATION FUNCTIONS AND INVARIANT MEASURES IN CONTINUOUS CONTACT MODEL

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Abstract

We study the continuous version of the contact model. Using an analytic approach we construct the non-equilibrium contact process as a Markov process on configuration space. The construction is based on the analysis of correlation functions evolution. The problem concerning invariant measures as well as asymptotic of correlation functions are also studied.

Keywords: configuration space; contact model; non-equilibrium Markov process.

1 Introduction

Lattice contact models form an important class of interacting particle systems with reach mathematical properties and many essential applications, see, e.g., [14]. A continuous version of these models was introduced recently in [10]. In the latter paper the existence problem for corresponding spatial Markov process was analyzed in details. This process is a special case of the general birth-and-death processes in the continuum. Namely, we consider configurations, i.e., locally finite subsets $\gamma \subset \mathbb{R}^d$ as values of the process. During the stochastic evolution the points of a configuration create independently new ones distributed in the space accordingly to a dispersion probability density $0 \le a \in \mathrm{L}^1(\mathbb{R}^d)$ which is an even function. Any existing point has an independent exponentially distributed random life time. The contact process generator is given then on proper functions $F(\gamma)$ by the expression

$$(LF)(\gamma) := \sum_{x\in\gamma}[F(\gamma \setminus x) - F(\gamma)] + \varkappa \int_{\mathbb{R}^d} \sum_{y\in\gamma} a(x-y)[F(\gamma \cup x) - F(\gamma)]dx,$$

where $\varkappa > 0$ is a birth intensity parameter.

The main problem considered in the present paper concerns asymptotic properties of the contact process. First of all, we construct the time evolution of correlation functions for the contact process started with an initial distribution from a large class of initial states. Corresponding infinite system of evolution equations for correlation functions has a recurrent form and admits simple analysis. We note, that the intensity parameter has a critical value $\varkappa = 1$. For all other values of this parameter the density of the system tends to ∞ or 0 with the time and we cannot expect an appearing of a limiting invariant state.

For the critical value $\varkappa = 1$ and the dimension $d \ge 3$ we prove the existence of a continuous family of invariant measures parameterized by the density values. These invariant measures are described by a simple recurrent relation between their correlation functions and create a concrete class of random point fields which, up to our knowledge, never before was considered in the literature. A specific point of this class is an extremal growth w.r.t. the number of correlation functions. Actually, this growth is a maximal possible one such that the uniqueness of the corresponding measure is still valid. We show that, starting with an admissible initial state, the critical contact process converges to the equilibrium measure uniquely defined by the density of the initial state.

Let us note, that the contact models in the continuum may be used in the epidemiology to model an infection spreading process as well as in the spatial plant ecology where they describe independent growth of a population with a given mortality rate. In such ecological models the case $d = 2$ has a special concrete motivation. As we have mentioned above, invariant measures for our model for $d = 2$ do not exist and the root of this effect is very easy. Namely, in the two-dimensional case correlations between population members are growing in time too fast and the limiting correlation function of second order will diverge to the infinity. To avoid this divergence we may include an additional free Kawasaki dynamics for points of the configuration (see [5]). This dynamics includes an independent random walk in $\mathbb{R}^d$ for each population member. Then, assuming long tile jumps for the individual random walk process, we can assure the existence of invariant measures for such infinite particle stochastic dynamics. The resulting contact model with Kawasaki dynamics may be used naturally for the study of plankton stochastic dynamics, cf. [19]. Detailed analysis of the discussed model will be given in our forthcoming paper [7].

2 Preliminaries

We consider Euclidian space $\mathbb{R}^d$. By $\mathcal{B}(\mathbb{R}^d)$ we denote the family of all Borel sets in $\mathbb{R}^d$. $\mathcal{B}_b(\mathbb{R}^d)$ denotes the system of all sets in $\mathcal{B}(\mathbb{R}^d)$ which are bounded.

The space of *n-point configuration* is

$$\Gamma_0^{(n)} = \Gamma_{0,\mathbb{R}^d}^{(n)} := \left\{ \eta \subset \mathbb{R}^d \,\middle|\, |\eta| = n \right\}, \quad n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\},$$

where $|A|$ denotes the cardinality of the set A.

The space $\Gamma_\Lambda^{(n)} = \Gamma_{0,\Lambda}^{(n)}$ for $\Lambda \in \mathcal{B}_b(\mathbb{R}^d)$ is defined analogously to the space $\Gamma_0^{(n)}$. As a set $\Gamma_0^{(n)}$ is equivalent to the symmetrization of

$$\widetilde{(\mathbb{R}^d)^n} = \left\{ (x_1, \ldots, x_n) \in (\mathbb{R}^d)^n \,|\, x_k \neq x_l \text{ if } k \neq l \right\},$$

i.e. to the $\widetilde{(\mathbb{R}^d)^n}/S_n$, where S_n is the permutation group of $\{1, \ldots, n\}$. Hence, one can introduce the corresponding topology and Borel σ-algebra, which we denote by $\mathcal{O}(\Gamma_0^{(n)})$ and $\mathcal{B}(\Gamma_0^{(n)})$, respectively.

The space of *finite configurations*

$$\Gamma_0 := \bigsqcup_{n \in \mathbb{N}_0} \Gamma_0^{(n)}$$

is equipped with the topology $\mathcal{O}(\Gamma_0)$ of disjoint union. Let $\mathcal{B}(\Gamma_0)$ denotes the corresponding Borel σ-algebra.

A *set* $B \in \mathcal{B}(\Gamma_0)$ is called *bounded* if there exists $\Lambda \in \mathcal{B}_b(\mathbb{R}^d)$ and $N \in \mathbb{N}$ such that $B \subset \bigsqcup_{n=0}^N \Gamma_\Lambda^{(n)}$.

We would like to emphasize that due to the structure of Γ_0, any function on Γ_0 can be interpreted as a system of symmetrical functions on each component $\Gamma_0^{(n)}$ of Γ_0.

The *configuration space*

$$\Gamma := \left\{ \gamma \subset \mathbb{R}^d \middle| \, |\gamma \cap \Lambda| < \infty, \text{ for all } \Lambda \in \mathcal{B}_b(\mathbb{R}^d) \right\}$$

is equipped with the *vague* topology $\mathcal{O}(\Gamma)$. It is Polish space (see e.g. [6]). $\mathcal{B}(\Gamma)$ denotes the corresponding Borel σ-algebra. The *filtration* on Γ with a base set $\Lambda \in \mathcal{B}_b(\mathbb{R}^d)$ is given by

$$\mathcal{B}_\Lambda(\Gamma) := \sigma \left(N_{\Lambda'} \middle| \Lambda' \in \mathcal{B}_b(\mathbb{R}^d), \ \Lambda' \subset \Lambda \right),$$

where $N_\Lambda : \Gamma_0 \to \mathbb{N}_0$ is such that $N_\Lambda(\eta) := |\eta \cap \Lambda|$. For short we write $\eta_\Lambda := \eta \cap \Lambda$.

For every $\Lambda \in \mathcal{B}_b(\mathbb{R}^d)$ the *projection* $p_\Lambda : \Gamma \to \Gamma_\Lambda := \bigsqcup_{n \geq 0} \Gamma_\Lambda^{(n)}$ is defined as

$$p_\Lambda(\gamma) := \gamma_\Lambda$$

One can show that Γ is the projective limit of the spaces $\{\Gamma_\Lambda\}_{\Lambda \in \mathcal{B}_b(\mathbb{R}^d)}$ w.r.t. this projections.

In the sequel we will use the following classes of function on Γ_0:

- $L^0(\Gamma_0)$ - the set of all *measurable functions* on Γ_0;
- $L^0_{\text{ls}}(\Gamma_0)$ - the set of measurable *functions* with *local support*, i.e. $G \in L^0_{\text{ls}}(\Gamma_0)$ if there exists $\Lambda \in \mathcal{B}_b(\mathbb{R}^d)$ such that $G \restriction_{\Gamma_0 \setminus \Gamma_\Lambda} = 0$;
- $L^0_{\text{bs}}(\Gamma_0)$ - the set of measurable *functions* with *bounded support*, i.e. $G \in L^0_{\text{bs}}(\Gamma_0)$ if there exists $\Lambda \in \mathcal{B}_b(\mathbb{R}^d)$ and $N \in \mathbb{N}$ such that $G \restriction_{\Gamma_0 \setminus \bigsqcup_{n=0}^N \Gamma_\Lambda^{(n)}} = 0$;
- $B(\Gamma_0)$ - the set of *bounded* measurable *functions*
- $B_{\text{bs}}(\Gamma_0)$ - the set of *bounded functions* with *bounded support*;

On Γ we consider the set of a *cylinder functions* $\mathcal{F}L^0(\Gamma)$, i.e. the set of all measurable function $G \in L^0(\Gamma)$ which are measurable w.r.t. $\mathcal{B}_\Lambda(\Gamma)$ for some $\Lambda \in \mathcal{B}_b(\mathbb{R}^d)$. These functions are characterized by the following relation:

$$F(\gamma) = F \restriction_{\Gamma_\Lambda} (\gamma_\Lambda).$$

Those cylinder functions which are measurable w.r.t. $\mathcal{B}_\Lambda(\Gamma)$ for fixed $\Lambda \in \mathcal{B}_b(\mathbb{R}^d)$ we will denote by $\mathcal{F}L^0(\Gamma, \ \mathcal{B}_\Lambda(\Gamma))$.

Next we would like to describe some facts from harmonic analysis on configuration space based on [4].

The following mapping between functions on Γ_0, and functions on Γ, plays a key role in our further considerations:

$$KG(\gamma) := \sum_{\xi \Subset \gamma} G(\xi), \quad G \in L^0_{\text{ls}}(\Gamma_0) \quad \gamma \in \Gamma,$$

see e.g. [12, 13]. The summation in the latter expression is taken over all finite subconfigurations of γ, which is denoted by symbol $\xi \Subset \gamma$.

K-transform is linear, positivity preserving, and invertible, with

$$K^{-1}F(\eta) := \sum_{\xi \subset \eta} (-1)^{|\eta \setminus \xi|} F(\xi), \quad F \in \mathcal{F}L^0(\Gamma) \quad \eta \in \Gamma_0. \tag{1}$$

It is easy to see that for any $\Lambda \in \mathcal{B}_b(\mathbb{R}^d)$ and arbitrary $F \in \mathcal{F}L^0(\Gamma, \ \mathcal{B}_\Lambda(\Gamma))$

$$K^{-1}F(\eta) = 1\!\!1_{\Gamma_\Lambda}(\eta) K^{-1}F(\eta), \quad \forall \eta \in \Gamma_0. \tag{2}$$

The map K, as well as map K^{-1}, can be extended to more wide classes of functions. For details and further properties of map K see, e.g. [4].

One can introduce a *convolution*

$$\begin{array}{rcl} \star : L^0(\Gamma_0) \times L^0(\Gamma_0) & \to & L^0(\Gamma_0) \\ (G_1, G_2) & \mapsto & (G_1 \star G_2)(\eta) \end{array} \tag{3}$$

$$:= \sum_{(\xi_1,\xi_2,\xi_3) \in \mathcal{P}^3_\emptyset(\eta)} G_1(\xi_1 \cup \xi_2) \ G_2(\xi_2 \cup \xi_3),$$

where $\mathcal{P}^3_\emptyset(\eta)$ denotes the set of all partitions (ξ_1, ξ_2, ξ_3) of η in 3 parts, i.e., all triples (ξ_1, ξ_2, ξ_3) with $\xi_i \subset \eta$, $\xi_i \cap \xi_j = \emptyset$ if $i \neq j$, and $\xi_1 \cup \xi_2 \cup \xi_3 = \eta$.

It has the property that for $G_1, G_2 \in L^0_{\mathrm{ls}}(\Gamma_0)$

$$K(G_1 \star G_2) = KG_1 \cdot KG_2.$$

Due to this convolution we can interpret K-transform as Fourier transform in configuration space analysis, see also [1].

Let $\mathcal{M}^1_{\mathrm{fm}}(\Gamma)$ be the set of all probability measures μ which have *finite local moments* of all orders, i.e.

$$\int_\Gamma |\gamma_\Lambda|^n \mu(d\gamma) < +\infty$$

for all $\Lambda \in \mathcal{B}_b(\mathbb{R}^d)$ and $n \in \mathbb{N}_0$.

A measure ρ on Γ_0 is called *locally finite* if $\rho(A) < \infty$ for all bounded sets A from $\mathcal{B}(\Gamma_0)$. The set of such measures is denoted by $\mathcal{M}_{\mathrm{lf}}(\Gamma_0)$.

A measure $\rho \in \mathcal{M}_{\mathrm{lf}}(\Gamma_0)$ is called *positive definite* if

$$\int_{\Gamma_0} (G \star \overline{G})(\eta)\rho(d\eta) \geq 0, \ \ \forall G \in B_{\mathrm{bs}}(\Gamma_0),$$

where $\overline{G}$ is a complex conjugate of G.

A measure ρ is called *normalized* iff $\rho(\{\emptyset\}) = 1$.

One can define a transform $K^* : \mathcal{M}^1_{\mathrm{fm}}(\Gamma) \to \mathcal{M}_{\mathrm{lf}}(\Gamma_0)$, which is *dual* to the K-transform, i.e., for every $\mu \in \mathcal{M}^1_{\mathrm{fm}}(\Gamma)$, $G \in \mathcal{B}_{\mathrm{bs}}(\Gamma_0)$ we have

$$\int_\Gamma KG(\gamma)\mu(d\gamma) = \int_{\Gamma_0} G(\eta)\,(K^*\mu)(d\eta).$$

The measure $\rho_\mu := K^*\mu$ is called the *correlation measure* of μ. As shown in [4] for $\mu \in \mathcal{M}^1_{\mathrm{fm}}(\Gamma)$ and any $G \in L^1(\Gamma_0, \rho_\mu)$ the series

$$KG(\gamma) := \sum_{\eta \Subset \gamma} G(\eta), \tag{4}$$

is μ-a.s. absolutely convergent. Furthermore, $KG \in L^1(\Gamma, \mu)$ and

$$\int_{\Gamma_0} G(\eta)\,\rho_\mu(d\eta) = \int_\Gamma (KG)(\gamma)\,\mu(d\gamma). \tag{5}$$

Fix a non-atomic and locally finite measure σ on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$. For any $n \in \mathbb{N}$ the product measure $\sigma^{\otimes n}$ can be considered by restriction as a measure on $\widetilde{(\mathbb{R}^d)^n}$ and hence on $\Gamma_0^{(n)}$. The measure on $\Gamma_0^{(n)}$ we denote by $\sigma^{(n)}$.

The *Lebesgue-Poisson measure* $\lambda_{z\sigma}$ on Γ_0 is defined as

$$\lambda_{z\sigma} := \sum_{n=0}^{\infty} \frac{z^n}{n!} \sigma^{(n)}.$$

Here $z > 0$ is the so-called activity parameter. The restriction of $\lambda_{z\sigma}$ to Γ_Λ will be also denoted by $\lambda_{z\sigma}$. We write λ_z instead of $\lambda_{z\sigma}$, if measure σ is considered to be fixed.

The *Poisson measure* $\pi_{z\sigma}$ on $(\Gamma, \mathcal{B}(\Gamma))$ is given as the projective limit of the family of measures $\{\pi^\Lambda_{z\sigma}\}_{\Lambda \in \mathcal{B}_b(\mathbb{R}^d)}$, where $\pi^\Lambda_{z\sigma}$ is the measure on Γ_Λ defined by $\pi^\Lambda_{z\sigma} := e^{-z\sigma(\Lambda)}\lambda_{z\sigma}$.

A measure $\mu \in \mathcal{M}^1_{\mathrm{fm}}(\Gamma)$ is called *locally absolutely continuous w.r.t.* $\pi_{z\sigma}$ iff $\mu_\Lambda := \mu \circ p_\Lambda^{-1}$ is absolutely continuous with respect to $\pi^\Lambda_{z\sigma} = \pi_{z\sigma} \circ p_\Lambda^{-1}$ for all $\Lambda \in \mathcal{B}_b(\mathbb{R}^d)$. In this case, $\rho_\mu := K^*\mu$ is absolutely continuous w.r.t $\lambda_{z\sigma}$. Let $k_\mu : \Gamma_0 \to \mathbb{R}_+$ be the corresponding Radon-Nikodym derivative, i.e.

$$k_\mu(\eta) := \frac{d\rho_\mu}{d\lambda_{z\sigma}}(\eta), \quad \eta \in \Gamma_0.$$

Remark 2.1 *The functions*

$$k_\mu^{(n)} : (\mathbb{R}^d)^n \longrightarrow \mathbb{R}_+ \tag{6}$$

$$k_\mu^{(n)}(x_1, \ldots, x_n) := \begin{cases} k_\mu(\{x_1, \ldots, x_n\}), & \text{if } (x_1, \ldots, x_n) \in \widetilde{(\mathbb{R}^d)^n} \\ 0, & \text{otherwise} \end{cases}$$

are the well known correlation functions in statistical physics, see e.g [17], [18].

For the technical purposes we also recall the following result:

Lemma 2.1 *Let $n \in \mathbb{N}$, $n \geq 2$, and $z > 0$ be given. Then*

$$\int_{\Gamma_0} \dots \int_{\Gamma_0} G(\eta_1 \cup \dots \cup \eta_n) H(\eta_1, \dots, \eta_n) d\lambda_{z\sigma}(\eta_1) \dots d\lambda_{z\sigma}(\eta_n) =$$

$$= \int_{\Gamma_0} G(\eta) \sum_{(\eta_1,\dots,\eta_n) \in \mathcal{P}_n(\eta)} H(\eta_1, \dots, \eta_n) d\lambda_{z\sigma}(\eta)$$

for all measurable functions $G : \Gamma_0 \mapsto \mathbb{R}$ and $H : \Gamma_0 \times \dots \times \Gamma_0 \mapsto \mathbb{R}$ with respect to which both sides of the equality make sense. Here $\mathcal{P}_n(\eta)$ denotes the set of all ordered partitions of η in n parts, which may be empty.

This lemma is known in the literature as *Minlos lemma* (cf., [9], [15]) and it will be crucial for calculations in many places in the next sections.

3 Generators. The symbol of the Glauber generator on the space of finite configurations

Let the activity parameter z be equal to 1 and let $0 \leq a \in \mathrm{L}^1(\mathbb{R}^d)$ be an arbitrary even function such that

$$\int_{\mathbb{R}^d} a(x) dx = 1.$$

We consider a Markov pre-generator which corresponds to the contact model on the configuration space Γ, the action of which is given by

$$(LF)(\gamma) := \sum_{x \in \gamma} D_x^- F(\gamma) + \varkappa \int_{\mathbb{R}^d} \sum_{y \in \gamma} a(x-y) D_x^+ F(\gamma) dx, \quad F \in \mathcal{F}L^0(\Gamma),$$

where $D_x^- F(\gamma) = F(\gamma \setminus x) - F(\gamma)$, $D_x^+ F(\gamma) = F(\gamma \cup x) - F(\gamma)$ and $\varkappa > 0$.

Proposition 3.1 *The image of L under the K-transform (or symbol of the operator L) on functions $G \in B_{bs}(\Gamma_0)$ has the following form*

$$(\widehat{L}G)(\eta) := (K^{-1} L K G)(\eta) = -|\eta| G(\eta) +$$

$$+ \varkappa \int_{\mathbb{R}^d} \sum_{y \in \eta} a(x-y) G((\eta \setminus y) \cup x) dx + \varkappa \int_{\mathbb{R}^d} \sum_{y \in \eta} a(x-y) G(\eta \cup x) dx$$

Proof. According to the definition of the operator $\widehat{L}$ we have

$$(\widehat{L}G)(\eta) = I_1(\eta) + I_2(\eta),$$

where

$$I_1(\eta) := K^{-1}\left(\sum_{x\in\cdot}[KG(\cdot\setminus x) - KG(\cdot)]\right)(\eta) =$$

$$= K^{-1}\left(\sum_{x\in\cdot}\left[\sum_{\xi\subset(\cdot\setminus x)}G(\xi) - \sum_{\xi\subset\cdot}G(\xi)\right]\right)(\eta) =$$

$$= K^{-1}\left(\sum_{x\in\cdot}\left[-\sum_{\xi\subset(\cdot\setminus x)}G(\xi\cup x)\right]\right)(\eta) =$$

$$= -\sum_{\zeta\subset\eta}(-1)^{|\eta\setminus\zeta|}\sum_{x\in\zeta}\sum_{\xi\subset\zeta\setminus x}G(\xi\cup x) = -\sum_{\zeta\subset\eta}(-1)^{|\eta\setminus\zeta|}\sum_{x\in\zeta}K\left(G(\cdot\cup x)\right)(\zeta\setminus x).$$

Changing summation in the last expression we get

$$I_1(\eta) = -\sum_{x\in\eta}\sum_{\zeta\in\eta\setminus x}(-1)^{|\eta\setminus(\zeta\cup x)|}K\left(G(\cdot\cup x)\right)(\zeta) =$$

$$= -\sum_{x\in\eta}K^{-1}\left(KG(\cdot\cup x)\right)(\eta\setminus x) = -\sum_{x\in\eta}G(\eta) = -|\eta|G(\eta);$$

Now we compute the second term of $\widehat{L}$

$$I_2(\eta) := K^{-1}\left(\varkappa\int_{\mathbb{R}^d}\sum_{y\in\cdot}a(x-y)\left[KG(\cdot\cup x) - KG(\cdot)\right]dx\right)(\eta) =$$

$$= \varkappa\sum_{\zeta\subset\eta}(-1)^{|\eta\setminus\xi|}\int_{\mathbb{R}^d}\sum_{y\in\zeta}a(x-y)\sum_{\rho\subset\zeta}G(\rho\cup x)dx.$$

Using the fact that

$$\sum_{y\in\zeta}a(x-y) = K\left(a(x-\cdot)1\!\!1_{\{|\cdot|=1\}}(\cdot)\right)(\zeta)$$

we obtain

$$I_2(\eta) = \varkappa\int_{\mathbb{R}^d}K^{-1}\left(K\left(\left(a(x-\cdot)1\!\!1_{\{|\cdot|=1\}}(\cdot)\right)\star G(\cdot\cup x)\right)\right)(\eta)dx =$$

$$= \varkappa\int_{\mathbb{R}^d}\left(\left(a(x-\cdot)1\!\!1_{\{|\cdot|=1\}}(\cdot)\right)\star G(\cdot\cup x)\right)(\eta)dx.$$

By the definition of the convolution, the latter expression can be written as follows

$$\varkappa \int_{\mathbb{R}^d} \sum_{(\xi_1,\xi_2,\xi_3)\in\mathcal{P}^3_\emptyset(\eta)} \big(a(x-\cdot)1\!\!1_{\{|\cdot|=1\}}(\cdot)\big)(\xi_1\cup\xi_2)G(\xi_2\cup\xi_3\cup x)dx.$$

Now, we note that there are only two cases when summands in the last expression are not equal to zero. These cases are $|\xi_1| = 1$, $\xi_2 = \emptyset$ and $\xi_1 = \emptyset$, $|\xi_2| = 1$. Therefore,

$$I_2(\eta) = \varkappa \int_{\mathbb{R}^d} \sum_{y\in\eta} a(x-y)G((\eta\backslash y)\cup x)dx + \varkappa \int_{\mathbb{R}^d} \sum_{y\in\eta} a(x-y)G((\eta\backslash y)\cup y\cup x)dx.$$

The latter fact proves the assertion of the lemma. ■

4 Construction of the contact process associated with the generator L

4.1 The adjoint operator to the symbol of L

Let measure $\rho \in \mathcal{M}_{\mathrm{lf}}(\Gamma_0)$ be absolutely continuous with respect to the Lebesgue-Poisson measure λ. By $k(\eta)$, $\eta \in \Gamma_0$ we denote the corresponding density.

Proposition 4.1 *Assume that*

$$k(\eta) \le C^{|\eta|}, \quad \eta \in \Gamma_0 \tag{7}$$

for some $C > 0$. Then, $\widehat{L}(B_{bs}(\Gamma_0))$ is a subset of $\mathrm{L}^1(\Gamma_0, \rho)$.

Proof. Let $G \in B_{bs}(\Gamma_0)$ be arbitrary and fixed. A direct application of Lemma 2.1 to the calculation of L^1-norm of $\widehat{L}G$ with respect to the measure ρ gives us the necessary result. ■

The adjoint operator $\widehat{L}^*$ to the operator $\widehat{L}$ on the space of correlation functions is defined via the duality given by the scalar product in $\mathrm{L}^2(\Gamma_0, \lambda)$

$$\int_{\Gamma_0} \widehat{L}G(\eta)\rho(d\eta) = \langle \widehat{L}G, k\rangle_{\mathrm{L}^2(\Gamma_0,\lambda)} = \langle G, \widehat{L}^*k\rangle_{\mathrm{L}^2(\Gamma_0,\lambda)}.$$

In the next proposition we give an explicit form of the adjoint operator of $\widehat{L}$.

Proposition 4.2 *The adjoint operator $\widehat{L}^*$ of $\widehat{L}$ on the space of functions which satisfy (7) has the following form:*

$$(\widehat{L}^* k)(\eta) = -|\eta| k(\eta) + \varkappa \sum_{x\in\eta} k(\eta\backslash x) \sum_{y\in\eta\backslash x} a(x-y) + \varkappa \sum_{x\in\eta} \int_{\mathbb{R}^d} a(x-y) k((\eta\backslash x)\cup y) dy.$$

Proof. Using the same notation as in Proposition 3.1 we get

$$\int_{\Gamma_0} I_1(\eta) k(\eta) \lambda(d\eta) = -\int_{\Gamma_0} |\eta| G(\eta) k(\eta) \lambda(d\eta) = \int_{\Gamma_0} G(\eta) \left[-|\eta| k(\eta)\right] \lambda(d\eta).$$

For the second part of $\widehat{L}$ we have

$$\int_{\Gamma_0} I_2(\eta) k(\eta) \lambda(d\eta) = J_1 + J_2,$$

where

$$J_1 := \varkappa \int_{\Gamma_0} \int_{\mathbb{R}^d} \sum_{y\in\eta} a(x-y) G((\eta\setminus y)\cup x) dx k(\eta) \lambda(d\eta)$$

and

$$J_2 := \varkappa \int_{\Gamma_0} \int_{\mathbb{R}^d} G(\eta\cup x) \sum_{y\in\eta} a(x-y) k(\eta) dx \lambda(d\eta).$$

Using Lemma 2.1 we obtain

$$J_1 = \varkappa \int_{\Gamma_0} \int_{\mathbb{R}^d} k(\eta\cup y) \left[\int_{\mathbb{R}^d} a(x-y) G(\eta\cup x) dx\right] dy \lambda(d\eta) =$$

$$= \varkappa \int_{\Gamma_0} \int_{\mathbb{R}^d} G(\eta\cup x) \left[\int_{\mathbb{R}^d} a(x-y) k(\eta\cup y) dy\right] dx \lambda(d\eta).$$

Using Lemma 2.1 again for the last expression and for the integral J_2, finally we get

$$J_1 = \varkappa \int_{\Gamma_0} G(\eta) \left[\sum_{x\in\eta} \int_{\mathbb{R}^d} a(x-y) k((\eta\setminus x)\cup y) dy\right] \lambda(d\eta).$$

$$J_2 = \varkappa \int_{\Gamma_0} G(\eta) \left[\sum_{x\in\eta} k(\eta\setminus x) \sum_{y\in\eta\backslash x} a(x-y)\right] \lambda(d\eta).$$

This concludes the proof of the proposition. ■

4.2 Time evolution of correlation functions

In this subsection we investigate the evolutional equation associated with the operator $\widehat{L}^*$. It has the following form

$$\frac{\partial k_t}{\partial t}(\eta) = \widehat{L}^* k_t(\eta) = -|\eta| k_t(\eta) + \varkappa \sum_{x\in\eta} k_t(\eta\setminus x) \sum_{y\in\eta\setminus x} a(x-y) +$$

$$+\varkappa \sum_{x\in\eta} \int_{\mathbb{R}^d} a(x-y) k_t((\eta\setminus x)\cup y) dy.$$

Having in mind relation between functions on the space of finite configurations and collection of symmetrical functions on each component $\Gamma_0^{(n)}$, $n \geq 0$, we rewrite this equation as a system of equations.

$$\frac{\partial k_t^{(n)}}{\partial t}(x_1,\ldots,x_n) = -n\, k_t^{(n)}(x_1,\ldots,x_n) +$$

$$+\varkappa \sum_{i=1}^{n} k_t^{(n-1)}(x_1,\ldots,\check{x}_i,\ldots,x_n) \sum_{j:\, j\neq i} a(x_i - x_j) +$$

$$+\varkappa \sum_{i=1}^{n} \int_{\mathbb{R}^d} a(x_i - y) k_t^{(n)}(x_1,\ldots,x_{i-1},y,x_{i+1},\ldots,x_n) dy =$$

$$= \widehat{L}_n^* k_t^{(n)}(x_1,\ldots,x_n) + f_t^{(n)}(x_1,\ldots,x_n), \quad n \geq 1,$$

where

$$\widehat{L}_n^* k_t^{(n)}(x_1,\ldots,x_n) := -n\, k_t^{(n)}(x_1,\ldots,x_n) +$$

$$+\varkappa \sum_{i=1}^{n} \int_{\mathbb{R}^d} a(x_i - y) k_t^{(n)}(x_1,\ldots,x_{i-1},y,x_{i+1},\ldots,x_n) dy, \quad n \geq 1$$

and

$$f_t^{(n)}(x_1,\ldots,x_n) := \varkappa \sum_{i=1}^{n} k_t^{(n-1)}(x_1,\ldots,\check{x}_i,\ldots,x_n) \sum_{j:\, j\neq i} a(x_i - x_j), \quad n \geq 2,$$

$$f_t^{(1)} \equiv 0.$$

Let $n \in \mathbb{N}$ be arbitrary and fixed. We consider the linear Cauchy problem

$$\frac{\partial k_t^{(n)}}{\partial t}(x_1,\ldots,x_n) = \widehat{L}_n^* k_t^{(n)}(x_1,\ldots,x_n) + f_t^{(n)}(x_1,\ldots,x_n), \;\; t \geq 0, \qquad (8)$$

$$k_t^{(n)}(x_1,\ldots,x_n)\Big|_{t=0} := k_0^{(n)}(x_1,\ldots,x_n),$$

in a Banach space X_n.

In our model we will consider Banach space X_n as $\mathrm{L}^\infty((\mathbb{R}^d)^n, \sigma^{\otimes n})$, where σ is a Lebesgue measure on $\mathbb{R}^d$ and $\sigma^{\otimes n}$ is a product measure on $(\mathbb{R}^d)^n$.

Remark 4.1 *The operator $\widehat{L}_n^*$ in X_n can be written also in another way*

$$\widehat{L}_n^* k^{(n)}(x_1, \ldots, x_n) = n(\varkappa - 1)\, k^{(n)}(x_1, \ldots, x_n) + \sum_{i=1}^{n} L_a^i k^{(n)}(x_1, \ldots, x_n),$$

where for each $1 \le i \le n$,

$$L_a^i k^{(n)}(x_1, \ldots, x_n) =$$

$$= \varkappa \int_{\mathbb{R}^d} a(x_i - y) \left[k^{(n)}(x_1, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_n) - k^{(n)}(x_1, \ldots, x_n) \right] dy$$

is a generator of a Markov process on $(\mathbb{R}^d)^n$ (see [2]), which describes the jump of the particle placed at the point $(x_1, \ldots, x_i, \ldots, x_n) \in (\mathbb{R}^d)^n$ into the point $(x_1, \ldots, y, \ldots, x_n) \in (\mathbb{R}^d)^n$ with intensity equal to $a(x_i - y)$.

Lemma 4.1 *Let $a \in \mathrm{L}^1(\mathbb{R}^d)$. Then, for any $n \ge 1$ the operator $\widehat{L}_n^*$ is bounded linear operator in X_n as well as in $\mathrm{L}^1((\mathbb{R}^d)^n)$. Moreover, for each $1 \le i \le n$, the operator L_a^i is a generator of a contraction semigroup on X_n and $\mathrm{L}^1((\mathbb{R}^d)^n)$.*

Proof. The first part of this theorem is trivial. The second one in the case of the space X_n follows directly from the Remark 4.1 and in the case of $\mathrm{L}^1((\mathbb{R}^d)^n)$ it is a consequence of Beurling-Deny criterion, see e.g. [16]. ■

This Lemma in its turn implies the following result (see e.g. [3]).

Proposition 4.3 *Let $n \ge 1$ be arbitrary and fixed. The solution to the Cauchy problem (8) in the Banach space X_n is given by*

$$k_t^{(n)}(x_1, \ldots, x_n) = e^{n(\varkappa-1)t} \left[\bigotimes_{i=1}^{n} e^{tL_a^i} \right] k_0^{(n)}(x_1, \ldots, x_n) + \varkappa e^{n(\varkappa-1)t} \times \quad (9)$$

$$\times \int_0^t e^{-n(\varkappa-1)s} \left[\bigotimes_{i=1}^{n} e^{(t-s)L_a^i} \right] \sum_{i=1}^{n} k_s^{(n-1)}(x_1, \ldots, \check{x}_i, \ldots, x_n) \sum_{j:\, j \ne i} a(x_i - x_j) ds.$$

Next proposition establish a priori estimates for the evolution of correlation functions

Proposition 4.4 *Let $a \in \mathrm{L}^1(\mathbb{R}^d) \cap \mathrm{L}^\infty(\mathbb{R}^d)$ be an arbitrary non-negative function. Suppose that there exists a constant $C > 0$ (independent of n) such that for any $(x_1, \ldots, x_n) \in \mathbb{R}^d$*

$$k_0^{(n)}(x_1, \ldots, x_n) \leq n!\, C^n, \quad \textit{for all} \;\; n \geq 0.$$

Then, for any $t \geq 0$ and a.a. $(x_1, \ldots, x_n) \in \mathbb{R}^d$ w.r.t. Lebesgue measure

$$k_t^{(n)}(x_1, \ldots, x_n) \leq \varkappa(t)^n (1 + A)^n e^{n(\varkappa - 1)t} (C + t)^n n!, \tag{10}$$

where

$$A = ||a||_{\mathrm{L}^\infty(\mathbb{R}^d)} \quad \textit{and} \quad \varkappa(t) := \max\left[1,\, \varkappa,\, \varkappa e^{-(\varkappa-1)t}\right]$$

holds for all $n \geq 0$.

Proof. The proof uses mathematical induction with respect to n. The first induction step (the fulfilment of (10) in the case of $n = 1$) follows from Proposition 4.3. Now assume that for any $t \geq 0$ bound (10) holds for $n - 1$. Using formula (9) and Remark 4.1 we get

$$k_t^{(n)}(x_1, \ldots, x_n) \leq e^{n(\varkappa-1)t} C^n n! +$$

$$+\varkappa(n-1)n!(1+A)^n e^{n(\varkappa-1)t} \int_0^t e^{-n(\varkappa-1)s} \varkappa(s)^{n-1} e^{(n-1)(\varkappa-1)s} (C+s)^{n-1} ds \leq$$

$$\leq e^{n(\varkappa-1)t} C^n n! + \varkappa(n-1)n!(1+A)^n \varkappa(t)^{n-1} e^{n(\varkappa-1)t} \int_0^t e^{-(\varkappa-1)s} (C+s)^{n-1} ds.$$

Using estimate

$$e^{-(\varkappa-1)s} \leq \max\{1,\, e^{-(\varkappa-1)t}\}, \quad \text{for all} \;\; s \in [0,\, t],\; \varkappa > 0$$

we obtain

$$k_t^{(n)}(x_1, \ldots, x_n) \leq \varkappa(t)^n (1 + A)^n e^{n(\varkappa-1)t} (C + t)^n n!,$$

that concludes the proof of this Proposition. ■

Corollary 4.1 *Let $0 \leq a \in \mathrm{L}^1(\mathbb{R}^d) \cap C(\mathbb{R}^d)$ be an arbitrary even function such that*

$$\int_{\mathbb{R}^d} a(x) dx = 1 \quad \text{and} \quad a(x) \to 0, \quad |x| \to \infty$$

and let $k_{t,\,a}^{(n)}$ be a solution to the Cauchy problem (8) in X_n. Suppose, that conditions of the Proposition 4.4 are fulfilled, then there exists a sequence $\{a_l\}_{l \geq 1} \subset C_0(\mathbb{R}^d)$ such that

$$k_{t,\,a_l}^{(n)} \to k_{t,\,a}^{(n)}, \;\; l \to \infty \;\; \text{in} \;\; \mathrm{X}_n.$$

Proof. There exists a sequence $\{a_l\}_{l\geq 1} \subset C_0(\mathbb{R}^d)$ such that

$$a_l \to a, \;\; l \to \infty \;\text{ in }\; \mathrm{X}_n. \tag{11}$$

The rest proof of the corollary we will perform using mathematical induction method. For $n = 1$ the statement is trivial. Now, using induction step $(n-1) \to n$ we estimate the L^∞-norm of the difference

$$k^{(n)}_{t,\,a_l}(x_1, \ldots, x_n) - k^{(n)}_{t,\,a}(x_1, \ldots, x_n) = \tag{12}$$

$$= e^{n(\varkappa-1)t}\left(\left[\bigotimes_{i=1}^n e^{tL^i_{a_l}}\right] k^{(n)}_0(x_1, \ldots, x_n) - \left[\bigotimes_{i=1}^n e^{tL^i_{a}}\right] k^{(n)}_0(x_1, \ldots, x_n)\right) +$$

$$+\varkappa e^{n(\varkappa-1)t} \times$$

$$\int_0^t e^{n(1-\varkappa)s}\left(\left[\bigotimes_{i=1}^n e^{(t-s)L^i_{a_l}}\right] \sum_{i=1}^n k^{(n-1)}_{s,\,a_l}(x_1, \ldots, \check{x}_i, \ldots, x_n) \sum_{j:\,j\neq i} a_l(x_i - x_j) - \right.$$

$$\left. - \left[\bigotimes_{i=1}^n e^{(t-s)L^i_{a}}\right] \sum_{i=1}^n k^{(n-1)}_{s,\,a}(x_1, \ldots, \check{x}_i, \ldots, x_n) \sum_{j:\,j\neq i} a(x_i - x_j)\right) ds.$$

Due to Proposition 4.3 the first summand of (12) converges to 0 in X_n. Indeed, strong convergence of semigroups in L^∞ space with a bounded generators follows from the corresponding convergence of generators. But the latter fact is trivial because of convergence (11).

In order to check the convergence of the second summand of (12) to 0 in X_n, let us note that

$$\sum_{i=1}^n k^{(n-1)}_{s,\,a_l}(x_1, \ldots, \check{x}_i, \ldots, x_n) \sum_{j:\,j\neq i} a_l(x_i - x_j)$$

converges to

$$\sum_{i=1}^n k^{(n-1)}_{s,\,a}(x_1, \ldots, \check{x}_i, \ldots, x_n) \sum_{j:\,j\neq i} a(x_i - x_j),$$

in X_n as $l \to \infty$.

Due to Proposition 4.4

$$\left\|\left[\bigotimes_{i=1}^n e^{(t-s)L^i_{a_l}}\right] \sum_{i=1}^n k^{(n-1)}_{s,\,a_l}(x_1, \ldots, \check{x}_i, \ldots, x_n) \sum_{j:\,j\neq i} a_l(x_i - x_j)\right\|_{\mathrm{X}_n}$$

is uniformly bounded in s. The latter two facts imply the convergence of the second summand of (12) to 0 in X_n. ■

Next we solve the following problem: suppose that $(k_0^{(n)})_{n\geq 0}$ is the system of correlation functions which means, that there exists a probability measure $\mu_0 \in \mathcal{M}^1_{\mathrm{fm}}(\Gamma)$, locally absolutely continuous with respect to Poisson measure, whose correlation functions are exactly $(k_0^{(n)})_{n\geq 0}$. We would like to investigate now whether the evolution of $(k_0^{(n)})_{n\geq 0}$ in time preserves the property described above. Namely, whether $(k_t^{(n)})_{n\geq 0}$, for any moment of time $t > 0$, be the system of correlation functions or not.

In order to answer this question, one can apply, for example, the result about characterization of correlation functions, which has been proposed by A. Lenard in [11] (see also [4]). For the readers' convenience below we give the conditions which have to be checked

- (Lenard positivity) for any $G \in B_{bs}(\Gamma_0)$ with $KG \geq 0$

$$\int_{\Gamma_0} G(\eta)\rho(d\eta) \geq 0, \tag{13}$$

  where $\rho \in \mathcal{M}(\Gamma_0)$ is a correlation measure which corresponds to the system of correlation functions $(k^{(n)})_{n\geq 0}$ and additionally it is supposed to be locally finite and normalized, i.e $\rho(\{\emptyset\}) = 1$.

Remark 4.2 *The condition of (Lenard positivity) ensures the existence of* $\mu \in \mathcal{M}_{\mathrm{fm}}(\Gamma)$ *such that the corresponding correlation measure* $\rho_\mu = \rho$.

- (moment growth) for any bounded set $\Lambda \subset \mathbb{R}^d$ and $j \geq 0$

$$\sum_{n=0}^{\infty} \left(m^{\Lambda}_{n+j}\right)^{-\frac{1}{n}} = \infty,$$

  where

$$m^{\Lambda}_n := (n!)^{-1} \int_{\Lambda} \cdots \int_{\Lambda} k^{(n)}(x_1, \ldots, x_n) dx_1 \ldots dx_n.$$

Remark 4.3 *The condition of (moment growth) ensures the uniqueness of* $\mu \in \mathcal{M}_{\mathrm{fm}}(\Gamma)$ *such that the corresponding correlation measure* $\rho_\mu = \rho$.

Further steps will be devoted to the verification of the latter conditions.

Lemma 4.2 *For any moment of time* $t > 0$*, the function* k_t *is positive in the sense of (13).*

Proof. See **Appendix 1**.

Remark 4.4 *The condition of the moment growth for the system of functions $\left\{k^{(n)}\right\}_{n\geq 1}$ is fulfilled if there exists a constant $C > 0$ (independent of n) such that*

$$k^{(n)}(x_1, \ldots, x_n) \leq n!\, C^n, \quad for\ \ all\ \ n \geq 0.$$

Proof. The statement of the remark follows from direct calculations.

For any system of functions $(k^{(n)})_{n\geq 0}$ we define the generating functional $\mathcal{L}_k : \mathcal{F}_k \to \mathbb{C}$:

$$\mathcal{L}_k(\theta) := \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} \theta(x_1) \ldots \theta(x_n) k^{(n)}(x_1, \ldots, x_n) dx_1 \ldots dx_n, \quad (14)$$

where $\mathcal{F}_k$ are such functions that sum (14) exists.

Remark 4.5 *Let us define for arbitrary $\delta > 0$*

$$U_\delta^1 := \left\{ \theta \in \mathrm{L}^1(\mathbb{R}^d) \mid ||\theta||_{\mathrm{L}^1(\mathbb{R}^d)} \leq \delta \right\}.$$

If system of functions $(k^{(n)})_{n\geq 0}$ satisfies the assumption of Remark 4.4, then the functional $\mathcal{L}_k$ is holomorphic in U_δ^1 for some $\delta > 0$.

Remark 4.6 *Assume that system of functions $(k^{(n)})_{n\geq 0}$ is a system of correlation functions for some measure $\mu \in \mathcal{M}^1_{\mathrm{fm}}(\Gamma)$. In this case there exists a connection between generating functional (14) and measure μ (see e.g. [4]) given by*

$$\mathcal{L}_k(\theta) = \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} \theta(x_1) \ldots \theta(x_n) k^{(n)}(x_1, \ldots, x_n) dx_1 \ldots dx_n =$$

$$= \int_\Gamma K \left(\prod_{x\in\gamma} \theta(x) \right) \mu(d\gamma) = \int_\Gamma \prod_{x\in\gamma} (1 + \theta(x))\, \mu(d\gamma), \quad \theta \in \mathcal{F}_k$$

The latter functional

$$\mathcal{L}_\mu(\theta) := \int_\Gamma \prod_{x\in\gamma} (1 + \theta(x))\, \mu(d\gamma), \quad (15)$$

is called Bogoliubov functional of measure μ.

Now, we set

$$\mathcal{M}^1_{hol}(\Gamma) := \left\{ \mu \in \mathcal{M}^1(\Gamma) \,\middle|\, \exists\, \delta > 0 : \ \mathcal{L}_\mu(\theta) \ \text{ is holomorphic in } \ U_\delta^1 \right\}.$$

Theorem 4.1 *Let $0 \le a \in L^1(\mathbb{R}^d) \cap C(\mathbb{R}^d)$ be an even function such that*

$$\int_{\mathbb{R}^d} a(x)dx = 1 \quad \text{and} \quad a(x) \to 0, \quad |x| \to \infty.$$

Then, for any $\mu \in \mathcal{M}^1_{hol}(\Gamma)$ there exists a Markov function $X^\mu_t \in \Gamma$ with initial distribution μ associated with generator L, such that for any $t > 0$, the corresponding distribution of X^μ_t is given by $\mu_t \in \mathcal{M}^1_{hol}(\Gamma)$.

Proof. Using the theorem about reconstruction of probability measures by correlation functions, Lemma 4.2, and Lemma 4.4 we are able to define evolution on $\mathcal{M}^1(\Gamma)$, which corresponds to the adjoint operator L^*. Then, using the standard scheme, one can immediately construct all finite dimensional distributions of X^μ_t.

4.3 Invariant measures

In this subsection we would like to describe invariant measures of the contact process on Γ constructed in the previous subsection.

In order to explain the expected asymptotics for the time evolution of correlation functions of our model let us consider the translation invariant case and evolution of the first correlation function. Namely, we will assume that first correlation function does not depend on $x \in \mathbb{R}^d$:

$$k^1_t(x) =: \rho_t, \ \text{ for all } \ t \ge 0.$$

The function ρ_t is called density. In this case, due to the results in the previous subsection the time evolution of the first correlation function is given by

$$\frac{\partial \rho_t}{\partial t} = (\varkappa - 1)\rho_t,$$
$$\rho_t|_{t=0} = \rho_0.$$

The solution to this equation can be written as follows:

$$\rho_t = \exp\{(\varkappa - 1)t\}\rho_0.$$

We distinguish the following cases

1. **Subcritical** ($\varkappa < 1$): $\rho_t \to 0$, as t tends to ∞;
2. **Supercritical** ($\varkappa > 1$): $\rho_t \to \infty$, as t tends to ∞;
3. **Critical** ($\varkappa = 1$): $\rho_t = \rho_0 = \rho$.

Remark 4.7 *For the case $\varkappa < 1$, the bound in Proposition 4.4 implies*

$$k^{(n)}_t \to 0, \quad t \to \infty$$

for any $n \in \mathbb{N}$.

Now, coming back to the purpose of this subsection it becomes clear that invariant measures may exist only in the critical case. Moreover, due to the Theorem 4.1, all invariant measures of the contact process can be described in terms of corresponding system of correlation functions as solutions to the following system of equations

$$\frac{\partial k_t^{(n)}}{\partial t}(x_1, \ldots, x_n) = 0, \quad n \geq 0.$$

This statement can be formulated more precisely:

Proposition 4.5 *If measure $\mu \in \mathcal{M}^1(\Gamma)$ is invariant measure for the contact process $X_t^\mu \in \Gamma$, then the system of corresponding correlation functions of this measure is a solution to the recurrent systems of equation*

$$n\, k^{(n)}(x_1, \ldots, x_n) = \sum_{i=1}^{n} k^{(n-1)}(x_1, \ldots, \check{x}_i, \ldots, x_n) \sum_{j:\, j \neq i} a(x_i - x_j) +$$

$$+ \sum_{i=1}^{n} \int_{\mathbb{R}^d} a(x_i - y) k^{(n)}(x_1, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_n) dy, \quad n \geq 1. \qquad (16)$$

Now, we give below the answer to the inverse problem and prove some kind of ergodicity result for our process in the translation invariant case. In this connection, for any $n \in \mathbb{N}$ we will be interested in asymptotics of the solution to an auxiliary Cauchy problem

$$\frac{\partial k_t^{(n)}}{\partial t}(x_1, \ldots, x_n) = \widehat{L}_n^* k_t^{(n)}(x_1, \ldots, x_n), \;\; t \geq 0, \qquad (17)$$

$$k_t^{(n)}(x_1, \ldots, x_n)\Big|_{t=0} := k_0^{(n)}(x_1, \ldots, x_n),$$

in the Banach space X_n .

Theorem 4.2 *Let $d \geq 3$ be arbitrary and fixed and let $0 \leq a \in \mathrm{L}^1(\mathbb{R}^d)$ be an arbitrary even continuous function such that*

1. $\int_{\mathbb{R}^d} a(x)dx = 1$,
2. $\int_{\mathbb{R}^d} x_k x_j\, a(x)dx < \infty$, for all $1 \leq k,\, j \leq d$,
3. $\hat{a}(p) := \int_{\mathbb{R}^d} e^{-i(p,\,x)} a(x)dx \in \mathrm{L}^1(\mathbb{R}^d)$.

Then, for any $\rho \in \mathbb{R}_+$ there exists a unique measure $\mu^\rho \in \mathcal{M}^1(\Gamma)$ such that its system of correlation functions $\left\{k^{(n),\,\rho}\right\}_{n\geq 0}$ are translation invariant, solve equation (16) and satisfy the following estimate

$$||k^{(n),\,\rho}||_{\mathrm{X}_n} \leq C(\rho)^n (n!)^2, \quad n \geq 1,$$

for some positive constant $C(\rho)$. Moreover, the first correlation function (density) of μ^ρ is exactly $\rho \in \mathbb{R}_+$.

Let μ_t be the distribution of $X_t^{\mu_0}$, $\mu_0 \in \mathcal{M}^1_{hol}(\Gamma)$ at time $t \geq 0$ and let $\left\{k_t^{(n)}\right\}_{n\geq 0}$ denote the system of correlation functions of μ_t.

Then, in the critical case ($\varkappa = 1$) the following condition are fulfilled

1. $k_t^{(1)} = k_0^{(1)} =: \rho$.
2. *for any $n \geq 2$ and $\varphi \in \mathrm{L}^1((\mathbb{R}^d)^n)$*

$$\left(k_t^{(n)},\, \varphi\right)_{\mathrm{L}^2((\mathbb{R}^d)^n)} \to \left(k^{(n),\,\rho},\, \varphi\right)_{\mathrm{L}^2((\mathbb{R}^d)^n)} \quad as \quad t \to \infty.$$

Proof. We have to show that under assumptions of the Theorem 4.2 the equation (16) has solution for any initial $k^{(1)} = \rho$, $\rho \in \mathbb{R}_+$ which satisfy conditions of moment growth and Lenard positivity (13). For the moment growth condition it is enough to show that solution has the following property

$$||k^{(n)}||_{\mathrm{X}_n} \leq C^n (n!)^2.$$

The proof we will perform using the mathematical induction method. Let us first consider the case $n = 2$. As it was pointed out before we consider translation invariant case. Therefore,

$$k^{(2)}(x_1, x_2) = k^{(2)}(x_1 - x_2, 0) =: k(x_1 - x_2),$$

where k is even function on $\mathbb{R}^d$. Hence, equation (16) in this case has form

$$(a \star k)(x_1 - x_2) - k(x_1 - x_2) = -\rho a(x_1 - x_2), \tag{18}$$

where

$$(a \star k)(x) := \int_{\mathbb{R}^d} a(x - y)k(y)dy.$$

Clear, that the best method for the investigation of equations with convolutions is the Fourier transform method. Suppose equation (18) has solution $v \in \mathrm{L}^1(\mathbb{R}^d)$. Then, Fourier transform of v satisfies the following equation

$$\hat{v}(p) = \frac{\rho \hat{a}(p)}{1 - \hat{a}(p)}.$$

If $d \geq 3$ and conditions of Theorem 4.4 are fulfilled, then $\hat{v}(p)$ has integrable peculiarity $\frac{1}{|p|^2}$ at $p = 0$, i.e $\hat{v}(p) \in \mathrm{L}^1(\mathbb{R}^d)$. Therefore,

$$v(x) := \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{i(p,x)} \frac{\rho \hat{a}(p)}{1 - \hat{a}(p)} dp \ \in \ \mathrm{L}^\infty(\mathbb{R}^\mathrm{d}). \tag{19}$$

Remark 4.8 *Let us consider translation invariant case. Suppose that solution to (18) is a second correlation function. Then, application of Fourier transform directly to (18) does not have any physical sense (in general second correlation function is not integrable function). Contrary to the second correlation function, the second Ursell function $u^{(2)}$ in majority of physical applications is integrable in one coordinate. Namely, function u which is defined by*

$$u(x_1 - x_2) := u^{(2)}(x_1 - x_2, 0) = k(x_1 - x_2) - \rho^2$$

is integrable on $\mathbb{R}^d$. It is easy to check that equation for the function u has the same form as (18) for the function k (the set of constants belong to the kernel of operator $\widehat{L}_2^*$). Namely,

$$(a \star u)(x_1 - x_2) - u(x_1 - x_2) = -\rho a(x_1 - x_2).$$

Having in mind Remark 4.8 and (19) one can easily check that

$$k^{(2)}(x_1, x_2) = k(x_1 - x_2) = v(x_1 - x_2) + \rho^2$$

is a solution to (18) in X_2. Moreover,

$$k^{(2)}(x_1, x_2) \leq \rho A + \rho^2 \leq C^2 (2!)^2,$$

where

$$A = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \frac{|\hat{a}(p)|}{1 - \hat{a}(p)} dp$$

and constant

$$C \geq \frac{\sqrt{\rho(A + \rho)}}{2}$$

will be chosen later.

Let us consider now equation (16) for $n \geq 3$

$$\widehat{L}_n^* k^{(n)}(x_1, \ldots, x_n) = \tag{20}$$

$$= -\sum_{i=1}^{n} k^{(n-1)}(x_1, \ldots, \check{x}_i, \ldots, x_n) \sum_{j:\, j \neq i} a(x_i - x_j) =: -f^{(n)}(x_1, \ldots, x_n).$$

The following function is a solution to this equation in the Banach space X_n:

$$k^{(n)}(x_1,\ldots,x_n) = \int_0^\infty \left(e^{t\widehat{L}_n^*}f^{(n)}\right)(x_1,\ldots,x_n)dt, \tag{21}$$

provided

$$\int_0^\infty \left(e^{t\widehat{L}_n^*}f^{(n)}\right)(x_1,\ldots,x_n)dt < \infty, \quad \text{for a.a. } (x_1,\ldots,x_n) \in (\mathbb{R}^d)^n$$

and

$$e^{t\widehat{L}_n^*}f^{(n)} \to 0, \qquad t \to \infty.$$

Therefore, in order to clarify the existence of the solution to (20) we should check whether right hand side of (21) has sense in X_n. Using the mathematical induction step $(n-1) \to n$ and the Markov property of $e^{t\widehat{L}_n^*}$, we get

$$\int_0^\infty \left(e^{t\widehat{L}_n^*}f^{(n)}\right)(x_1,\ldots,x_n)dt \leq \tag{22}$$

$$\leq \int_0^\infty \left(e^{t\widehat{L}_n^*}C^{n-1}\left((n-1)!\right)^2\sum_{i=1}^n\sum_{j:\,j\neq i} a(\cdot_i - \cdot_j)\right)(x_1,\ldots,x_n)dt =$$

$$= C^{n-1}\left((n-1)!\right)^2\sum_{i=1}^n\sum_{j:\,j\neq i}\int_0^\infty \left(e^{t\left(L_a^i+L_a^j\right)}a(\cdot_i-\cdot_j)\right)(x_i,\, x_j)dt. \tag{23}$$

The contraction property of the semigroup $e^{tL_a^j}$ implies: there exists $N \subset \mathbb{R}^d$, the Lebesgue measure of which is zero, such that for any $x_j \in \mathbb{R}^d \setminus N$

$$\int_0^\infty \left(e^{t\left(L_a^i+L_a^j\right)}a(\cdot_i-\cdot_j)\right)(x_i,\, x_j)dt \leq$$

$$\leq \int_0^\infty \sup_{x_j\in\mathbb{R}^d\setminus N}\left(e^{tL_a^i}a(\cdot_i - x_j)\right)(x_i)dt \leq$$

$$\leq \frac{1}{(2\pi)^d}\int_0^\infty \sup_{x_j\in\mathbb{R}^d\setminus N}\int_{\mathbb{R}^d}\left|\left(e^{tL_a^i}\widehat{a(\cdot_i - x_j)}\right)(p)\right| dp\, dt =$$

$$= \frac{1}{(2\pi)^d}\int_0^\infty \sup_{x_j\in\mathbb{R}^d\setminus N}\int_{\mathbb{R}^d} e^{t(\hat{a}(p)-1)}\left|\int_{\mathbb{R}^d} e^{-i(p,x)}a(x-x_j)dx\right| dp\, dt =$$

$$= \frac{1}{(2\pi)^d}\int_0^\infty \sup_{x_j\in\mathbb{R}^d\setminus N}\int_{\mathbb{R}^d} e^{t(\hat{a}(p)-1)}\left|e^{-i(p,x_j)}\hat{a}(p)\right| dp\, dt \leq$$

$$\leq \frac{1}{(2\pi)^d}\int_0^\infty \int_{\mathbb{R}^d} e^{t(\hat{a}(p)-1)}\left|\hat{a}(p)\right| dp\, dt.$$

For any $p \in \mathbb{R}^d \setminus \{0\}$

$$\int_0^\infty e^{t(\hat{a}(p)-1)} dt = \frac{1}{1-\hat{a}(p)}.$$

Moreover, because of (19)

$$\int_{\mathbb{R}^d} \frac{|\hat{a}(p)|}{1-\hat{a}(p)} dp < \infty.$$

Therefore, the Fubini theorem for non-negative functions implies that

$$\int_0^\infty \int_{\mathbb{R}^d} e^{t(\hat{a}(p)-1)} \, |\hat{a}(p)| \, dp \, dt < \infty.$$

Finally, using the result obtained for the case $n = 2$, under conditions of the Theorem 4.2, for almost all $(x_1, \ldots, x_n) \in (\mathbb{R}^d)^n$ w.r.t. Lebesgue measure we get

$$\int_0^\infty \left(e^{t\widehat{L}_n^*} f^{(n)} \right) (x_1, \ldots, x_n) dt \leq C^{n-1} A (n!)^2 \leq C^n (n!)^2,$$

where

$$C = \max \left\{ A, \, \frac{\sqrt{\rho(A+\rho)}}{2} \right\}.$$

The remaining statement of the first part of the theorem which has to be proved is Lenard positivity for the system of functions $\{k^{(n)}\}_{n \geq 0}$. But it follows directly from the second part of the theorem which describes asymptotic of the time evolution for the considered system of correlation functions.

The first statement of the second part of the theorem is trivial. In order to prove the second one, let us consider the following difference

$$k_t^{(n)}(x_1, \ldots, x_n) - k^{(n)}(x_1, \ldots, x_n) = \tag{24}$$

$$= \left[e^{t\widehat{L}_n^*} - 1\!\!1 \right] k^{(n)}(x_1, \ldots, x_n) + e^{t\widehat{L}_n^*} \left[k_0^{(n)}(x_1, \ldots, x_n) - k^{(n)}(x_1, \ldots, x_n) \right] +$$

$$+ \int_0^t e^{s\widehat{L}_n^*} f_{t-s}^{(n)}(x_1, \ldots, x_n) ds,$$

where $\{k^{(n)}\}_{n \geq 0}$ is a solution to (16). Since

$$\left[e^{t\widehat{L}_n^*} - 1\!\!1 \right] k^{(n)}(x_1, \ldots, x_n) = \int_0^t e^{s\widehat{L}_n^*} \widehat{L}_n^* k^{(n)}(x_1, \ldots, x_n) ds =$$

$$= - \int_0^t e^{s\widehat{L}_n^*} f^{(n)}(x_1, \ldots, x_n),$$

the difference (24) can be rewritten in the form

$$k_t^{(n)}(x_1,\ldots,x_n) - k^{(n)}(x_1,\ldots,x_n) = \tag{25}$$

$$= e^{t\widehat{L}_n^*}\left[k_0^{(n)}(x_1,\ldots,x_n) - k^{(n)}(x_1,\ldots,x_n)\right] +$$

$$+\int_0^t e^{s\widehat{L}_n^*}\left[f_{t-s}^{(n)}(x_1,\ldots,x_n) - f^{(n)}(x_1,\ldots,x_n)\right] ds.$$

Similar to the observations which were proposed above for the investigation of the right hand side of (21) and due to Proposition 4.4, one can easily claim that

$$\int_0^\infty e^{s\widehat{L}_n^*} f^{(n)}(x_1,\ldots,x_n) ds \in \mathrm{X}_n \tag{26}$$

and

$$\int_0^t e^{s\widehat{L}_n^*} f_{t-s}^{(n)}(x_1,\ldots,x_n) ds \in \mathrm{X}_n.$$

As a next step we use method of mathematical induction. For $n = 1$ the second statement of the second part of the theorem is trivial. Let us assume the inductive step $(n-1) \to n$. Namely, let

$$k_t^{(n-1)} \to k^{(n-1)}, \;\; t \to \infty \;\text{ in }\; \mathrm{X}_n.$$

This immediately implies convergence of

$$f_{t-s}^{(n)} \to f^{(n)}, \;\; t \to \infty \;\text{ in }\; \mathrm{X}_n. \tag{27}$$

Therefore, taking into account Proposition 4.4 and (26), one can find a constant $K > 0$ such that for any $t \geq 0$

$$||k_t^{(n-1)}||_{X_n} \leq K||k^{(n-1)}||_{X_n}.$$

Now, for arbitrary $\varepsilon > 0$ there exists $T > 0$ such that for all $t \geq T$

$$\int_T^t e^{s\widehat{L}_n^*}\left[f_{t-s}^{(n)}(x_1,\ldots,x_n) - f^{(n)}(x_1,\ldots,x_n)\right] ds \leq$$

$$\leq \int_T^t e^{s\widehat{L}_n^*}\left[|f_{t-s}^{(n)}(x_1,\ldots,x_n)| + |f^{(n)}(x_1,\ldots,x_n)|\right] ds \leq$$

$$\leq 2K\int_T^t ||k^{(n-1)}||_{X_n} \sum_{i=1}^n \sum_{j:\, j\neq i} \left(e^{s\left(L_a^i + L_a^j\right)} a(\cdot_i - \cdot_j)\right)(x_i,\, x_j) ds \leq$$

$$\leq 2K\int_T^\infty ||k^{(n-1)}||_{X_n} \sum_{i=1}^n \sum_{j:\, j\neq i} \left(e^{s\left(L_a^i + L_a^j\right)} a(\cdot_i - \cdot_j)\right)(x_i,\, x_j) ds < \varepsilon$$

In the latter estimate we have used (22) and bound for (26).

The convergence (27) and contraction property of the semigroup $e^{t\widehat{L}_n^*}$ implies

$$\int_0^T e^{s\widehat{L}_n^*}\left[f_{t-s}^{(n)}(x_1,\dots,x_n)-f^{(n)}(x_1,\dots,x_n)\right]ds\to 0,\ \ t\to\infty\ \text{ in }\ \mathrm{X}_n.$$

Finally,

$$\int_0^t e^{s\widehat{L}_n^*}\left[f_{t-s}^{(n)}(x_1,\dots,x_n)-f^{(n)}(x_1,\dots,x_n)\right]ds\to 0,\ \ t\to\infty\ \text{ in }\ \mathrm{X}_n.$$

Assuming that

$$e^{t\widehat{L}_n^*}\left[k_0^{(n)}(x_1,\dots,x_n)-k^{(n)}(x_1,\dots,x_n)\right]\to 0,\ \ t\to\infty,\ \text{ in }\ \mathrm{X}_n. \tag{28}$$

and due to (25), the second statement of the second part of the theorem is becoming now obvious.

Now, let us come back to the assumption (28). This assumption means that asymptotics, as t tends to ∞, of the solution to the Cauchy problem

$$\frac{\partial k_t^{(n)}}{\partial t}(x_1,\dots,x_n)=\widehat{L}_n^* k_t^{(n)}(x_1,\dots,x_n),\ \ t\geq 0, \tag{29}$$

$$k_t^{(n)}(x_1,\dots,x_n)\Big|_{t=0}:=k_0^{(n)}(x_1,\dots,x_n)\in\mathrm{X}_n,$$

does not depend on the initial data. The boundness of the operator $\widehat{L}_n^*$ in X_n implies that the solution $k_t^{(n)}=e^{t\widehat{L}_n^*}k_0^{(n)}$ to the Cauchy problem (29) is a function from X_n. The latter fact gives us possibility to look at the solution of (29) in the class of generalized functions $(\mathrm{L}^1)'((\mathbb{R}^d)^n)\subset \mathrm{S}'((\mathbb{R}^d)^n)$ (where $\mathrm{S}'((\mathbb{R}^d)^n)$ linear continuous functionals on the class of rapidly decreasing functions on $(\mathbb{R}^d)^n$). The Cauchy problem (29) in terms of generalized functions can be written as

$$\left(\frac{\partial k_t^{(n)}}{\partial t},\varphi\right)_{\mathrm{L}^2((\mathbb{R}^d)^n)}=-n\left(k_t^{(n)},\varphi\right)_{\mathrm{L}^2((\mathbb{R}^d)^n)}+ \tag{30}$$

$$+\int_{\mathbb{R}^d}\cdots\int_{\mathbb{R}^d}\sum_{i=1}^n\int_{\mathbb{R}^d}a(x_i-y)k_t^{(n)}(x_1,...,y,...,x_n)\varphi(x_1,...,x_n)dy\,dx_1\dots dx_n,$$

$$k_t^{(n)}\Big|_{t=0}:=k_0^{(n)}(x_1,\dots,x_n),\ \ \varphi\in\mathrm{S}((\mathbb{R}^d)^n).$$

The well-definiteness of Fourier transform for the class $\mathrm{S}'((\mathbb{R}^d)^n)$ gives possibility to rewrite (29) in Fourier coordinates

$$\left(\frac{\partial k_t^{(n)}}{\partial t}, \widehat{\varphi}\right)_{\mathrm{L}^2((\mathbb{R}^d)^n)} = -n\left(k_t^{(n)}, \widehat{\varphi}\right)_{\mathrm{L}^2((\mathbb{R}^d)^n)} + \tag{31}$$

$$+\int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} \sum_{i=1}^n \int_{\mathbb{R}^d} a(x_i - y) k_t^{(n)}(x_1, ..., y, ..., x_n)\widehat{\varphi}(x_1, ..., x_n) dy\, dx_1 \ldots dx_n,$$

$$k_t^{(n)}\Big|_{t=0} := k_0^{(n)}(x_1, \ldots, x_n), \quad \varphi \in \mathrm{S}((\mathbb{R}^d)^n).$$

We would like to stress that function

$$\sum_{i=1}^n \widehat{a}(x_i)\varphi(x_1, ..., x_n) \in \mathrm{L}^1((\mathbb{R}^d)^n)$$

and hence its Fourier transform

$$\sum_{i=1}^n \int_{\mathbb{R}^d} a(x_i - y)\widehat{\varphi}(x_1, ..., x_n) dx_i.$$

is also an element of $\mathrm{L}^1((\mathbb{R}^d)^n)$. Therefore,

$$\left(\frac{\partial \widehat{k_t^{(n)}}}{\partial t}, \varphi\right) = -n\left(\widehat{k_t^{(n)}}, \varphi\right) + \left(\widehat{k_t^{(n)}}, \sum_{i=1}^n \widehat{a}(\cdot_i)\varphi\right) = \tag{32}$$

$$= \left(\left(\sum_{i=1}^n \widehat{a}(\cdot_i) - n\right)\widehat{k_t^{(n)}}, \varphi\right), \quad \varphi \in \mathrm{S}((\mathbb{R}^d)^n),$$

$$k_t^{(n)}\Big|_{t=0} := k_0^{(n)} \in (\mathrm{L}^1)'((\mathbb{R}^d)^n).$$

It is easy to see that function

$$\widehat{k_t^{(n)}}(p_1, ..., p_n) := \exp\left\{t\left(\sum_{i=1}^n \widehat{a}(p_i) - n\right)\right\}\widehat{k_0^{(n)}}(p_1, ..., p_n) \in (\mathrm{L}^1)'((\mathbb{R}^d)^n)$$

is a solution to the Cauchy problem (32). Moreover, for any $\varphi \in \mathrm{S}((\mathbb{R}^d)^n)$

$$\left(\widehat{k_t^{(n)}}, \varphi\right) \to 0, \quad \text{as } t \to 0.$$

But the latter convergence implies that

$$\left(k_t^{(n)}, \varphi\right) \to 0, \quad \text{as } t \to 0.$$

Since $S((\mathbb{R}^d)^n)$ is dense in $L^1((\mathbb{R}^d)^n)$ and

$$||k_t^{(n)}||_{\left(L^1((\mathbb{R}^d)^n)\right)'} = ||k_t^{(n)}||_{L^\infty((\mathbb{R}^d)^n)} \leq ||k_0^{(n)}||_{L^\infty((\mathbb{R}^d)^n)}$$

we have: for any $\varphi \in L^1((\mathbb{R}^d)^n)$

$$\left(k_t^{(n)}, \varphi\right) \to 0, \quad \text{as } t \to 0.$$

This fact concludes the proof of the main theorem. ■

The system of equations (17) corresponds to the infinite number of independent random walks on $\mathbb{R}^d$. Similar to the observations proposed in [5], the investigation of the asymptotic for (17) in the sense of convergence in norm requires some restrictions on initial correlation functions. It is more convenient to formulate these restrictions in terms of Ursell functions, which corresponds to correlation functions. Ursell functions are defined as follows

$$u(\eta) := \sum_{i=1}^{|\eta|} \sum_{(\xi_1,\ldots,\xi_i)\in\mathcal{P}_\emptyset^i(\eta)} (-1)^{i-1}(i-1)!\, k(\xi_1)\cdots k(\xi_k), \quad \eta \in \Gamma_0,$$

$$u(\eta) = k(\eta), \text{ if } |\eta| = 1.$$

The inverse relation is given by

$$k(\eta) := \sum_{i=1}^{|\eta|} \sum_{(\xi_1,\ldots,\xi_i)\in\mathcal{P}_\emptyset^i(\eta)} u(\xi_1)\cdots u(\xi_k), \quad \eta \in \Gamma_0. \tag{33}$$

Remark 4.9 *Let us denote by $K_{\text{addmissible}}$ the class of translation invariant correlation functions (or the class of corresponding measures), whose Ursell functions satisfy the following assumptions:*

1. *for any $n \geq 2$*
$$\sup_{x\in\mathbb{R}^d} u^{x,(n-1)} \in L^1((\mathbb{R}^d)^{n-1}),$$
where
$$u^{x,(n-1)}(x_1, ..., x_{n-1}) := u(\{x_1, ..., x_{n-1}, x\});$$
2. *for any $n \geq 2$*
$$\sup_{x\in\mathbb{R}^d} \widehat{u^{x,(n-1)}} \in L^1((\mathbb{R}^d)^{n-1}).$$

Under conditions of the Theorem 4.2 and for any $\left\{k_0^{(n)}\right\}_{n\geq 0} \in K_{\text{addmissible}}$

$$k_t^{(n)} \to k^{(n),\rho} \;\; \textit{in } X_n \;\; \textit{as} \;\; t \to \infty.$$

Proof. Below we will show, that for the class of initial correlation functions from $K_{\text{admissible}}$, the assumption (28) is fulfilled. The first important observation, which follows from the definition of Ursell function, is that evolutional equation for the n-th Ursell function $u^{(n)}$, which corresponds to the equation (29), is of the same type. Namely,

$$\frac{\partial u_t^{(n)}}{\partial t}(x_1,\ldots,x_n) = \widehat{L}_n^* u_t^{(n)}(x_1,\ldots,x_n), \;\; t \geq 0, \tag{34}$$

$$u_t^{(n)}(x_1,\ldots,x_n)\Big|_{t=0} := u_0^{(n)}(x_1,\ldots,x_n),$$

Since operator $\widehat{L}_n^*$ preserves translation invariant functions and initial function $u_0 \in K_{\text{admissible}}$ is considered to be translation invariant, the evolution of u_t will be also translation invariant.

Let $x \in \mathbb{R}^d$ be an arbitrary and fixed. The definition of the class $K_{\text{admissible}}$ implies

$$u_t^{x,\,(n-1)} \in \mathrm{L}^1((\mathbb{R}^d)^{n-1}), \quad n \geq 2.$$

Therefore, the Fourier transform of $u_t^{x,\,(n-1)}$ exists and the equation (34) for this function in Fourier coordinates has the following form

$$\frac{\partial \widehat{u_t^{x,\,(n-1)}}}{\partial t}(p_1,\ldots,p_{n-1}) = \int_{\mathbb{R}^d} a(x-y)\widehat{u_t^{y,\,(n-1)}}(p_1,\ldots,p_{n-1})dy + \tag{35}$$

$$+ \left[\sum_{i=1}^{n-1} \hat{a}(p_i) - n\right] \widehat{u_t^{x,\,(n-1)}}(p_1,\ldots,p_{n-1}).$$

For fixed $(p_1,\ldots,p_{n-1}) \in (\mathbb{R}^d)^{n-1}$, we define

$$\tilde{u}_t(x) := \widehat{u_t^{x,\,(n-1)}}(p_1,\ldots,p_{n-1}).$$

In terms of this function, the equation (35) has form

$$\frac{\partial \tilde{u}_t}{\partial t}(x) = \int_{\mathbb{R}^d} a(x-y)\tilde{u}_t(y)dy + \left[\sum_{i=1}^{n-1} \hat{a}(p_i) - n\right] \tilde{u}_t(x). \tag{36}$$

Due to the definition of $K_{\text{admissible}}$, function $\tilde{u}_t(x)$ is bounded. Moreover,

$$\tilde{u}_t(x) := \exp\left\{ t\left[\sum_{i=1}^{n-1} \hat{a}(p_i) - (n-1)\right]\right\} \left(e^{t\widehat{L}_1^*}\tilde{u}_0\right)(x)$$

is a solution to (36).

Now,

$$||u_t^{(n)}||_{\mathrm{X}_n} \le \sup_{x\in\mathbb{R}^d} \int_{\mathbb{R}^d} ... \int_{\mathbb{R}^d} |\widehat{u_t^{x,(n-1)}}(p_1, \dots, p_{n-1})| dp_1 ... dp_{n-1} \le$$

$$\le \int_{\mathbb{R}^d} ... \int_{\mathbb{R}^d} \exp\left\{ t \left[\sum_{i=1}^{n-1} \hat{a}(p_i) - (n-1) \right] \right\} \sup_{x\in\mathbb{R}^d} |\widehat{u_0^{x,(n-1)}}(p_1, \dots, p_{n-1})| dp_1 ... dp_{n-1}.$$

Since

$$\int_{\mathbb{R}^d} ... \int_{\mathbb{R}^d} \sup_{x\in\mathbb{R}^d} |\widehat{u_0^{x,(n-1)}}(p_1, \dots, p_{n-1})| dp_1 ... dp_{n-1} < \infty,$$

the following norm

$$||u_t^{(n)}||_{\mathrm{X}_n} \to 0,$$

as t tends to ∞. ■.

Appendix 1

According to the definition of positive definiteness it is enough to check that

$$\int_{\Gamma_0} G(\eta)\rho_t(d\eta) :=$$

$$= \sum_{n\ge 0} \frac{1}{n!} \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} G^{(n)}(x_1, \dots, x_n) k_t^{(n)}(x_1, \dots, x_n) dx_1 \dots dx_n \ge 0, \quad (37)$$

for all $G \in B_{\mathrm{bs}}(\Gamma_0)$, such that $KG \ge 0$. Moreover, due to the Corollary 4.1 it is enough to check the latter inequality only in the case of $a \in C_0(\mathbb{R}^d)$.

Let $\mu \in \mathcal{M}^1_{\mathrm{fm}}(\Gamma)$ be locally absolutely continuous w.r.t. Poisson measure whose system of correlation functions $\{k^{(n)}\}_{n\ge 0}$ satisfies assumption of Remark 4.4.

As it was shown in [10], there exists a Markov process X_t^γ on configuration space Γ with the corresponding generator L in the case of $a \in C_0(\mathbb{R}^d)$.

Next, we consider the following functions on Γ

$$F^{(n)}(\gamma) = \sum_{\{x_1,\dots,x_n\}\subset\gamma} e^{-\beta|x_1|} \cdots e^{-\beta|x_n|}, \ \ \beta > 0, \ \ n \in \mathbb{N}, \ \ |\gamma| \ge n.$$

Note, that

$$\int_\Gamma F^{(n)}(\gamma)\mu(d\gamma) =$$

$$= \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} e^{-\beta|x_1|} \cdots e^{-\beta|x_n|} k^{(n)}(x_1, \ldots, x_n) dx_1 \ldots dx_n < \infty, \tag{38}$$

because of (i) of Remark 4.4. Using the same direct computation as in [10] we obtain

$$LF^{(n)}(\gamma) \leq C_1 F^{(n)}(\gamma) + C_2 F^{(n-1)}(\gamma)$$

for some constants C_1, $C_2 > 0$. The use of the latter estimate for the function

$$\mathbb{L}^{(N)}(\gamma) := \sum_{n=1}^{N} F^{(n)}(\gamma)$$

gives us the bound

$$L\mathbb{L}^{(N)}(\gamma) \leq C\mathbb{L}^{(N)}(\gamma), \quad C > 0.$$

The application of martingale representation together with the Gronwall inequality implies

$$\mathrm{E}\left[\mathbb{L}^{(N)}(X_t^\gamma)\right] \leq \mathbb{L}^{(N)}(\gamma) e^{Ct}. \tag{39}$$

Let $\{\mu_t\}_{t\geq 0}$ be the corresponding evolution of μ_0 described by the dual Kolmogorov equation

$$\frac{\partial \mu_t}{\partial t} = L^\star \mu_t,$$
$$\mu_t\,|_{t=0} = \mu_0.$$

Then (39) and the bound

$$\mathbb{L}^{(N)}(\gamma) \geq \mathbb{L}^{(N)}(\gamma_\Lambda) \geq \min_{x\in\Lambda}\{e^{-\beta|x|}\} \sum_{k=1}^{N} C^k_{|\gamma_\Lambda|} \geq$$

$$\geq \min_{x\in\Lambda}\{e^{-\beta|x|}\} \times \begin{cases} 2^{|\gamma_\Lambda|} - 1, & \text{if } |\gamma_\Lambda| \leq N, \\ C_N |\gamma_\Lambda|^N, & \text{otherwise,} \end{cases} \qquad \Lambda \in \mathcal{B}_b(\mathbb{R}^d)$$

with

$$0 < C_N < \frac{1}{N^N}$$

imply

$$\int_\Gamma |\gamma_\Lambda|^N \mu_t(d\gamma) \leq \left(\min_{x\in\Lambda}\{e^{-\beta|x|}\}\right)^{-1} N^N \int_\Gamma \mathrm{E}\left[\mathbb{L}^{(N)}(X_t^\gamma)\right] \mu_0(d\gamma) \leq$$

$$\leq \left(\min_{x\in\Lambda}\{e^{-\beta|x|}\}\right)^{-1} N^N e^{Ct} \int_\Gamma \mathbb{L}^{(N)}(\gamma)\mu_0(d\gamma) < \infty,$$

where the latter integral is finite because of (38). Therefore, the evolution of states $\{\mu_t\}_{t\geq 0} \subset \mathcal{M}^1_{\mathrm{fm}}(\Gamma)$, which means that there exists a Markov evolution of corresponding correlation measures (or corresponding correlation functions) on $\mathcal{M}_{\mathrm{lf}}(\Gamma_0)$ associated with the generator L. The fulfilment of (37) is now obvious because of the Markov property of the semigroup which corresponds to the evolution of states. ■

References

[1] Yu. M. Berezansky, Yu. G. Kondratiev, T. Kuna, E.V. Lytvynov, *On a spectral representation for correlation measures in configuration space analysis*, *Methods Funct. Anal. Topology* **5, no.4** (1999) 87-100.

[2] I. I. Gikhman, A. V. Skorohod **The Theory of Stochastic Processes, Vol. II**, *Springer Verlag, 1975*

[3] K. Ito, F. Kappel, Evolution equations and approximations (Series on advances in mathematics for applied sciencies - Vol. 61, World Scientific, 2002)

[4] Yu. G. Kondratiev and T. Kuna, *Harmonic analysis on configuration space I. General theory*, *Infinite Dimensional Analysis, Quantum Probability and Related Topics* **5, no.2** (2002) 201-233.

[5] Yu. G. Kondratiev, T. Kuna, M. J. Oliveira, J. L. da Silva and L. Streit, Hydrodynamic limits for the free Kawasaki dynamics of continuous particle systems *SFB-701 Preprint, University of Bielefeld, Bielefeld, Germany* (2007).

[6] Yu. G. Kondratiev, O. V. Kutoviy, On the metrical properties of the configuration space *Math. Nachr.* **279**, No.7, (2006) 774–783.

[7] Yu. G. Kondratiev, O. V. Kutoviy, S. Struckmeier, Contact model with Kawasaki dynamics in continuum *SFB-701 Preprint, University of Bielefeld, Bielefeld, Germany* (2007).

[8] Yu. G. Kondratiev, E. Lytvynov, M. Röckner, Equilibrium Kawasaki dynamics of continuous particle systems. *Infin. Dimens. Anal. Quantum Probab. Relat. Top.*, **10**, No.2, (2007) 185–209.

[9] Yu. G. Kondratiev, R. Minlos, and E. Zhizhina, One-particle subspaces of the generator of Glauber dynamics of continuous particle systems. *Rev. Math. Phys.*, **16**, No.9, (2004) 1-42.

[10] Yu. G. Kondratiev and A. Skorokhod, On contact processes in continuum. *Infinite Dimensional Analysis, Quantum Probabilities and Related Topics*, **9**, No.2, (2006) 187-198.

[11] A. Lenard, *Correlation Functions and the Uniqueness of the state in Classical Statistical Machanics. I*, *Commun. math. Phys.* **30** (1973) 35-44.

[12] A. Lenard, *States of classical statistical mechanical systems of infinitely many particles. I*, *Arch. Rational Mech. Anal.* **59** (1975) 219-239.

[13] A. Lenard, *States of classical statistical mechanical systems of infinitely many particles. II*, *Arch. Rational Mech. Anal.* **59** (1975) 241-256.

[14] T. M. Liggett, **Interacting Particle Systems**, Springer-Verlag, 1985.

[15] R. A. Minlos, **Lectures on statistical physics**, *UMN* (Russian) No.1, (1968) 133-190.

[16] M. Reed, B. Simon, **Methods of modern mathematical physics**, **4. Analysis of Operators**, (New York, London: Academic Press, 1978).

[17] D. Ruelle, **Statistical Mechanics** (New York, Benjamin, 1969).

[18] D. Ruelle, *Superstable interactions in classical statistical mechanics*, *Commun. Math. Phys.* **18** (1970) 127-159.

[19] W. R. Young, A. J. Roberts and G. Stuhne *Reproductive pair correlations and the clustering of organisms*, *Nature* **412** (2001) 328-331.