AN ILLUSTRATED THEORY OF HYPERBOLIC VIRTUAL POLYTOPES

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Abstract. The paper gives an illustrated introduction to the theory of hyperbolic virtual polytopes and related counterexamples to A.D. Alexandrov's conjecture.

1. Introduction

Sometimes it is reasonable to treat mostly non-convex objects instead of convex ones. For instance, to consider hyperbolic (i. e., saddle) virtual polytopes in contrast to convex ones, or to embed graphs such that the embedding looks as non-convex as possible. Probably it is better to start with the figures rather than with definitions: just have a look at the **hyperbolic polytope** with 8 horns (Fig. 13) and its fan (Fig. 14).

This motto appeared independently in a natural way in different fields (chronologically, in computer science, graph embedding problems, and classical convex geometry) and for quite different reasons.

The paper is focused on the latter subject, namely, on A.D. Alexandrov's conjecture and hyperbolic virtual polytopes.

We aim at a mostly elementary introductive description, trying nevertheless to keep complete proofs and constructions. By this reason, we organize the paper as follows.

We give first necessary remindments on Minkowski addition and support functions of convex polytopes (Section 2). Instead of giving the complete theory of virtual polytopes (for which the reader should be referred to [3] and [6]), we give a shortcut to the notion of virtual polytopes (Section 3).

Hyperbolic virtual polytopes are discussed in Section 4. Dislike the earlier papers [6] and [8], we give two explicit examples of hyperbolic virtual polytopes (Section 5) rather than existence-type theorems. Namely, we describe the construction of hyperbolic polytopes with 8 and 6 horns. The latter arose originally as an auxiliary tool for constructing counterexamples to A.D. Alexandrov's conjecture (discussed in Section 6). The exact coordinates are given in section 7.

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2. Minkowski addition, support function

Minkowski addition

Let $\mathbb{R}^3$ be the real space with the origin O. We identify points in $\mathbb{R}^3$ with their radius-vectors.

Denote by $\mathcal{P}$ the set of all compact convex polytopes in $\mathbb{R}^3$. Degenerate polytopes are also included, so a closed segment and a point are polytopes.

Definition 2.1. Let K, L be polytopes. Their *Minkowski sum* is defined by $K \otimes L = \{\mathbf{x} + \mathbf{y} : \mathbf{x} \in K, \mathbf{y} \in L\}$.

The set $\mathcal{P}$ endowed with Minkowski addition forms a semigroup with the unite element $E = \{O\}$.

Support function

Definition 2.2. The support function of a polytope K is the function $h_K : \mathbb{R}^3 \to \mathbb{R}$ defined by

$$h_K(\mathbf{x}) = max_{y \in K}(\mathbf{x}, \mathbf{y}),$$

where $(\mathbf{x}, \mathbf{y})$ stands for the scalar product.

Example 2.3. *Let $\boldsymbol{a}$ be a point. Then its support function is linear:*

$$h_{\{\boldsymbol{a}\}}(\boldsymbol{x}) = (\boldsymbol{a}, \boldsymbol{x}).$$

Propositon 2.4. *The support function of a convex polytope K is*

(1) *continuous;*
(2) *positively homogeneous, namely,*

$$h_K(\lambda x) = \lambda h_K(x)$$

for $\lambda \geq 0$ (in particular, this implies that $h_K(O) = 0$);
(3) *convex;*
(4) *piecewise linear. The domains of linearity correspond to the vertices of the polytope K (for the maximum of the above scalar product is achieved at one of the vertices). These domains tile $\mathbb{R}^3$ into a union of polytopal cones with the apex at O. This tiling is called the fan of the polytope K.* □

In the sequel, we sometimes speak of (and draw) the intersection of the fan with the unit sphere S^2 centered at O. This yields the *spherical fan.* The cones of the fan correspond to spherical polytopes of the spherical fan. The latter are called the *cells* of the fan. The 0-dimensional cells are called the *vertices* of the spherical fan.

A polytope and its fan are combinatorially dual. In particular, vertices of K correspond to cones of the fan (consequently, to the 2-dimensional cells of the spherical fan).

Denote by $\mathcal{G}$ the set of functions that are:

- defined on $\mathbb{R}^3$;

- convex;
- continuous;
- piecewise linear with respect to some fan;
- equal 0 at the origin O.

This set (together with the pointwise addition) forms a semigroup which is known to be isomorphic to $\mathcal{P}$. The canonical isomorphism $s : \mathcal{P} \to \mathcal{G}$ maps each polytope k to its support function h_K.

3. Virtual polytopes

Virtual polytopes

Virtual polytopes were defined by A. Pukhlikov and A. Khovanskii [3].

Consider the Grothendieck group $\mathcal{P}^*$ of the semigroup $\mathcal{P}$. Remind that it is defined to be the group of all formal expressions of type $K \oplus L^{-1}$ subject to the usual cancelation law: $(K \oplus M) \oplus (L \oplus M)^{-1} = K \oplus L^{-1}$. The group $\mathcal{P}^*$ is called *the group of virtual polytopes.*

The semigroup isomorphism s induces a group isomorphism s^*:

$$s^* : \mathcal{P}^* \to \mathcal{G}^*.$$

Here $\mathcal{G}^*$ is the group of functions that are:

- defined on $\mathbb{R}^3$;
- continuous;
- piecewise linear with respect to some fan;
- equal 0 at the origin O.

(In comparison with $\mathcal{G}$, the convexity property disappears.)

Definition 3.1. Let K be a virtual polytope. By the *support function* h_K of K we mean the function $s^*(K)$. In other words, if $K = L \oplus M^{-1}$, then $h_K = h_L - h_M$.

Similarly to the convex case, the support function of a virtual polytope K is piecewise linear with respect to some conical tiling of $\mathbb{R}^3$, which is called the *fan* of K. Dislike the convex case, the tiles of such a fan might be non-convex.

Virtual polytopes related to a polytopal surface

Theorem 3.2. *Let C be a closed polytopal surface in $\mathbb{R}^3$ (possibly non-convex, with self-intersections).*

Suppose there exists a collection of normal vectors ξ_i of its facets T_i and a spherical fan Σ, such that:

- *the set of vertices of the fan Σ equals the set of endpoints of $\{\xi_i\}$;*
- *the fan Σ is combinatorially dual to the surface C. (In particular, this means that the points ξ_i and ξ_j are connected by an edge of Σ if and only if T_i and T_j share an edge in C.)*

In this case, the pair (C, Σ) canonically defines some virtual polytope K whose fan equals Σ. □

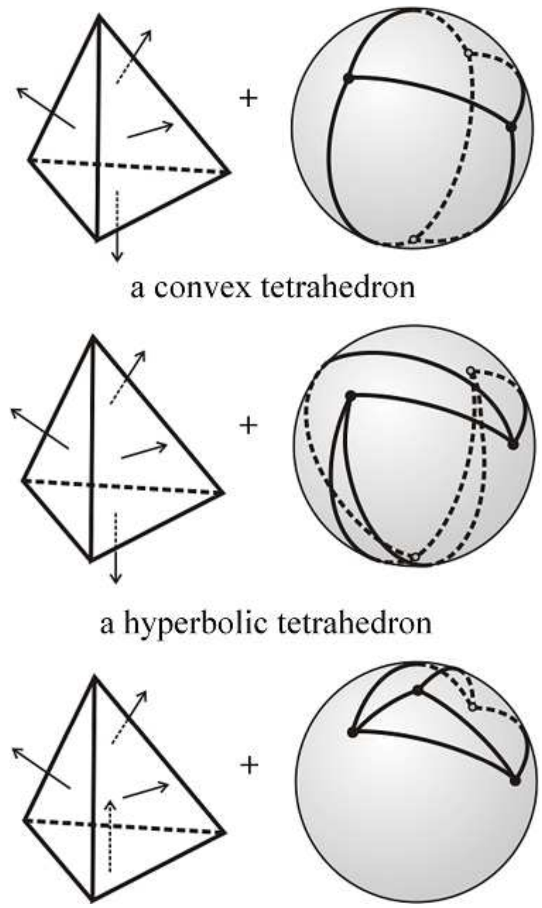

Figure 1. Three examples of virtual tetrahedra

Given a surface C, sometimes there exist several fans, satisfying the above conditions. This means, that for each fan we have a virtual polytope, related to C. Different fans give different virtual polytopes.

Example 3.3. *Figure 1 depicts a tetrahedron with three associated fans. This yields three different virtual tetrahedra.*

The surface of a tetrahedron can be associated with 52 (!) different fans (and therefore, with 52 virtual polytopes). The complete list is depicted in Figure 2 (by Vlad Scherbina).

However, this example is misleading: for most of polytopal surfaces, there are no associated virtual polytopes.

4. Hyperbolic virtual polytopes

Let K be a virtual polytope and let $h = h_K$ be its support function. For $\xi \in S^2$, let $e(\xi)$ be the plane defined by the equation $(x, \xi) = 1$. Consider the restriction of h to the plane $e(\xi)$ and denote by $\mathcal{F} = \mathcal{F}_K(\xi)$ the graph of the restriction. The surface $\mathcal{F}$ is piecewise linear. Its vertices and edges

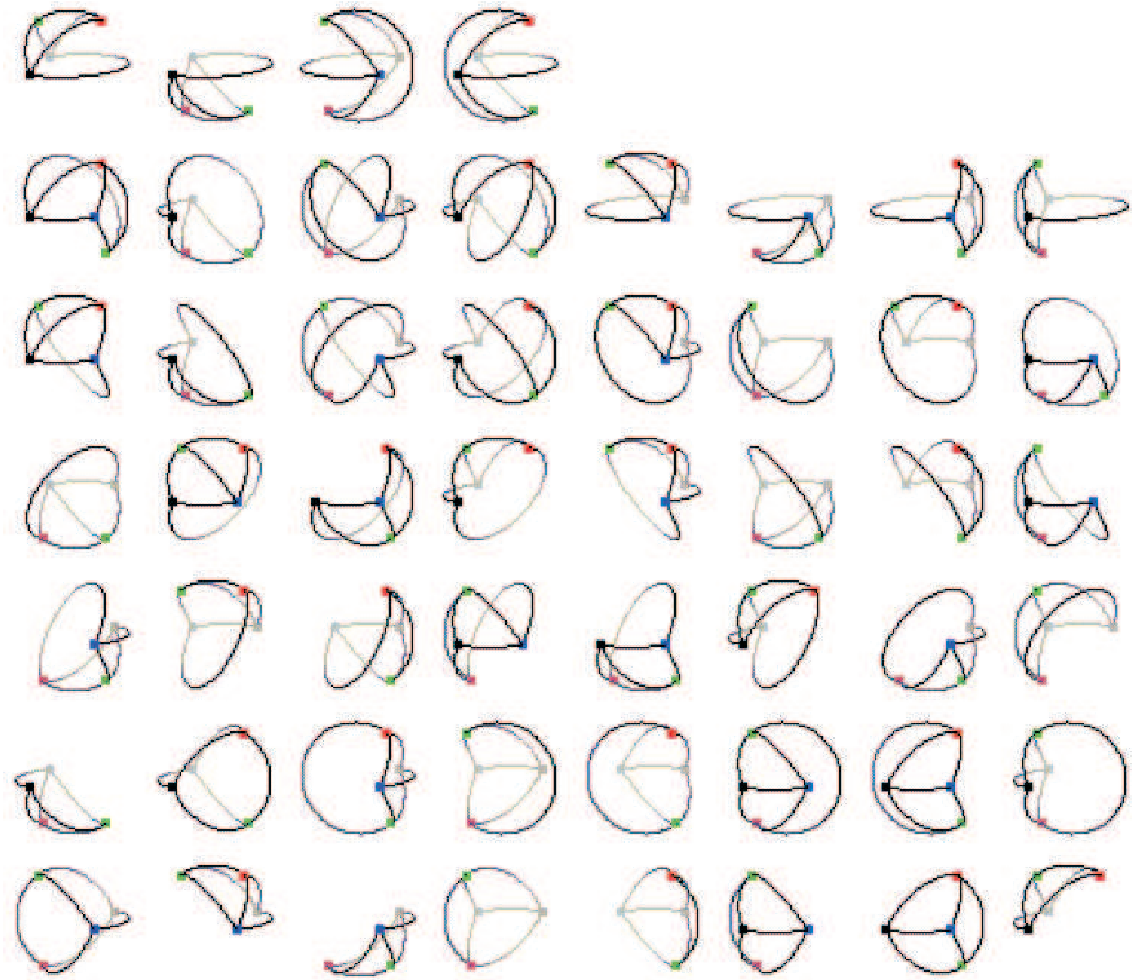

FIGURE 2. The complete list of virtual tetrahedra

correspond to those of the fan Σ_K intersected with the open hemisphere with the pole at ξ.

The virtual polytope K is convex (i.e. $K \in \mathcal{P}$) if and only if the surface $\mathcal{F}_K(\xi)$ is concave down for any ξ.

Definition 4.1. Let F be a surface in $\mathbb{R}^3$, $x \in F$. The point x is called *saddle*, if any plane passing through x locally intersects F in more than one point. The surface is *saddle* if all its points are saddle.

Definition 4.2. A virtual polytope K is called *hyperbolic* if $\mathcal{F}_K(\xi)$ is a saddle surface for any $\xi \in S^2$. In the sequel, we call such virtual polytopes, for short *hyperbolic polytopes*.

Definition 4.3. A vertex ξ of a spherical fan Σ is *pointed*, if there exists an adjacent to ξ angle greater than π. A fan is *pointed*, if each its vertex is pointed (Fig. 3).

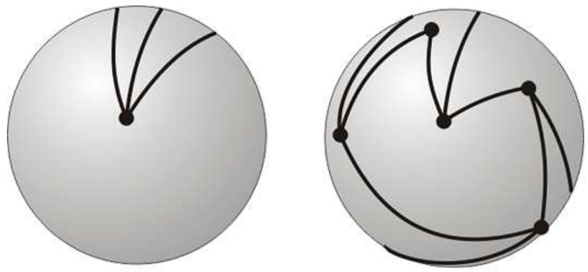

FIGURE 3. A pointed vertex of the fan and a pointed fan

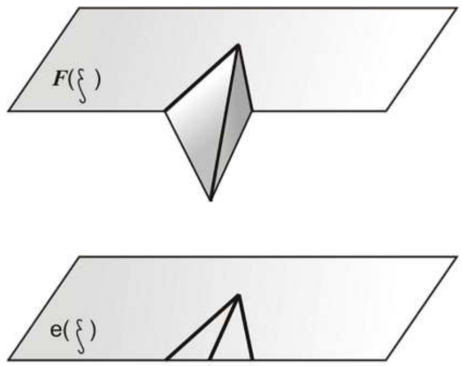

FIGURE 4. The part of the surface $\mathcal{F}_K(\xi)$ and the plane e

All below examples of hyperbolic polytopes are constructed in the framework of Theorem 3.2. This means that each time we construct a polytopal surface and an associated fan. These polytopal surfaces have non-saddle vertices. Such vertices are called *horns*.

Lemma 4.4. *Let K be a virtual polytope with a pointed spherical fan Σ_K. Then the polytope K is hyperbolic.*

Proof. The vertices of $\mathcal{F}_K(\xi)$ correspond to those of Σ_K. At each of its vertices, the surface $\mathcal{F}_K(\xi)$ has an adjacent face with an angle greater than π (Fig. 4). Thus it is pointed at each of its vertices. □

Corollary 4.5. *The second virtual polytope from Example 3.3 is hyperbolic.* □

5. NON-TRIVIAL EXAMPLES OF HYPERBOLIC POLYTOPES WITH 6 AND 8 HORNS

The existence of hyperbolic polytopes with any number of horns greater than 3 was proved in [6] and [8]. Here we present two explicit examples according to the following scheme.

(1) First we construct a polytopal surface with a boundary (the double star).

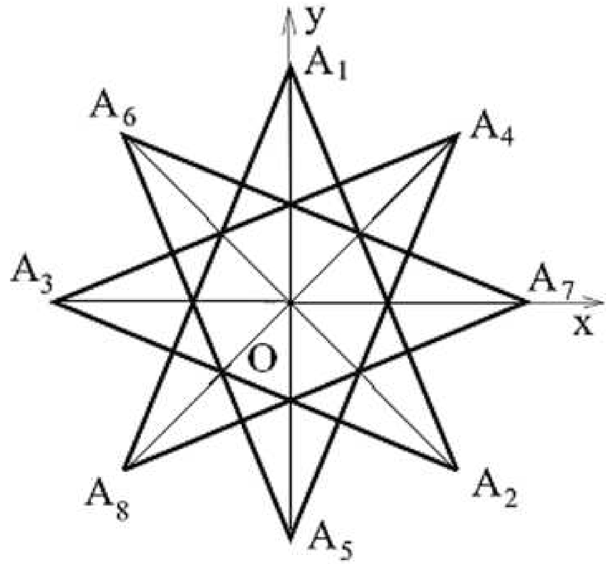

Figure 5. Planar star with 8 vertices

(2) Then we patch up some cross-caps along its boundary and obtain a closed polytopal surface with oriented facets. Each cross-cap gives a horn.
(3) We depict an associated spherical fan. Due to Theorem 3.2, this yields a virtual polytope.
(4) By construction, the fan is pointed. By Lemma 4.4 this means that the polytope is hyperbolic.

A hyperbolic polytope with 8 horns

(1) Start with the planar star with 8 vertices in the plane $(x, y) \in \mathbb{R}^3$ (Fig. 5).

Make the star 3-dimensional by shifting up its even vertices and shifting down the odd ones. This gives a polytopal surface $S_1 \in \mathbb{R}^3$ (Fig. 6), consisting of all triangles of type $(OA_iA_{i+1}), i = 1, ..., 8$. (We assume that $A_9 = A_1$.)

Take the star S_1 and its mirror image S_2 with respect to the plane (x, y). Figure 7 depicts them separately. Together they form a complicated object - the double star (Fig. 8).

We choose normal vectors of the facets of S_1 (respectively, S_2) looking upwards (respectively, downwards).

(2) We patch now 8 cross-caps to this double star (each of them gives a horn). A *cross-cap* is a collection of 4 oriented triangles (Fig. 9).

Keeping in mind Theorem 3.2, we depict a cross-cap with orientation of its faces together with the cell of the fan (to be constructed below) corresponding to the vertex H. The cell is a spherical 4-gon with only two convex angles.

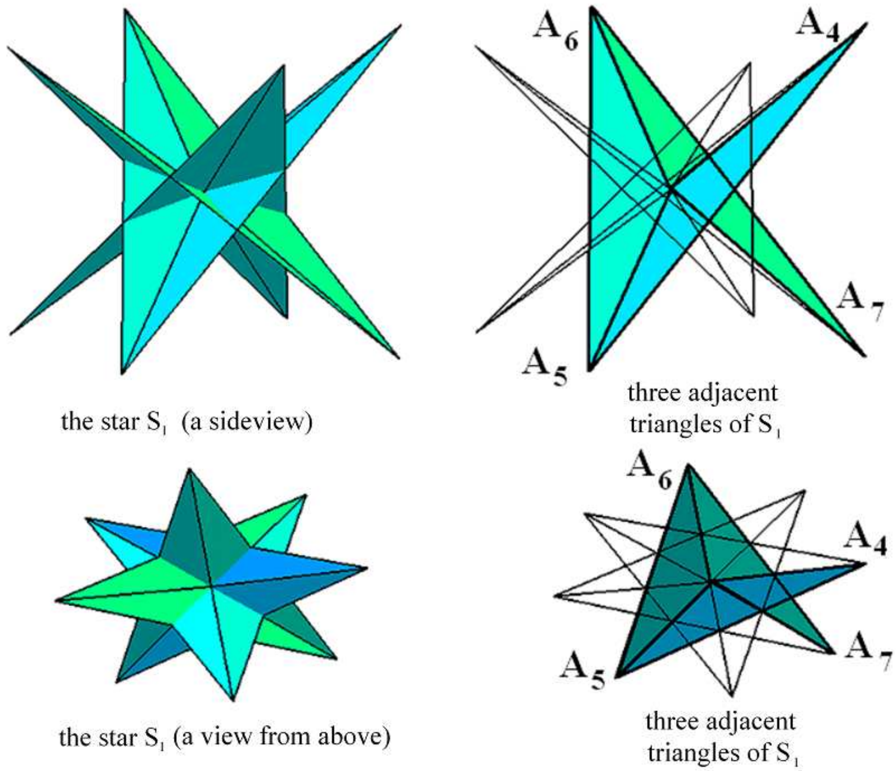

FIGURE 6. 3D star with 8 vertices

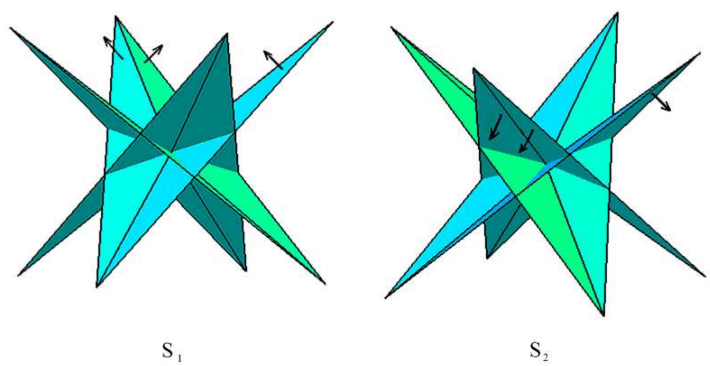

FIGURE 7. Stars S_1 and S_2

The boundary of the double star splits into a union of "crosses". A "cross" consists of two symmetric segments: an edge of the star S_1 (marked blue) and an edge of the star S_2 (marked red).

The cross-cap has also a cross (blue + red), which indicates the patching rule (Fig. 10)

An orange edge of the cross-cap is patched up to the orange edge of the neighbor cross-caps (Fig. 11).

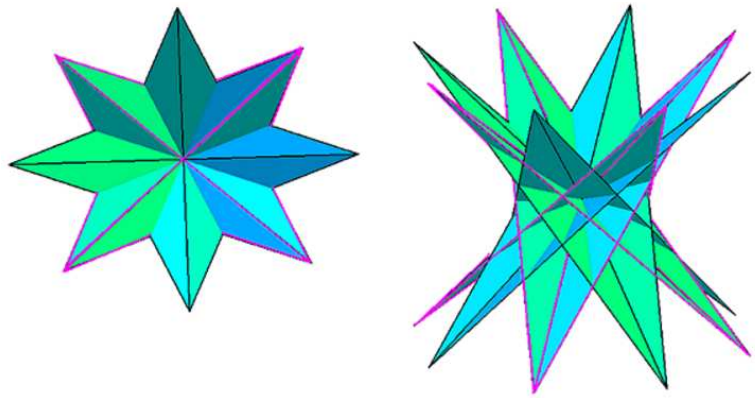

FIGURE 8. The double star

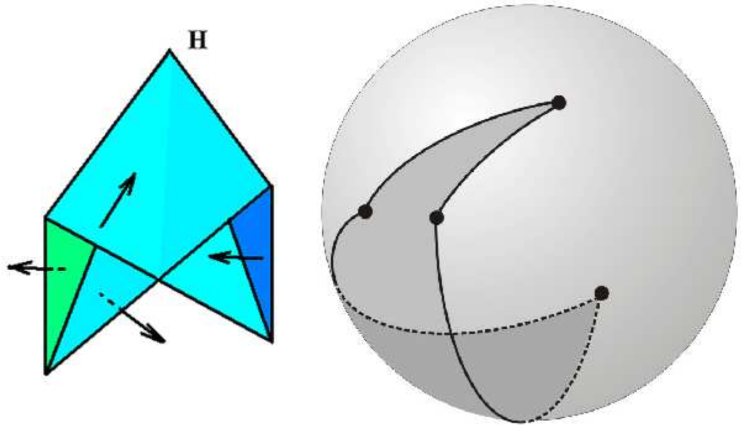

FIGURE 9. A cross-cap and the cell of the fan, corresponding to the vertex H

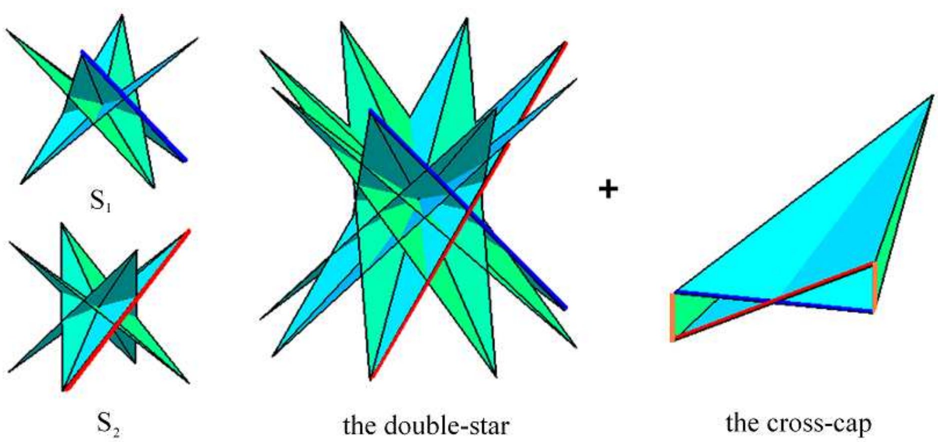

FIGURE 10. Patching rule

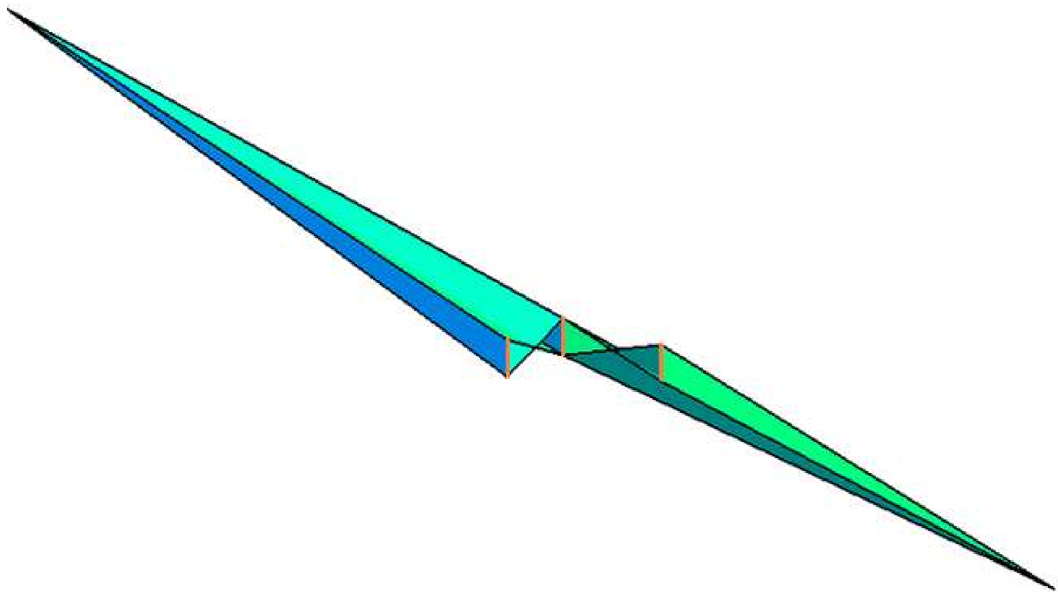

FIGURE 11. Two adjacent cross-caps

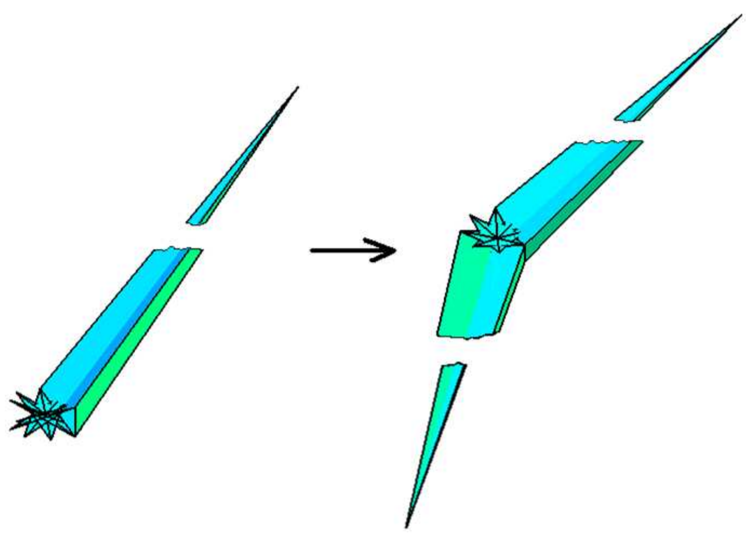

FIGURE 12. The process of patching

Finally, we get a closed polytopal surface, which is saddle at each of its vertices except for the 8 horns (Fig. 13)

(3) Figure14 depicts the associated fan.

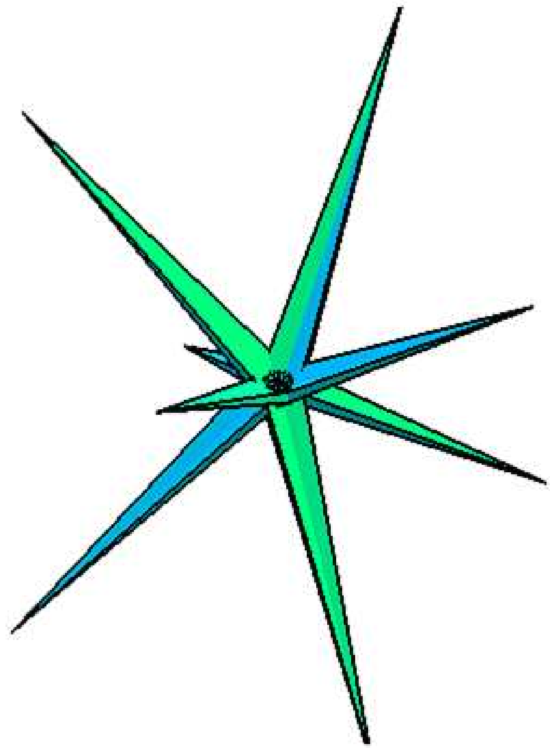

FIGURE 13. Hyperbolic polytope with 8 horns

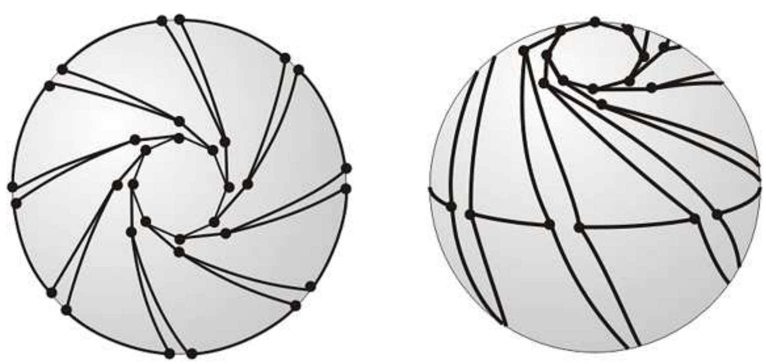

FIGURE 14. The fan of the hyperbolic polytope with 8 horns

(4) This fan is pointed. Therefore we have a hyperbolic virtual polytope with 8 horns.

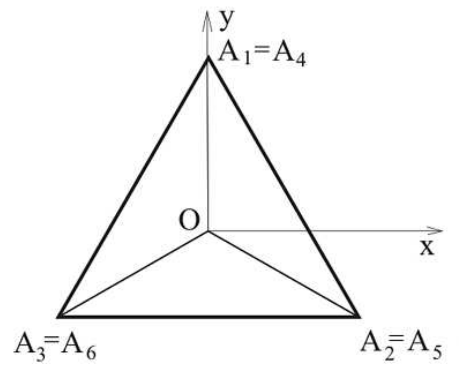

FIGURE 15. Planar star with 6 vertices

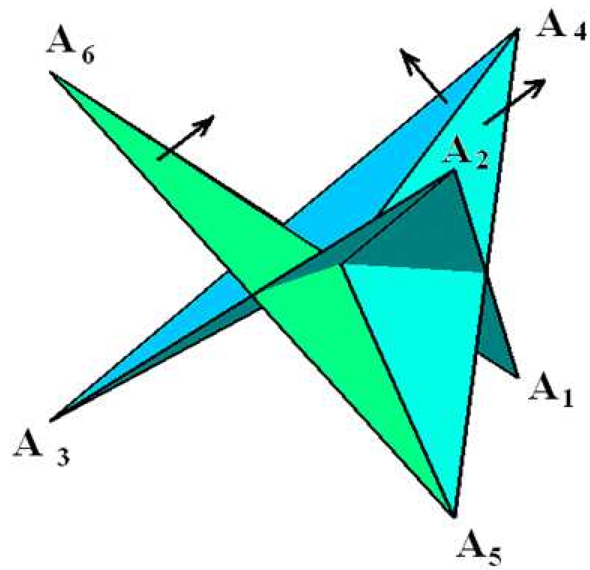

FIGURE 16. The star S_3

A hyperbolic polytope with 6 horns

The construction is similar to the previous example, but still there is a difference.

(1) Start with the planar star with 6 vertices in the plane (x, y). It is a doubly covered triangle (Fig. 15).

Similarly to the previous example, we make the star 3-dimensional by shifting up its even vertices and shifting down the odd ones. This gives a polytopal surface S_3 (Fig. 16).

We take the surface S_3 and its copy S_4 (this time the surface constructed is symmetric with respect to the plane (x, y)). We choose different orientation for the two surfaces: the normal vectors of S_3 look upwards, whereas the normal vectors of S_4 look downwards. Taken together, these stars form a

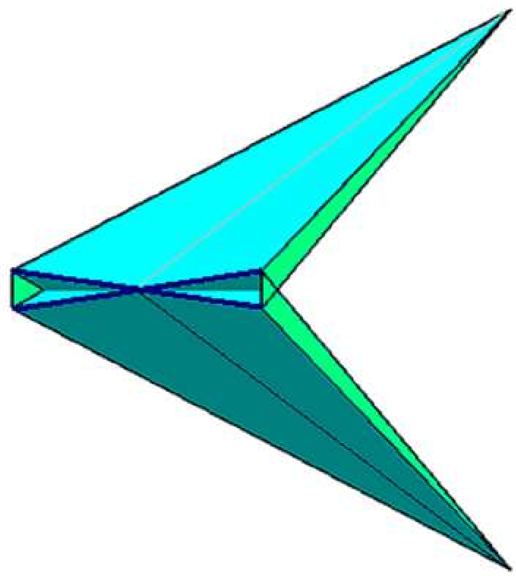

FIGURE 17. Two cross-caps patched to coinciding crosses

double star. This time the double star has not only self-intersections but also self-overlappings.

(2) The boundary of the double star again splits into a union of crosses. Note that some of the crosses coincide. We patch up a cross-cap to each of these crosses. So, dislike the example with 8 horns, the cross-caps, patched up to two coinciding crosses, are placed one over another (Fig. 17)

Finally, we get a hyperbolic polytope 6 horns (Fig. 18).

(3) The associated fan looks analogously to the previous example, but this time the fan has a 6-gon in the center instead of an 8-gon. Thus, we have a virtual polytope.

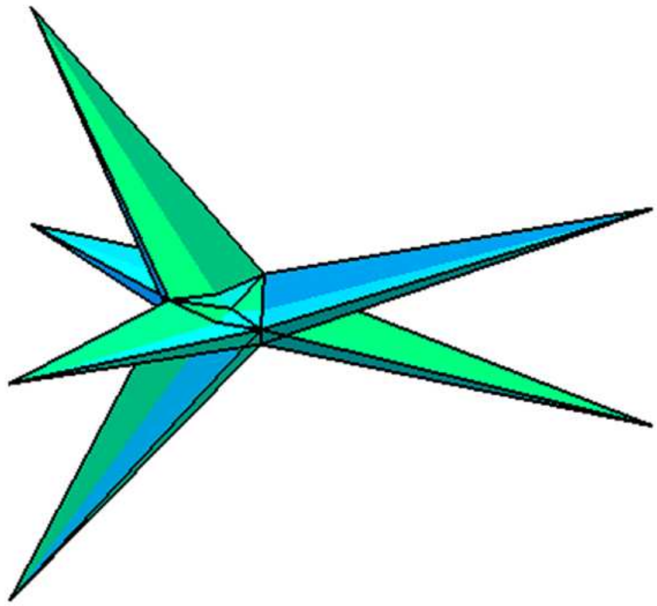

FIGURE 18. Hyperbolic polytope with 6 horns

6. A.D. Alexandrov's conjecture

Non-trivial hyperbolic virtual polytopes appeared originally as an auxiliary construction for various counterexamples to the following conjecture.

A.D. Alexandrov's conjecture

Let $K \subset \mathbb{R}^3$ be a smooth convex body. If for a constant C, at each point of ∂K, we have $R_1 \leq C \leq R_2$, then K is a ball. (R_1 and R_2 stand for the principal curvature radii of ∂K).

It was proven for analytic surfaces by A.D. Alexandrov [1].

For a long time mathematicians were certain about correctness of the conjecture but obtained only some partial results. Recently, Y. Martinez-Maure has given a counterexample. First, he demonstrated that each smooth *hyperbolic hérisson* generates a desired counterexample. Next, he presented such an example. It is a smooth hyperbolic surface with four horns, given by an explicit formula [4].

Due to the theory of hyperbolic virtual polytopes, this counterexample proved to be not unique [6], [7], [8].

Here is a way of constructing (unexpectedly diverse) counterexamples to the above conjecture [6].

- Construct a hyperbolic virtual polytope (this is the most difficult step).
- Smoothen its support function h (preserving saddle property).

- Add to h the support function of a ball (which is sufficiently large to make the sum convex). The result is the support function of a counterexample to the conjecture.

7. Exact coordinates of the vertices

The hyperbolic polytope with 8 horns.

The vertices of the double star are:
$O(0,0,0), (0,1,-\frac{12\sqrt{93}}{841}), (\frac{\sqrt{2}}{2},-\frac{\sqrt{2}}{2},\frac{12\sqrt{93}}{841}), (-1,0,-\frac{12\sqrt{93}}{841}), (\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2},\frac{12\sqrt{93}}{841})$,
$(0,-1,-\frac{12\sqrt{93}}{841}), (-\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2},\frac{12\sqrt{93}}{841}), (1,0,-\frac{12\sqrt{93}}{841}), (-\frac{\sqrt{2}}{2},-\frac{\sqrt{2}}{2},\frac{12\sqrt{93}}{841}), (0,1,\frac{12\sqrt{93}}{841})$,
$(\frac{\sqrt{2}}{2},-\frac{\sqrt{2}}{2},-\frac{12\sqrt{93}}{841}), (-1,0,\frac{12\sqrt{93}}{841}), (\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2},-\frac{12\sqrt{93}}{841}), (0,-1,\frac{12\sqrt{93}}{841}), (-\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2},-\frac{12\sqrt{93}}{841})$,
$(1,0,\frac{12\sqrt{93}}{841})$ and $(-\frac{\sqrt{2}}{2},-\frac{\sqrt{2}}{2},-\frac{12\sqrt{93}}{841})$.

The horns are:
$(\frac{12.3\sqrt{2+\sqrt{2}}}{2}+\frac{\sqrt{2}}{4}, \frac{12.3\sqrt{2-\sqrt{2}}}{2}+\frac{\sqrt{2-\sqrt{2}}}{4}, 4.241)$,
$(\frac{-12.3\sqrt{2-\sqrt{2}}}{2}-\frac{2-\sqrt{2}}{4}, \frac{-12.3\sqrt{2+\sqrt{2}}}{2}-\frac{\sqrt{2}}{4}, -4.241)$,
$(\frac{-12.3\sqrt{2-\sqrt{2}}}{2}-\frac{2-\sqrt{2}}{4}, \frac{12.3\sqrt{2+\sqrt{2}}}{2}+\frac{\sqrt{2}}{4}, 4.241)$,
$(\frac{12.3\sqrt{2+\sqrt{2}}}{2}+\frac{\sqrt{2}}{4}, \frac{-12.3\sqrt{2-\sqrt{2}}}{2}-\frac{2-\sqrt{2}}{4}, -4.241)$,
$(\frac{-12.3\sqrt{2+\sqrt{2}}}{2}-\frac{\sqrt{2}}{4}, \frac{-12.3\sqrt{2-\sqrt{2}}}{2}-\frac{2-\sqrt{2}}{4}, 4.241)$,
$(\frac{12.3\sqrt{2-\sqrt{2}}}{2}+\frac{2-\sqrt{2}}{4}, \frac{12.3\sqrt{2+\sqrt{2}}}{2}+\frac{\sqrt{2}}{4}, -4.241)$,
$(\frac{12.3\sqrt{2-\sqrt{2}}}{2}+\frac{2-\sqrt{2}}{4}, \frac{-12.3\sqrt{2+\sqrt{2}}}{2}-\frac{\sqrt{2}}{4}, 4.241)$,
$(\frac{-12.3\sqrt{2+\sqrt{2}}}{2}-\frac{\sqrt{2}}{4}, \frac{12.3\sqrt{2-\sqrt{2}}}{2}+\frac{2-\sqrt{2}}{4}, -4.241)$.

The hyperbolic polytope with 6 horns.

The vertices of the double star are:
$O(0,0,0), (0,1,-\frac{4}{29}), (\frac{\sqrt{3}}{2},-\frac{1}{2},\frac{4}{29}), (-\frac{\sqrt{3}}{2},-\frac{1}{2},-\frac{4}{29}), (0,1,\frac{4}{29}), (\frac{\sqrt{3}}{2},-\frac{1}{2},-\frac{4}{29})$ and $(-\frac{\sqrt{3}}{2},-\frac{1}{2},\frac{4}{29})$.

The horns are:
$(\frac{87+4\sqrt{3}}{16}, \frac{4+29\sqrt{3}}{16}, \frac{261\sqrt{570}}{3040})$,
$(\frac{87+4\sqrt{3}}{16}, \frac{4+29\sqrt{3}}{16}, -\frac{261\sqrt{570}}{3040})$,
$(-\frac{87+4\sqrt{3}}{16}, \frac{4+29\sqrt{3}}{16}, \frac{261\sqrt{570}}{3040})$,
$(-\frac{87+4\sqrt{3}}{16}, \frac{4+29\sqrt{3}}{16}, -\frac{261\sqrt{570}}{3040})$,
$(0, -\frac{4+29\sqrt{3}}{16}, -\frac{261\sqrt{570}}{3040})$,
$(0, -\frac{4+29\sqrt{3}}{16}, \frac{261\sqrt{570}}{3040})$.

R E F E R E N C E S

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