WEAKLY COMMENSURABLE ARITHMETIC GROUPS, LENGTHS OF CLOSED GEODESICS AND ISOSPECTRAL LOCALLY SYMMETRIC SPACES

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Dedicated to the memory of A.R.'s mother Izabella B. Rapinchuk

1. Introduction

The goal of this paper is two-fold. First, we introduce and analyze a new relationship between (Zariski-dense) abstract subgroups of the group of F-rational points of a connected semi-simple algebraic group defined over a field F, which we call *weak commensurability.* This relationship is expressed in terms of the eigenvalues of individual elements, and does not involve any structural connections between the subgroups. Nevertheless, it turns out that weakly commensurable S-arithmetic subgroups always split into finitely many commensurability classes, and that in certain types of groups, any two weakly commensurable S-arithmetic subgroups are actually commensurable. Second, we use results and conjectures in transcendental number theory to relate weak commensurability with interesting differential geometric problems on length-commensurable, and isospectral, locally symmetric spaces, and to settle a series of open questions in this area by applying our results on weakly commensurable arithmetic (and more general) subgroups. These applications lead us to believe that the notion of weak commensurability is likely to be useful in investigation of a variety of problems in geometry and ergodic theory.

We begin with the definition of weak commensurability. Let G be a connected semi-simple algebraic group defined over a field F.

Definitions. 1. Two semi-simple elements g_1, g_2 of $G(F)$ are said to be *weakly commensurable* if there exist maximal F-tori T_1, T_2 of G such that $g_i \in T_i(F)$, and for some characters χ_i of T_i (defined over an algebraic closure $\overline{F}$ of F), we have

$$\chi_1(g_1) = \chi_2(g_2) \neq 1.$$[1]

2. Two subgroups Γ_1 and Γ_2 of $G(F)$ are *weakly commensurable* if every semi-simple element $\gamma_1 \in \Gamma_1$ of infinite order is weakly commensurable to

[1]In other words, the subgroup of $\overline{F}^{\times}$ generated by the eigenvalues (in the adjoint representation) of g_1 intersects the subgroup generated by the eigenvalues of g_2 nontrivially.

some semi-simple element $\gamma_2 \in \Gamma_2$, and conversely, every semi-simple element $\gamma_2 \in \Gamma_2$ of infinite order is weakly commensurable to some semi-simple element $\gamma_1 \in \Gamma_1$.

Our first basic result is the following.

Theorem A. *Let G be a connected absolutely simple algebraic group defined over a field F of characteristic zero. Let Γ_1 and Γ_2 be two finitely generated Zariski-dense subgroups of $G(F)$, and K_{Γ_i} be the subfield of F generated by* $\mathrm{tr}\,\mathrm{Ad}\gamma$ *for $\gamma \in \Gamma_i$. If Γ_1 and Γ_2 are weakly commensurable, then $K_{\Gamma_1} = K_{\Gamma_2}$.*

Most of the results of this paper are on arithmetic subgroups. In fact, the central issue for us is what can be said about two forms over number fields, of a connected absolutely simple F-group G, given that these forms contain weakly commensurable Zariski-dense S-arithmetic subgroups. To give the precise statements (see Theorems B–E), we need to describe our set-up more carefully. Let G be a connected absolutely simple algebraic group defined over a field F of characteristic zero. Suppose we are given a number field K, an embedding $K \hookrightarrow F$, and an algebraic K-group G_0 such that the F-group ${}_F G_0$ obtained by extension of scalars $K \hookrightarrow F$, is F-isomorphic to G (in other words, G_0 is an F/K-form of G). Then we have the embedding $\iota \colon G_0(K) \hookrightarrow G(F)$, which is well-defined up to an F-automorphism of G. Now let S be a finite set of places of K which contains the set V_∞^K of all archimedean places, but does not contain any nonarchimedean place where G_0 is anisotropic. Let $\mathcal{O}_K(S)$ denote the ring of S-integers in K (with $\mathcal{O}_K = \mathcal{O}_K(V_\infty^K)$ denoting the ring of algebraic integers in K), and let $G_0(\mathcal{O}_K(S))$ denote the corresponding S-arithmetic subgroup defined in terms of a fixed K-embedding $G_0 \hookrightarrow \mathrm{GL}_n$, i.e., $G_0(\mathcal{O}_K(S)) = G_0(K) \cap \mathrm{GL}_n(\mathcal{O}_K(S))$. We will say that two subgroups Γ', Γ'' of $G(F)$ are commensurable up to an F-automorphism of G if there exists an F-automorphism σ of G such that $\sigma(\Gamma')$ and Γ'' are commensurable in the usual sense (i.e., their intersection has finite index in both of them). Then any subgroup Γ of $G(F)$ which is commensurable with $\iota(G_0(\mathcal{O}_K(S)))$ up to an F-automorphism of G, will be called a (G_0, K, S)-*arithmetic subgroup*[2]. As usual, (G_0, K, V_∞^K)-arithmetic subgroups will simply be called (G_0, K)-arithmetic. Now, let $\Gamma_i \subset G(F)$, where $i = 1, 2$, be Zariski-dense (G_i, K_i, S_i)-arithmetic subgroups. The key question for us is when does the fact that Γ_1 and Γ_2 are weakly commensurable imply that they are commensurable up to an F-automorphism of G, i.e., $K_1 = K_2$, $S_1 = S_2$ and G_1 and G_2 are K-isomorphic (cf. Proposition 2.5). Theorems B, C, D and E address this question.

[2]Notice that if G_0 is anisotropic over K_v, where v is a nonarchimedean place of K, then $G_0(\mathcal{O}_K(S))$ is commensurable with $G_0(\mathcal{O}_K(S \cup \{v\}))$, so the classes of S- and $(S \cup \{v\})$-arithmetic subgroups coincide. Thus, the above assumption on S is necessary if one wants to recover S from a given S-arithmetic subgroup.

Theorem B. *If Zariski-dense (G_i, K_i, S_i)-arithmetic subgroups Γ_i of $G(F)$ are weakly commensurable, where $i = 1, 2$, then $K_1 = K_2$ and $S_1 = S_2$.*

Examples 6.5 and 6.6 show that the existence of weakly commensurable S-arithmetic subgroups does not guarantee that G_1 and G_2 are always isomorphic over $K := K_1 = K_2$. In the next theorem we list the cases where it can be asserted that G_1 and G_2 are K-isomorphic, and then give a general finiteness result for the number of K-isomorphism classes.

Theorem C. *Suppose G is not of type A_n $(n > 1)$, D_n $(n \geqslant 4)$ or E_6. If $G(F)$ contains Zariski-dense weakly commensurable (G_i, K, S)-arithmetic subgroups Γ_i for $i = 1, 2$, then $G_1 \simeq G_2$ over K, and hence Γ_1 and Γ_2 are commensurable up to an F-automorphism of G.*

Theorem D. *Let Γ_1 be a Zariski-dense (G_1, K, S)-arithmetic subgroup of $G(F)$. Then the set of K-isomorphism classes of K-forms G_2 of G such that $G(F)$ contains a Zariski-dense (G_2, K, S)-arithmetic subgroup weakly commensurable to Γ_1, is finite. In other words, the set of all (K, S)-arithmetic subgroups of $G(F)$ which are weakly commensurable to a given (K, S)-arithmetic subgroup is a union of finitely many commensurability classes.*

A noteworthy fact about weak commensurability is that it has the following implication for the existence of unipotent elements in arithmetic subgroups (even though it is formulated entirely in terms of semi-simple ones).

Theorem E. *Assume that $G(F)$ contains Zariski-dense (G_1, K, S)- and (G_2, K, S)-arithmetic subgroups which are weakly commensurable. Then the Tits indices of G_1/K and G_2/K, and for every place v of K, the Tits indices of G_1/K_v and G_2/K_v, are isomorphic. In particular, $\mathrm{rk}_K\, G_1 = \mathrm{rk}_K\, G_2$, and consequently if G_1 is K-isotropic, then so is G_2.*

(For a description of Tits index of a semi-simple algebraic group, see §7.)

The following result asserts that a lattice which is weakly commensurable with an S-arithmetic group is arithmetic.

Theorem F. *Let G be a connected absolutely simple algebraic group over a nondiscrete locally compact field F of characteristic zero, and let Γ_1 and Γ_2 be two Zariski-dense lattices in $G(F)$. Assume that Γ_1 is (K, S)-arithmetic. If Γ_1 and Γ_2 are weakly commensurable, then Γ_2 is also (K, S)-arithmetic.*

To prove these theorems, we develop further the ideas introduced in our earlier papers [26]-[28], in particular the notion of irreducible tori, and the techniques used there for proving the existence of semi-simple elements with some "amazing" properties in a given Zariski-dense subgroup. These techniques relied primarily on algebraic and number-theoretic considerations, but turned out to be quite useful for a variety of questions of geometric nature. The results on weak commensurability stated above were motivated by, and actually enabled us to settle, some problems about the lengths of closed geodesics in, and isospectrality of, arithmetically defined locally symmetric spaces. We now proceed to describe these geometric applications.

For a Riemannian manifold M, the *length spectrum* $\mathcal{L}(M)$ (resp., the *weak length spectrum* $L(M)$) is defined to be the set of lengths of closed geodesics in M with multiplicities (resp., without multiplicities), cf. [15]. The following question has received considerable attention: to what extent do $\mathcal{L}(M)$, $L(M)$, or the spectrum of the Laplace operator, determine M? It turns out that all these sets are interrelated: for example, two compact hyperbolic 2-manifolds are isospectral[3] if and only if they have the same length spectrum, cf. [18]; two hyperbolic 3-manifolds are isospectral if and only if they have the same *complex-length* spectrum, cf. [3] or [12]. Furthermore, it is known that isospectral compact locally symmetric spaces of nonpositive curvature have the same weak length spectrum, see Theorem 10.1 below. The first examples of isospectral but not isometric (although commensurable[4]) compact hyperbolic 2- and 3-manifolds were given in [37]. Recently, in [16], isospectral noncommensurable locally symmetric spaces have been constructed. On the other hand, in 1985 Sunada [34] described a general method for producing examples of nonisometric (but commensurable) isospectral manifolds. Variations of Sunada's construction has been used in [15] to give examples of hyperbolic manifolds with equal weak length spectra but different volumes. Earlier, in [31], the same approach was used to produce nonisometric hyperbolic 3-manifolds with equal weak length spectra. It should be pointed out that Sunada's construction, which is the only known general method for constructing manifolds with the same (weak) length, or Laplace operator, spectra, *always* produces commensurable manifolds (in particular, the examples in [15] and [31] are commensurable). So, the following question was raised:

(1) *Let M_1 and M_2 be two (hyperbolic) manifolds (of finite volume or even compact). Suppose $L(M_1) = L(M_2)$. Are M_1 and M_2 necessarily commensurable?*

(cf., for example, [31]). One may generalize this question by introducing the notion of length commensurability, which in particular allows us to replace the manifolds under consideration with commensurable ones: we say that M_1 and M_2 are *length-commensurable* if $\mathbb{Q} \cdot L(M_1) = \mathbb{Q} \cdot L(M_2)$. Now, (1) can be reformulated as follows:

(2) *Suppose M_1 and M_2 are length-commensurable. Are they commensurable?*

[3]Two compact Riemannian manifolds are said to be *isospectral* if their Laplace operators have the same eigenvalues with the same multiplicities, cf. §10.

[4]Two manifolds are called *commensurable* if they admit a common finite-sheeted cover.

In [31], an affirmative answer (to (1)) was given for arithmetically defined hyperbolic 2-manifolds, and very recently in [7] a similar result has been obtained for hyperbolic 3-manifolds. The results of this paper provide an affirmative answer to (2) for arithmetically defined even-dimensional hyperbolic manifolds, but a negative answer for complex hyperbolic manifolds and real hyperbolic manifolds of dimension $4k+1$. In fact, we analyze the problem in the general context of arithmetically defined locally symmetric spaces.

Let G be a connected semi-simple real algebraic subgroup of SL_n, $\mathcal{G}$ be $G(\mathbb{R})$ considered as a Lie group, and let $\mathcal{K}$ be a maximal compact subgroup of $\mathcal{G}$. Then $\mathfrak{X} = \mathcal{K}\backslash\mathcal{G}$ is the symmetric space of $\mathcal{G}$. Given a discrete torsion-free subgroup Γ of $\mathcal{G}$, the quotient $\mathfrak{X}_\Gamma = \mathfrak{X}/\Gamma$ is a locally symmetric space. We say that $\mathfrak{X}_\Gamma$ is *arithmetically defined* if Γ is an arithmetic subgroup of $\mathcal{G}$ (cf. [17], Ch. IX). According to the following theorem, length-commensurability of locally symmetric spaces is closely related to weak commensurability of the corresponding discrete subgroups.

Theorem 8.7. *Let Γ_1, Γ_2 be discrete torsion-free subgroups of $\mathcal{G}$. If Γ_1 and Γ_2 are not weakly commensurable, then, possibly after interchanging them, the following assertions hold:*

(i) *If $\mathrm{rk}_{\mathbb{R}}\, G = 1$, then there exists $\lambda_1 \in L(\mathfrak{X}_{\Gamma_1})$ such that for* any $\lambda_2 \in L(\mathfrak{X}_{\Gamma_2})$, *the ratio λ_1/λ_2 is irrational.*

(ii) *If there exists a number field K such that both Γ_1 and Γ_2 can be conjugated into $\mathrm{SL}_n(K)$, and Schanuel's conjecture holds, then there exists $\lambda_1 \in L(\mathfrak{X}_{\Gamma_1})$ which is algebraically independent from* any $\lambda_2 \in L(\mathfrak{X}_{\Gamma_2})$.

In either case, (under the above assumptions) $\mathfrak{X}_{\Gamma_1}$ and $\mathfrak{X}_{\Gamma_2}$ are not length-commensurable.

We would like to emphasize that while the results below for rank one locally symmetric spaces (which include hyperbolic spaces of all types) are unconditional, the results for spaces of higher rank depend on the validity of the well-known conjecture in transcendental number theory due to Schanuel (see §8 for the statement); needless to say that our results in §§ 2-7, 9 on weak commensurability (in particular, Theorems A-F) *do not* involve *any* transcendental number theory.

Henceforth, we will assume that G is a connected absolutely simple real algebraic group; the corresponding real Lie group $\mathcal{G} = G(\mathbb{R})$ will then also be called "absolutely simple". Let Γ_1 and Γ_2 be torsion-free discrete subgroups of $\mathcal{G}$ such that the associated locally symmetric spaces $\mathfrak{X}/\Gamma_1$ and $\mathfrak{X}/\Gamma_2$ are length-commensurable. Then Theorem 8.7 implies that Γ_1 and Γ_2 are weakly commensurable. Now observing that $\mathfrak{X}_{\Gamma_1}$ and $\mathfrak{X}_{\Gamma_2}$ are commensurable as manifolds if and only if Γ_1 and Γ_2 are commensurable up to an $\mathbb{R}$-automorphism of G, from Theorems C and D, we obtain

Theorem 8.9. *Each class of length-commensurable arithmetically defined locally symmetric spaces of $\mathcal{G} = G(\mathbb{R})$ is a union of finitely many commensurability classes. It in fact consists of a single commensurabilty class if G is not of type A_n $(n > 1)$, D_n $(n \geqslant 4)$, or E_6.*

Furthermore, Theorem E implies the following rather surprising result which has so far defied attempts to prove it purely geometrically.

Theorem 8.12. *Let $\mathfrak{X}_{\Gamma_1}$ and $\mathfrak{X}_{\Gamma_2}$ be two arithmetically defined locally symmetric spaces of the same absolutely simple real Lie group $\mathcal{G}$. If they are length-commensurable, then the compactness of one of them implies the compactness of the other.*

Theorem A shows that length-commensurability provides some information about the fundamental groups even without any assumptions of arithmeticity.

Theorem 8.13. *Let $\mathfrak{X}_{\Gamma_1}$ and $\mathfrak{X}_{\Gamma_2}$ be two locally symmetric spaces of the same absolutely simple real Lie group $\mathcal{G}$, modulo finitely generated torsion-free Zariski-dense discrete subgroups Γ_1 and Γ_2. Denote by K_{Γ_i} the field generated by $\mathrm{tr}\,\mathrm{Ad}\,\gamma$ for $\gamma \in \Gamma_i$. If $\mathfrak{X}_{\Gamma_1}$ and $\mathfrak{X}_{\Gamma_2}$ are length-commensurable, then $K_{\Gamma_1} = K_{\Gamma_2}$.*

In §9, we present a general cohomological construction which, in particular, enables one to come up with examples of length-commensurable, but not commensurable, arithmetically defined locally symmetric spaces associated to an absolutely simple Lie group of any of the following types: A_n, D_{2n+1} $(n > 1)$, or E_6, see 9.14 (thus, the second assertion of Theorem 8.9 definitely cannot be extended to these types). Towards this end, we establish a new local-global principle for the existence of an embedding of a given K-torus as a maximal torus in a absolutely simple simply connected K-group (for the precise assertion, see Theorem 9.5). Using this local-global principle, we show that there exist nonisomorphic K-forms G_1 and G_2 of an absolutely simple K-group of each of the types A_n, D_{2n+1} $(n > 1)$, or E_6, such that (i) G_1 is isomorphic to G_2 over K_v, for all places v of K, and (ii) given a maximal K-torus T_i of G_i, there is an isomorphism $G_i \to G_{3-i}$ whose restriction to T_i is defined over K. Such K-forms are likely to be of interest in Langlands program. Given such nonisomorphic K-forms G_1 and G_2, any arithmetic subgroup Γ_1 of $G_1(K)$ is weakly commensurable, but not commensurable, to any arithmetic subgroup Γ_2 of $G_2(K)$, and the associated locally symmetric spaces $\mathfrak{X}_{\Gamma_1}$ and $\mathfrak{X}_{\Gamma_2}$ are length commensurable but not commensurable (see Proposition 9.13 and 9.14).

Since isospectral compact locally symmetric spaces of nonpositive curvature are length-commensurable (Theorem 10.1), the following theorem, which answers Mark Kac's famous question "Can one hear the shape of a drum?" for arithmetically defined compact locally symmetric spaces, is a consequence of Theorem 8.9.

Theorem 10.4. *Any two arithmetically defined compact isospectral locally symmetric spaces of an absolutely simple real Lie group of type other than* A_n $(n > 1)$, D_n $(n \geqslant 4)$, *and* E_6, *are commensurable to each other.*

We finally mention some results dealing with arithmeticity. If Γ_1 and Γ_2 are torsion-free lattices in $\mathcal{G}$ such that $\mathfrak{X}_{\Gamma_1}$ and $\mathfrak{X}_{\Gamma_2}$ are length-commensurable, then Theorem 8.7 implies that Γ_1 and Γ_2 are weakly commensurable, and according to Theorem F, if one of them is arithmetic, then so is the other (cf. Theorem 8.8). Moreover, if Γ_1 and Γ_2 are torsion-free cocompact discrete subgroups of $\mathcal{G}$ such that $\mathfrak{X}_{\Gamma_1}$ and $\mathfrak{X}_{\Gamma_2}$ are isospectral, then by Theorem 10.1 they are length-commensurable, and consequently we again see that if one of the Γ_i's is arithmetic then so is the other (cf. Theorem 10.3).

Notations and conventions. Unless stated otherwise, all our fields will be of characteristic zero. For a number field K, we let V^K (resp., V_∞^K and V_f^K) denote the set of all places (resp., the subsets of archimedean and nonarchimedean places). For a torus T, we let $X(T)$ denote the character group, and for a morphism $\pi\colon T_1 \to T_2$ between two tori, we let $\pi^*\colon X(T_2) \to X(T_1)$ denote the induced homomorphism of the character groups. If T is defined over K, then K_T will denote the (minimal) splitting field of T over K and $X(T)$ will be considered as a module over the Galois group $\mathrm{Gal}(K_T/K)$.

In the sequel, all number fields are assumed to be contained in the field $\mathbb{C}$ of complex numbers. For a subfield K (resp., K_i) of $\mathbb{C}$, $\overline{K}$ (resp., $\overline{K}_i$) will denote its algebraic closure in $\mathbb{C}$. For a place v of a number field K (resp., K_i), $\overline{K}_v$ (resp., $\overline{K}_{iv}$) will denote an algebraic closure of the completion K_v (resp., K_{iv}) of K (resp., K_i) at v. In particular, $\overline{\mathbb{Q}}_p$ will denote an algebraic closure of $\mathbb{Q}_p$.

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2. Preliminaries

We begin with a simple comment on the notion of weak commensurability of semi-simple elements.

Lemma 2.1. *Let $\gamma_1, \gamma_2 \in G(F)$ be semi-simple elements. The following conditions are equivalent:*

(1) *γ_1 and γ_2 are weakly commensurable, i.e.,* there exist *maximal F-tori T_i of G for $i = 1, 2$ such that $\gamma_i \in T_i$ and $\chi_1(\gamma_1) = \chi_2(\gamma_2) \neq 1$ for some characters $\chi_i \in X(T_i)$;*

(2) for any *maximal F-tori T_i of G such that $\gamma_i \in T_i$, there exist characters $\chi_i \in X(T_i)$ such that $\chi_1(\gamma_1) = \chi_2(\gamma_2) \neq 1$.*

While (2) trivially implies (1), the opposite implication follows from the fact that if C_i is the Zariski-closure of the subgroup generated by γ_i, then for *any* torus T_i containing γ_i, the restriction map $X(T_i) \to X(C_i)$ is surjective (cf. [4], 8.2).

Corollary 2.2. *For $i = 1, 2$, let K_i be a subfield of F, G_i be an F/K_i-form of G, and $\gamma_i \in G_i(K_i) \hookrightarrow G(F)$ be a semi-simple element. Then γ_1 and γ_2 are weakly commensurable if and only if there exist maximal K_i-tori T_i of G_i such that $\chi_1(\gamma_1) = \chi_2(\gamma_2) \neq 1$ for some $\chi_i \in X(T_i)$.*

This follows from the lemma because every semi-simple $\gamma_i \in G_i(K_i)$ is contained in a maximal K_i-torus of G_i.

We continue with two elementary lemmas on weak commensurability of subgroups. The first lemma enables one to replace each of the two weakly commensurable subgroups with a commensurable subgroup.

Lemma 2.3. *Let Γ_1 and Γ_2 be two weakly commensurable finitely generated Zariski-dense subgroups of $G(F)$. For $i = 1,\ 2$, if Δ_i is a subgroup of $G(F)$ commensurable with Γ_i for $i = 1, 2$, then the subgroups Δ_1 and Δ_2 are weakly commensurable.*

Proof. We recall that a subgroup Δ of $\mathrm{GL}_n(K)$ is *neat* if for every $\delta \in \Delta$, the subgroup of $\overline{K}^{\times}$ generated by the eigenvalues of Δ is torsion-free. According to a result proved by Borel (cf. [30], Theorem 6.11) every finitely generated subgroup of $\mathrm{GL}_n(K)$ contains a neat subgroup of finite index. We fix a neat subgroup of finite index $\Theta \subset \Gamma_1 \cap \Delta_1$; then $[\Delta_1 : \Theta] < \infty$. Given a semi-simple element $\delta_1 \in \Delta_1$ of infinite order we can pick $n_1 \geqslant 1$ so that $\gamma_1 := \delta_1^{n_1} \in \Theta$. Since Γ_1 and Γ_2 are weakly commensurable, one can find $\gamma_2 \in \Gamma_2$ so that

$$\chi_1(\gamma_1) = \chi_2(\gamma_2) \neq 1$$

for some characters χ_i of the maximal F-tori T_i with the property $\gamma_i \in T_i$ for $i = 1, 2$. Now, pick $n_2 \geqslant 1$ so that $\delta_2 := \gamma_2^{n_2} \in \Gamma_2 \cap \Delta_2$. Then

$$(n_2\chi_1)(\gamma_1) = ((n_1 n_2)\chi_1)(\delta_1) = \chi_2(\delta_2). \tag{1}$$

It remains to observe that since $\chi_1(\gamma_1) \neq 1$ belongs to the subgroup generated by the eigenvalues of γ_1, which by our construction is torsion-free, it is not a root of unity. This implies that the common value in (1) is $\neq 1$, and therefore δ_1 and δ_2 are weakly commensurable. Thus, every semi-simple $\delta_1 \in \Delta_1$ of infinite order is weakly commensurable to some semi-simple $\delta_2 \in \Delta_2$, and by symmetry, every semi-simple $\delta_2 \in \Delta_2$ of infinite order is weakly commensurable to some semi-simple $\delta_2 \in \Delta_2$, which makes Δ_1 and Δ_2 weakly commensurable. □

The next lemma shows that in the analysis of weak commensurability of subgroups, one can replace the ambient algebraic group with an isogenous group.

Lemma 2.4. *Let $\pi\colon G \to G'$ be an F-isogeny of connected semi-simple algebraic F-groups, and let Γ_1, Γ_2 be two finitely generated Zariski-dense subgroups of $G(F)$. Then Γ_1 and Γ_2 are weakly commensurable if and only if their images $\Gamma'_1 = \pi(\Gamma_1)$ and $\Gamma'_2 = \pi(\Gamma_2) \subset G'(F)$ are weakly commensurable.*

Proof. One direction is almost immediate. Namely, suppose Γ'_1 and Γ'_2 are weakly commensurable. Then for a given semi-simple element γ_1 of Γ_1 of infinite order, there exists a semi-simple element $\gamma_2 \in \Gamma_2$ so that for some maximal F-tori $T'_i \ni \pi(\gamma_i)$ of G', there exist characters $\chi'_i \in X(T'_i)$ such that

$$\chi'_1(\pi(\gamma_1)) = \chi'_2(\pi(\gamma_2)) \neq 1.$$

Then $T_i := \pi^{-1}(T'_i)$ are maximal F-tori of G containing γ_i $(i = 1, 2)$, and for their characters $\chi_i = \pi^*(\chi'_i)$ we have

$$\chi_1(\gamma_1) = \chi_2(\gamma_2) \neq 1.$$

This, combined with a "symmetric" argument, implies that Γ_1 and Γ_2 are weakly commensurable.

Conversely, suppose that Γ_1 and Γ_2 are weakly commensurable, and for $i = 1$, 2, pick neat subgroups Δ_i of Γ_i of finite index. By Lemma 2.3, it is enough to show that $\pi(\Delta_1)$ and $\pi(\Delta_2)$ are weakly commensurable. Let δ_1 be a nontrivial semi-simple element of Δ_1. Then there exists $\delta_2 \in \Delta_2$ such that for some maximal F-tori $T_i \ni \delta_i$ of G, and some characters χ_i of T_i, we have

$$\chi_1(\delta_1) = \chi_2(\delta_2) \neq 1. \tag{2}$$

Set $T'_i = \pi(T_i) \ni \pi(\gamma_i)$. If $m = |\ker \pi|$, then there exist characters $\chi'_i \in X(T'_i)$ such that $m\chi_i = \pi^*(\chi'_i)$. Since Δ_1 is neat, the common value in (2) is not an m-th root of unity, and then

$$\chi'_1(\pi(\delta_1)) = \chi'_2(\pi(\delta_2)) = \chi_1(\delta_1)^m \neq 1.$$

This, together with a "symmetric" argument, implies that $\pi(\Delta_1)$ and $\pi(\Delta_2)$ are weakly commensurable. □

Next, we prove the following (known) proposition which characterizes commensurable S-arithmetic subgroups. Since we have not been able to find a reference for its proof, we give a complete argument.

Proposition 2.5. *Let G be a connected absolutely simple algebraic group over a field F of characteristic zero, and let $\Gamma_i \subset G(F)$, for $i = 1, 2$, be a Zariski-dense (G_i, K_i, S_i)-arithmetic subgroup. Then Γ_1 and Γ_2 are commensurable up to an F-automorphism of G if and only if $K_1 = K_2 =: K$, $S_1 = S_2$, and G_1 and G_2 are K-isomorphic.*

Proof. Let $\iota_i \colon G_i \to G$ be an F-isomorphism used to define (G_i, K_i, S_i)-arithmetic subgroups, where G_i is defined over a subfield K_i of F (technically, ι_i should have been written as $\iota_i \colon {}_F G_i \to G$, but our simplified notation will not lead to a confusion). One implication is obvious. Namely, suppose $K_1 = K_2 =: K$, $S_1 = S_2 =: S$, and let $\tau \colon G_1 \to G_2$ be a K-isomorphism. Then $\tau(G_1(\mathcal{O}_K(S)))$ is commensurable with $G_2(\mathcal{O}_K(S))$, and $\sigma := \iota_2 \circ \tau \circ \iota_1^{-1}$ is an F-automorphism of G. Clearly, $\sigma(\iota_1(G_1(\mathcal{O}_K(S))))$ is commensurable with $\iota_2(G_2(\mathcal{O}_K(S)))$, implying that $\sigma(\Gamma_1)$ is commensurable with Γ_2, as required.

Conversely, suppose σ is an F-automorphism of G such that $\sigma(\Gamma_1)$ and Γ_2 are commensurable. Then to prove that $K_1 = K_2$ we apply the following assertion to $\sigma(\Gamma_1) \cap \Gamma_2$, which is both, (G_1, K_1, S_1)- and (G_2, K_2, S_2)-arithmetic.

Lemma 2.6. *Let G be a connected absolutely simple algebraic group over a field F of characteristic zero, and $\Gamma \subset G(F)$ be a Zariski-dense (G_0, K, S)-arithmetic subgroup. Then the subfield K_Γ of F generated by $\mathrm{tr}\, \mathrm{Ad}_G(\gamma)$ for $\gamma \in \Gamma$ coincides with K.*

Proof. We will assume (as we may) that the group G is adjoint and is realized as a matrix group by means of the adjoint representation on the Lie algebra $\mathfrak{g}$ of G. By definition, there exists an F-isomorphism $\iota \colon G_0 \simeq G$, and we will use its differential $d\iota$ to identify the Lie algebra $\mathfrak{g}_0$ of G_0 with $\mathfrak{g}$. Set

$$\Gamma_0 := \iota^{-1}(\Gamma) \subset G_0(F).$$

Then Γ_0 is a (K, S)-arithmetic subgroup. As G_0 is of adjoint type, Γ_0 is contained in $G_0(K)$ (see, for example, Proposition 1.2 of [5]). This implies that $\mathrm{tr}\mathrm{Ad}_{G_0}(\Gamma_0) = \mathrm{tr}\mathrm{Ad}_G(\Gamma) \subset K$, hence the inclusion $K_\Gamma \subset K$. Conversely, according to Theorem 1 of Vinberg [38], there exists a basis of $\mathfrak{g}_0$ in which Γ_0 is represented by matrices with entries in K_Γ, and then G_0 is actually defined over K_Γ. Let $A \subset \mathrm{End}\, \mathfrak{g}$ be the linear span of Γ_0. Then A is invariant under conjugation by Γ_0, hence by G_0, so we can consider the corresponding (faithful) representation $\rho \colon G_0 \to \mathrm{GL}(A)$. Obviously, one can pick a basis of A which is contained in the K_Γ-span A_0 of Γ_0. Furthermore, any subgroup of finite index $\Gamma_0' \subset \Gamma_0$ has the same Zariski-closure as Γ_0 (viz., G_0), and hence the same K_Γ-span (viz., A_0). Since for any $g \in G_0(K)$, the intersection $\Gamma_0 \cap g^{-1}\Gamma_0 g$ is of finite index in Γ_0, we see that A_0 is invariant under conjugation by $G_0(K)$, and therefore $\rho(G_0(K))$ is represented by matrices with entries in K_Γ. Thus, $G_0(K_\Gamma) = G_0(K)$, so our claim is an immediate consequence of the following lemma.

Lemma 2.7. *Let G be a connected reductive algebraic group of positive dimension defined over an infinite field K. Then for any nontrivial extension F/K, $G(K) \neq G(F)$.*

Proof. We may assume that G is connected. It is known that G is unirational over K (cf. [4], Theorem 18.2), i.e., there exists a dominant K-rational map $f\colon \mathbb{A}^n \to G$. We pick a line ℓ in $\mathbb{A}^n$, defined over K, such that f restricts to a nonconstant map on ℓ. Let C be the Zariski-closure of $f(\ell(K))$. Then C is a curve defined over K; furthermore, by Lüroth's theorem, C is rational over K, i.e., it is K-isomorphic to an open subvariety of $\mathbb{A}^1$. This immediately implies that $C(K) \neq C(F)$, and our claim follows. □

Now, consider the F-isomorphism $\tau = \iota_2^{-1} \circ \sigma \circ \iota_1$ between G_1 and G_2. We can obviously choose subgroups Δ_i of $G_i(\mathcal{O}_{K_i}(S_i))$ of finite index so that $\sigma(\iota_1(\Delta_1)) = \iota_2(\Delta_2)$, and then $\tau(\Delta_1) = \Delta_2$. Since Δ_i is a Zariski-dense subgroup of $G_i(K)$, where $K := K_1 = K_2$, we see that τ is in fact defined over K. Next, take any $v \notin S_1$. Since the closure of Δ_1 in $G_1(K_v)$ is compact, we obtain that the closure of $\Delta_2 = \tau(\Delta_1)$ in $G_2(K_v)$ is also compact. If we assume that $v \in S_2$, then the facts that $G_2(K_v)$ is noncompact by our construction, and Δ_2 is a lattice in $\prod_{w \in S_2} G_2(K_w)$, yield a contradiction. Thus, $v \notin S_2$, proving the inclusion $S_2 \subset S_1$. The opposite inclusion is proved similarly, so $S_1 = S_2$. □

Remark 2.8. The assertion of Lemma 2.7 remains true also over a finite field K for any connected reductive group G which is not a torus. Indeed, in this case G is quasi-split over K (cf. [4], Proposition 16.6), and therefore it contains a 1-dimensional split torus C. Clearly, $C(K) \neq C(F)$, implying that $G(K) \neq G(F)$.

Let now $G = T$ be a torus over $K = \mathbb{F}_q$, and let $F = \mathbb{F}_{q^m}$ with $m > 1$. It follows from ([39], 9.1) that

$$|T(K)| = \prod_{i=1}^{d}(q - \lambda_i) \quad \text{and} \quad |T(F)| = \prod_{i=1}^{d}(q^m - \lambda_i^m),$$

where λ_i are certain complex roots of unity and $d = \dim T$. We have

$$|q - \lambda_i| \leqslant q + 1 \quad \text{and} \quad |q^m - \lambda_i^m| \geqslant q^m - 1,$$

so if $q^m - q > 2$, which is always the case unless $q = 2 = m$, then $|T(F)| > |T(K)|$. Suppose now that $q = 2 = m$. Clearly, $|T(K)| = |T(F)|$ is possible only if $|q - \lambda_i| = q + 1$, i.e., $\lambda_i = -1$, for all i. This means that $T \simeq \left(R^{(1)}_{F/K}(\mathrm{GL}_1)\right)^d$, where $R^{(1)}_{F/K}(\mathrm{GL}_1)$ is the norm one torus associated with the extension $F/K = \mathbb{F}_4/\mathbb{F}_2$. For these tori we have $T(K) = T(F)$, and our argument shows that these are the only exceptions to Lemma 2.7 over finite fields.

3. Results on irreducible tori

A pivotal role in the proof of Theorems A-F is played by a reformulation of Theorem 3 of [27]. To explain this reformulation, we need to introduce some additional notations. Since the results given below will be applied to different groups over different fields (G_1, G_2 over K_1, K_2 etc.), to avoid any confusion, we will use script letters in this section.

Let $\mathcal{K}$ be an infinite field and $\mathcal{G}$ be a connected absolutely simple algebraic $\mathcal{K}$-subgroup of GL_n. Let $\mathcal{T}$ be a maximal $\mathcal{K}$-torus of $\mathcal{G}$. As usual, $\Phi = \Phi(\mathcal{G}, \mathcal{T})$ will denote the root system of $\mathcal{G}$ with respect to $\mathcal{T}$, and $W(\Phi)$, or $W(\mathcal{G}, \mathcal{T})$, the Weyl group of Φ. We shall denote by $\mathcal{K}_{\mathcal{T}}$ the (minimal) splitting field of $\mathcal{T}$ in a fixed algebraic closure $\overline{\mathcal{K}}$. Then there exists a natural injective homomorphism $\theta_{\mathcal{T}}\colon \mathrm{Gal}(\mathcal{K}_{\mathcal{T}}/\mathcal{K}) \to \mathrm{Aut}(\Phi)$. The following result is a strengthening of Theorem 3(i) of [27], which does not require any significant changes in the proof.

Theorem 3.1. *Let $\mathcal{G}$ be a connected absolutely simple algebraic group defined over a finitely generated field $\mathcal{K}$ of characteristic zero, and $\mathcal{L}$ be a finitely generated field containing $\mathcal{K}$. Let r be the number of nontrivial conjugacy classes of the Weyl group of $\mathcal{G}$, and suppose that we are given r inequivalent (nontrivial) discrete valuations $v_1, \ldots, v_r$ of $\mathcal{K}$ such that the completion $\mathcal{K}_{v_i}$ is locally compact and contains $\mathcal{L}$, and $\mathcal{G}$ splits over $\mathcal{K}_{v_i}$, for $i = 1, \ldots, r$. There exist maximal $\mathcal{K}_{v_i}$-tori $\mathcal{T}(v_i)$ of $\mathcal{G}$, one for each $i \in \{1, \ldots, r\}$, with the property that for any maximal $\mathcal{K}$-torus $\mathcal{T}$ of $\mathcal{G}$ which is conjugate to $\mathcal{T}(v_i)$ by an element of $\mathcal{G}(\mathcal{K}_{v_i})$ for all $i = 1, \ldots, r$, we have*

$$\theta_{\mathcal{T}}(\mathrm{Gal}(\mathcal{L}_{\mathcal{T}}/\mathcal{L})) \supset W(\mathcal{G}, \mathcal{T}), \tag{3}$$

where $\mathcal{L}_{\mathcal{T}} = \mathcal{K}_{\mathcal{T}}\mathcal{L}$ is the splitting field of $\mathcal{T}$ over $\mathcal{L}$ so that $\mathrm{Gal}(\mathcal{L}_{\mathcal{T}}/\mathcal{L})$ can be identified with a subgroup of $\mathrm{Gal}(\mathcal{K}_{\mathcal{T}}/\mathcal{K})$.

We will now derive a series of corollaries that will be used in the subsequent sections.

Corollary 3.2. *Let $\mathcal{G}$, $\mathcal{K}$ and $\mathcal{L}$ be as in Theorem 3.1, and let $\mathcal{V}$ be a finite set of nontrivial valuations of $\mathcal{K}$ such that for each $v \in \mathcal{K}$, the completion $\mathcal{K}_v$ is locally compact. Suppose that for each $v \in \mathcal{V}$ we are given a maximal $\mathcal{K}_v$-torus $\mathcal{T}(v)$ of $\mathcal{G}$. Then there exists a maximal $\mathcal{K}$-torus $\mathcal{T}$ of $\mathcal{G}$ for which* (3) *holds and which is conjugate to $\mathcal{T}(v)$ by an element of $\mathcal{G}(\mathcal{K}_v)$, for all $v \in \mathcal{V}$.*

Proof. Let r denote the number of nontrivial conjugacy classes in the Weyl group of $\mathcal{G}$. Enlarging $\mathcal{L}$ if necessary, we assume that $\mathcal{G}$ splits over $\mathcal{L}$. By Proposition 1 of [27], there exists an infinite set Π of rational primes such that for each $p \in \Pi$ there exists an embedding $\iota_p\colon \mathcal{L} \to \mathbb{Q}_p$. It follows that one can pick r distinct primes $p_1, \ldots, p_r \in \Pi$ so that for the valuations

v_i of $\mathcal{K}$ obtained as pullbacks of the p_i-adic valuations v_{p_i} on $\mathbb{Q}_{p_i}$, the set $\mathcal{R} = \{v_1, \ldots, v_r\}$ is disjoint from $\mathcal{V}$. Now, let $\mathcal{T}(v_i)$, where $i = 1, \ldots, r$, be the tori constructed in Theorem 3.1. Since the completions $\mathcal{K}_v$ for $v \in \mathcal{R} \cup \mathcal{V}$ are locally compact, it follows from the Implicit Function Theorem that the tori in the $\mathcal{G}(\mathcal{K}_v)$-conjugacy class of $\mathcal{T}(v)$ correspond to points of an open subset of $\mathcal{T}(K_v)$, where $\mathcal{T}$ is the variety of maximal tori of $\mathcal{G}$. Since $\mathcal{T}$ has the weak approximation property (cf. [21], Corollary 3 in §7.2), there exists a maximal $\mathcal{K}$-torus $\mathcal{T}$ which is conjugate to $\mathcal{T}(v)$ by an element of $\mathcal{G}(\mathcal{K}_v)$ for all $v \in \mathcal{R} \cup \mathcal{V}$. It follows from our construction that this torus has the desired properties. □

To reformulate the above results for individual elements instead of tori, we need the following lemma. We will call a subset of a topological group *solid* if it intersects every open subgroup of that group.

Lemma 3.3. *Let v be a nontrivial valuation of $\mathcal{K}$ with locally compact completion $\mathcal{K}_v$, and let $\mathcal{T}$ be a maximal $\mathcal{K}_v$-torus of $\mathcal{G}$. Consider the map*

$$\varphi\colon \mathcal{G} \times \mathcal{T} \longrightarrow \mathcal{G}, \quad (g, t) \mapsto gtg^{-1}.$$

Then

$$\mathcal{U}(\mathcal{T}, v) := \varphi(\mathcal{G}(\mathcal{K}_v), \mathcal{T}_{\mathrm{reg}}(\mathcal{K}_v)),$$

where $\mathcal{T}_{\mathrm{reg}}$ is the Zariski-open subvariety of $\mathcal{T}$ of regular elements, is a solid open subset of $\mathcal{G}(\mathcal{K}_v)$.

Proof. Indeed, one easily verifies that the differential $d_{(g,t)}\varphi$ is surjective for any $(g, t) \in \mathcal{G}(\mathcal{K}_v) \times \mathcal{T}_{\mathrm{reg}}(\mathcal{K}_v)$, so the openness of $\mathcal{U}(\mathcal{T}, v)$ follows from the Implicit Function Theorem. Furthermore, for any open subgroup Ω of $\mathcal{G}(\mathcal{K}_v)$, the set $\mathcal{T}(\mathcal{K}_v) \cap \Omega$ is Zariski-dense in $\mathcal{T}$ (cf. [21], Lemma 3.2), and therefore it contains an element of $\mathcal{T}_{\mathrm{reg}}(\mathcal{K}_v)$. So, $\mathcal{U}(\mathcal{T}, v) \cap \Omega \neq \emptyset$. □

Corollary 3.4. *Let $\mathcal{G}$, $\mathcal{K}$, $\mathcal{L}$ and r be as in Theorem 3.1. Furthermore, let $v_1, \ldots, v_r$ be r valuations of $\mathcal{K}$ with the properties specified in Theorem 3.1, and let*

$$\delta\colon \mathcal{G}(\mathcal{K}) \hookrightarrow \prod_{i=1}^{r} \mathcal{G}(\mathcal{K}_{v_i}) =: \mathcal{H}$$

be the diagonal embedding. Then there exists a solid open subset $\mathcal{U} \subset \mathcal{H}$ such that any $\gamma \in \mathcal{G}(\mathcal{K})$ satisfying $\delta(\gamma) \in \mathcal{U}$ is regular semi-simple, and for the torus $\mathcal{T} = Z_{\mathcal{G}}(\gamma)^{\circ}$, condition (3) *holds.*

Indeed, let $\mathcal{T}(v_i)$, where $i = 1, \ldots, r$, be the tori constructed in Theorem 3.1. Then it is easy to see that the set

$$\mathcal{U} = \prod_{i=1}^{r} \mathcal{U}(\mathcal{T}(v_i), v_i)$$

(notations as in Lemma 3.3), satisfies all our requirements.

In [26], a $\mathcal{K}$-torus $\mathcal{T}$ was called $\mathcal{K}$-*irreducible* if it has no proper $\mathcal{K}$-subtori, which is equivalent to the condition that the absolute Galois group $\mathrm{Gal}(\overline{\mathcal{K}}/\mathcal{K})$ acts irreducibly on the $\mathbb{Q}$-vector space $X(\mathcal{T})\otimes_{\mathbb{Z}}\mathbb{Q}$. It follows that, in our previous notation, a maximal $\mathcal{K}$-torus $\mathcal{T}$ of $\mathcal{G}$ such that $\theta_{\mathcal{T}}(\mathrm{Gal}(\mathcal{K}_{\mathcal{T}}/\mathcal{K})) \supset W(\mathcal{G},\mathcal{T})$ is $\mathcal{K}$-irreducible (cf. [6], Ch. VI, §1, n° 2). We need the following general fact about irreducible tori.

Lemma 3.5. *Let $\mathcal{T}$ be a $\mathcal{K}$-irreducible torus, and $\mathcal{K}_{\mathcal{T}}$ be its splitting field over $\mathcal{K}$. Let $t \in \mathcal{T}(\mathcal{K})$ be an element of infinite order, and $\chi \in X(\mathcal{T})$ be a nontrivial character. Then for $\lambda := \chi(t)$, the Galois conjugates $\sigma(\lambda)$, with $\sigma \in \mathrm{Gal}(\mathcal{K}_{\mathcal{T}}/\mathcal{K})$, generate $\mathcal{K}_{\mathcal{T}}$ over $\mathcal{K}$.*

Proof. We need to show that if $\tau \in \mathrm{Gal}(\mathcal{K}_{\mathcal{T}}/\mathcal{K})$ is such that

$$\tau(\sigma(\lambda)) = \sigma(\lambda) \text{ for all } \sigma \in \mathrm{Gal}(\mathcal{K}_{\mathcal{T}}/\mathcal{K}),$$

then $\tau = \mathrm{id}$. For such a τ we have

$$(\sigma^{-1}\tau\sigma)(\chi(t)) = ((\sigma^{-1}\tau\sigma)(\chi))(t) = \chi(t).$$

Hence, the character $(\sigma^{-1}\tau\sigma)(\chi) - \chi$ takes the value 1 at t, and therefore, $\tau(\sigma(\chi)) = \sigma(\chi)$ because t generates a Zariski-dense subgroup of $\mathcal{T}$ (as does any $\mathcal{K}$-rational element of infinite order in a $\mathcal{K}$-irreducible torus). But the fact that $\mathcal{T}$ is $\mathcal{K}$-irreducible implies that the characters $\sigma(\chi)$, for $\sigma \in \mathrm{Gal}(\mathcal{K}_{\mathcal{T}}/\mathcal{K})$, span $X(\mathcal{T}) \otimes_{\mathbb{Z}} \mathbb{Q}$, so $\tau = \mathrm{id}$. □

4. The isogeny theorem

In this section, K will be a field of arbitrary characteristic and $K^{\mathfrak{s}}$ a fixed separable closure of K. Let G be a connected absolutely simple algebraic K-group. Let T and T' be two maximal tori of G, and L any field extension of K such that both the tori are defined and split over it. Given systems $\Delta \subset \Phi(G,T)$ and $\Delta' \subset \Phi(G,T')$ of simple roots, there exists $g \in G(L)$ such that the corresponding inner automorphism i_g of G maps T onto T', and the induced homomorphism $i_g^*\colon X(T') \to X(T)$ of the character groups maps Δ' onto Δ. Such a g is determined uniquely up to an element of $T(L)$, which implies that the identification $\Delta \simeq \Delta'$ induced by i_g^* is *canonical* (i.e., it is independent of the choice of g). We will always employ this canonical identification of Δ with Δ' in the sequel.

Now let T be a maximal K-torus of G. Fix a system $\Delta \subset \Phi(G,T)$ of simple roots. Then for any $\sigma \in \mathrm{Gal}(K^{\mathfrak{s}}/K)$, there exists a unique $w_\sigma \in W(G,T)$ such that $w_\sigma(\sigma(\Delta)) = \Delta$. The correspondence $\alpha \mapsto w_\sigma(\sigma(\alpha))$ defines an action of $\mathrm{Gal}(K^{\mathfrak{s}}/K)$ on Δ, which is called the $*$-*action* (cf. [35]) .

The following lemma describes some properties of the $*$-action, and of the aforementioned identification of Δ with Δ', which will be used later in the paper.

Lemma 4.1. (a) *Let T and T' be two maximal K-tori of G, and let $\Delta \subset \Phi(G,T)$ and $\Delta' \subset \Phi(G,T')$ be two systems of simple roots. Pick $g \in G(K^{\mathfrak{s}})$*

so that $i_g(T) = T'$ *and* $i_g^*(\Delta') = \Delta$. *Then* i_g^* *commutes with the* $*$*-action of* $\mathrm{Gal}(K^{\mathfrak{s}}/K)$ *on* Δ' *and* Δ *respectively. In particular, it carries the orbits of the* $*$*-action on* Δ' *to the orbits of the* $*$*-action on* Δ.

(b) *The following conditions are equivalent:*

(i) G *is an inner form (i.e. an inner twist of the split group) over* K;

(ii) $*$*-action is trivial for some (equivalently, any) maximal* K*-torus* T *and a system of simple roots* $\Delta \subset \Phi(G, T)$;

(iii) $\theta_T(\mathrm{Gal}(K_T/K)) \subset W(G, T)$ *for some (equivalently, any) maximal* K*-torus* T *of* G.

(c) *The minimal Galois extension* L *of* K *over which* G *becomes an inner form admits the following (equivalent) characterizations:*

(i) $L = (K^{\mathfrak{s}})^{\mathcal{H}}$, *where* $\mathcal{H}$ *is the kernel of the* $*$*-action;*

(ii) $L = (K_T)^{\mathcal{H}_T}$, *where* $\mathcal{H}_T = \theta_T^{-1}(\theta_T(\mathrm{Gal}(K_T/K)) \cap W(G, T))$.

Proof. (a): Let $\sigma \in \mathrm{Gal}(K^{\mathfrak{s}}/K)$, and pick $w_\sigma \in W(G, T)$ and $w'_\sigma \in W(G, T')$ so that $w_\sigma(\sigma(\Delta)) = \Delta$ and $w'_\sigma(\sigma(\Delta')) = \Delta'$. We need to show that

$$i_g^*(w'_\sigma(\sigma(\alpha'))) = w_\sigma(\sigma(i_g^*(\alpha'))) \quad \text{for all} \quad \alpha' \in \Delta'. \tag{4}$$

Since both T and T' are defined over K, we have $g^{-1}\sigma(g) \in N_G(T)$, and we let u_σ denote the corresponding element of $W(G, T)$. Then

$$\sigma(i_g^*(\alpha')) = u_\sigma(i_g^*(\sigma(\alpha'))).$$

Now, we observe that both $i_g^* \circ w'_\sigma \circ \sigma$ and $w_\sigma \circ \sigma \circ i_g^* = w_\sigma \circ u_\sigma \circ i_g^* \circ \sigma$ take Δ' to Δ. This means that

$$\widetilde{w} := (i_g^*)^{-1} \circ u_\sigma^{-1} \circ w_\sigma^{-1} \circ i_g^* \circ w'_\sigma$$

leaves the system of simple roots $\sigma(\Delta')$ invariant. On the other hand, $\widetilde{w} \in W(G, T')$. So, $\widetilde{w} = 1$, and (4) follows.

(b): It follows from (a) that if the $*$-action is trivial on some $\Delta \subset \Phi(G, T)$ for some maximal K-torus T, then it is trivial on any $\Delta' \subset \Phi(G, T')$ for any maximal K-torus T'. On the other hand, it follows from the description of the $*$-action on $\Delta \subset \Phi(G, T)$ that its triviality is equivalent to the following:

$$\theta_T(\mathrm{Gal}(K_T/K)) \subset W(G, T). \tag{5}$$

This shows that (ii) and (iii) are equivalent. It remains to show that (i) is equivalent to the inclusion (5). For this, we assume, as we clearly may, that G is adjoint. Let G_0 be the K-split adjoint group of the same type as G, and T_0 be a K-split maximal torus of G_0. Pick an isomorphism $\varphi\colon G_0 \to G$ such that $\varphi(T_0) = T$. Then

$$\alpha_\sigma = \varphi^{-1} \circ \sigma(\varphi) \quad \text{for} \quad \sigma \in \mathrm{Gal}(K^{\mathfrak{s}}/K),$$

defines a 1-cocycle $\alpha \in Z^1(K, \mathrm{Aut}\, G_0)$ associated to G. For any $\chi \in X(T)$ we have $\chi \circ \varphi \in X(T_0)$, and therefore, $\sigma(\chi \circ \varphi) = \chi \circ \varphi$ as T_0 is K-split. An easy computation then shows that

$$\sigma(\chi) = \chi \circ (\varphi \circ \alpha_\sigma^{-1} \circ \varphi^{-1}). \tag{6}$$

Next, (i) amounts to the assertion that α is cohomologous to a $\mathrm{Int}\, G_0(K^{\mathfrak{s}})$-valued Galois cocycle $\beta : \sigma \mapsto \beta_\sigma$, $\sigma \in \mathrm{Gal}(K^{\mathfrak{s}}/K)$, i.e., there exists $\gamma \in \mathrm{Aut}\, G_0$ such that $\alpha_\sigma = \gamma^{-1} \circ \beta_\sigma \circ \sigma(\gamma)$, for all $\sigma \in \mathrm{Gal}(K^{\mathfrak{s}}/K)$. Let us show that then in fact

$$\alpha_\sigma \in \mathrm{Int}\, G_0 \ \text{ for all } \ \sigma \in \mathrm{Gal}(K^{\mathfrak{s}}/K). \tag{7}$$

Indeed, it is well-known that

$$\mathrm{Aut}\, G_0 = \mathrm{Int}\, G_0 \rtimes \Psi(T_0, B_0),$$

where $\Psi(T_0, B_0)$ is a subgroup of the group of all K-rational automorphisms of G_0 that leave invariant T_0 and a Borel K-subgroup B_0 containing T_0. Since all the elements of $\Psi(T_0, B_0)$ are K-rational, by writing γ in the form $\gamma = \delta \circ \psi$ with $\delta \in \mathrm{Int}\, G_0$ and $\psi \in \Psi(T_0, B_0)$, we obtain that

$$\alpha_\sigma = \psi^{-1} \circ (\delta^{-1} \circ \beta_\sigma \circ \sigma(\delta)) \circ \psi.$$

So, since $\mathrm{Int}\, G_0 \lhd \mathrm{Aut}\, G_0$, we obtain (7). In addition, since both T_0 and T are defined over K, we have $\alpha_\sigma(T_0) = T_0$, and therefore for $\varkappa_\sigma := \varphi \circ \alpha_\sigma^{-1} \circ \varphi^{-1}$ ($\in \mathrm{Aut}\, G$), $\varkappa_\sigma(T) = T$. Thus, if G is an inner form, then $\varkappa_\sigma$ is an inner automorphism of G which leaves T invariant. Then its restriction $\varkappa_\sigma | T$ is given by an element of the Weyl group $W(G, T)$, so (6) yields the inclusion (5). Conversely, (5) in conjunction with (6) implies that $\varkappa_\sigma | T$ is induced by an element of $W(G, T)$. But then $\varkappa_\sigma$ itself is inner, which implies that G is an inner form.

(*c*): Characterization (i) immediately follows from part (*b*). For (ii), let $F = (K_T)^{\mathcal{H}_T}$. Since G is an inner form over L and splits over K_T, by (*b*), for $L_T = K_T$ we have

$$\theta_T(\mathrm{Gal}(L_T/L)) \subset W(G, T),$$

implying that $F \subset L$. On the other hand, using the definition of F we see that

$$\theta_T(\mathrm{Gal}(F_T/F)) \subset W(G, T).$$

Then, again by (*b*), G is an inner form over F, and therefore $L \subset F$. Thus, $L = F$, as claimed. □

Theorem 4.2. (Isogeny theorem.) *Let G be an absolutely simple algebraic group over an infinite field F. For $i = 1, 2$, let G_i be a form of G over an infinite subfield K of F, and let L_i be the minimal Galois extension of K over which G_i is an inner form of a split group. Suppose that for $i = 1, 2$, we are given a semi-simple element $\gamma_i \in G_i(K)$ contained in a maximal K-torus T_i of G_i. Assume that γ_1 has infinite order and that $\theta_{T_1}(\mathrm{Gal}(K_{T_1}/K)) \supset W(G_1, T_1)$. If γ_1 and γ_2 are weakly commensurable, then there exists a K-isogeny $\pi \colon T_2 \to T_1$ which carries $\gamma_2^{m_2}$ to $\gamma_1^{m_1}$ for some integers $m_1, m_2 \geqslant 1$.*

Moreover, if $L_1 = L_2$[5]*, then* $\pi^*\colon X(T_1)\otimes_{\mathbb{Z}}\mathbb{Q} \to X(T_2)\otimes_{\mathbb{Z}}\mathbb{Q}$ *has the property* $\pi^*(\mathbb{Q}\cdot\Phi(G_1,T_1)) = \mathbb{Q}\cdot\Phi(G_2,T_2)$*, and in fact, if* G *is of type different from* $B_2 = C_2$*,* F_4 *or* G_2*, a suitable rational multiple of* π^* *maps* $\Phi(G_1,T_1)$ *onto* $\Phi(G_2,T_2)$*.*

Proof. By Lemma 2.1, there exist characters $\chi_i \in X(T_i)$ such that

$$\chi_1(\gamma_1) = \chi_2(\gamma_2) =: \lambda \neq 1.$$

We will proceed by showing first that T_2 is irreducible and that the splitting fields K_{T_1} and K_{T_2} coincide. The first assertion requires the following.

Lemma 4.3. *Let* Φ *be an irreducible root system, and let* H_1 *be a subgroup of* $\mathrm{Aut}(\Phi)$ *which contains the Weyl group* $W(\Phi)$*. Then any subgroup* H_2 *of* $\mathrm{Aut}(\Phi)$ *which admits a surjective homomorphism* $H_2 \overset{\delta}{\twoheadrightarrow} H_1$*, acts irreducibly on the* $\mathbb{Q}$*-vector space* $\mathbb{Q}[\Phi]$ *spanned by* Φ*.*

Proof. If $\mathrm{Aut}(\Phi) = W(\Phi)$, then our assumption implies that $H_2 = W(\Phi)$, and there is nothing to prove. Next, we consider the cases $\Phi = A_n$ $(n > 1)$ or E_6 where $\mathrm{Aut}(\Phi) = W(\Phi)\times S$, with $S = \{\pm I\}$ (cf. [6], Tables I and V). It is enough to show that $H_2S = \mathrm{Aut}(\Phi)$ as then any H_2-invariant subspace would be $\mathrm{Aut}(\Phi)$-invariant. But $H_2S \neq \mathrm{Aut}(\Phi)$ can occur only when δ is an isomorphism of H_2 onto $H_1 = W(\Phi)$ and $S \subset H_2$. Then $\delta(S) \subset W(\Phi)$ would be a central subgroup of $\mathrm{Aut}(\Phi)$ of order two, hence $\delta(S) = S$ by Schur's Lemma, which is impossible. It remains to consider the case $\Phi = D_n$ $(n \geqslant 4)$. First, suppose that $n \geqslant 5$. Then $\mathrm{Aut}(\Phi)/W(\Phi)$ has order two and $W(\Phi) = D \rtimes S_n$, where in terms of a suitable basis $e_1, \ldots, e_n$ of $V = \mathbb{Q}[\Phi]$, the group D consists of $\mathrm{diag}(\varepsilon_1, \ldots, \varepsilon_n)$ with $\varepsilon_i = \pm 1$ and $\varepsilon_1 \cdots \varepsilon_n = 1$, and S_n permutes the basic vectors (cf. [6], Table IV). Then $H_2 \cap D$ has order 2^{n-1} or 2^{n-2}, and therefore has at least $n-2$ distinct weight subspaces. At least one of those subspaces is 1-dimensional, hence is spanned by a basic vector e_i. So, if the action of H_2 is not irreducible, there is a proper invariant subspace $W \subset V$ containing e_i. But $H_2 \cap S_n$ has index $\leqslant 2$ in S_n, hence it contains A_n. Since A_n acts on the basic vectors transitively, we obtain $W = V$ – a contradiction. It remains to consider the case $\Phi = D_4$. In this case also the above description of $W(\Phi)$ remains valid, but $\mathrm{Aut}(\Phi)/W(\Phi) \simeq S_3$. It is well-known that a Sylow 2-subgroup P of S_4 acts on $\{1,2,3,4\}$ transitively, which easily implies that the Sylow 2-subgroup $W(\Phi)_2 = D \rtimes P$ acts on V irreducibly. Pick a Sylow 2-subgroup $A \subset H_2$ and let B be a Sylow 2-subgroup of $\mathrm{Aut}(\Phi)$ that contains A. Clearly, $[B : A] \leqslant 2$, so it follows from the irreducibility of B and Clifford's Lemma that if V is not H_2-irreducible, then $V = V_1 \oplus V_2$, where the V_i's are 2-dimensional H_2-invariant subspaces. But then the image of H_2 in each of $\mathrm{GL}(V_i)$s would be conjugate to a subgroup of $\mathrm{O}_2(\mathbb{R})$, hence it is cyclic or dihedral, implying that H_2 has derived length $\leqslant 2$, which is not the case. Thus, the action of H_2 is irreducible. □

[5]cf. Theorem 6.3(2)

Clearly, T_1 is K-irreducible, so according to Lemma 3.5, the conjugates $\sigma(\lambda)$ with $\sigma \in \mathrm{Gal}(K^{\mathfrak{s}}/K)$, generate the splitting field K_{T_1}. At the same time, since K_{T_2}/K is a Galois extension, all these conjugates belong to K_{T_2}, yielding the inclusion $K_{T_1} \subset K_{T_2}$ and hence a surjective homomorphism

$$\mathrm{Gal}(K_{T_2}/K) \longrightarrow \mathrm{Gal}(K_{T_1}/K).$$

It now follows from Lemma 4.3 that $\theta_{T_2}(\mathrm{Gal}(K_{T_2}/K))$ acts irreducibly on $\mathbb{Q}[\Phi(G_2, T_2)] \simeq X(T_2) \otimes_{\mathbb{Z}} \mathbb{Q}$, implying that T_2 is K-irreducible. Now, γ_2 has infinite order as $\gamma_2^m = 1$ would imply $(m\chi_1)(\gamma_1) = 1$, which is impossible since γ_1 generates a Zariski-dense subgroup of T_1. It follows that γ_2 generates a Zariski-dense subgroup of T_2, and therefore the conjugates $\sigma(\lambda)$, where $\sigma \in \mathrm{Gal}(K^{\mathfrak{s}}/K)$, generate K_{T_2} as well, yielding $K_{T_1} = K_{T_2} =: \mathcal{K}$.

Let $\mathcal{G} = \mathrm{Gal}(\mathcal{K}/K)$. We next show that there is an isomorphism of $\mathbb{Q}[\mathcal{G}]$-modules

$$\rho\colon \mathbb{Q} \otimes_{\mathbb{Z}} X(T_1) \longrightarrow \mathbb{Q} \otimes_{\mathbb{Z}} X(T_2)$$

that takes χ_1 to χ_2. For this, we consider

$$\nu_i\colon \mathbb{Q}[\mathcal{G}] \to \mathbb{Q} \otimes_{\mathbb{Z}} X(T_i), \quad \sum a_\sigma \sigma \mapsto \sum a_\sigma \sigma(\chi_i);$$

clearly, $\nu_i(\mathbb{Z}[\mathcal{G}]) \subset X(T_i)$. The irreducibility of T_i implies that ν_i is surjective for $i = 1, 2$, so it is enough to show that

$$\mathrm{Ker}\, \nu_1 = \mathrm{Ker}\, \nu_2. \tag{8}$$

For this we observe that given $a = \sum a_\sigma \sigma \in \mathbb{Z}[\mathcal{G}]$, we have

$$\nu_1(a)(\gamma_1) = \prod \sigma(\chi_1)(\gamma_1)^{a_\sigma} = \prod (\sigma(\lambda))^{a_\sigma} = \prod \sigma(\chi_2)(\gamma_2)^{a_\sigma} = \nu_2(a)(\gamma_2).$$

Since for $i = 1, 2$, γ_i generates a Zariski-dense subgroup of T_i, the above computation shows that $\nu_1(a) = 0$ is equivalent to $\nu_2(a) = 0$, and (8) follows.

The subgroup $\Theta := \nu_1(\mathbb{Z}[\mathcal{G}])$ has finite index, say d, in $X(T_1)$. Then the multiplication by d followed by ρ defines the homomorphism

$$\pi^*\colon X(T_1) \to \nu_2(\mathbb{Z}[\mathcal{G}]) \subset X(T_2)$$

of $\mathcal{G}$-modules such that $\pi^*(\chi_1) = d\chi_2$. Let $\pi\colon T_2 \to T_1$ be the K-isogeny corresponding to π^*. Then $\chi_1(\pi(\gamma)) = \chi_2(\gamma)^d$ for every $\gamma \in T_2$, and in particular,

$$\chi_1(\pi(\gamma_2)) = \chi_2(\gamma_2)^d = \chi_1(\gamma_1^d).$$

Applying the elements of $\mathcal{G}$, we see that $\chi(\pi(\gamma_2)) = \chi(\gamma_1^d)$ for all $\chi \in \Theta$, and therefore,

$$\chi(\pi(\gamma_2)^d) = \chi(\gamma_1^{d^2}) \ \text{ for every } \ \chi \in X(T_1).$$

Thus, $\pi(\gamma_2)^d = \gamma_1^{d^2}$, so the first assertion of the theorem holds with $m_1 = d^2$, $m_2 = d$. The second assertion of Theorem 4.2 will be deduced from:

Lemma 4.4. *For $i = 1, 2$, let Φ_i be an irreducible reduced root system contained in, and spanning, the $\mathbb{Q}$-vector space V_i. We assume that Φ_1 is isomorphic to Φ_2, and there is an isomorphism $\mu\colon W(\Phi_1) \to W(\Phi_2)$ of the corresponding Weyl groups, and a linear isomorphism $\lambda\colon V_1 \to V_2$ compatible* *with μ (i.e., $\lambda(w(v)) = \mu(w)(\lambda(v))$ for all $v \in V_1$ and $w \in W(\Phi_1)$). Then $\lambda(\mathbb{Q} \cdot \Phi_1) = \mathbb{Q} \cdot \Phi_2$, and in fact, if Φ_1 and Φ_2 are not of type $B_2 = C_2$, F_4 or G_2, a suitable rational multiple of λ maps Φ_1 onto Φ_2.*

Proof. We equip V_i with a positive definite $W(\Phi_i)$-invariant inner product scaled so that the short (long) roots in Φ_1 and Φ_2 have the same length in the respective spaces. We note that as V_i is an absolutely irreducible $W(\Phi_i)$-module, any two $W(\Phi_i)$-invariant inner products on V_i are multiples of each other, see [6], Ch. VI, §1, Proposition 7. This implies, in particular, that λ is a multiple of an isometry. For a root $\alpha \in \Phi_1$, let $w_\alpha \in W(\Phi_1)$ be the corresponding reflection. Then $\mu(w_\alpha)$ is the reflection of V_2 with respect to $\lambda(\alpha)$. On the other hand, $\mu(w_\alpha) \in W(\Phi_2)$, so it follows from ([6], Ch. V, §3, Cor. in n° 2) that $\mu(w_\alpha) = w_{\bar{\alpha}}$ for some $\bar{\alpha} \in \Phi_2$. So, $\lambda(\alpha) = t_\alpha \bar{\alpha}$ for some $t_\alpha \in \mathbb{Q}$, and our first assertion follows.

Now fix an arbitrary (resp., an arbitrary short) root $\alpha_0 \in \Phi_1$ if all all roots have the same length (resp., if Φ_1 contains roots of unequal lengths). Replacing λ with $t_{\alpha_0}^{-1}\lambda$, we assume that $\lambda(\alpha_0) = \bar{\alpha}_0$. If all roots have the same length, then $W(\Phi_1) \cdot \alpha_0 = \Phi_1$ and $W(\Phi_2) \cdot \bar{\alpha}_0 = \Phi_2$ ([6], Ch. VI, §1, Proposition 11), yielding $\lambda(\Phi_1) = \Phi_2$. It remains now to deal with the root systems of types B_n and C_n with $n > 2$. Then $W(\Phi_1) \cdot \alpha_0$ is the subset Φ_1^{short} of all short roots, and $W(\Phi_2) \cdot \bar{\alpha}_0$ is either Φ_2^{short} or Φ_2^{long} depending on whether $\bar{\alpha}_0$ is short or long (cf. *loc. cit.*). But for the types under consideration, $|\Phi_1^{\text{short}}| \neq |\Phi_2^{\text{long}}|$, and therefore, $\lambda(\Phi_1^{\text{short}}) = \Phi_2^{\text{short}}$. Since α_0 and $\lambda(\alpha_0)$ have the same length, λ is an isometry. So, if $\beta_0 \in \Phi_1^{\text{long}}$, then writing $\lambda(\beta_0) = t_{\beta_0}\bar{\beta}_0$ and observing that the squared-length of β_0 is twice the squared-length of any short root, we conclude that $\bar{\beta}_0$ cannot be a short root. Therefore, $\bar{\beta}_0$ is long, $t_{\beta_0} = \pm 1$, and it follows that $\lambda(\Phi_1) = \Phi_2$. □

Set $L := L_1 = L_2$. Then it follows from Lemma 4.1 that

$$\theta_{T_1}(\mathrm{Gal}(L_{T_1}/L)) = W(G_1, T_1) \text{ and } \theta_{T_2}(\mathrm{Gal}(L_{T_2}/L)) \subset W(G_2, T_2).$$

Since $L_{T_1} = L_{T_2}$, we see that the composite map

$$\mu \colon W(G_1, T_1) \xrightarrow{\theta_{T_1}^{-1}} \mathrm{Gal}(L_{T_1}/L) = \mathrm{Gal}(L_{T_2}/L) \xrightarrow{\theta_{T_2}} W(G_2, T_2)$$

is an isomorphism of the Weyl groups compatible with $\pi^* \colon \mathbb{Q} \otimes_{\mathbb{Z}} X(T_1) \to \mathbb{Q} \otimes_{\mathbb{Z}} X(T_2)$. Now, the second assertion of Theorem 4.2 follows from Lemma 4.4. □

Remark 4.5. The second assertion of Theorem 4.2 has the following consequence. We assume π^* scaled so that $\pi^*(\Phi(G_1, T_1)) = \Phi(G_2, T_2)$. Then it induces a K-isomorphism $\bar{\pi} \colon \overline{T}_2 \to \overline{T}_1$ of the corresponding tori in the adjoint groups $\overline{G}_i$, which still has the property $\bar{\pi}(\bar{\gamma}_2^{\bar{m}_2}) = \bar{\gamma}_1^{\bar{m}_1}$ for some integers $\bar{m}_1, \bar{m}_2 \geqslant 1$, where $\bar{\gamma}_i$ is the image of γ_i in $\bar{T}_i$. Furthermore, if Y_i is the dual in V_i (where V_i is as in Lemma 4.4) of the lattice X_i spanned by Φ_i, then

Y_i is the character group of the maximal K-torus $\widetilde{T}_i$, corresponding to the maximal torus T_i, of the simply connected cover $\widetilde{G}_i$ of G_i, and π^* induces an isomorphism $Y_1 \to Y_2$, which in turn induces a K-isomorphism $\widetilde{\pi}\colon \widetilde{T}_2 \to \widetilde{T}_1$. Both $\widetilde{\pi}$ and $\bar{\pi}$ extend to $K^{\mathfrak{s}}$-isomorphisms $\widetilde{G}_2 \to \widetilde{G}_1$ and $\overline{G}_2 \to \overline{G}_1$. Also, if Δ_2 is a system of simple roots in $\Phi(G_2, T_2)$, and $\Delta_1 = \pi^*(\Delta_2)$, then π^* commutes with the $*$-action of $\mathrm{Gal}(K^{\mathfrak{s}}/K)$ on Δ_1 and Δ_2 respectively.

5. Proof of Theorems A, B and F

We begin this section with a proof of Theorem A, and will then prove Theorems B and F. First, we will establish the following two auxiliary assertions, the first of which is a variant of Proposition 1 in [27].

Proposition 5.1. *Let $\mathcal{F}_1 \subsetneqq \mathcal{F}_2 \subset \mathcal{E}$ be a tower of finitely generated fields of characteristic zero, and let $\mathcal{R} \subset \mathcal{E}$ be a finitely generated subring. Then there exists an infinite set of rational primes Π such that for each $p \in \Pi$, there are embeddings $\iota', \iota''\colon \mathcal{E} \to \mathbb{Q}_p$ with the following properties:*

(1) *both $\iota'(\mathcal{R})$ and $\iota''(\mathcal{R})$ are contained in $\mathbb{Z}_p$;*
(2) *$\iota'|\mathcal{F}_1 = \iota''|\mathcal{F}_1$, but $\iota'|\mathcal{F}_2 \neq \iota''|\mathcal{F}_2$.*

Proof. First, we observe that there exists a transcendence basis $t_1, \ldots, t_n$ of $\mathcal{E}$ over $\mathbb{Q}$ such that for $\mathcal{K} := \mathbb{Q}(t_1, \ldots, t_n)$ we have $\mathcal{K}\mathcal{F}_1 \neq \mathcal{K}\mathcal{F}_2$. Indeed, let $t_1, \ldots, t_{n_1}$ be an arbitrary transcendence basis of $\mathcal{F}_1$ over $\mathbb{Q}$, and $t_{n_1+1}, \ldots, t_{n_2}$ be a transcendence basis of $\mathcal{F}_2$ over $\mathcal{F}_1$ such that

$$\mathcal{F}_2 \neq \mathcal{F}_1(t_{n_1+1}, \ldots, t_{n_2}).$$

Then, for $\mathcal{K}_0 := \mathbb{Q}(t_1, \ldots, t_{n_2})$, we have

$$\mathcal{K}_0\mathcal{F}_1 = \mathcal{F}_1(t_{n_1+1}, \ldots, t_{n_2}) \neq \mathcal{F}_2.$$

Now, let $t_{n_2+1}, \ldots, t_n$ be a transcendence basis of $\mathcal{E}$ over $\mathcal{F}_2$. Then, of course, $(\mathcal{K}_0\mathcal{F}_1)(t_{n_2+1}, \ldots, t_n) \neq \mathcal{F}_2(t_{n_2+1}, \ldots, t_n)$, and therefore

$$\mathcal{K}\mathcal{F}_1 = (\mathcal{K}_0\mathcal{F}_1)(t_{n_2+1}, \ldots, t_n) \neq \mathcal{F}_2(t_{n_2+1}, \ldots, t_n) = \mathcal{K}\mathcal{F}_2,$$

as required.

Obviously, $\mathcal{E}$ is a finite extension of $\mathcal{K}\mathcal{F}_1$, and we let $\mathcal{M}$ denote the Galois closure of $\mathcal{E}$ over $\mathcal{K}\mathcal{F}_1$. Then there exists $\sigma \in \mathrm{Gal}(\mathcal{M}/\mathcal{K}\mathcal{F}_1)$ which acts nontrivially on $\mathcal{K}\mathcal{F}_2$, and hence on $\mathcal{F}_2$. Let $\mathcal{R}_0$ be the subring generated by $\mathcal{R}$ and $\sigma(\mathcal{R})$. Since $\mathcal{M}$ is a finitely generated field and $\mathcal{R}_0$ is a finitely generated ring, by Proposition 1 of [27], one can find an infinite set of rational primes Π such that for $p \in \Pi$ there exists an embedding $\iota_p\colon \mathcal{M} \to \mathbb{Q}_p$ with the property $\iota_p(\mathcal{R}_0) \subset \mathbb{Z}_p$. Then the embeddings

$$\iota' = \iota_p|\mathcal{E} \quad \text{and} \quad \iota'' = (\iota_p \circ \sigma)|\mathcal{E}$$

satisfy both of our conditions. □

Proposition 5.2. *Let G be a connected absolutely simple adjoint algebraic group defined over a field $\mathcal{F}$ of characteristic zero. Let $\mathcal{E}$ be an extension of $\mathcal{F}$, $\Gamma \subset G(\mathcal{E})$ be a Zariski-dense subgroup, and $\mathcal{K}_\Gamma$ be the subfield generated by the traces* $\mathrm{tr}\mathrm{Ad}\gamma$ *for* $\gamma \in \Gamma$. *Given two embeddings $\iota^{(1)}, \iota^{(2)} \colon \mathcal{E} \to \mathbb{Q}_p$ such that $\iota^{(1)}|\mathcal{F} = \iota^{(2)}|\mathcal{F} := \iota$, we consider G as a $\mathbb{Q}_p$-group via extension of scalars $\iota : \mathcal{F} \to \mathbb{Q}_p$, and let $\rho^{(1)}, \rho^{(2)} \colon G(\mathcal{E}) \to G(\mathbb{Q}_p)$ denote the homomorphisms induced by $\iota^{(1)}$ and $\iota^{(2)}$, respectively. If*

(a) $\rho^{(i)}(\Gamma)$ *is relatively compact for* $i = 1, 2$;

(b) $\iota^{(1)}|\mathcal{K}_\Gamma \neq \iota^{(2)}|\mathcal{K}_\Gamma$;

then the closure of the image of the diagonal homomorphism

$$\rho \colon \Gamma \to G(\mathbb{Q}_p) \times G(\mathbb{Q}_p), \quad \gamma \mapsto (\rho^{(1)}(\gamma), \rho^{(2)}(\gamma)),$$

in the p-adic topology, is open.

Proof. We begin by showing that the image of ρ is Zariski-dense in $G \times G$.

Lemma 5.3. *Let G be a connected simple adjoint algebraic group, and let $\rho_i \colon \Gamma \to G$, where $i = 1, 2$, be two homomorphisms of a group Γ with Zariski-dense images. Then either*

$$\mathrm{tr}\ \mathrm{Ad}\, \rho_1(\gamma) = \mathrm{tr}\ \mathrm{Ad}\, \rho_2(\gamma) \quad \textit{for all} \quad \gamma \in \Gamma, \tag{9}$$

or the image of the homomorphism

$$\rho \colon \Gamma \to G \times G, \quad \gamma \mapsto (\rho_1(\gamma), \rho_2(\gamma)),$$

is Zariski-dense in $G \times G$.

Proof. Let H be the Zariski-closure of $\rho(\Gamma)$ in $G \times G$, and assume that $H \neq G \times G$. Since both ρ_1 and ρ_2 have Zariski-dense images, for the corresponding projections we have

$$\mathrm{pr}_i(H) = G, \quad i = 1, 2.$$

Set $H_i = H \cap \ker \mathrm{pr}_i$. Then $\mathrm{pr}_2(H_1)$ is a normal subgroup of G, and therefore it is either G or is trivial. Furthermore, if it equals G, then as $\mathrm{pr}_1(H) = G$, we easily see that $H = G \times G$. Similarly, $\mathrm{pr}_1(H_2)$ is ether G or is trivial, and in the former case $H = G \times G$. Thus, since $H \neq G \times G$, we see that H_i is trivial for $i = 1, 2$. This means that pr_i induces an isomorphism $\epsilon_i \colon H \to G$ for $i = 1, 2$. Then $\sigma := \epsilon_2 \circ \epsilon_1^{-1}$ is an automorphism of G, and

$$H = \{\, (g, \sigma(g)) \mid g \in G\}.$$

It follows that $\rho_2 = \sigma \circ \rho_1$, which implies (9). □

We now return to the notations introduced in the statement of Proposition 5.2, and denote by $\mathcal{H}$ the closure of $\rho(\Gamma)$ in $G(\mathbb{Q}_p) \times G(\mathbb{Q}_p)$ in the p-adic topology. Then $\mathcal{H}$ is a p-adic Lie group (cf. [6], ch. III, §8, Theorem 2), and we let $\mathfrak{h}$ denote its Lie algebra. It follows from condition (b) that (9) does not hold, and hence by Lemma 5.3, $\rho(\Gamma)$ is Zariski-dense in $G \times G$. This immediately implies (cf. [21], Prop. 3.4) that $\mathfrak{h}$ is an ideal of $\mathfrak{g} \times \mathfrak{g}$, where $\mathfrak{g}$ is the Lie algebra of $G(\mathbb{Q}_p)$ as a p-adic Lie group. If $\mathfrak{h}$ has the zero projection to, say, the first component then the image of $\rho^{(1)}$ would be discrete, hence finite (in view of condition (a)), which is impossible. Thus, $\mathfrak{h}$ has nonzero projections to both components, and therefore, being an ideal of $\mathfrak{g} \times \mathfrak{g}$, must coincide with $\mathfrak{g} \times \mathfrak{g}$ since $\mathfrak{g}$ is simple. But this means that $\mathcal{H}$ is open in $G(\mathbb{Q}_p) \times G(\mathbb{Q}_p)$. □

We will now prove Theorem A. Without any loss of generality, we assume (as we may) that the group G is adjoint and fix its matrix realization given by the adjoint representation. Since Γ_i is finitely generated, it is contained in $\mathrm{GL}_n(F_i)$ for some finitely generated field F_i. Then the field $K_i := K_{\Gamma_i}$ is a subfield of F_i, and therefore it is finitely generated, for $i = 1, 2$. By symmetry, it is enough to establish the inclusion $K_1 \subset K_2$. Assume the contrary, and set $K = K_1K_2$. By Theorem 1 of Vinberg [38], one can choose a basis (which we fix for the rest of the proof) of the Lie algebra of G so that Γ_2 is represented by matrices with entries in K_2. Then G is defined over K_2 (hence also over K) and $\Gamma_2 \subset G(K_2)$. Now, pick a finitely generated extension L of K which contains the splitting field of a maximal K-torus of G and has the property that $\Gamma_1 \subset G(L)$. Furthermore, pick a finitely generated subring $\mathcal{R}$ of L such that $\Gamma_1 \subset G(\mathcal{R})$. Let r be the number of nontrivial conjugacy classes of the Weyl group of G. By Proposition 1 of [27], there exist rational primes $p_1, \ldots, p_r$ and embeddings $\iota_j \colon L \to \mathbb{Q}_{p_j}$ such that $\iota_j(\mathcal{R}) \subset \mathbb{Z}_{p_j}$. Let $\rho_j \colon \Gamma_1 \to G(\mathbb{Z}_{p_j})$ be the corresponding homomorphisms. Then according to Lemma 2 of [27], the closure of the image of the homomorphism

$$\delta \colon \Gamma_1 \to G(\mathbb{Z}_{p_1}) \times \cdots \times G(\mathbb{Z}_{p_r}), \quad \gamma \mapsto (\rho_1(\gamma), \ldots, \rho_r(\gamma)),$$

is open. Furthermore, by Corollary 3.4, there exists a solid open subset $U \subset G(\mathbb{Z}_{p_1}) \times \cdots \times G(\mathbb{Z}_{p_r})$ such that any $\gamma \in \Gamma_1$ ($\subset G(L)$) satisfying $\delta(\gamma) \in U$, is regular semi-simple and for the L-torus $T = Z_G(\gamma)^\circ$, we have

$$\theta_T(\mathrm{Gal}(L_T/L)) \supset W(G, T), \tag{10}$$

where L_T/L is the splitting field of T.

Next, applying Proposition 5.1 to the tower

$$K_2 \subsetneqq K \subset L$$

we find a prime $p \notin \{p_1, \ldots, p_r\}$ such that there exists a pair of embeddings $\iota^{(1)}, \iota^{(2)} \colon L \to \mathbb{Q}_p$ that have the same restriction to K_2, but different restrictions to K, hence K_1, and also satisfy $\iota^{(i)}(\mathcal{R}) \subset \mathbb{Z}_p$ for $i = 1, 2$. Consider the resulting homomorphisms $\rho^{(1)}, \rho^{(2)} \colon \Gamma_1 \to G(\mathbb{Z}_p)$ (as in Proposition 5.2, G is considered to be a $\mathbb{Q}_p$-group by extension of scalars $K_2 \to \mathbb{Q}_p$ in terms of the embedding $\iota^{(1)}|K_2 = \iota^{(2)}|K_2$). Since $\iota^{(1)}$ and $\iota^{(2)}$ have different restrictions to $K_1 = K_{\Gamma_1}$, by Proposition 5.2, the closure of the image of the homomorphism

$$\Gamma_1 \to G(\mathbb{Z}_p) \times G(\mathbb{Z}_p), \quad \gamma \mapsto (\rho^{(1)}(\gamma), \rho^{(2)}(\gamma)),$$

is open in $G(\mathbb{Z}_p) \times G(\mathbb{Z}_p)$. Since $p \notin \{p_1, \ldots, p_r\}$, it follows that the closure of the image of

$$\rho\colon \Gamma_1 \to G(\mathbb{Z}_{p_1}) \times \cdots \times G(\mathbb{Z}_{p_r}) \times G(\mathbb{Z}_p) \times G(\mathbb{Z}_p),$$
$$\gamma \mapsto (\rho_1(\gamma), \ldots, \rho_r(\gamma), \rho^{(1)}(\gamma), \rho^{(2)}(\gamma)) = (\delta(\gamma), \rho^{(1)}(\gamma), \rho^{(2)}(\gamma)),$$

is open as well. Since $L \subset \mathbb{Q}_p$, G splits over $\mathbb{Q}_p$, so one can choose a maximal $\mathbb{Q}_p$-split torus $\mathcal{T}_1$ of G. On the other hand, by [21], Theorem 6.21 (for a different proof, see [8], §2.4), G contains a $\mathbb{Q}_p$-anisotropic maximal torus $\mathcal{T}_2$. For $i = 1, 2$, let $U_i = \mathcal{U}(\mathcal{T}_i, v_p)$ in the notation of Lemma 3.3, where v_p is the p-adic valuation on $\mathbb{Q}_p$. Since the sets U, U_1 and U_2 are solid in the corresponding groups, it follows from our preceding observation about the openness of the closure of $\operatorname{Im}\rho$ that there exists $\gamma_1 \in \Gamma_1$ such that

$$\rho(\gamma_1) \in U \times U_1 \times U_2.$$

Let $T_1 = Z_G(\gamma_1)^\circ$. Since Γ_1 and Γ_2 are weakly commensurable, there exist a maximal K_2-torus T_2 of G, and $\gamma_2 \in \Gamma_2 \cap T_2(K_2)$ such that

$$\chi_1(\gamma_1) = \chi_2(\gamma_2) =: \lambda \neq 1$$

for some characters $\chi_i \in X(T_i)$. Since $\gamma_2 \in T_2(K_2)$, λ is algebraic over K_2. Furthermore, even though γ_1 may not have entries in K_1, by Vinberg's theorem, it is conjugate to a matrix with entries in K_1. It follows that the torus T_1 is definable over K_1 and $\gamma_1 \in T_1(K_1)$, hence λ is algebraic over K_1 as well. For $i = 1, 2$, let $\mathcal{K}_i$ be the field generated over K_i by all the conjugates $\sigma(\lambda)$ with $\sigma \in \mathrm{Gal}(\overline{K}_i/K_i)$, and let $\mathcal{L}$ be the field generated over L by the conjugates $\sigma(\lambda)$ with $\sigma \in \mathrm{Gal}(\overline{L}/L)$. We claim that

$$\mathcal{K}_1 L = \mathcal{L} = \mathcal{K}_2 L. \tag{11}$$

By looking at the minimal polynomials of λ over K_i and L, we immediately see that $\mathcal{L} \subset \mathcal{K}_i L$ for $i = 1, 2$. For the opposite inclusion, we first observe that as $\delta(\gamma_1) \in U$, it follows from (10) that T_1 is L-irreducible, and therefore by Lemma 3.5, $\mathcal{L}$ coincides with L_{T_1}, the splitting field of T_1 over L. Thus, again from (10),

$$|\mathrm{Gal}(\mathcal{L}/L)| \geqslant |W(G_1, T_1)|. \tag{12}$$

On the other hand, for both $i = 1, 2$, the field $\mathcal{K}_i L$ is contained in the splitting field $L_{T_i}$of T_i over L, and since G_i is of inner type over L, we obtain from Lemma 4.1(b) that $\theta_{T_i}(\mathrm{Gal}(L_{T_i}/L)) \subset W(G_i, T_i)$. Thus,

$$|\mathrm{Gal}(\mathcal{K}_i L/L)| \leqslant |W(G_i, T_i)|;$$

combining this with (12), we obtain (11).

To complete the argument, we let v_1 and v_2 denote the valuations of K_1 obtained by pulling back the p-adic valuation on $\mathbb{Q}_p$ under the embeddings $\iota^{(1)}|K_1$ and $\iota^{(2)}|K_1$ of K_1 into $\mathbb{Q}_p$, respectively. Then, of course, the completion K_{1v_i} can be identified with $\mathbb{Q}_p$ for $i = 1, 2$. It follows from the description of the open sets U_i that as $\mathcal{T}_1$ splits over K_{1v_1}, T_1 also splits over K_{1v_1}, and as $\mathcal{T}_2$ is anisotropic over K_{1v_2}, so is T_1. Therefore, given a nontrivial character $\chi \in X(T_1)$, there exists $\sigma \in \mathrm{Gal}(\overline{K}_{1v_2}/K_{1v_2})$ such that $\sigma(\chi) \neq \chi$. Then, in view of the Zariski-density of the subgroup generated by γ_1, we have

$$\sigma(\chi)(\gamma_1) = \sigma(\chi(\gamma_1)) \neq \chi(\gamma_1),$$

and consequently,

$$\chi(\gamma_1) \notin K_{1v_2} \ \text{ for } \textit{any} \text{ nontrivial } \ \chi \in X(T_1). \tag{13}$$

Now, we extend our original embeddings $\iota^{(1)}, \iota^{(2)} \colon L \to \mathbb{Q}_p$ to embeddings $\tilde{\iota}^{(1)}, \tilde{\iota}^{(2)} \colon \mathcal{L} \to \overline{\mathbb{Q}}_p$. As T_1 splits over K_{1v_1},

$$\sigma(\lambda) = \sigma(\chi_1)(\gamma_1) \in K_{1v_1} \ \text{ for all } \ \sigma \in \mathrm{Gal}(\overline{K}_1/K_1),$$

and therefore, $\tilde{\iota}^{(1)}(\mathcal{K}_1) \subset \mathbb{Q}_p$. Then $\tilde{\iota}^{(1)}(\mathcal{L}) \subset \mathbb{Q}_p$, which, in view of (11), implies that $\tilde{\iota}^{(1)}(\mathcal{K}_2) \subset \mathbb{Q}_p$. On the other hand, it follows from (13) that $\tilde{\iota}^{(2)}(\mathcal{K}_1) \not\subset \mathbb{Q}_p$, so $\tilde{\iota}^{(2)}(\mathcal{K}_2) \not\subset \mathbb{Q}_p$. But $\tilde{\iota}^{(1)}$ and $\tilde{\iota}^{(2)}$ have the same restriction to K_2, and since $\mathcal{K}_2/K_2$ is a Galois extension, the restrictions $\tilde{\iota}^{(1)}|\mathcal{K}_2$ and $\tilde{\iota}^{(2)}|\mathcal{K}_2$ differ by an element of $\mathrm{Gal}(\mathcal{K}_2/K_2)$, which shows that the conditions

$$\tilde{\iota}^{(1)}(\mathcal{K}_2) \subset \mathbb{Q}_p \ \text{ and } \ \tilde{\iota}^{(2)}(\mathcal{K}_2) \not\subset \mathbb{Q}_p$$

are incompatible. A contradiction, which shows that our assumption that $K_1 \not\subset K_2$ is false, and therefore, $K_1 \subset K_2$. This proves Theorem A.

Remark 5.4. As we will prove soon, weakly commensurable Zariski-dense S-arithmetic subgroups share not only the field of definition, but also many other important characteristics (cf. Theorems B, C and E). For arbitrary finitely generated Zariski-dense subgroups, however, we cannot say much beyond Theorem A. One of the reasons is that at this point, classification results for semi-simple groups over general fields are very scarce. Here is one intriguing basic question in this direction: *let D_1 and D_2 be two quaternion algebras over a field K. Assume that D_1 and D_2 are* weakly isomorphic, *i.e., have the same maximal subfields. Are they isomorphic?* The answer is easily seen to be in the affirmative when K is a global field. On the other hand, M. Rost has informed us that over large fields (like those used in the proof of the Merkurjev-Suslin theorem), the answer can be negative. However, for finitely generated fields (and the fields that arise in the context of the present paper are finitely generated), the question remains open (apparently, even for such fields as $K = \mathbb{Q}(x)$). Furthermore, if the answer turns out to be negative, one would like to know if every class of weakly isomorphic quaternion algebras splits into finitely many isomorphism classes (for a finitely generated field K). Of course, one can ask similar questions for other types of algebraic groups (defining two K-forms of the same group to be *weakly isomorphic* if they have the same maximal K-tori).

Proof of Theorem B. For $i = 1, 2$, let Γ_i, be a Zariski-dense (G_i, K_i, S_i)-arithmetic subgroup of $G(F)$, and assume that Γ_1 and Γ_2 are weakly commensurable. By Lemma 2.6, the field K_{Γ_i} generated by $\mathrm{tr}\,\mathrm{Ad}\gamma$ for $\gamma \in \Gamma_i$, coincides with K_i. Since the Γ_is are finitely generated (cf. [21], Theorem 6.1), we can now use Theorem A to conclude that

$$K_1 = K_{\Gamma_1} = K_{\Gamma_2} = K_2 =: K.$$

In view of the obvious symmetry, to prove that $S_1 = S_2$, it is enough to prove the inclusion $S_1 \subset S_2$. Suppose there exists $v_0 \in S_1 \setminus S_2$. Our restrictions on S_i imply that the group G_1 is K_{v_0}-isotropic, so there exists a maximal K_{v_0}-torus $T(v_0)$ of G_1 which is K_{v_0}-isotropic. Then by Corollary 3.2, there exists a maximal K-torus T_1 of G_1 for which

$$\theta_{T_1}(\mathrm{Gal}(K_{T_1}/K)) \supset W(G_1, T_1). \tag{14}$$

and which is conjugate to $T(v_0)$ under an element of $G_1(K_{v_0})$, hence is K_{v_0}-isotropic.

Clearly, T_1 is K-anisotropic, so the quotient $T_{1S_1}/T_1(\mathcal{O}_K(S_1))$ is compact, where $T_{1S_1} = \prod_{v \in S_1} T_1(K_v)$ (cf. [21], Theorem 5.7), which implies that the quotient of $T_1(K_{v_0})$ by the closure C of $T_1(K_{v_0})$ in T_{1S_1} is also compact. But as T_1 is K_{v_0}-isotropic, the group $T_1(K_{v_0})$ is noncompact, and we conclude C is noncompact as well. Since $T_1(\mathcal{O}_K(S_1))$ is a finitely generated abelian group (cf. [21], Theorem 5.12), this implies that there exists $\gamma_1 \in T_1(\mathcal{O}_K(S_1))$ such that the closure of the cyclic group $\langle\gamma_1\rangle$ in $T_1(K_{v_0})$ is noncompact. We can in fact assume that $\gamma_1 \in \Gamma_1 \cap T_1(\mathcal{O}_K(S_1))$. By our assumption, γ_1 is weakly commensurable to a semi-simple element γ_2 of Γ_2. Let T_2 be a maximal K-torus containing γ_2. Then according to Theorem 4.2, there exists a K-isogeny $\pi\colon T_2 \to T_1$ such that $\pi(\gamma_2^{m_2}) = \gamma_1^{m_1}$ for some integers $m_1, m_2 \geqslant 1$. π induces a continuous homomorphism $\pi_{v_0}\colon T_2(K_{v_0}) \to T_1(K_{v_0})$. But since $v_0 \notin S_2$ and Γ_2 is S_2-arithmetic, the subgroup $\langle\gamma_2\rangle$ has compact closure in $T_2(K_{v_0})$, and we obtain that $\langle\gamma_1^{m_1}\rangle$, and hence $\langle\gamma_1\rangle$, has compact closure in $T_1(K_{v_0})$; a contradiction. □

Proof of Theorem F.[6] We will assume (as we may) that G is adjoint and is realized as a matrix group via the adjoint representation on its Lie algebra $\mathfrak{g}$. Suppose that Γ_1 is (G_1, K, S)-arithmetic; then, in particular, $\Gamma_1 \subset G_1(K)$ as G_1 is adjoint (see, for example, [5], Proposition 1.2). Let v_0 be the valuation of K obtained as the pullback of the normalized valuation on F using the embedding $K \hookrightarrow F$. Then of course $K_{v_0} \subset F$. Furthermore, $v_0 \in S$. Indeed, if $v_0 \notin S$, then v_0 is nonarchimedean and the group $G_1(\mathcal{O}_K(S))$ is relatively compact in $G_1(K_{v_0})$. Since Γ_1 is commensurable with $G_1(\mathcal{O}_K(S))$, it would also be relatively compact in $G(F)$. However, as Γ_1 is discrete, it would be finite, which would contradict its Zariski-density. Moreover, being commensurable with Γ_1, $G_1(\mathcal{O}_K(S))$ is discrete in $G_1(K_{v_0})$. Combining this with the fact that $G_1(\mathcal{O}_K(S))$ is a lattice in $G_{1S} := \prod_{v \in S} G_1(K_{1v})$, we obtain that the group $G_1(K_v)$ is compact for all $v \in S\setminus\{v_0\}$ (so, in particular,

[6] Of course, if $\mathrm{rk}_F\, G \geqslant 2$, then Γ_2 is automatically arithmetic by Margulis' Arithmeticity Theorem (cf. [17], Ch. IX), so we only need to consider the case $\mathrm{rk}_F\, G = 1$. Our argument, however, does not depend on $\mathrm{rk}_F\, G$.

$K_{1v} = \mathbb{R}$ for all archimedean $v \in S \setminus \{v_0\}$). Because of our convention regarding S, we see that there are in fact only two possibilities: (1) $S = V_\infty^K$, or (2) $v_0 \notin V_\infty^K$ and $S = V_\infty^K \cup \{v_0\}$. Furthermore, as we have already noted above, Γ_1 is relatively compact in $G_1(K_v)$ for any $v \notin S$. Thus, for any $\gamma_1 \in \Gamma_1$, the cyclic subgroup $\langle \gamma_1 \rangle$ is relatively compact in $G_1(K_v)$ for all $v \in V^K \setminus \{v_0\}$.

Let K_{Γ_i} denote the field generated by the traces of all elements $\gamma \in \Gamma_i$. Being lattices, Γ_1 and Γ_2 are finitely generated, and therefore Theorem A applies. Combining the latter with Lemma 2.6, we conclude that

$$K_{\Gamma_1} = K = K_{\Gamma_2}.$$

By Vinberg's theorem [37], there exists a basis of $\mathfrak{g}$ in which Γ_2 is represented by matrices with entries in K, and we fix this basis for the rest of the proof. Then G has a K-form G_2 such that $\Gamma_2 \subset G_2(K)$. In the sequel, the groups of points of G_2 over subrings of K will be understood in terms of the realization of $G_2(K)$ as a matrix group using the basis of $\mathfrak{g}$ fixed above. We claim that Γ_2 is commensurable with $G_2(\mathcal{O}_K(S))$, which will prove our claim. For this it is enough to establish the following two assertions:

(a) $G_2(K_v)$ is compact for all $v \in V_\infty^K \setminus \{v_0\}$.

(b) Γ_2 is bounded in $G_2(K_v)$ for all $v \in V_f^K \setminus \{v_0\}$.

Indeed, since Γ_2 is finitely generated, and therefore it is contained in $G_2(\mathcal{O}_v)$ for all but finitely many $v \in V_f^K$, we derive from (b), in either possibility for S, that we have

$$[\Gamma_2 : \Gamma_2 \cap G_2(\mathcal{O}_K(S))] < \infty;$$

in particular, $\Gamma_2 \cap G_2(\mathcal{O}_K(S))$ is a lattice in $G(F)$, and hence in $G_2(K_{v_0})$. On the other hand, it follows from (a) that again, in either possibility for S, the subgroup $G_2(\mathcal{O}_K(S))$ is a lattice in $G_2(K_{v_0})$, implying that $[G_2(\mathcal{O}_K(S)) : \Gamma_2 \cap G_2(\mathcal{O}_K(S))] < \infty$.

Both the assertions, (a) and (b), will be proved using the following argument. Let $v \in V^K \setminus \{v_0\}$ be such that the respective assertion fails. We will then find a regular semi-simple element $\gamma_2 \in \Gamma_2$ of infinite order such that the closure of $\langle \gamma_2 \rangle$ in $G_2(K_v)$ is noncompact and for the unique maximal K-torus T_2 of G_2 containing γ_2 we have

$$\theta_{T_2}(\mathrm{Gal}(K_{T_2}/K)) \supset W(G_2, T_2). \tag{15}$$

Since Γ_1 and Γ_2 are weakly commensurable, there exists a semi-simple element $\gamma_1 \in \Gamma_1$ which is weakly commensurable to γ_2. Then, if T_1 is a maximal K-torus of G_1 containing γ_1, by Theorem 4.2 there exists a K-isogeny $\pi \colon T_1 \to T_2$ which carries $\gamma_1^{m_1}$ to $\gamma_2^{m_2}$ for some integers $m_1, m_2 \geqslant 1$. The isogeny π induces a continuous group homomorphism of the closures $\overline{\langle \gamma_1^{m_1} \rangle} \to \overline{\langle \gamma_2^{m_2} \rangle}$ of the cyclic subgroups generated by $\gamma_1^{m_1}$ and $\gamma_2^{m_2}$ in $G_1(K_v)$ and $G_2(K_v)$ respectively. As we observed above, $\overline{\langle \gamma_1^{m_1} \rangle}$ is compact, so $\overline{\langle \gamma_2^{m_2} \rangle}$ must also be compact, a contradiction.

To find a $\gamma_2 \in \Gamma_2$ with the desired properties we will use the results of [27]. First, let $v \in V_\infty^K \setminus \{v_0\}$ be such that $G_2(K_v) = G_2(\mathbb{R})$ is noncompact (or, equivalently, $\mathrm{rk}_{K_v} G_2 > 0$). It was shown in [27] (cf. the proof of Theorem 2) that there exists a regular $\mathbb{R}$-regular[7] semi-simple element $\gamma_2 \in \Gamma_2$ for which the corresponding torus T_2 satisfies (15). Since the fact that γ_2 is $\mathbb{R}$-regular clearly implies that the closure of $\langle\gamma_2\rangle$ is noncompact, we see that γ_2 has the desired properties, proving (a).

To prove (b), we need to find a $\gamma_2 \in \Gamma_2$, with the properties described above, assuming that $v \in V_f^K$ and Γ_2 is unbounded in $G_2(K_v)$. For this, we will use the results of [27] in conjunction with the following result of Weisfeiler ([40], Theorem 10.5): there exists a finite subset $\mathcal{S}$ of V^K containing V_∞^K such that (i) the subgroup $\widetilde{\Gamma}_2 := \Gamma_2 \cap G_2(\mathcal{O}(\mathcal{S}))$ is Zariski-dense in G_2, (ii) for every $v \in V^K \setminus \mathcal{S}$, the closure of $\widetilde{\Gamma}_2$ in $G_2(K_v)$ is open, and (iii) for any $v \in \mathcal{S} \setminus V_\infty^K$, the subgroup $\widetilde{\Gamma}_2$ is discrete in $G_2(K_v)$. Pick such a set $\mathcal{S}$, and first consider the case where $v \in \mathcal{S} \setminus V_\infty^K$. Since $\widetilde{\Gamma}_2$ is Zariski-dense, by [27], there exists a regular semi-simple element $\gamma_2 \in \widetilde{\Gamma}_2$ of infinite order such that the corresponding torus T_2 satisfies (15). But since $\widetilde{\Gamma}_2$ is discrete in $G_2(K_v)$, the subgroup $\langle\gamma_2\rangle$ is automatically unbounded. Now, let $v \in V^K \setminus \mathcal{S}$, and suppose that Γ_2 is unbounded in $G_2(K_v)$. Then G_2 is K_v-isotropic and the closure of Γ_2 in $G_2(K_v)$ is unbounded and open, so it contains the normal subgroup $G_2(K_v)^+$ of $G_2(K_v)$ generated by the unipotent elements (cf. [23]), which is known to be an open subgroup of $G_2(K_v)$ of finite index (cf. [21], Theorem 3.3 and Proposition 3.17). Now we fix a maximal K_v-torus T_2^v of G_2 which contains a maximal K_v-split torus of the latter. Consider the solid open subset $\mathcal{U} = \mathcal{U}(T_2^v, v)$ of $G_2(K_v)$ constructed in Lemma 3.3. Then $\Omega_2^v := \mathcal{U} \cap G_2(K_v)^+$ is a nonempty open subset of $G_2(K_v)^+$. On the other hand, $\Gamma_2 \cap G_2(K_v)^+$ is dense in $G_2(K_v)^+$. So, one can pick a $y \in \Gamma_2 \cap \Omega_2^v$. Then an argument similar to the one used to prove Theorem 2 in [27] (where instead of using Lemma 3.5 of [24], we use Proposition 2.6 of [22]) shows that there exists $x \in \Gamma_2$ such that, for a suitable large positive integer n, $\gamma_2 := xy^n$ is regular K_v-regular, and for the unique maximal K-torus T_2 of G_2 containing γ_2, (15) holds. At the same time, since γ_2 is K_v-regular, the subgroup $\langle\gamma_2\rangle$ is unbounded in $G_2(K_v)$. Thus, γ_2 is as required, and the proof of (b) is now complete.

Remark 5.5. Let Γ be a torsion-free Zariski-dense subgroup of $G(F)$. For any positive integer m, the normal subgroup $\Gamma^{(m)}$ of Γ, generated by the m-th powers of the elements in Γ, is weakly commensurable with Γ. On the other hand, it is known, see [19], that if Γ is a cocompact lattice in a real semi-simple Lie group of real rank 1, then there exists an integer m such that $\Gamma^{(m)}$ is of infinite index in Γ. This shows that the requirement that Γ_2

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[7]Given a connected semi-simple algebraic group G defined over a local field L, an element $x \in G(L)$ is called L-regular if the number of eigenvalues, counted with multiplicity, of modulus 1 of $\mathrm{Ad}\, x$ is minimum possible.

be a lattice in Theorem F cannot be omitted in case G is of F-rank 1. The question whether or not a (discrete) subgroup weakly commensurable to an irreducible lattice (which is, of course, automatically arithmetic) in a real semi-simple Lie group of real rank > 1, is itself a lattice, remains open. We would like to point out, however, that no variation of the above method for constructing counter-examples is likely to work in the higher rank case.

More precisely, let again Γ be a torsion-free Zariski-dense subgroup of $G(F)$. Given a map $\varphi\colon \Gamma \to \mathbb{N}$, we let Γ_{φ} denote the subgroup of Γ generated by $\gamma^{\varphi(\gamma)}$ for all $\gamma \in \Gamma$. This subgroup is obviously weakly commensurable to Γ for *any* choice of φ. However, in contrast to the case of cocompact lattices in rank one groups groups discussed in the previous paragraph, or even finite index subgroups of $SL_2(\mathbb{Z})$, where the subgroup $\Gamma^{(m)}$ (which corresponds to $\varphi \equiv m$) has infinite index in Γ for a suitable m, the subgroup Γ_{φ} always has finite index in Γ if Γ is "boundedly generated" (this fact was pointed out to us by Thomas Delzant). On the other hand, several non-cocompact arithmetic lattices in the higher rank case are known to be boundedly generated (see [11] for the definition of and most recent results on "bounded generation"), and for them considering subgroups of the form Γ_{φ} will never lead to a weakly commensurable subgroup of infinite index.

5.6. A question. Given two Zariski-dense weakly commensurable subgroups of $G(F)$ (where F is a nondiscrete locally compact field), is it true that discreteness of one of them implies that of the other?

6. The invariance of rank and the proof of Theorems C and D

In view of Theorem B, weakly commensurable Zariski-dense S-arithmetic subgroups necessarily have the same field of definition K and correspond to the same set of places S. So now the focus of our study of such subgroups shifts to identifying common characteristics of the K-forms G_i used to construct them.

Proposition 6.1. *Let V_0 be a finite set of places of K. Let Γ_i be a Zariski-dense (G_i, K, S)-arithmetic subgroup of $G(F)$ for $i = 1, 2$. Let L_i be the smallest Galois extension of K over which G_i is inner. If Γ_1 and Γ_2 are weakly commensurable, then there exists a maximal K-torus T_1 of G_1 which contains a maximal K_{v_0}-split torus of G_1 for all $v_0 \in V_0$, a maximal K-torus T_2 of G_2, and a K-isogeny $\pi : T_2 \to T_1$. Moreover, if $L_1 = L_2$, and G is either simply connected or adjoint, and it is not of type $B_2 = C_2$, F_4, or G_2, then we can assume that π is an isomorphism, and $\pi^*(\Phi(G_1, T_1)) = \Phi(G_2, T_2)$.*

Proof. Using Corollary 3.2, we can find a maximal K-torus T_1 of G_1 which contains a maximal K_v-split torus of G_1 for every $v \in S \cup V_0$, and for which

$$\theta_{T_1}(\mathrm{Gal}(K_{T_1}/K)) \supset W(G_1, T_1).$$

Then the group $T_{1S} = \prod_{v \in S} T_1(K_v)$ is noncompact, and since the quotient $T_{1S}/T_1(\mathcal{O}_K(S))$ is compact as T_1 is K-anisotropic, we infer that $T_1(\mathcal{O}_K(S))$ is infinite. Therefore, $\Gamma_1 \cap T_1(K)$ contains an element γ_1 of infinite order. By our assumption, γ_1 is weakly commensurable to some semi-simple $\gamma_2 \in \Gamma_2 \cap G_2(K)$. Let T_2 be a maximal K-torus of G_2 that contains γ_2. According to Theorem 4.2, there exists a K-isogeny $\pi\colon T_2 \to T_1$. The second assertion of the proposition follows from Theorem 4.2 and Remark 4.5. □

Theorem 6.2. *Let Γ_i be a Zariski-dense (G_i, K, S)-arithmetic subgroup of $G(F)$ for $i = 1, 2$. If Γ_1 and Γ_2 are weakly commensurable, then*

$$\mathrm{rk}_{K_v}\, G_1 = \mathrm{rk}_{K_v}\, G_2 \quad \textit{for all} \quad v \in V^K.$$

Proof. Fix $v_0 \in V^K$. By symmetry, it is enough to show that

$$\mathrm{rk}_{K_{v_0}}\, G_1 \leqslant \mathrm{rk}_{K_{v_0}}\, G_2.$$

Applying the preceding proposition to $V_0 = \{v_0\}$, for $i = 1, 2$, we can find a maximal K-torus T_i of G_i such that T_1 contains a maximal K_{v_0}-split torus of G_1, and there is a K-isogeny $\pi : T_2 \to T_1$. From this we see that

$$\mathrm{rk}_{K_{v_0}}\, G_1 = \mathrm{rk}_{K_{v_0}}\, T_1 = \mathrm{rk}_{K_{v_0}}\, T_2 \leqslant \mathrm{rk}_{K_{v_0}}\, G_2.$$

□

For a connected absolutely simple algebraic group G_0 defined over a number field K, we let $\Sigma(G_0, K)$ (resp., $\Sigma^q(G_0, K)$) be the set of places $v \in V^K$ such that G_0 is split (resp., is quasi-split but not split) over K_v (of course, $\Sigma^q(G_0, K)$ is empty if G_0 is an inner form of a split group over K).

Theorem 6.3. *Let Γ_i be a Zariski-dense (G_i, K, S)-arithmetic subgroup of $G(F)$ for $i = 1, 2$. If Γ_1 and Γ_2 are weakly commensurable, then*

(1) $\Sigma(G_1, K) = \Sigma(G_2, K)$;

(2) *if L_i is the minimal Galois extension of K over which G_i becomes an inner form (of a split group), then $L_1 = L_2$;*

(3) $\Sigma^q(G_1, K) = \Sigma^q(G_2, K)$.

Proof. Assertion (1) immediately follows from the preceding theorem. To prove (2), by symmetry it is enough to show that $L_1 \subset L_2$. Assume, if possible, that L_1 is not contained in L_2. Then L_1L_2 is a Galois extension of K that properly contains L_2. It follows from Chebotarev's Density Theorem that there are infinitely many $v \in V_f^K$ that split completely in L_2 but not in L_1. Also, G_2 is quasi-split over K_v for all but finitely many $v \in V_f^K$, cf. [21], Theorem 6.7. So there exists a $v \in V_f^K$ which splits completely in L_2 but not in L_1, and G_2 is quasi-split over K_v. Then G_2 actually splits over K_v, i.e., $v \in \Sigma(G_2, K)$, but since v does not split in L_1, we have $v \notin \Sigma(G_1, K)$, which contradicts assertion (1). Now assertion (3) follows at once from Theorem 6.2. □

Remark 6.4. Technically, Theorem 6.2 and Theorem 6.3, parts (1) and (3), are consequences of the assertion in Theorem E (to be proved in the next section) that in the situation at hand, the Tits indices of G_1 and G_2 over K_v are identical, for all $v \in V^K$. We decided to include the above straightforward proofs for the following two reasons: first, the assertions of Theorems 6.2 and 6.3 are actually used in the proof of Theorem E, and second, we would like to show the reader that all theorems *except Theorem E* can be obtained without using the technical results involving Tits index.

Before we proceed to the proofs of Theorems C and D, we briefly recall the classification of absolutely simple algebraic groups of a given type over a field K (cf. [32], [36]). Any such group is an inner twist of a K-quasi-split group of the given type. So, fix a K-quasi-split group G_0. Notice that G_0 is completely determined by specifying (in addition to its Lie type) the minimal Galois extension L/K over which it splits; this extension necessarily has degree 1 (which means that G_0 splits over K) if the type is different from A_n $(n > 1)$, D_n $(n \geqslant 4)$, or E_6, can have degree 1 or 2 for the types A_n, D_n and E_6, and can also be either a cyclic extension of degree 3 or a Galois extension with the Galois group S_3 for type D_4. Furthermore, the K-isomorphism classes of inner twists of G_0 correspond bijectively to the elements lying in the image of the natural map

$$H^1(K, \overline{G}_0) \longrightarrow H^1(K, \mathrm{Aut}\,\overline{G}_0),$$

where $\overline{G}_0$ is the adjoint group of G_0 identified with its group of inner automorphisms. When K is a number field, one considers the natural "global-to-local" map

$$H^1(K, \overline{G}_0) \stackrel{\omega}{\longrightarrow} \bigoplus_{v \in V^K} H^1(K_v, \overline{G}_0),$$

and also the truncated maps

$$H^1(K, \overline{G}_0) \stackrel{\omega_S}{\longrightarrow} \bigoplus_{v \notin S} H^1(K_v, \overline{G}_0),$$

for every finite subset S of V^K. It is known that ω is injective (cf. [21], Theorem 6.22) and $\mathrm{Ker}\,\omega_S$ is finite (cf. [32], Theorem 7 in Ch. III, §4.6).

Proof of Theorem C. Let G_0 be the corresponding split group. For the groups of the types under consideration we have $\mathrm{Aut}\; G_0 = \overline{G}_0$, so the group G_i for $i = 1, 2$ is obtained from G_0 by twisting with a Galois cocycle representing an appropriate element c_i of $H^1(K, \overline{G}_0)$. We need to show that $c_1 = c_2$. For this we notice that according to Theorem 6.2, we have $\mathrm{rk}_{K_v}\, G_1 = \mathrm{rk}_{K_v}\, G_2$, for all $v \in V^K$. But for the types considered this implies that

$$G_1 \simeq G_2 \quad \text{over} \quad K_v. \tag{16}$$

Indeed, for v real, this follows from the classification of real forms of absolutely simple Lie algebras / real algebraic groups (cf. [14], Ch. X, §6, or

[36]). For v nonarchimedean, one can either consult Table II in [36] again, or use the fact that for the types under consideration, the center Z of the corresponding simply connected group is a subgroup of μ_2, the kernel of the endomorphism $x \mapsto x^2$ of GL_1. In view of the bijection between $H^1(K_v, \overline{G}_0)$ and $H^2(K_v, Z)$ (cf. [21], Corollary to Theorem 6.20), we see that $|H^1(K_v, \overline{G}_0)| \leqslant 2$, which means that there exists at most one nonsplit form, and therefore the equality of ranks implies the isomorphism between the forms. If we now let

$$\omega_v \colon H^1(K, \overline{G}_0) \longrightarrow H^1(K_v, \overline{G}_0)$$

denote the restriction map, then the isomorphism (16) implies that $\omega_v(c_1) = \omega_v(c_2)$, for all $v \in V^K$. Thus, $\omega(c_1) = \omega(c_2)$, and therefore, $c_1 = c_2$, as required. □

Proof of Theorem D. By Theorem 6.3 (2), the groups G_1 and G_2 have the same minimal Galois extension L/K over which they become inner forms. Let G_0 be the unique quasi-split inner twist of G_1 over K. Next, let

$$V_i = V^K \setminus (\Sigma(G_i, K) \cup \Sigma^q(G_i, K))$$

be the set of places v of K where G_i is not quasi-split. It is well-known that V_i is finite (cf. [21], Theorem 6.7). Furthermore, it follows from Theorem 6.3, (1) and (3), that $V_1 = V_2 =: V$. Thus, by fixing G_1 we automatically fix a finite set of places V such that any G_2 as in the statement of the theorem is quasi-split outside V. Now, consider $\xi_2 \in H^1(K, \overline{G}_0)$ which twists G_0 into G_2. Then for all $v \notin V$, the group G_2 is quasi-split over K_v, hence it is K_v-isomorphic to G_0, which means that $\omega_v(\xi_2)$ is trivial. (Here we use the fact that for a quasi-split group G_0 over any field F, the map $H^1(F, \overline{G}_0) \to H^1(F, \mathrm{Aut}\, G_0)$ has trivial kernel, which follows from the observation that $\mathrm{Aut}\, G_0$ is a semi-direct product over F, of $\overline{G}_0$ and a finite F-group of automorphisms corresponding to the symmetries of the Dynkin diagram.) Thus, $\xi_2 \in \mathrm{Ker}\, \omega_V$, so the finiteness of the kernel yields the finiteness of the number of K-isomorphism classes of possible K-groups G_2 with the properties described in the theorem. □

We conclude this section with two explicit examples demonstrating that in groups of type A_n, $n > 1$, the collection of weakly commensurable arithmetic subgroups may consist of more than one commensurability classes. Later, in §9, the idea underlying these examples will be developed into a new general technique for constructing nonisomorphic K-groups of type A_n, D_{2n+1} ($n > 1$) and E_6 which contain weakly commensurable arithmetic subgroups.

Example 6.5. Take $G = \mathrm{SL}_d$, where $d > 2$, over $F = \mathbb{C}$ (so that G is of type A_n with $n = d - 1 > 1$), and fix a number field K. Pick four arbitrary nonarchimedean places $v_1, v_2, v_3, v_4 \in V_f^K$. Let D_1 and D_2 be the central division algebras of degree d over K whose local invariants ($\in \mathbb{Q}/\mathbb{Z}$) are respectively

$$n_v^{(1)} = \begin{cases} 0\,, & v \neq v_i,\ i \leqslant 4 \\ 1/d\,, & v = v_1 \text{ or } v_2 \\ -1/d\,, & v = v_3 \text{ or } v_4 \end{cases} \quad \text{and} \quad n_v^{(2)} = \begin{cases} 0\,, & v \neq v_i,\ i \leqslant 4 \\ 1/d\,, & v = v_1 \text{ or } v_3 \\ -1/d\,, & v = v_2 \text{ or } v_4. \end{cases}$$

Then as $d > 2$, the algebras D_1 and D_2 are neither isomorphic nor anti-isomorphic. So the algebraic K-groups $G_1 = \mathrm{SL}_{1,D_1}$ and $G_2 = \mathrm{SL}_{1,D_2}$, which are inner K-forms of G, are not K-isomorphic. Thus, for any finite $S \subset V^K$, containing V_∞^K, the corresponding (G_i, K, S)-arithmetic subgroups $\Gamma_i \subset G(F)$ are not commensurable (cf. Proposition 2.5). On the other hand, if D is a central division algebra of degree d over K, then an extension L/K of degree d is isomorphic to a maximal subfield of D if and only if for every $v \in V^K$, and any extension $w|v$, the local degree $[L_w : K_v]$ annihilates the corresponding local invariant n_v (cf. [20], Corollary b in §18.4). It follows that the maximal subfields of either D_1 or D_2 are characterized as those extensions L/K of degree d for which $[L_{w_i} : K_{v_i}] = d$ for $i = 1, 2, 3, 4$. Thus, D_1 and D_2 have the *same* maximal subfields, which easily implies that Γ_1 and Γ_2 are weakly commensurable. Indeed, let $\gamma_1 \in \Gamma_1$ be a semi-simple element of infinite order, and let T_1 be a maximal K-torus of G_1 that contains γ_1. Since D_1 and D_2 have the same maximal subfields, there exists a K-isomorphism $T_1 \stackrel{\varphi}{\simeq} T_2$ with a maximal K-torus T_2 of G_2. Then the subgroup $\varphi(T_1 \cap \Gamma_1)$ is an S-arithmetic subgroup of T_2, so there exists $n > 0$ such that $\gamma_2 := \varphi(\gamma_1)^n \in \Gamma_2$. Let $\chi_1 \in X(T_1)$ be a character such that $\chi_1(\gamma_1)$ is not a root of unity. Then for $\chi_2 = (\varphi^*)^{-1}(\chi_1) \in X(T_2)$ we have

$$(n\chi_1)(\gamma_1) = \chi_1(\gamma_1)^n = \chi_2(\gamma_2) \neq 1,$$

which implies that Γ_1 and Γ_2 are weakly commensurable.

This example can be refined in two ways. First, by picking a sufficiently large number of nonarchimedean places and modifying the above construction accordingly, one can construct an arbitrarily large number of noncommensurable weakly commensurable S-arithmetic subgroups of the group $G(F) = \mathrm{SL}_d(\mathbb{C})$. Second, suppose $d > 2$ is even, and consider the real algebraic group $G = \mathrm{SL}_{d/2,\mathbb{H}}$, where $\mathbb{H}$ is the division algebra of Hamiltonian quaternions. Assume that K is a number field that admits a real embedding $K \hookrightarrow \mathbb{R} =: F$, and we let v_∞ denote the real place corresponding to this embedding. In addition to the four places $v_1, v_2, v_3, v_4 \in V_f^K$ fixed in the above example, we pick a fifth place $v_5 \in V_f^K \setminus \{v_1, v_2, v_3, v_4\}$, and consider the central division algebras D_1 and D_2 of degree d over K with the same local invariants at v_1, v_2, v_3, v_4 as above, and having the invariants $1/2$ at v_∞ and v_5, and 0 everywhere else. Then for any finite $S \subset V^K$ containing V_∞^K (in particular, for $S = V_\infty^K$ itself), the corresponding (G_i, K, S)-arithmetic subgroups are weakly commensurable, but not commensurable, and in addition are contained in $G(F) = \mathrm{SL}_{d/2}(\mathbb{H})$. Furthemore, by increasing the number of places picked, we can construct an arbitrarily large number of noncommensurable weakly commensurable S-arithmetic subgroups of $\mathrm{SL}_{d/2}(\mathbb{H})$.

The above construction implemented for $K = \mathbb{Q}$ and $d = 4$ has the following geometric significance. Over $\mathbb{R}$, the group G is isomorphic to the spinor group of a real quadratic form with signature $(5, 1)$, and therefore the associated symmetric space is the real hyperbolic 5-space. So, the noncommensurable arithmetic subgroups constructed above give rise to noncommensurable length-commensurable compact hyperbolic 5-manifolds (cf. Remark 8.11). We will elaborate on this observation in §9, where, in particular, noncommensurable length-commensurable compact hyperbolic manifolds will be constructed in any dimension of the form $4k + 1$.

Example 6.6. Let K be a number field and L be a quadratic extension of K. For $i = 1$, 2, let v_i be a nonarchimedean place of K which splits in L, and v'_i, v''_i be the places of L lying over v_i. Let $d > 1$ be an odd integer. Let D_1 and D_2 be the division algebra over L of degree d whose local invariants are respectively

$$n_v^{(1)} = \begin{cases} 1/d\ , & v = v'_1 \ \text{ or } \ v'_2 \\ -1/d\ , & v = v''_1 \ \text{ or } \ v''_2, \end{cases} \quad \text{and} \quad n_v^{(2)} = \begin{cases} 1/d\ , & v = v'_1 \ \text{ or } \ v''_2 \\ -1/d\ , & v = v''_1 \ \text{ or } \ v'_2, \end{cases}$$

and whose local invariant at every other place of L is zero. Then for $i = 1, 2$, the algebra D_i admits an involution σ_i of the second kind such that the fixed field L^{σ_i} coincides with K. Let G_i be an absolutely simple K-group with

$$G_i(K) = \{x \in D_i^{\times} \mid x\sigma_i(x) = 1,\ \mathrm{Nrd}\, x = 1\};$$

it is well-known that G_i is an outer form of type A_n with $n = d - 1 > 1$. For simplicity, let us assume that the involutions are chosen so that G_1 and G_2 are quasi-split at every real place of K which does not split in L (then G_1 and G_2 are automatically split at all other real places of K). Furthermore, since d is odd, G_1 and G_2 are automatically quasi-split at every nonarchimedean place of K which does not split in L. Thus, it follows from Proposition A.2 of Appendix A in [25] and the subsequent discussion that for an extension P/L of degree d provided with an automorphism τ of order two which induces the nontrivial automorphism of L/K, an embedding $(P, \tau) \to (D_i, \sigma_i)$ as algebras with involution exists if an only if $[P_w : K_{v_j}] = d$ for $j = 1, 2$ and $w|v_j$. This easily implies that the maximal σ_1-invariant subfields in D_1 are the same as the maximal σ_2-invariant subfields in D_2, and therefore G_1 and G_2 have the same maximal K-tori. Then as in the previous example, we conclude that for any S, the S-arithmetic subgroups of G_1 and G_2 are weakly commensurable. On the other hand, it follows from our choice of local invariants that G_1 and G_2 are not isomorphic even over L, so the constructed S-arithmetic subgroups are not commensurable. A suitable variation of this construction (applied to $K = \mathbb{Q}$, $L = \mathbb{Q}(i)$) enables one to construct length-commensurable, but not commensurable, compact complex hyperbolic $(d-1)$-manifolds, providing thereby a negative answer to Question (2) of the introduction for complex hyperbolic manifolds of any even dimension. We will not give the details here as the general construction described in §9 yields counterexamples for *all* dimensions.

7. Proof of Theorem E

Tits index of a semi-simple algebraic group (cf. [35], or [33], §15.5). Let G be a connected semi-simple algebraic K-group. To describe the Tits index of G/K, we pick a maximal K-split torus T_s of G and a maximal K-torus T of G containing T_s. Furthermore, we choose an ordering on the vector space $X(T_s) \otimes_{\mathbb{Z}} \mathbb{R}$, lift it to an ordering on $X(T) \otimes_{\mathbb{Z}} \mathbb{R}$ (cf. [33], §15.5–we will call such orderings on these vector spaces *coherent*), and let $\Delta \subset \Phi(G, T)$ denote the system of simple roots associated with this ordering. Then the *Tits index* of G/K is the data consisting of Δ (or the corresponding Dynkin diagram), the subset of *distinguished* roots, and the orbits of the $*$-action. We recall that a root $\alpha \in \Delta$ (or the corresponding vertex in the Dynkin diagram) is *distinguished* if its restriction to T_s is nontrivial. If $\alpha \in \Delta$ is distinguished then every root in the orbit Ω of α is distinguished; this is indicated by circling together all the vertices corresponding to the roots in Ω, and the latter is referred to as a *distinguished orbit.* We note that $\mathrm{rk}_K\, G$ equals the number of distinguished orbits, and G is quasi-split over K if and only if every root in Δ is distinguished.

For a subset $\Theta \subset \Delta$, we let P_Θ denote the corresponding standard parabolic subgroup which contains the centralizer of $\left(\bigcap_{\beta \in \Theta} \ker \beta\right)^\circ$ as a Levi subgroup. Then for a subset $\Omega \subset \Delta$, the subgroup $P_{\Delta \setminus \Omega}$ is defined over K if and only if Ω is $*$-invariant and consists entirely of distinguished roots (in other words, it is a union of distinguished orbits). In particular, a root $\alpha \in \Delta$ is distinguished if and only if for its $*$-orbit Ω the subgroup $P_{\Delta \setminus \Omega}$ is K-defined.

In the proof of Theorem E, we will need to work with the Tits indices of a given connected absolutely simple algebraic K-group G over various completions of K. For this purpose, we fix a maximal K-torus T of G and a system of simple roots $\Delta \subset \Phi(G, T)$. Given a field extension L/K, we choose a maximal L-torus T' containing a maximal L-split torus T'_s of G, and a system of simple roots $\Delta' \subset \Phi(G, T')$ determined by some coherent orderings on $X(T'_s) \otimes_{\mathbb{Z}} \mathbb{R}$ and $X(T') \otimes_{\mathbb{Z}} \mathbb{R}$. We say that $\alpha \in \Delta$ *corresponds to a distinguished vertex in the Tits index of* G/L if the root $\alpha' \in \Delta'$ that corresponds to α under the *canonical* identification $\Delta \simeq \Delta'$ (as described at the beginning of §4), is distinguished. The set of all $\alpha \in \Delta$ which correspond to distinguished vertices in the Tits index of G/L will be denoted $\Delta^{(d)}(L)$. It follows from Lemma 4.1(a) and the above discussion that $\alpha \in \Delta^{(d)}(L)$ if and only if for the $*$-orbit Ω of α, a suitable conjugate of $P_{\Delta \setminus \Omega}$ is defined over L. More generally, for an arbitrary subset $\Omega \subset \Delta$, a suitable conjugate of $P_{\Delta \setminus \Omega}$ is defined over L if and only if Ω is invariant under the $*$-action of $\mathrm{Gal}(\overline{L}/L)$ and contained in $\Delta^{(d)}(L)$. Thus, $\mathrm{rk}_L\, G$ equals the number of orbits of the $*$-action of $\mathrm{Gal}(\overline{L}/L)$ on $\Delta^{(d)}(L)$, and G is quasi-split over L if and only if $\Delta^{(d)}(L) = \Delta$.

The following proposition, which is proved using some results of [29], will not only play a crucial role in the proof of Theorem E, it is also of independent interest.

Proposition 7.1. *Let G be a connected absolutely simple algebraic group over a number field K. Fix a maximal K-torus T, a system of simple roots $\Delta \subset \Phi(G,T)$, and let $\Omega_1, \ldots, \Omega_n$ be the orbits of the $*$-action of* $\mathrm{Gal}(\overline{K}/K)$ *on Δ. Furthermore, let $\Omega_{i_1}, \ldots, \Omega_{i_r}$ be all the orbits satisfying the following property:*

$$\Omega_{i_j} \subset \Delta^{(d)}(K_v) \ \ \textit{for all} \ \ v \in V^K. \tag{17}$$

Then $r = \mathrm{rk}_K\, G$, and $\Omega_{i_1}, \ldots, \Omega_{i_r}$ are precisely the orbits contained in $\Omega^{(d)}(K)$.

Proof. Without any loss of generality, we may (and do) assume that G is adjoint and T contains a maximal K-split torus of G. Clearly, the distinguished orbits in the Tits index of G/K are amongst $\Omega_{i_1}, \ldots, \Omega_{i_r}$, yielding the inequality $\mathrm{rk}_K\, G \leqslant r$. To prove the opposite inequality, it is enough to show that $\Omega_{i_1}, \ldots, \Omega_{i_r}$ are the distinguished orbits in the Tits index of G/K. For this, we set

$$\Omega = \Omega_{i_1} \cup \cdots \cup \Omega_{i_r},$$

and let $P_{\Delta \setminus \Omega}$ be the corresponding parabolic subgroup. It suffices to prove that the conjugacy class of $P_{\Delta \setminus \Omega}$ contains a subgroup defined over K. The group G is an inner twist of a unique quasi-split K-group G_0. Let T_0 be the centralizer of a maximal K-split torus T_0^s of G_0. Furthermore, let $\Delta_0 \subset \Phi(G_0, T_0)$ be the system of simple roots with respect to some coherent orderings on $X(T_0) \otimes_{\mathbb{Z}} \mathbb{R}$ and $X(T_0^s) \otimes_{\mathbb{Z}} \mathbb{R}$ (then, in particular, all the roots in Δ_0 are distinguished). Since G is an inner twist of G_0, we can pick a $\overline{K}$-isomorphism $f \colon G_0 \to G$ so that the associated Galois cocycle

$$\sigma \mapsto \xi_\sigma := f^{-1} \circ \sigma(f), \quad \sigma \in \mathrm{Gal}(\overline{K}/K),$$

is of the form

$$\xi_\sigma = i_{g_\sigma}$$

where i_x denotes the inner automorphism of G_0 corresponding to $x \in G_0$, and $g = \{g_\sigma\}$ is a Galois cocycle with values in $G_0(\overline{K})$. Adjusting f by an inner automorphism, we can assume that $f(T_0) = T$ and $f^*(\Delta) = \Delta_0$. Having chosen f this way, we set $\Omega_0 = f^*(\Omega)$. Then for the corresponding parabolic K-subgroup $P_{\Delta_0 \setminus \Omega_0}$ of G_0, we have $f(P_{\Delta_0 \setminus \Omega_0}) = P_{\Delta \setminus \Omega}$. Let H_0 be a Levi K-subgroup of $P_{\Delta_0 \setminus \Omega_0}$.

Take an arbitrary $v \in V^K$. In view of (17), there exists $a_v \in G(\overline{K}_v)$ such that $P^{(v)}_{\Delta \setminus \Omega} := a_v P_{\Delta \setminus \Omega} a_v^{-1}$ is defined over K_v. Set $b_v = f^{-1}(a_v)$ and $f_v = f \circ i_{b_v}$. Then $f_v(P_{\Delta_0 \setminus \Omega_0}) = P^{(v)}_{\Delta \setminus \Omega}$, and since both $P_{\Delta_0 \setminus \Omega_0}$ and $P^{(v)}_{\Delta \setminus \Omega}$ are defined over K_v, for any $\sigma \in \mathrm{Gal}(\overline{K}_v/K_v)$, the automorphism

$$\xi^{(v)}_\sigma := f_v^{-1} \circ \sigma(f_v) = i_{b_v}^{-1} \circ \xi_\sigma \circ i_{\sigma(b_v)} = i_{b_v^{-1} g_\sigma \sigma(b_v)}$$

leaves $P_{\Delta_0\setminus\Omega_0}$ invariant. As $P_{\Delta_0\setminus\Omega_0}$ coincides with its normalizer in G_0 (cf. [4], Theorem 11.16), we conclude that $b_v^{-1}g_\sigma\sigma(b_v)$ lies in $P_{\Delta_0\setminus\Omega_0}(\overline{K}_v)$. Furthermore, since the unipotent radical of $P_{\Delta_0\setminus\Omega_0}$ has trivial Galois cohomology, we conclude that the cocycle $\sigma \mapsto b_v^{-1}g_\sigma\sigma(b_v)$ is cohomologous to a $H(\overline{K}_v)$-valued Galois cocycle $h^{(v)}$. Thus, the image of the cohomology class x corresponding to the cocycle g, under the restriction map $\rho_v \colon H^1(K, G_0) \to H^1(K_v, G_0)$, is equal to the image of the cohomology class in $H^1(K_v, H_0)$, corresponding to $h^{(v)}$, under the map $H^1(K_v, H_0) \to H^1(K_v, G_0)$.

Now, let L be the minimal Galois extension of K over which G_0 splits, and set $P = L$ if $[L : K] \neq 6$, and let P be any cubic subextension of L otherwise. Pick $v_0 \in V_f^K$ which does not split in P (i.e., $P \otimes_K K_{v_0}$ is a field). As we showed in the proof of Theorem 1 in [29], it follows from Theorem 2 therein that there exists $y \in H^1(K, H_0)$ which maps to $(\rho_v(x))$ under the composite of the following two maps

$$H^1(K, H_0) \xrightarrow{\omega} H^1(K, G_0) \xrightarrow{\rho} \bigoplus_{v \neq v_0} H^1(K_v, G_0).$$

But according to Theorem 3 in [29], ρ is injective, so $x = \omega(y)$. This means that there exists $c \in G_0(\overline{K})$ such that

$$c^{-1}g_\sigma\sigma(c) \in H_0(\overline{K}) \quad \text{for all} \quad \sigma \in \mathrm{Gal}(\overline{K}/K). \tag{18}$$

We claim that the subgroup $f(c)P_{\Delta\setminus\Omega}f(c)^{-1} = f(cP_{\Delta_0\setminus\Omega_0}c^{-1})$ is defined over K. Indeed, for $\sigma \in \mathrm{Gal}(\overline{K}/K)$ we have

$$\begin{aligned}\sigma(f(cP_{\Delta_0\setminus\Omega_0}c^{-1})) &= \sigma(f)(\sigma(c)P_{\Delta_0\setminus\Omega_0}\sigma(c)^{-1}) \\ &= f(g_\sigma\sigma(c)P_{\Delta_0\setminus\Omega_0}\sigma(c)^{-1}g_\sigma^{-1}) = f(cP_{\Delta_0\setminus\Omega_0}c^{-1})\end{aligned}$$

in view of (18), proving our claim and completing the proof of the proposition. □

Corollary 7.2. *Let G be an absolutely simple K-group of one of the following types: B_n $(n \geqslant 2)$, C_n $(n \geqslant 2)$, E_7, E_8, F_4 or G_2. If G is isotropic over K_v for all real $v \in V_\infty^K$, then G is isotropic over K. Additionally, if G is as above, but not of type E_7, then*

$$\mathrm{rk}_K\, G = \min_{v \in V^K} \mathrm{rk}_{K_v}\, G. \tag{19}$$

Proof. The groups of these types do not have outer automorphisms, so given any two maximal K-tori T and T' of G, and systems of simple roots $\Delta \subset \Phi(G, T)$ and $\Delta' \subset \Phi(G, T')$, there is a unique isomorphism between $\Phi(G, T)$ and $\Phi(G, T')$ that carries Δ to Δ'. It necessarily coincides with the canonical identification as defined in the beginning of §4. Using this remark and inspecting Table II in [36], we see that for the types listed in the statement, if for every real place v of K, G is isotropic over K_v, then there is a vertex in the Tits index of G/K which corresponds to a distinguished vertex in the Tits index of G/K_v, for all $v \in V^K$. Then it follows from the proposition that this vertex is distinguished in the Tits index of G/K, and therefore G

is K-isotropic. Moreover, if G is not of type E_7, then it follows from the tables that the total number of vertices which are distinguished in the Tits index of G/K_v for all $v \in V^K$ is $\min_{v \in V^K} \mathrm{rk}_{K_v} G$, so (19) follows from the proposition. □

Proof of Theorem E. If G is of type $B_2 = C_2$, F_4, or G_2, then its Tits index over any extension L/K is uniquely determined by its L-rank. So, since $\mathrm{rk}_{K_v} G_1 = \mathrm{rk}_{K_v} G_2$ according to Theorem 6.2 and consequence $\mathrm{rk}_K G_1 = \mathrm{rk}_K G_2$ according to Corollary 7.2, all our assertions follow. So, we assume that G is not of any of the above three types, and in addition is adjoint.

We pick a finite set V_0 of places of K such that for every $v \notin V_0$, both G_1 and G_2 are quasi-split over K_v. By Theorem 6.3(2), we have $L_1 = L_2$, so we can use Proposition 6.1 to find maximal K-tori T_i of G_i such that T_1 contains a maximal K_v-split torus T_{1s}^v of G_1 for all $v \in V_0$, and a K-isogeny (actually, a K-isomorphism) $\pi\colon T_2 \to T_1$ such that $\pi^*(\Phi(G_1, T_1)) = \Phi(G_2, T_2)$. Since $\mathrm{rk}_{K_v} G_1 = \mathrm{rk}_{K_v} G_2$ for all v, we see that T_2 also contains a maximal K_v-split torus T_{2s}^v of G_2, for all $v \in V_0$. Notice that if we choose any system of simple roots Δ_1 in $\Phi(G_1, T_1)$ and set $\Delta_2 = \pi^*(\Delta_1)$ then because π^* commutes with the action of $\mathrm{Gal}(\overline{K}/K)$ and the corresponding Weyl groups, it also commutes with the $*$-action of $\mathrm{Gal}(\overline{F}/F)$ for any extension F/K. Now, let $v \in V_0$, and let Δ_1^v be a system of simple roots in $\Phi(G_1, T_1)$ that corresponds to a coherent choice of orderings on $X(T_{1s}^v) \otimes_{\mathbb{Z}} \mathbb{R}$ and $X(T_1) \otimes_{\mathbb{Z}} \mathbb{R}$. Then $\Delta_2^v = \pi^*(\Delta_1^v)$ corresponds to the coherent orderings on $X(T_{2s}^v) \otimes_{\mathbb{Z}} \mathbb{R}$ and $X(T) \otimes_{\mathbb{Z}} \mathbb{R}$. Furthermore, since π induces an isomorphism between T_{2s}^v and T_{1s}^v, we see that $\alpha \in \Delta_1^v$ has nontrivial restriction to T_{1s}^v, i.e. is distinguished in the Tits index of G_1/K_v if and only if $\pi^*(\alpha)$ has nontrivial restriction to T_{2s}^v, i.e. is distinguished in the Tits index of G_2/K_v. This shows that the Tits indices of G_2/K_v and G_2/K_v are isomorphic for all $v \in V_0$. They are also isomorphic for any $v \in V^K \setminus V_0$ because then G_1 and G_2 are quasi-split, which completes the proof of the "local" part of Theorem E.

It remains to prove that the Tits indices of G_1/K and G_2/K are isomorphic. For this, we fix a system of simple roots Δ_1 of $\Phi(G_1, T_1)$ and set $\Delta_2 = \pi^*(\Delta_1)$. If $\Delta_1' \subset \Phi(G_1, T_1)$ is another system of simple roots and $\Delta_2' = \pi^*(\Delta_1)$ then the fact that π^* commutes with the action of the corresponding Weyl groups implies that π^* transports the canonical identification $\Delta_1 \simeq \Delta_1'$ to the canonical identification $\Delta_2 \simeq \Delta_2'$ (another way to see this is to observe that according to Remark 4.5, π extends to a $\overline{K}$-isomorphism $f\colon G_2 \to G_1$). So, by symmetry, it is enough to prove that if $\Omega \subset \Delta_1$ is an orbit of the $*$-action of $\mathrm{Gal}(\overline{K}/K)$ which corresponds to a distinguished orbit in the Tits index of G_1/K then $\pi^*(\Omega)$ (which is also a $*$-orbit) corresponds to a distinguished orbit in the Tits index of G_2/K. According to Proposition 7.1, it is enough to show that

$$\pi^*(\Omega) \subset \Delta_2^{(d)}(K_v) \tag{20}$$

for all $v \in V^K$. By our construction, $\Delta_2^{(d)}(K_v) = \Delta_2$ for all $v \in V^K \setminus V_0$, so we only need to establish (20) for $v \in V_0$. But since π^* induces a bijection between distinguished vertices in Δ_1^v and Δ_2^v in the above notations, we see that

$$\Delta_2^{(d)}(K_v) = \pi^*(\Delta_1^{(d)}(K_v)),$$

and (20) follows, completing the proof of Theorem E. □

The following interesting result is an immediate consequence of Theorems 6.3(2), C, D, and E.

Theorem 7.3. *Let K be a number field and G be a connected absolutely simple K-group. Let L be the smallest Galois extension of K over which G is an inner form of a split group. Let $\mathfrak{F}$ be a collection of K-forms G' of G such that the set of K-isomorphism classes of maximal K-tori of G' equals the set of K-isomorphism classes of maximal K-tori in G. Then*

(1) *For any group belonging to $\mathfrak{F}$, L is the smallest Galois extension of K over which it is an inner form of a split group.*
(2) *For any $G' \in \mathfrak{F}$, the Tits indices of G/K and G'/K, and for every place v of K, the Tits indices of G/K_v and G'/K_v, are isomorphic.*
(3) *If G is not of type A_n, D_n, or E_6, then every $G' \in \mathfrak{F}$ is K-isomorphic to G.*
(4) *$\mathfrak{F}$ consists of finitely many K-isomorphism classes.*

Proof. Fix $G' \in \mathfrak{F}$ and pick a finite set S of places of K containing all the archimedean ones so that $\prod_{v\in S} G(K_v)$ and $\prod_{v\in S} G'(K_v)$ are noncompact. Let Γ and Γ' be some S-arithmetic subgroup of $G(K)$ and $G'(K)$, respectively. As G and G' have the same K-tori, it immediately follows from the definition of weak commensurability that Γ and Γ' are weakly commensurable. Now all the four assertions of the present theorem follow from Theorems 6.3(2), C, D and E. □

Remark 7.4. In section 9 we will show that assertion (3) of the preceding theorem is false in general if G is of type A_n, D_{2n+1} $(n > 1)$ or E_6.

8. Lengths of closed geodesics, length-commensurable locally symmetric spaces and Schanuel's conjecture

Let G be a connected semi-simple real algebraic group, $\mathcal{G} = G(\mathbb{R})$, and let $\mathcal{K}$ be a maximal compact subgroup of $\mathcal{G}$. We let $\mathfrak{g}$ and $\mathfrak{k}$ denote the Lie algebras of $\mathcal{G}$ and $\mathcal{K}$ respectively, and let $\mathfrak{p}$ denote the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ relative to the Killing form $\langle\ ,\ \rangle$, so that $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is a Cartan decomposition of $\mathfrak{g}$. The corresponding symmetric space $\mathfrak{X} = \mathcal{K}\backslash\mathcal{G}$ is a Riemannian manifold with the metric induced by the restriction of the Killing form to $\mathfrak{p}$ (see [14] for the details).

A character χ of an $\mathbb{R}$-torus T is said to be *positive* if for every $x \in T(\mathbb{R})$, the value $\chi(x)$ is a positive real number. Any positive character of T is defined over $\mathbb{R}$. Given an arbitrary character $\chi \in X(T)$, the character $\chi + \bar{\chi}$, where $\bar{\chi}$ is the character obtained by applying the complex conjugation to χ, satisfies

$$(\chi + \bar{\chi})(x) = \chi(x)\overline{\chi(x)} = |\chi(x)|^2$$

for all $x \in T(\mathbb{R})$. Thus, for any character χ and any $x \in T(\mathbb{R})$, the square of the absolute value of $\chi(x)$ is the value assumed by the positive character $\chi + \bar{\chi}$ of T at x.

Let S be an $\mathbb{R}$-split torus and T be a $\mathbb{R}$-torus containing S. Then every character of S is defined over $\mathbb{R}$. Given a character α of S, let χ be a complex character of T whose restriction to S equals α. Then the restriction of the positive character $\chi + \overline{\chi}$ to S is 2α. Thus every character lying in the subgroup $2X(S)$ of the character group $X(S)$ of S extends to a positive character of any $\mathbb{R}$-torus containing S.

Let $\mathfrak{a}$ be a Cartan subspace contained in $\mathfrak{p}$, and $\mathcal{A} = \exp \mathfrak{a}$ be the connected abelian subgroup of $\mathcal{G}$ with Lie algebra $\mathfrak{a}$. Let S be the Zariski-closure of $\mathcal{A}$. Then S is a maximal $\mathbb{R}$-split torus of G and $\mathcal{A} = S(\mathbb{R})^\circ$. We fix a closed Weyl chamber $\mathfrak{a}^+$ in $\mathfrak{a}$. Let $\{\alpha_1, \ldots, \alpha_r\}$, where $r = \mathrm{rk}_{\mathbb{R}}\, G = \dim S$, be the basis of the root system of G, with respect to S, determined the Weyl chamber $\mathfrak{a}^+$, and let $\beta_i = 2\alpha_i$. Then $\beta_1, \ldots, \beta_r$ are linearly independent positive characters. In the sequel, we will identify $\mathfrak{a}$ with $\mathbb{R}^r$ by identifying $X \in \mathfrak{a}$ with $(d\beta_1(X), \ldots, d\beta_r(X))$, where, for $i \in \{1, \ldots, r\}$, $d\beta_i$ denotes the differential of β_i at the identity.

We will now make some brief comments on the Lyapunov map and its relations with weak commensurability, and will then proceed to the core issue of the lengths of closed geodesics and length-commensurable locally symmetric spaces.

Lyapunov map. For an element $g \in G$, we let $g = g_s g_u$ be its Jordan decomposition. Let now $g \in \mathcal{G}$. Set $s = g_s$, and let T be a maximal $\mathbb{R}$-torus of G containing s. Let $\mathcal{C}$ be the maximal compact subgroup of $T(\mathbb{R})$ and T_s be the maximal $\mathbb{R}$-split subtorus of T. Then $T(\mathbb{R})$ is a direct product of $\mathcal{C}$ and $T_s(\mathbb{R})^\circ$, so we can write $s = s_e \cdot s_h$, with $s_e \in \mathcal{C}$, and $s_h \in T_s(\mathbb{R})^\circ$. The elements s_e and s_h are called the *elliptic* and the *hyperbolic* components of s (or of g). There is an element $z \in \mathcal{G}$ which conjugates $\mathcal{C}$ into $\mathcal{K}$ and $T_s(\mathbb{R})^\circ$ into $\mathcal{A}$ such that $z s_h z^{-1} = \exp X$, with $X \in \mathfrak{a}^+$. The element X is the unique element of $\mathfrak{a}^+$ such that the hyperbolic component s_h of g is a conjugate of $\exp X$, and we will denote it by $\ell(g)$. Thus we get a map (the Lyapunov map) $\ell : \mathcal{G} \to \mathfrak{a}^+$. Clearly, for any $g \in \mathcal{G}$ we have $\ell(g) = \ell(g_s)$, and moreover, for any positive integer n, $\ell(g^n) = n\ell(g)$.

Continuing with the above notations, we let χ_i, for $i \in \{1, \ldots, r\}$, be the unique positive character of T extending the character $\mathrm{Int}\, z^{-1} \cdot \beta_i|_{T_s}$, and let $d\chi_i$ denote its differential at the identity. Since $\chi_i(s) = \chi_i(s_h)$, we have

$$\ell(s) = (d\chi_1(\mathrm{Ad}\, z^{-1}(X)), \ldots, d\chi_r(\mathrm{Ad}\, z^{-1}(X))) = (\log \chi_1(s), \ldots, \log \chi_r(s)).$$

For a subgroup Γ of $\mathcal{G}$, let Γ^{ss} denote the set of semi-simple elements of Γ. From the above description of the Lyapunov map, the following proposition is obvious.

Proposition 8.1. *If Γ_1 and Γ_2 are two subgroups of $\mathcal{G}$ such that $\mathbb{Q}\cdot\ell(\Gamma_1^{\mathrm{ss}}) = \mathbb{Q}\cdot\ell(\Gamma_2^{\mathrm{ss}})$, then Γ_1 and Γ_2 are weakly commensurable.*

Now if Γ is an arithmetic subgroup of $\mathcal{G}$ and $g \in \Gamma$, then it is not difficult to see that there exists an integer $n = n(g)$ such that $g_u^n \in \Gamma$. Then g_s^n lies in Γ, and from this we conclude that $\mathbb{Q}\cdot\ell(\Gamma) = \mathbb{Q}\cdot\ell(\Gamma^{\mathrm{ss}})$.

Lengths of closed geodesics on locally symmetric spaces. Given a discrete torsion-free subgroup Γ of $\mathcal{G}$, the quotient $\mathfrak{X}_\Gamma := \mathfrak{X}/\Gamma$ is a Riemannian locally symmetric space. We first need to recall some facts about closed geodesics in $\mathfrak{X}_\Gamma$, and in particular the formula for their length, given in [28]. Closed geodesics in $\mathfrak{X}_\Gamma$ correspond to semi-simple elements in Γ, and are obtained by a construction similar to the one used to define the Lyapunov map. More precisely, let γ be a fixed semi-simple element of Γ, and let T be a maximal $\mathbb{R}$-torus of G containing γ. As we mentioned above, $T(\mathbb{R})$ is a direct product of $\mathcal{C}$ and $T_s(\mathbb{R})^\circ$, where $\mathcal{C}$ is the maximal compact subgroup of $T(\mathbb{R})$ and T_s is the maximal $\mathbb{R}$-split subtorus of T. Take *any* $z \in \mathcal{G}$ such that zTz^{-1} is invariant under the Cartan involution associated with the decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, and consequently

$$z\mathcal{C}z^{-1} \subset \mathcal{K} \quad \text{and} \quad zT_s(\mathbb{R})^\circ z^{-1} \subset \exp\mathfrak{p}. \tag{21}$$

Thus, here we do not require the inclusion $zT_s(\mathbb{R})^\circ z^{-1} \subset \exp\mathfrak{a}^+$, however, all the z's satisfying (21) lie in the same coset modulo $\mathcal{K}$, and therefore define the same point $\mathcal{K}z \in \mathfrak{X}$. So, if we write $\gamma = \gamma_e \cdot \gamma_h$ with $\gamma_e \in \mathcal{C}$ and $\gamma_h \in T_s(\mathbb{R})^\circ$, and then $\gamma_h = z^{-1}\exp(X)z$ for some $X \in \mathfrak{p}$ that commutes with $z\gamma_e z^{-1}$; moreover, it follows from the above discussion that X is a conjugate of $\ell(\gamma)$ under an element of $\mathrm{Ad}\,\mathcal{K}$. With these notations, the curve $\tilde{c}_\gamma$ parametrized by $\widetilde{\varphi}\colon t \mapsto \mathcal{K}\exp(tX)z$ for $t \in \mathbb{R}$, is a geodesic on $\mathfrak{X}$ which passes through the point $\mathcal{K}z$. Furthermore,

$$\widetilde{\varphi}(t)\cdot\gamma = \mathcal{K}\exp(tX)\cdot z\gamma_e z^{-1}\cdot z\gamma_h z^{-1}\cdot z = \mathcal{K}\exp(tX)\cdot\exp(X)\cdot z = \widetilde{\varphi}(t+1),$$

implying that the map $\varphi\colon \mathbb{R} \to \mathfrak{X}_\Gamma$, obtained by composing $\widetilde{\varphi}$ with the natural covering map $\pi\colon \mathfrak{X} \to \mathfrak{X}_\Gamma$, is periodic with period 1, and hence its smallest period is of the form $1/n_\gamma$ for some integer $n_\gamma \geqslant 1$. It follows that the image c_γ of $\tilde{c}_\gamma$ in $\mathfrak{X}_\Gamma$ is a closed geodesic, and since

$$\langle\varphi'(t), \varphi'(t)\rangle = \langle\widetilde{\varphi}'(t), \widetilde{\varphi}'(t)\rangle = \langle X, X\rangle,$$

for all $t \in \mathbb{R}$, we see that the length of c_γ is $(1/n_\gamma)\langle X, X\rangle$.

Proposition 8.2. (i) *Every closed geodesic in $\mathfrak{X}_\Gamma$ is of the form c_γ for some semi-simple $\gamma \in \Gamma$.*

(ii) *The length of* c_γ *is* $(1/n_\gamma)\lambda_\Gamma(\gamma)$ *where* n_γ *is an integer* $\geqslant 1$ *and* $\lambda_\Gamma(\gamma)$ *is given by the following formula:*

$$\lambda_\Gamma(\gamma)^2 = \langle \ell(\gamma), \ell(\gamma)\rangle = \left(\sum (\log|\alpha(\gamma)|)^2\right), \tag{22}$$

where the summation is over all roots of G *with respect to* T *and* log *denotes the natural logarithm.*

Thus,

$$\mathbb{Q}\cdot L(\mathfrak{X}_\Gamma) = \mathbb{Q}\cdot\{\lambda_\Gamma(\gamma) \mid \gamma\in\Gamma \quad \textit{semi-simple}\},$$

where $\lambda_\Gamma(\gamma)$ *is given by* (22).

Proof. (i) Any closed geodesic c in $\mathfrak{X}_\Gamma$ is obtained as the image under π of a geodesic $\tilde{c}$ in $\mathfrak{X}$. Fix a point $\mathcal{K}z \in \tilde{c}$. It is known that $\tilde{c}$ admits a parametrization of the form

$$\widetilde{\varphi}(t) = \mathcal{K}\exp(tX)z$$

for some $X\in\mathfrak{p}$ (cf. [14], Theorem 3.3(iii) in Ch. IV). After replacing X by a suitable positive-real multiple, we can assume that $\pi(\widetilde{\varphi}(0)) = \pi(\widetilde{\varphi}(1))$, and $d_{\widetilde{\varphi}(0)}\pi(\widetilde{\varphi}'(0)) = d_{\widetilde{\varphi}(1)}\pi(\widetilde{\varphi}'(1))$. Then, in particular, $\widetilde{\varphi}(1) = \widetilde{\varphi}(0)\gamma$ for some $\gamma\in\Gamma$. Since the map

$$\mathcal{K}\times\mathfrak{p}\to\mathcal{G}, \quad (\kappa, Y)\mapsto \kappa\exp(Y),$$

is a diffeomorphism, the element $z\gamma z^{-1}$ can be uniquely written in the form $z\gamma z^{-1} = \kappa\exp(Y)$. Then $\tilde{c}(1) = \tilde{c}(0)\gamma$ yields $X = Y$, i.e.

$$z\gamma z^{-1} = \kappa\exp(X). \tag{23}$$

Furthermore, the curves in $\mathfrak{X}$ with the parametrizations

$$\widetilde{\varphi}_1(t) = \widetilde{\varphi}(t)\cdot\gamma \quad\text{and}\quad \widetilde{\varphi}_2(t) = \tilde{\varphi}(t+1)$$

are both geodesics in $\mathfrak{X}$ such that

$$\widetilde{\varphi}_1(0) = \widetilde{\varphi}(0)\cdot\gamma = \widetilde{\varphi}(1) = \widetilde{\varphi}_2(0) =: p.$$

Since $\pi(\widetilde{\varphi}_1(t)) = \pi(\widetilde{\varphi}(t))$, we have

$$d_p\pi(\widetilde{\varphi}_1'(0)) = d_{\widetilde{\varphi}(0)}\pi(\widetilde{\varphi}'(0)) = d_{\widetilde{\varphi}(1)}\pi(\widetilde{\varphi}'(1)) = d_p\pi(\widetilde{\varphi}_2'(0)).$$

Thus, $\widetilde{\varphi}_1'(0) = \widetilde{\varphi}_2'(0)$, hence by the uniqueness of a geodesic through a given point in a given direction, we get $\widetilde{\varphi}_1(t) = \widetilde{\varphi}_2(t)$ for all t. Combining the definitions of $\widetilde{\varphi}$, $\widetilde{\varphi}_1$ and $\widetilde{\varphi}$ with (23), we now obtain that

$$\mathcal{K}\exp(tX)\kappa = \mathcal{K}\exp(t(\operatorname{Ad}\kappa^{-1}(X))) = \mathcal{K}\exp(tX),$$

which implies that κ commutes with $\exp(tX)$ for all t. Since the elements κ and $\exp(X)$ are semi-simple, we conclude that $\gamma = z^{-1}(\kappa\exp(X))z$ is semi-simple. Moreover, κ and $\exp(X)$ are contained in a maximal $\mathbb{R}$-torus of T_0 of G invariant under the Cartan involution. Then $T = z^{-1}T_0z$ contain γ, and $\gamma_e = z^{-1}\kappa z$ and $\gamma_h = z^{-1}\exp(X)z$ in the notations introduced prior to the statement of the proposition. It is now obvious that c coincides with the geodesic c_γ. As we already explained, its length is $(1/n_\gamma)\langle X, X\rangle^{1/2}$, where n_γ is the integer $\geqslant 1$ such that $1/n_\gamma$ is the smallest positive period of $\varphi(t) = \pi(\tilde{\varphi}(t))$.

(ii) We need to show that $\lambda_\Gamma(\gamma) := \langle X, X\rangle^{1/2}$ $(= \langle \ell(\gamma), \ell(\gamma)\rangle^{1/2})$ is given by the equation (22). Since the Killing form is invariant under the adjoint action of $\mathcal{G}$ on $\mathfrak{g}$, we have $\langle X, X\rangle = \langle X', X'\rangle$, where $X' = \mathrm{Ad} z^{-1}(X)$ so that $\gamma_h = \exp(X')$. In a suitable basis of $\mathfrak{g}$, $\mathrm{Ad}\,\gamma_h$ is represented by a diagonal matrix whose diagonal entries are 1 (repeated $\dim T$ times) and $\alpha(\gamma_h)$ for all $\alpha \in \Phi(G, T)$; notice that all these numbers are real and positive. In the same basis, $\mathrm{ad}\, X'$ is represented by a diagonal matrix with the diagonal entries 0 (repeated $\dim T$ times) and $d\alpha(X')$ for all $\alpha \in \Phi(G, T)$. For every α we clearly have

$$|\alpha(\gamma)| = |\alpha(\gamma_h)| = \exp(d\alpha(X')).$$

So,

$$\langle X, X\rangle = \langle X', X'\rangle = \sum_{\alpha\in\Phi(G,T)} (d\alpha(X'))^2 = \sum_{\alpha\in\Phi(G,T)} (\log|\alpha(\gamma)|)^2,$$

and (22) follows. □

In order to relate the notion of length commensurability with that of weak commensurability, we need to recast formula (22) in a slightly different form. As a root α of G with respect to T is a character of T, $|\alpha(\gamma)|^2$ is the value assumed by a positive character of T, and therefore,

$$\lambda_\Gamma(\gamma)^2 = \sum_{i=1}^{p} s_i(\log \chi_i(\gamma))^2, \tag{24}$$

where $\chi_1, \ldots, \chi_p$ are certain positive characters of T and $s_1, \ldots, s_p$ are positive rational numbers (whose denominators are divisors of 4).

We will now elaborate on (24) in the rank one case.

Lemma 8.3. *Assume that* $\mathrm{rk}_{\mathbb{R}}\, G = 1$, *and let* Γ *be a discrete torsion-free subgroup of* $\mathcal{G} = G(\mathbb{R})$. *Let* $\gamma \in \Gamma$ *be a semi-simple element* $\neq 1$, *and let* T *be a maximal* $\mathbb{R}$*-torus containing it. Then*

(1) $\mathrm{rk}_{\mathbb{R}}\, T = 1$, *so the group of positive characters of* T *is cyclic with a generator, say,* χ.
(2) $\chi(\gamma) \neq 1$.
(3) *There exists* $t > 0$, *depending only on* G, *but not on* γ, Γ *or* T *such that*
$$\lambda_\Gamma(\gamma) = t|\log \chi(\gamma)|.$$

Proof. (1): $\mathrm{rk}_{\mathbb{R}}\, T = 0$ would imply that $T(\mathbb{R})$ is compact, so the discreteness of $\langle\gamma\rangle$ would imply its finiteness. Since Γ is torsion-free, we would get $\gamma = 1$, a contradiction.

(2): Proved similarly using the fact that $(\ker \chi)(\mathbb{R})$ is compact.

(3): This follows from (22) and (24) combined with the fact that any two maximal $\mathbb{R}$-tori of G having real rank one are conjugate under an element of $\mathcal{G}$. □

Corollary 8.4. *Assume that* $\mathrm{rk}_{\mathbb{R}}\, G = 1$. *Let K be a number field contained in $\mathbb{R}$, and assume that G_1 and G_2 are two K-forms of G having the same set of K-isomorphism classes of maximal K-tori. Furthermore, for $i = 1, 2$, let Γ_i be a discrete torsion-free (G_i, K)-arithmetic subgroup of $\mathcal{G}$. Then*

$$\mathbb{Q} \cdot \lambda_{\Gamma_1}(\Gamma_1^{ss}) = \mathbb{Q} \cdot \lambda_{\Gamma_2}(\Gamma_2^{ss}), \tag{25}$$

and consequently, $\mathfrak{X}_{\Gamma_1}$ and $\mathfrak{X}_{\Gamma_2}$ are length-commensurable.

Indeed, let $\gamma_1 \in \Gamma_1^{ss} \setminus \{1\}$, and let T_1 be a maximal K-torus of G_1 containing γ_1. By our assumption, for a suitable maximal K-torus T_2 of G_2, there exists a K-isomorphism $\varphi\colon T_1 \to T_2$. Since $\varphi(T_1 \cap \Gamma_1)$ is an arithmetic subgroup of T_2, there exists $n > 0$ such that $\gamma_2 := \varphi(\gamma_1)^n \in T_2 \cap \Gamma_2$. Let $\chi^{(1)}$ be a generator of the group of positive characters of T_1 (cf. Lemma 8.3)(1)). Then $\chi^{(2)} := (\varphi^*)^{-1}(\chi_1)$ is a generator of the group of positive characters of T_2, and $\chi^{(2)}(\gamma_2) = \chi^{(1)}(\gamma_1)^n$. It follows from Lemma 8.3(3) that

$$|\lambda_{\Gamma_2}(\gamma_2)/\lambda_{\Gamma_1}(\gamma_1)| = n,$$

yielding the inclusion

$$\mathbb{Q} \cdot \lambda_{\Gamma_1}(\Gamma_1^{ss}) \subset \mathbb{Q} \cdot \lambda_{\Gamma_2}(\Gamma_2^{ss}).$$

By symmetry, we get (25). The last assertion follows from (25) and Proposition 8.2.

To deal with the higher rank case, we need the following.

Lemma 8.5. *Let $\gamma_1, \gamma_2 \in G(\mathbb{R})$ be two semi-simple elements contained in the maximal $\mathbb{R}$-tori T_1 and T_2 of G, respectively. Given two collections of characters $\chi_1^{(1)}, \ldots, \chi_{d_1}^{(1)} \in X(T_1)$ and $\chi_1^{(2)}, \ldots, \chi_{d_2}^{(2)} \in X(T_2)$, we set*

$$S_i = \{\log|\chi_1^{(i)}(\gamma_i)|, \ldots, \log|\chi_{d_i}^{(i)}(\gamma_i)|\}.$$

If γ_1, γ_2 are not weakly commensurable and each of the sets (of real numbers) S_1 and S_2 is linearly independent over $\mathbb{Q}$, then so is their union $S_1 \cup S_2$.

Proof. According to the above discussion, there exist positive characters $\theta_1^{(1)}, \ldots, \theta_{d_1}^{(1)} \in X(T_1)$ and $\theta_1^{(2)}, \ldots, \theta_{d_2}^{(2)} \in X(T_2)$ such that

$$\theta_j^{(i)}(x) = |\chi_j^{(i)}(x)|^2 \quad \text{for all} \;\; x \in T_i(\mathbb{R}).$$

If the set $S_1 \cup S_2$ is linearly dependent over $\mathbb{Q}$, there exist integers $s_1, \ldots, s_{d_1}$, $t_1, \ldots, t_{d_2}$, not all zero, such that

$$s_1 \log\theta_1^{(1)}(\gamma_1) + \cdots + s_{d_1}\log\theta_{d_1}^{(1)}(\gamma_1) + t_1\log\theta_1^{(2)}(\gamma_2) + \cdots + t_{d_2}\log\theta_{d_2}^{(2)}(\gamma_2) = 0.$$

Consider the characters

$$\psi_1 = s_1\theta_1^{(1)} + \cdots + s_{d_1}\theta_{d_1}^{(1)} \;\text{ of }\; T_1 \;\text{ and }\; \psi_2 = -(t_1\theta_1^{(2)} + \cdots + t_{d_2}\theta_{d_2}^{(2)}) \;\text{ of }\; T_2.$$

Then $\psi_1(\gamma_1) = \psi_2(\gamma_2)$, and hence,

$$\psi_1(\gamma_1) = 1 = \psi_2(\gamma_2)$$

because γ_1 and γ_2 are not commensurable. This means that

$$s_1 \log \theta_1^{(1)}(\gamma_1) + \cdots + s_{d_1} \log \theta_{d_1}^{(1)}(\gamma_1) = 0 = t_1 \log \theta_1^{(2)}(\gamma_2) + \cdots + t_{d_2} \log \theta_{d_2}^{(2)}(\gamma_2),$$

and therefore all the coefficients are zero because the sets S_1 and S_2 are linearly independent. $\square$

Some of our results depend on the validity of Schanuel's conjecture in transcendental number theory (cf. [1]), and we recall here its statement.

Schanuel's conjecture. *If $z_1, \ldots, z_n \in \mathbb{C}$ are linearly independent over $\mathbb{Q}$, then the transcendence degree (over $\mathbb{Q}$) of the field generated by*

$$z_1, \ldots, z_n;\ e^{z_1}, \ldots, e^{z_n}$$

is $\geqslant n$.

We will only use the fact that the truth of this conjecture implies that for algebraic numbers $z_1, \ldots, z_n$, (any values of) their logarithms

$$\log z_1, \ldots, \log z_n$$

are algebraically independent once they are linearly independent (over $\mathbb{Q}$).

Proposition 8.6. *Let G be a connected semi-simple real algebraic subgroup of SL_n and $\mathcal{G} = G(\mathbb{R})$. Let Γ_1, Γ_2 be two discrete torsion-free subgroups of $\mathcal{G}$. Suppose that nontrivial semi-simple elements $\gamma_1 \in \Gamma_1$ and $\gamma_2 \in \Gamma_2$ are not weakly commensurable. Then*

(i) *If $\mathrm{rk}_{\mathbb{R}}\, G = 1$, then $\theta = \lambda_{\Gamma_1}(\gamma_1)/\lambda_{\Gamma_2}(\gamma_2)$ is irrational. Moreover, if there exists a number field K such that Γ_1 and Γ_2 can be conjugated into $\mathrm{SL}_n(K)$, then θ is transcendental over $\mathbb{Q}$.*

(ii) *If there exists a number field K such that Γ_1 and Γ_2 can be conjugated into $\mathrm{SL}_n(K)$, and Schanuel's conjecture holds, then $\lambda_{\Gamma_1}(\gamma_1)$ and $\lambda_{\Gamma_2}(\gamma_2)$ are algebraically independent over $\mathbb{Q}$.*

Proof. We fix maximal $\mathbb{R}$-tori T_1 and T_2 of G which contain γ_1 and γ_2 respectively.

(i) Using Lemma 8.3, (1) and (2), for $i = 1, 2$, we can pick a generator $\chi^{(i)}$ of the group of positive characters of T_i so that $\chi^{(i)}(\gamma_i) > 1$. for $i = 1, 2$. Then by Lemma 8.3(3) we have

$$\lambda_{\Gamma_i}(\gamma_i) = t \log \chi^{(i)}(\gamma_i).$$

Since the elements γ_1 and γ_2 are not weakly commensurable, for every nonzero integers m, n, we have

$$\chi^{(1)}(\gamma_1)^m \neq \chi^{(2)}(\gamma_2)^n,$$

i.e., the ratio

$$\theta = \frac{\lambda_{\Gamma_1}(\gamma_1)}{\lambda_{\Gamma_2}(\gamma_2)} = \frac{\log \chi^{(1)}(\gamma_1)}{\log \chi^{(2)}(\gamma_2)}$$

is irrational. If Γ_1 and Γ_2 can be conjugated into $G(\overline{K})$, then the numbers $\chi^{(i)}(\gamma_i)$ are algebraic, and therefore by a theorem proved independently by Gel'fond and Schneider in 1934 (cf. [2]), θ is transcendental over $\mathbb{Q}$.

(ii) According to (24), we have the following expressions

$$\lambda_{\Gamma_1}(\gamma_1)^2 = \sum_{i=1}^{p} s_i^{(1)}(\log \chi_i^{(1)}(\gamma_1))^2 \quad \text{and} \quad \lambda_{\Gamma_2}(\gamma_2)^2 = \sum_{i=1}^{p} s_i^{(2)}(\log \chi_i^{(2)}(\gamma_2))^2$$

After renumbering the characters, we can assume that

$$a_1 := \log \chi_1^{(1)}(\gamma_1), \ldots, a_{m_1} := \log \chi_{m_1}^{(1)}(\gamma_1)$$

$$(\text{resp., } b_1 := \log \chi_1^{(2)}(\gamma_2), \ldots, b_{m_2} = \log \chi_{m_2}^{(2)}(\gamma_2))$$

for some $m_1, m_2 \leqslant p$, form a basis of the $\mathbb{Q}$-subspace of $\mathbb{R}$ spanned by $\log \chi_i^{(1)}(\gamma_1)$ (resp., $\log \chi_i^{(2)}(\gamma_2)$) for $i \leqslant p$ (notice that $m_1, m_2 \geqslant 1$ as otherwise the length of the corresponding geodesic would be zero, which is impossible). It follows from Lemma 8.5 that the numbers

$$a_1, \ldots, a_{m_1};\ b_1, \ldots, b_{m_2}$$

are linearly independent over $\mathbb{Q}$. Since by our assumption the subgroups Γ_1 and Γ_2 can be conjugated into $G(\overline{K})$, the values $\chi_i^{(j)}(\gamma_j)$ are algebraic numbers, so it follows from Schanuel's conjecture that $a_1, \ldots, a_{m_1},\ b_1, \ldots, b_{m_2}$ are algebraically independent over $\mathbb{Q}$. It remains to observe that $\lambda_{\Gamma_1}(\gamma_1)^2$ and $\lambda_{\Gamma_2}(\gamma_2)^2$ are given by nonzero homogeneous polynomials of degree two, with rational coefficients, in $a_1, \ldots, a_{m_1}$ and $b_1, \ldots, b_{m_2}$, respectively, and therefore they are algebraically independent. □

By combining Propositions 8.2 and 8.6 we obtain the following:

Theorem 8.7. *Let Γ_1, Γ_2 be discrete torsion-free subgroups of $\mathcal{G}$. If Γ_1 and Γ_2 are not weakly commensurable, then, possibly after interchanging them, the following assertions hold.*

(i) *If $\mathrm{rk}_{\mathbb{R}}\, G = 1$, then there exists $\lambda_1 \in L(\mathfrak{X}_{\Gamma_1})$ such that for* any *$\lambda_2 \in L(\mathfrak{X}_{\Gamma_2})$, the ratio λ_1/λ_2 is irrational.*

(ii) *If there exists a number field K such that both Γ_1 and Γ_2 can be conjugated into $\mathrm{SL}_n(K)$, and Schanuel's conjecture holds, then there exists $\lambda_1 \in L(\mathfrak{X}_{\Gamma_1})$ which is algebraically independent from* any *$\lambda_2 \in L(\mathfrak{X}_{\Gamma_2})$.*

In either case, (under the above assumptions) $\mathfrak{X}_{\Gamma_1}$ and $\mathfrak{X}_{\Gamma_2}$ are not length-commensurable.

The results in the rest of this section for locally symmetric spaces of rank > 1 *assume the truth of Schanuel's conjecture.*

Henceforth, we will study locally symmetric spaces of $\mathcal{G} = G(\mathbb{R})$, where G is an *absolutely simple* real algebraic group. It follows from Theorem 8.7 that length-commensurability of the locally symmetric spaces $\mathfrak{X}_{\Gamma_1}$ and $\mathfrak{X}_{\Gamma_2}$ implies weak commensurability of the subgroups Γ_1 and Γ_2. On the other hand, commensurability of Γ_1 and Γ_2 up to an $\mathbb{R}$-automorphism of G is equivalent to commensurability of $\mathfrak{X}_{\Gamma_1}$ and $\mathfrak{X}_{\Gamma_2}$. These observation combined with Theorem F immediately imply the following.

Theorem 8.8. *If* $\mathfrak{X}_{\Gamma_1}$ *and* $\mathfrak{X}_{\Gamma_2}$ *are of finite volume, length commensurable, and* Γ_1 *is arithmetic, then so is* Γ_2.

We will now focus on *arithmetically defined* locally symmetric spaces. Using the above observation and applying Theorems C and D, we obtain the following.

Theorem 8.9. *Each class of length-commensurable arithmetically defined locally symmetric spaces of* $\mathcal{G} = G(\mathbb{R})$ *is a union of finitely many commensurability classes. It in fact consists of a single commensurability class if* G *is not of type* A_n $(n > 1)$, D_n $(n \geqslant 4)$, *or* E_6.

To see what this theorem means for hyperbolic spaces, we recall that the even-dimensional real hyperbolic space $\mathbf{H}^{2n}$ is the symmetric space of a group of type B_n, the odd-dimensional real hyperbolic space $\mathbf{H}^{2n-1}$ - of a group of type D_n, the complex hyperbolic space $\mathbf{H}^n_{\mathbb{C}}$ - of a group of type A_n, and the quaternionic hyperbolic space $\mathbf{H}^n_{\mathbb{H}}$ - of a group of type C_{n+1}. Besides, all these spaces are of rank one, so Schanuel's conjecture is not needed in the analysis of the associated (arithmetic) locally symmetric spaces. Using Proposition 8.6(i), we get the following result.

Corollary 8.10. *Let* M *be either the even-dimensional real hyperbolic space* $\mathbf{H}^{2n}$ *or the quaternionic hyperbolic space* $\mathbf{H}^n_{\mathbb{H}}$ *of any dimension, and let* M_1 *and* M_2 *be two arithmetic quotients of* M. *If* M_1 *and* M_2 *are not commensurable, then after a possible interchange of* M_1 *and* M_2, *there exists* $\lambda_1 \in L(M_1)$ *such that for any* $\lambda_2 \in L(M_2)$, *the ratio* λ_1/λ_2 *is transcendental over* $\mathbb{Q}$.

Remark 8.11. In Example 6.6, we indicated that for the $\mathbb{R}$-group $G = \mathrm{SL}_{2,\mathbb{H}}$, one can construct two anisotropic $\mathbb{Q}$-forms G_1 and G_2 that have the same set of $\mathbb{Q}$-isomorphism classes of maximal $\mathbb{Q}$-tori. For $i = 1, 2$, fix a torsion-free $(G_i, \mathbb{Q})$-arithmetic subgroup Γ_i of $\mathcal{G}$. Since $G \simeq \mathrm{Spin}_6(q)$, where q is a real quadratic form of signature $(5, 1)$, the corresponding symmetric space $\mathfrak{X}$ is $\mathbf{H}^5$. Using Corollary 8.4, we now conclude that $\mathfrak{X}_{\Gamma_1}$ and $\mathfrak{X}_{\Gamma_2}$ are length-commensurable, but noncommensurable, compact hyperbolic 5-manifolds. A similar argument applied to a suitable modification of Example 6.6 enables one to construct examples of noncommensurable length-commensurable complex hyperbolic manifolds of any even dimension. These examples will be subsumed by general constructions in §9, which in particular, allow one to construct examples of this nature for real hyperbolic manifolds of any dimension of the form $4k+1$, and for complex hyperbolic manifolds of any dimension, cf. 9.14.

We now recall that given a discrete (G_i, K_i)-arithmetic subgroup $\Gamma_i \subset \mathcal{G}$, the compactness of the quotient $\mathcal{G}/\Gamma_i$, and hence of the locally symmetric subspace $\mathfrak{X}_{\Gamma_i}$, is equivalent to G_i being K_i-anisotropic (cf. [21], Theorem 4.17). Combining this with Theorem E, we obtain the following.

Theorem 8.12. *Let $\mathfrak{X}_{\Gamma_1}$ and $\mathfrak{X}_{\Gamma_2}$ be two arithmetically defined locally symmetric spaces of the same absolutely simple real Lie group $\mathcal{G}$. If they are length-commensurable, then the compactness of one of them implies the compactness of the other.*

We close this section with a result which applies also to nonarithmetic subgroups.

Theorem 8.13. *Let $\mathfrak{X}_{\Gamma_1}$ and $\mathfrak{X}_{\Gamma_2}$ be two locally symmetric spaces of the same absolutely simple real Lie group $\mathcal{G}$ modulo finitely generated torsion-free Zariski-dense discrete subgroups Γ_1 and Γ_2. Denote by K_{Γ_i} the field generated by* $\mathrm{trAd}\,\gamma$ *for $\gamma \in \Gamma_i$. If $\mathfrak{X}_{\Gamma_1}$ and $\mathfrak{X}_{\Gamma_2}$ are length-commensurable, then $K_{\Gamma_1} = K_{\Gamma_2}$.*

Indeed, by Theorem 8.7, Γ_1 and Γ_2 are weakly commensurable, so our claim follows from Theorem A.

9. Construction of nonisomorphic groups with the same tori and noncommensurable length-commensurable locally symmetric spaces of type A_n, D_n and E_6 .

According to Theorem 7.3, if K is a number field and G_1 and G_2 are two K-forms of a connected absolutely simple group of type different from A_n $(n > 1)$, D_n $(n \geqslant 4)$ and E_6, then the fact that every maximal K-torus T_1 of G_1 is K-isomorphic to some maximal K-torus T_2 of G_2, and vice versa, implies that G_1 and G_2 are K-isomorphic. The goal of this section is to describe a general construction of nonisomorphic K-forms of each of the types A_n, D_{2n+1}, $n > 1$, and E_6, which have the "same" systems of maximal K-tori in a very strong sense (see the definition of groups having coherently equivalent systems of maximal K-tori below). Furthermore, we show that arithmetic subgroups of the forms we construct lead to noncommensurable length-commensurable locally symmetric spaces, cf. Proposition 9.13.

We begin by recalling the well-known cohomological parametrization of the conjugacy classes of maximal K-tori of a given group. Let G be a connected semi-simple simply connected algebraic group over a number field K. Fix a maximal K-torus T^0 of G, and let $N = N_G(T^0)$ and $W = N/T^0$ denote its normalizer and the corresponding Weyl group. For any field extension $\mathcal{K}/K$, we let $\theta_{\mathcal{K}}\colon H^1(\mathcal{K}, N) \to H^1(\mathcal{K}, W)$ denote the canonical map, and let

$$\mathcal{C}_{\mathcal{K}} := \mathrm{Ker}(H^1(\mathcal{K}, N) \longrightarrow H^1(\mathcal{K}, G)).$$

The maximal $\mathcal{K}$-tori of G bijectively correspond to the $\mathcal{K}$-rational points of the variety $\mathcal{T} = G/N$ of maximal tori of G. Furthermore, G acts on $\mathcal{T}$ by left multiplication (which corresponds to the conjugation action of $G(\mathcal{K})$ on the set of maximal $\mathcal{K}$-tori), and the elements of the orbit set $G(\mathcal{K})\backslash\mathcal{T}(\mathcal{K})$ are in one-to-one correspondence with the $G(\mathcal{K})$-conjugacy classes of maximal $\mathcal{K}$-tori of G. The following is well-known.

Lemma 9.1. *There is a natural bijection $\delta_{\mathcal{K}}$ from $\mathcal{C}_{\mathcal{K}}$ onto $G(\mathcal{K})\backslash\mathcal{T}(\mathcal{K})$.*

We just recall the construction of $\delta_{\mathcal{K}}$. If $n : \sigma \mapsto n_\sigma$, $\sigma \in \mathrm{Gal}(\overline{\mathcal{K}}/\mathcal{K})$, is a $N(\overline{\mathcal{K}})$-valued Galois cocycle representing an element of $\mathcal{C}_{\mathcal{K}}$, then there exists $g \in G(\overline{\mathcal{K}})$ such that $n_\sigma = g^{-1}\sigma(g)$ for all $\sigma \in \mathrm{Gal}(\overline{\mathcal{K}}/\mathcal{K})$. Then the torus $T = gT^0g^{-1}$ is defined over $\mathcal{K}$, and $\delta_{\mathcal{K}}$ carries the cohomology class of n to the $G(\mathcal{K})$-conjugacy class of T.

We now establish a local-global principle pertaining to the description of maximal K-tori of G. To formulate it, we observe that there is an obvious map $W \longrightarrow \mathrm{Aut}\, T^0$, so for any $x \in H^1(\mathcal{K}, W)$, one can consider the corresponding twisted $\mathcal{K}$-torus ${}_xT^0$.

Theorem 9.2. *Fix $x \in H^1(K, W)$ and suppose that*

(i) *$x \in \theta_{K_v}(\mathcal{C}_{K_v})$ for all $v \in V^K$;*

(ii) *$Ш^2({}_xT^0) := \mathrm{Ker}(H^2(K, {}_xT^0) \longrightarrow \prod_{v\in V^K} H^2(K_v, {}_xT^0))$ is trivial (which, in particular, holds if there exists $v_0 \in V^K$ such that ${}_xT^0$ is K_{v_0}-anisotropic, cf.* [21], *Proposition 6.12.).*

Then $x \in \theta_K(\mathcal{C}_K)$.

Proof. Applying the constructions from [32], Ch. I, §5.6, to the exact sequence

$$1 \to T^0 \longrightarrow N \longrightarrow W \to 1,$$

we see that to any field extension $\mathcal{K}/K$, one can associate a natural cohomology class $\Delta_{\mathcal{K}}(x) \in H^2(\mathcal{K}, {}_xT^0)$ such that $x \in \theta_{\mathcal{K}}(H^1(\mathcal{K}, N))$ if and only if $\Delta_{\mathcal{K}}(x)$ is trivial. It follows from (i) that $\Delta_K(x) \in Ш^2({}_xT^0)$, which is trivial by (ii). Thus, $x = \theta_K(y)$ for some $y \in H^1(K, N)$. Furthermore, according to *loc.cit.,* §5.5, for any $\mathcal{K}/K$ there is a natural surjective map $\nu_{\mathcal{K}}\colon H^1(\mathcal{K}, {}_xT^0) \to \theta_{\mathcal{K}}^{-1}(x)$. For each $v \in V_\infty^K$, by (i), we can find $z_v \in \mathcal{C}_{K_v}$ such that $\theta_{K_v}(z_v) = x$, and then pick $t_v \in H^1(K_v, {}_xT^0)$ for which $\nu_{K_v}(t_v) = z_v$. By [21], Proposition 6.17, the diagonal map $H^1(K, {}_xT^0) \longrightarrow \prod_{v\in V_\infty^K} H^1(K_v, {}_xT^0)$ is surjective, so there is $t \in H^1(K, {}_xT^0)$ that maps to $(t_v)_{v\in V_\infty^K}$. Set $z = \nu_K(t)$. Then z maps onto $(z_v)_{v\in V_\infty^K}$ under the diagonal map $H^1(K, N) \longrightarrow \prod_{v\in V_\infty^K} H^1(K_v, N)$. Combining the fact that $z_v \in \mathcal{C}_{K_v}$ with the injectivity of the map $H^1(K, G) \longrightarrow \prod_{v\in V_\infty^K} H^1(K_v, G)$ ([21], Theorem 6.6), we obtain that $z \in \mathcal{C}_K$. Thus, $x = \theta_K(z) \in \theta_K(\mathcal{C}_K)$, as required. □

We now turn to the comparison of the sets of maximal K-tori of two absolutely simple simply connected K-groups G_1 and G_2. We assume that there exist maximal K-tori T_1^0 of G_1 and T_2^0 of G_2, and a $\overline{K}$-isomorphism $\varphi_0\colon G_1 \to G_2$ whose restriction to T_1^0 is an isomorphism onto T_2^0 defined over K, and we fix these T_1^0, T_2^0 and φ_0 for the rest of the section. Clearly, φ_0 induces an isomorphism between $N_1 = N_{G_1}(T_1^0)$ and $N_2 = N_{G_2}(T_2^0)$, and hence an isomorphism φ_0^W between the Weyl groups $W_1 = N_1/T_1^0$ and $W_2 = N_2/T_2^0$.

Lemma 9.3. *The map $\varphi_0^W : W_1 \to W_2$ is defined over K.*

Proof. Since $\varphi_0|T_1^0$ is defined over K, for any $n \in N_1(\overline{K})$, $t \in T_1^0(\overline{K})$ and any $\sigma \in \mathrm{Gal}(\overline{K}/K)$, we have

$$\varphi_0(\sigma(ntn^{-1})) = \sigma(\varphi_0(ntn^{-1})),$$

which implies that

$$\varphi_0(\sigma(n))\varphi_0(\sigma(t))\varphi_0(\sigma(n))^{-1} = \sigma(\varphi_0(n))\sigma(\varphi_0(t))\sigma(\varphi_0(n))^{-1}.$$

Since $\varphi_0(\sigma(t)) = \sigma(\varphi_0(t))$, we conclude that $\sigma(\varphi_0(n)) \equiv \varphi_0(\sigma(n))$ modulo $T_2^0(\overline{K})$. This means that φ_0^W commutes with every $\sigma \in \mathrm{Gal}(\overline{K}/K)$, hence it is defined over K. □

Lemma 9.3 enables us to define, for any field extension $\mathcal{K}/K$, the induced isomorphism $H^1(\mathcal{K}, W_1) \to H^1(\mathcal{K}, W_2)$, which will also be denoted by φ_0^W. This isomorphism will play a critical role in comparing the maximal K-tori of G_1 and G_2. More precisely, for a field extension $\mathcal{K}/K$ and $i = 1, 2$, we let $\theta_{\mathcal{K}}^{(i)}\colon H^1(\mathcal{K}, N_i) \to H^1(\mathcal{K}, W_i)$ be the map induced by the canonical homomorphism $N_i \to W_i$. Furthermore, let $\mathcal{C}_{\mathcal{K}}^{(i)} = \mathrm{Ker}(H^1(\mathcal{K}, N_i) \to H^1(\mathcal{K}, G_i))$, and let $\delta_{\mathcal{K}}^{(i)}\colon \mathcal{C}_{\mathcal{K}}^{(i)} \to G_i(\mathcal{K})\backslash\mathcal{T}_i(\mathcal{K})$ (where $\mathcal{T}_i$ is the variety of maximal tori of G_i) be the bijection provided by Lemma 9.1. Then the fact that G_1 and G_2 have the "same" maximal K-tori is basically equivalent to the fact that

$$\varphi_0^W(\theta_K^{(1)}(\mathcal{C}_K^{(1)})) = \theta^{(2)}(\mathcal{C}_K^{(2)}). \tag{26}$$

To give a precise interpretation of (26), we need to introduce the following definition.

Definition. Let $\mathcal{K}$ be a field extension of K and let T_1 be a maximal $\mathcal{K}$-torus of G_1. A $\mathcal{K}$-embedding $\iota\colon T_1 \to G_2$ will be called *coherent* (relative to φ_0) if there exists a $\overline{\mathcal{K}}$-isomorphism $\varphi\colon G_1 \to G_2$ of the form $\varphi = \mathrm{Int}\, h \circ \varphi_0$, where $h \in G_2(\overline{\mathcal{K}})$, such that $\iota = \varphi|T_1$. Furthermore, we say that G_1 and G_2 *have coherently equivalent systems of maximal K-tori* if every maximal K-torus T_1 of G_1 admits a coherent K-embedding into G_2, and every maximal K-torus T_2 of G_2 admits a coherent K-embedding into G_1.

Lemma 9.4. *Let T_1 be a maximal $\mathcal{K}$-torus of G_1, and let $x_1 \in \mathcal{C}_{\mathcal{K}}^{(1)}$ be the cohomology class that corresponds to T_1 under $\delta_{\mathcal{K}}^{(1)}$. Then T_1 admits a* *coherent (relative to φ_0) $\mathcal{K}$-embedding into G_2 if and only if $\varphi_0^W(\theta_{\mathcal{K}}^{(1)}(x_1)) \in \theta_{\mathcal{K}}^{(2)}(\mathcal{C}_{\mathcal{K}}^{(2)})$. Thus, (26) is equivalent to the fact that G_1 and G_2 have coherently equivalent systems of maximal K-tori.*

Proof. Pick $g_1 \in G_1(\overline{\mathcal{K}})$ so that $T_1 = g_1 T_1^0 g_1^{-1}$. Then x_1 is represented by the $N_1(\overline{\mathcal{K}})$-valued Galois cocycle $\sigma \mapsto \alpha_\sigma := g_1^{-1}\sigma(g_1)$, $\sigma \in \mathrm{Gal}(\overline{\mathcal{K}}/\mathcal{K})$, and therefore, $\varphi_0^W(\theta_{\mathcal{K}}^{(1)}(x_1))$ is represented by

$$\beta_\sigma := \varphi_0(g_1^{-1}\sigma(g_1))T_2^0 \in W_2. \tag{27}$$

Let $\varphi\colon G_1 \to G_2$ be an isomorphism of the form $\varphi = \mathrm{Int}\, h \circ \varphi_0$, where $h \in G_2(\overline{\mathcal{K}})$. Then $T_2 := \varphi(T_1)$ can be written in the form $T_2 = g_2 T_2^0 g_2^{-1}$, where $g_2 = h\varphi_0(g_1)$. So, T_2 is defined over $\mathcal{K}$ if and only if $g_2^{-1}\sigma(g_2) \in N_2(\overline{\mathcal{K}})$ for all $\sigma \in \mathrm{Gal}(\overline{\mathcal{K}}/\mathcal{K})$, in which case the class x_2 corresponding to T_2 is represented by the $N_2(\overline{\mathcal{K}})$-valued Galois cocycle $\sigma \mapsto g_2^{-1}\sigma(g_2)$. Then $\theta_{\mathcal{K}}^{(2)}(x_2)$ is represented by the cocycle

$$\sigma \mapsto \gamma_\sigma := g_2^{-1}\sigma(g_2)T_2^0 = \varphi_0(g_1)^{-1}h^{-1}\sigma(h)\sigma(\varphi_0(g_1))T_2^0 \in W_2. \tag{28}$$

Finally, notice that the fact that $\varphi|T_1$ is defined over $\mathcal{K}$ is equivalent to

$$\varphi(\sigma(g_1 t g_1^{-1})) = \sigma(\varphi(g_1 t g_1^{-1})) \ \text{ for all } \ t \in T^0(\overline{\mathcal{K}}) \ \text{ and } \ \sigma \in \mathrm{Gal}(\overline{\mathcal{K}}/\mathcal{K}). \tag{29}$$

The left- and right-hand sides of (29) can be expanded as follows:

$$\varphi(\sigma(g_1 t g_1^{-1})) = h\varphi_0(\sigma(g_1 t g_1^{-1}))h^{-1} = h\varphi_0(\sigma(g_1))\varphi_0(\sigma(t))\varphi_0(\sigma(g_1))^{-1}h^{-1}$$

and

$$\sigma(\varphi(g_1 t g_1^{-1})) = \sigma(h\varphi_0(g_1 t g_1^{-1})h^{-1}) = \sigma(h)\sigma(\varphi_0(g_1))\sigma(\varphi_0(t))\sigma(\varphi_0(g_1))^{-1}\sigma(h)^{-1}.$$

So, since $\varphi_0(\sigma(t)) = \sigma(\varphi_0(t))$, we see that (29) is equivalent to

$$\varphi_0(\sigma(g_1))^{-1}h^{-1}\sigma(h)\sigma(\varphi_0(g_1)) \in T_2^0 \ \text{ for all } \ \sigma \in \mathrm{Gal}(\overline{\mathcal{K}}/\mathcal{K}). \tag{30}$$

Now, suppose $\varphi|T_1$ is defined over $\mathcal{K}$, i.e., (30) holds. We claim that $\varphi_0^W(\theta_{\mathcal{K}}^{(1)}(x_1)) = \theta_{\mathcal{K}}^{(2)}(x_2) \in \theta_{\mathcal{K}}^{(2)}(\mathcal{C}_{\mathcal{K}}^{(2)})$. Indeed, combining (30) with (27) and (28), we see that

$$\gamma_\sigma = \varphi_0(g_1)^{-1}h^{-1}\sigma(h)\sigma(\varphi_0(g_1))T_2^0 = \varphi_0(g_1^{-1}\sigma(g_1))T_2^0 = \beta_\sigma,$$

as required.

Conversely, suppose $\varphi_0^W(\theta_{\mathcal{K}}^{(1)}(x_1)) \in \theta_{\mathcal{K}}^{(2)}(\mathcal{C}_{\mathcal{K}}^{(2)})$. This means that there exists $g_2 \in G_2(\overline{\mathcal{K}})$ such that

$$\beta_\sigma = g_2^{-1}\sigma(g_2)T_2^0 \ \text{ for all } \ \sigma \in \mathrm{Gal}(\overline{\mathcal{K}}/\mathcal{K}). \tag{31}$$

Set $h = g_2\varphi_0(g_1)^{-1}$ and $\varphi = \mathrm{Int}\, h \circ \varphi_0$. We need to show that $\varphi|T_1$ is defined over $\mathcal{K}$, in other words, (30) holds. But this is obtained directly by combining (27) with (31). □

Combining Theorem 9.2 with Lemma 9.4, we obtain the following local-global principle for the existence of a coherent K-embedding of a K-torus as a maximal torus in a semi-simple group.

Theorem 9.5. *Let G_1 and G_2 be two connected semi-simple simply connected algebraic groups over a number field K. Assume that*

(∗) *there exist maximal K-tori T_1^0 of G_1 and T_2^0 of G_2, and a $\overline{K}$-isomorphism $\varphi_0\colon G_1 \to G_2$ whose restriction to T_1^0 is an isomorphism onto T_2^0 defined over K.*

Let T_1 be a maximal K-torus of G_1 such that $Ш^2(T_1)$ is trivial (which automatically holds if there exists $v_0 \in V^K$ such that T_1 is K_{v_0}-anisotropic). If T_1 admits a coherent (relative to φ_0) K_v-embedding into G_2 for every $v \in V^K$, then it admits a coherent K-embedding into G_2.

The following lemma explains why coherent embeddings of tori are easier to analyze if the ambient group is not of type D_{2n}.

Lemma 9.6. *Assume that G_1 and G_2 are of type different from D_{2n}, and let $\mathcal{K}/K$ be a field extension. If T_1 is a maximal $\mathcal{K}$-torus of G_1 and $\varphi\colon G_1 \to G_2$ is a $\overline{\mathcal{K}}$-isomorphism such that $\iota := \varphi|T_1$ is defined over $\mathcal{K}$, then either ι, or ι', defined by $\iota'(t) = \iota(t)^{-1}$, is a* coherent *$\mathcal{K}$-embedding of T_1 into G_2 (in particular, T_1 admits such an embedding). Thus, if G_1 and G_2 are $\mathcal{K}$-isomorphic, then they have coherently equivalent systems of maximal $\mathcal{K}$-tori.*

Proof. Obviously, $T_2 := \varphi(T_1)$ is defined over $\mathcal{K}$. Let Φ_2 be the root system of G_2 with respect to T_2. Since G_2 is not of type D_{2n}, the quotient $\mathrm{Aut}(\Phi_2)/W(\Phi_2)$ is of order $\leqslant 2$, and in case it is of order 2, the automorphism $\alpha \mapsto -\alpha$ represents the nontrivial coset. Equivalently, $\mathrm{Aut}\, G_2/\mathrm{Int}\, G_2$ has order $\leqslant 2$, and in case it has order 2, there is an outer automorphism τ of G_2 defined over $\overline{\mathcal{K}}$ such that $\tau(t) = t^{-1}$ for all $t \in T_2$. Set $\varphi' = \tau \circ \varphi$, then $\varphi'|T_1 = \iota'$. Since one of φ or φ' is of the form $\mathrm{Int}\, h \circ \varphi_0$, the lemma follows. □

Combined with Theorem 9.5, this lemma yields the following.

Corollary 9.7. *Let G_1 and G_2 be two connected absolutely simple simply connected algebraic groups of type different from D_{2n}, and suppose that the condition (∗) of Theorem 9.5 holds. Assume in addition that $Ш^2$ is trivial for all maximal K-tori of G_1 and G_2 (which automatically holds if there exists a place v_0 of K such that G_i is K_{v_0}-anisotropic for $i = 1, 2$). If $G_1 \simeq G_2$ over K_v, for all $v \in V^K$, then G_1 and G_2 have coherently equivalent systems of maximal K-tori.*

Of course, if G_1 and G_2 are not of type A_n ($n > 1$), D_n or E_6, then the assumption that $G_1 \simeq G_2$ over K_v for all $v \in V^K$ implies that $G_1 \simeq G_2$ over K, and our assertion becomes vacuous (cf. Lemma 9.6). We will use Corollary 9.7 to show that for each of the types A_n, D_{2n+1}, or E_6, one can construct an arbitrarily large number of pairwise nonisomorphic absolutely simple simply connected K-groups of this type with coherently equivalent systems of maximal K-tori (cf. Theorem 9.11).

Let G_0 be a connected absolutely simple simply connected quasi-split K-group of one of the following types: A_n $(n > 1)$, D_{2n+1} or E_6. We first describe a general construction of nonisomorphic inner twists G_1 and G_2 of G_0 which are isomorphic over K_v for all $v \in V^K$. Let L be the minimal Galois extension of K over which G_0 splits, and let V_0 be the set of $v \in V_f^K$ that split in L. We let C denote the center of G_0; clearly, C is L-isomorphic to μ_ℓ, the group of ℓ-th roots of unity, where $\ell = n+1$ for G_0 of type A_n, $\ell = 4$ for type D_{2n+1}, and $\ell = 3$ for type E_6. Each $x \in G_0$ gives the inner automorphism $z \mapsto xzx^{-1}$ of G_0. This leads to the natural isomorphism i from the adjoint group $\overline{G}_0$ of G_0 onto the group of inner automorphisms $\mathrm{Int}\, G_0$. Any automorphism g of G_0 can be regarded as an automorphism of $\overline{G}_0$, and then for every $x \in \overline{G}_0$, we have $g \circ i(x) \circ g^{-1} = i(g(x))$ in $\mathrm{Aut}\, G_0$.

For a class $\mathfrak{c} \in H^1(K, \overline{G}_0)$, in the sequel we will let $\sigma \mapsto \mathfrak{c}_\sigma$, $\sigma \in \mathrm{Gal}(\overline{K}/K)$, denote a Galois cocycle representing $\mathfrak{c}$.

For any $v \in V^K$, we have the following commutative diagram

$$\begin{array}{ccc} H^1(K, \overline{G}_0) & \xrightarrow{\alpha} & H^1(K, \mathrm{Aut}\, G_0) \\ \gamma_v \downarrow & & \downarrow \beta_v \\ H^1(K_v, \overline{G}_0) & \xrightarrow{\alpha_v} & H^1(K_v, \mathrm{Aut}\, G_0), \end{array}$$

in which α and α_v are induced by i. Furthermore, for any extension $\mathcal{K}/K$ there is a natural map $\rho_{\mathcal{K}} \colon H^1(\mathcal{K}, \overline{G}_0) \to H^2(\mathcal{K}, C)$. We will also need the map $\mu \colon H^2(K, C) \to \bigoplus_v H^2(K_v, C)$.

Lemma 9.8. *Let* $\xi_1,\ \xi_2 \in H^1(K, \overline{G}_0)$.

(i) *If* $\rho_K(\xi_1) \neq \pm\rho_K(\xi_2)$ *then* $\alpha(\xi_1) \neq \alpha(\xi_2)$.

(ii) *If* $v \in V_f^K$ *and* $\rho_{K_v}(\gamma_v(\xi_1)) = \pm\rho_{K_v}(\gamma_v(\xi_2))$ *then* $\beta_v(\alpha(\xi_1)) = \beta_v(\alpha(\xi_2))$.

Proof. Notice that $\mathrm{Aut}\, G_0$ has the following semi-direct product decomposition

$$\mathrm{Aut}\, G_0 = \mathrm{Int}\, G_0 \rtimes \Sigma,$$

where Σ is an K-subgroup of order two, whose generator s is defined over K and acts on C as $c \mapsto c^{-1}$.

(i): Suppose $\alpha(\xi_1) = \alpha(\xi_2)$. Then there exists $g \in \mathrm{Aut}\, G_0$ such that

$$i(\xi_{2\,\sigma}) = g \circ i(\xi_{1\,\sigma}) \circ \sigma(g)^{-1} \quad \text{for all} \quad \sigma \in \mathrm{Gal}(\overline{K}/K).$$

If $g \in \mathrm{Int}\, G_0$, then $\xi_1 = \xi_2$, and therefore, $\rho_K(\xi_1) = \rho_K(\xi_2)$. Now, suppose $g \notin \mathrm{Int}\, G_0$. Then $g = hs$, $h \in \mathrm{Int}\, G_0$. The cohomology class in $H^1(K, \overline{G}_0)$ corresponding to the cocycle

$$\sigma \mapsto \xi'_{2\,\sigma} = s(\xi_{1\,\sigma}), \quad \sigma \in \mathrm{Gal}(\overline{K}/K),$$

is clearly ξ_2. As $s(c) = c^{-1}$ for $c \in C$, we conclude that

$$\rho_K(\xi_2) = \rho_K(\xi_2') = -\rho_K(\xi_1),$$

a contradiction.

(ii): Recall that ρ_{K_v} is a bijection for any $v \in V_f^K$ (cf. [21], Corollary of Theorem 6.20), so our claim is obvious if $\rho_{K_v}(\gamma_v(\xi_1)) = \rho_{K_v}(\gamma_v(\xi_2))$. Suppose now that $\rho_{K_v}(\gamma_v(\xi_1)) = -\rho_{K_v}(\gamma_v(\xi_2))$. Consider the $\overline{G}(\overline{K})$-valued Galois cocycle $\sigma \mapsto \xi'_{2\,\sigma} := s(\xi_{2\,\sigma})$, and let ξ_2' be the associated cohomology class. Then for $\sigma \in \mathrm{Gal}(\overline{K}/K)$ we have

$$i(\xi'_{2\,\sigma}) = s \circ i(\xi_{2\,\sigma}) \circ s^{-1} = s \circ i(\xi_{2\,\sigma}) \circ \sigma(s)^{-1},$$

so $\alpha(\xi_2') = \alpha(\xi_2)$. On the other hand,

$$\rho_{K_v}(\gamma_v(\xi_2')) = -\rho_{K_v}(\gamma_v(\xi_2)) = \rho_{K_v}(\gamma_v(\xi_1)).$$

Then $\gamma_v(\xi_2') = \gamma_v(\xi_1)$, and

$$\beta_v(\alpha(\xi_1)) = \beta_v(\alpha(\xi_2')) = \beta_v(\alpha(\xi_2)).$$

□

Let $\widehat{C}$ be the character group of C. Fix a generator χ of $\widehat{C}(K)$, and let d denote its order. For each $v \in V^K$, χ induces a character

$$\chi_v \colon H^2(K_v, C) \to H^2(K_v, \mathrm{GL}_1) \subset \mathbb{Q}/\mathbb{Z}.$$

If $v \in V_0$, then $H^2(K_v, C) \simeq \mathrm{Br}(K_v)_\ell$ is cyclic of order ℓ, and one can choose a generator $b_v \in H^2(K_v, C)$ such that $\chi_v(b_v) = 1/d$. Now, let V be a finite subset of V^K containing V_∞^K, and suppose that for each $v \in V$ we are given $\xi^{(v)} \in H^1(K_v, \overline{G}_0)$. Fix an integer $t \geqslant 1$, and pick $2(t+1)$ places

$$v_0',\, v_0'',\, v_1',\, v_1'',\, \ldots,\, v_t',\, v_t'' \in V_0 \setminus (V_0 \cap V).$$

Let $V_t = \{v_0',\, v_0'',\, v_1',\, v_1'',\, \ldots,\, v_t',\, v_t''\}$. Now pick $x_{v_0''} \in H^2(K_{v_0''}, C)$ so that

$$\sum_{v \in V} \chi_v(\rho_{K_v}(\xi^{(v)})) + \chi_{v_0'}(b_{v_0'}) + \chi_{v_0''}(x_{v_0''}) = 0.$$

Next, fix $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_t) \in E_t := \displaystyle\prod_{i=1}^{t} \{\pm 1\}$, and consider $(x(\varepsilon)_v) \in \bigoplus_v H^2(K_v, C)$ with the following components:

$$x(\varepsilon)_v = \begin{cases} \rho_{K_v}(\xi^{(v)}) & , \quad v \in V \\ b_{v_0'} & , \quad v = v_0' \\ x_{v_0''} & , \quad v = v_0'' \\ \varepsilon_j b_{v_j'} & , \quad v = v_j',\ j \geqslant 1 \\ -\varepsilon_j b_{v_j''} & , \quad v = v_j'',\ j \geqslant 1 \\ 0 & , \quad \text{for all other } v \end{cases} \tag{32}$$

We obviously have $\sum_v \chi_v(x(\varepsilon)_v) = 0$, so it follows from the Poitou-Tate theorems (cf. [32], Ch. II, §6) that there exists $x(\varepsilon) \in H^2(K, C)$ such that

$\mu(x(\varepsilon)) = (x(\varepsilon)_v)$. We now want to construct a maximal K-torus $\overline{T}_0$ of $\overline{G}_0$ (depending on V, $\xi^{(v)}$ for $v \in V$ and V_t) such that for all $\varepsilon \in E_t$, $x(\varepsilon)$ lifts to a class $\zeta(\varepsilon) \in H^1(K, \overline{T}_0)$ whose image in $H^1(K_v, \overline{G}_0)$ is $\xi^{(v)}$ for all $v \in V$.

For each real v, $\xi^{(v)}$ is given by an element $g_v \in \overline{G}_0(\overline{K}_v)$ such that $g_v \overline{g}_v = 1$, where $\overline{g}_v$ denotes the conjugate of g_v under the nontrivial automorphism of $\overline{K}_v/K_v = \mathbb{C}/\mathbb{R}$. It follows from the uniqueness of the Jordan decomposition that the semi-simple and the unipotent components g_v^s, g_v^u of g_v also define cocycles. If $g_v^u \neq 1$, then the 1-dimensional connected unipotent subgroup U generated by g_v^u is defined over $K_v = \mathbb{R}$. Using the fact that $H^1(K_v, U)$ is trivial, one sees that $\xi^{(v)}$ is the cohomology class given by g_v^s. So we can assume that g_v is semi-simple. Then g_v is contained in the connected centralizer $H = Z_{\overline{G}_0}(g_v)^\circ$ (cf. [4], Corollary 11.12) and H is defined over K_v. Hence, g_v is contained in a maximal K_v-torus $\overline{T}^{(v)}$ of H which is also a maximal torus of $\overline{G}_0$. For each $v \in (V \setminus V_\infty^K) \cup V_t$, we pick a maximal K_v-torus $\overline{T}^{(v)}$ of $\overline{G}_0$ which is anisotropic over K_v (see [21], Theorem 6.21, or [8], §2.4). Using the weak approximation property for the variety of maximal tori of $\overline{G}_0$ (cf. [21], Corollary 3 in §7.1), we can find a maximal K-torus $\overline{T}_0$ of $\overline{G}_0$ which is conjugate to $\overline{T}^{(v)}$ under an element of $\overline{G}_0(K_v)$ for all $v \in V \cup V_t$. Let $\pi\colon G_0 \to \overline{G}_0$ be the natural K-isogeny, and $T_0 = \pi^{-1}(\overline{T}_0)$.

Lemma 9.9. *For every $\varepsilon \in E_t$, there exists $\zeta(\varepsilon) \in H^1(K, \overline{T}_0)$ which maps onto $x(\varepsilon)$ under the coboundary map $H^1(K, \overline{T}_0) \to H^2(K, C)$, and whose image in $H^1(K_v, \overline{G}_0)$ equals $\xi^{(v)}$ for all $v \in V$.*

Proof. For any real v, as $\overline{T}_0$ is conjugate to $\overline{T}^{(v)}$ under an element of $\overline{G}_0(K_v)$, and $\xi^{(v)}$ is given by $g_v \in \overline{T}^{(v)}(\overline{K}_v)$, there exists a cohomology class $\xi'^{(v)}$ in $H^1(K_v, \overline{T}_0)$ which maps onto $\xi^{(v)}$ under the natural map $H^1(K_v, \overline{T}_0) \to H^1(K_v, \overline{G}_0)$. On the other hand, for every nonarchimedean $v \in V$, as $\overline{T}_0$ is anisotropic over K_v, the natural map $H^1(K_v, \overline{T}_0) \to H^1(K_v, \overline{G}_0)$ is onto (see the proof of Theorem 6.20 on p. 326 of [21]), there is a $\xi'^{(v)} \in H^1(K_v, \overline{T}_0)$ which maps onto $\xi^{(v)}$.

We have the following commutative diagram with exact rows:

$$\begin{array}{ccccc} H^1(K, \overline{T}_0) & \xrightarrow{\delta_1} & H^2(K, C) & \xrightarrow{\delta_2} & H^2(K, T_0) \\ \eta_1 \downarrow & & \eta_2 \downarrow & & \eta_3 \downarrow \\ \bigoplus_v H^1(K_v, \overline{T_0}) & \xrightarrow{\Delta_1} & \bigoplus_v H^2(K_v, C) & \xrightarrow{\Delta_2} & \bigoplus_v H^2(K, T_0) \end{array}$$

(notice that η_2 actually coincides with μ). First, we will show that $x(\varepsilon) \in \operatorname{Im} \delta_1 = \operatorname{Ker} \delta_2$. Observe that

$$x(\varepsilon)_v \in \operatorname{Im}(H^1(K_v, \overline{T}_0) \to H^2(K_v, C)) \tag{33}$$

for all v. This is obvious if $v \notin V \cup V_t$. For any real v, this follows from the fact that $x(\varepsilon)_v = \rho_{K_v}(\xi^{(v)})$, and $\xi^{(v)}$ is the image of $\xi'^{(v)} \in H^1(K_v, \overline{T}_0)$. For a nonarchimedean $v \in V \cup V_t$, by our construction T_0 is K_v-anisotropic, and it follows from the Nakayama-Tate Theorem (cf. [21], Theorem 6.2) that $H^2(K_v, T_0)$ is trivial. So the map $H^1(K_v, \overline{T}_0) \to H^2(K_v, C)$ is surjective, and (33) is automatic. Thus, $\eta_2(x(\varepsilon)) = (x(\varepsilon)_v) \in \mathrm{Im}\,\Delta_1$, so

$$\Delta_2(\eta_2(x(\varepsilon))) = \eta_3(\delta_2(x(\varepsilon))) = 0.$$

Since T_0 is anisotropic at every $v \in V_t$, we have that $\text{Ш}^2(T_0) = \mathrm{Ker}\,\eta_3$ is trivial, and hence $\delta_2(x(\varepsilon)) = 0$, as required. Fix $\zeta'(\varepsilon) \in H^1(K, \overline{T}_0)$ such that $\delta_1(\zeta'(\varepsilon)) = x(\varepsilon)$.

For an extension $\mathcal{K}/K$, we consider the natural homomorphism

$$\lambda_{\mathcal{K}} \colon H^1(\mathcal{K}, T_0) \to H^1(\mathcal{K}, \overline{T}_0),$$

and for $v \in V^K$, we let $\zeta'(\varepsilon)^{(v)}$ denote the image of $\zeta'(\varepsilon)$ under the restriction map $H^1(K, \overline{T}_0) \to H^1(K_v, \overline{T}_0)$. For each $v \in V$, the cohomology classes $\zeta'(\varepsilon)^{(v)}$ and $\xi'^{(v)}$ have the same image in $H^2(K_v, C)$, so there exists $\theta(\varepsilon)_v \in H^1(K_v, T_0)$ such that

$$\xi'^{(v)} = \lambda_{K_v}(\theta(\varepsilon)_v) \cdot \zeta'(\varepsilon)^{(v)}.$$

By ([21], Proposition 6.17), the map $H^1(K, T_0) \to \prod_{v \in V_\infty^K} H^1(K_v, T_0)$ is surjective. Pick $\theta(\varepsilon) \in H^1(K, T_0)$ which maps onto $(\theta(\varepsilon)_v)_{v \in V_\infty^K}$, and set $\zeta(\varepsilon) = \lambda_K(\theta(\varepsilon)) \cdot \zeta'(\varepsilon)$. Let $\zeta(\varepsilon)^{(v)}$ be the image of $\zeta(\varepsilon)$ under the map $H^1(K, \overline{T}_0) \to H^1(K_v, \overline{T}_0)$. Then $\delta_1(\zeta(\varepsilon)) = \delta_1(\zeta'(\varepsilon)) = x(\varepsilon)$ and $\zeta(\varepsilon)^{(v)} = \xi'^{(v)}$ for all $v \in V_\infty^K$. Finally, to show that the image of $\zeta(\varepsilon)^{(v)}$ in $H^1(K_v, \overline{G}_0)$ coincides with $\xi^{(v)}$ for nonarchimedean $v \in V$, we observe that these elements have the same image under ρ_{K_v}, which is a bijection for all $v \in V_f^K$ (Corollary in §6.4 of [21]). □

Let $\xi(\varepsilon)$ be the image of $\zeta(\varepsilon)$ under the natural map $H^1(K, \overline{T}_0) \to H^1(K, \overline{G}_0)$. Then $\rho_K(\xi(\varepsilon)) = x(\varepsilon)$ and $\gamma_v(\xi(\varepsilon)) = \xi^{(v)}$ for all $v \in V$. Fix two distinct $\varepsilon_1, \varepsilon_2 \in E_t$, and let $\xi_j = \xi(\varepsilon_j)$. Since each b_v has order $\ell > 2$, it follows from (32) that $\mu(\rho_K(\xi_1)) \neq \pm\mu(\rho_K(\xi_2))$, hence $\rho_K(\xi_1) \neq \pm\rho_K(\xi_2)$, so according to Lemma 9.8(i), $\alpha(\xi_1) \neq \alpha(\xi_2)$. On the other hand, we have

$$\rho_{K_v}(\gamma_v(\xi_1)) = 0 = \rho_K(\gamma_v(\xi_2)) \quad \text{for any} \quad v \in V^K \setminus (V \cup V_0),$$

$$\rho_{K_v}(\gamma_v(\xi_1)) = \pm\rho_{K_v}(\gamma_v(\xi_2)) \quad \text{for any} \quad v \in V_0,$$

and

$$\gamma_v(\xi_1) = \xi^{(v)} = \gamma_v(\xi_2) \quad \text{for any} \quad v \in V.$$

Using Lemma 9.8(ii), we now see that $\beta_v(\alpha(\xi_1)) = \beta_v(\alpha(\xi_2))$ for all $v \in V^K$. Thus, we obtain the following proposition.

Proposition 9.10. *The 2^t elements $\xi(\varepsilon) \in H^1(K, \overline{G}_0)$, $\varepsilon \in E_t$, have the following properties: the elements $\alpha(\xi(\varepsilon)) \in H^1(K, \mathrm{Aut}\, G_0)$ are pairwise distinct, while for any $v \in V^K$, the elements $\beta_v(\alpha(\xi(\varepsilon))) \in H^1(K_v, \mathrm{Aut}\, G_0)$ are all equal, and, in addition, $\gamma_v(\xi(\varepsilon)) = \xi^{(v)}$ for all $v \in V$.*

For $\xi(\varepsilon)$ as above, we let G_ε denote the form of G_0 obtained by twisting it by a cocycle representing $\alpha(\xi(\varepsilon))$. Since the cohomology classes $\alpha(\xi(\varepsilon))$, $\varepsilon \in E_t$, are pairwise distinct, the corresponding groups G_ε are pairwise nonisomorphic over K. Now, fix $\varepsilon_1, \varepsilon_2 \in E_t$, and set

$$\zeta_j = \zeta(\varepsilon_j) \in H^1(K, \overline{T}_0), \quad \xi_j = \xi(\varepsilon_j) \in H^1(K, \overline{G}_0) \ \text{ and } \ G_j = G_{\varepsilon_j}$$

for $j = 1, 2$. As ξ_j is the image of ζ_j under the natural map $H^1(K, \overline{T}_0) \to H^1(K, \overline{G}_0)$, there is a $\overline{T}_0(\overline{K})$-valued Galois cocycle $\sigma \mapsto z_{j\,\sigma}$, $\sigma \in \mathrm{Gal}(\overline{K}/K)$, representing ξ_j. Therefore, there exists a $\overline{K}$-isomorphism $\varphi_j \colon G_0 \to G_j$ such that $\varphi_j^{-1} \circ \sigma(\varphi_j) = i(z_{j\,\sigma})$, for all $\sigma \in \mathrm{Gal}(\overline{K}/K)$, where i is the natural isomorphism $\overline{G}_0 \to \mathrm{Int}\, G_0$. Then $\varphi_j | T_0$ is defined over K, and hence, $T_j^0 := \varphi_j(T_0)$ is a maximal K-torus of G_j. Now $\varphi_0 := \varphi_2 \circ \varphi_1^{-1}$ is a $\overline{K}$-isomorphism from G_1 onto G_2 whose restriction to T_1^0 is an isomorphism onto T_2^0 defined over K. Since $\beta_v(\alpha(\xi_1)) = \beta_v(\alpha(\xi_2))$, the groups G_1 and G_2 are K_v-isomorphic, for all $v \in V^K$. In addition, for each $j = 1, 2$ and any $v \in V$, the group G_j is K_v-isomorphic to the group ${}_{\xi^{(v)}}G_0$ obtained from G_0 by twisting over K_v by any cocycle representing $\alpha_v(\xi^{(v)})$. So, applying Corollary 9.7, we obtain the following.

Theorem 9.11. *Let $\xi(\varepsilon) \in H^1(K, \overline{G}_0)$, $\varepsilon \in E_t$, be the cohomology classes constructed in Proposition 9.10, and let G_ε be the group obtained by twisting G_0 by a cocycle representing $\xi(\varepsilon)$. Then G_ε, $\varepsilon \in E_t$, are pairwise nonisomorphic K-forms of G_0. Moreover, if for every $\varepsilon \in E_t$, and every maximal K-torus T of G_ε, we have $\text{Ш}^2(T) = 0$ (which is automatically the case if for some $v \in V$ the twist ${}_{\xi^{(v)}}G_0$ is K_v-anisotropic), then all the groups G_ε have coherently equivalent systems of maximal K-tori.*

Remark 9.12. If G is an inner K-form of type A_n, then the condition $\text{Ш}^2(T) = \{0\}$ is automatically satisfied for any maximal K-torus T of G. Indeed, T is of the form $T = R^{(1)}_{A/K}(\mathrm{GL}_1)$, where A is a commutative étale $(n+1)$-dimensional K-algebra. Letting $S = R_{A/K}(\mathrm{GL}_1)$, we have the exact sequence

$$1 \to T \longrightarrow S \longrightarrow \mathrm{GL}_1 \to 1,$$

which in conjunction with Hilbert's Theorem 90 induces the following commutative diagram with exact rows:

$$\begin{array}{ccccc} 0 & \longrightarrow & H^2(K, T) & \longrightarrow & H^2(K, S) \\ & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \bigoplus_v H^2(K_v, T) & \longrightarrow & \bigoplus_v H^2(K_v, S). \end{array}$$

Since the map $H^2(K, S) \longrightarrow \bigoplus_v H^2(K_v, S)$ is injective by the Albert-Hasse-Brauer-Noether Theorem, our assertion follows.

We observe that if G_1 and G_2 have coherently equivalent systems of maximal K-tori, then for any finite set $S \subset V_\infty^K$ containing V_∞^K, any (G_i, K, S)-arithmetic subgroups $\Gamma_i \subset G_i$ are weakly commensurable (see the argument in Example 6.5). It turns out that in this situation arithmetic subgroups result in length-commensurable locally symmetric spaces.

Proposition 9.13. *Let G be a connected semi-simple real algebraic group and $\mathfrak{X}$ be the symmetric space of $\mathcal{G} = G(\mathbb{R})$. For $i = 1, 2$, let Γ_i be a torsion-free (G_i, K)-arithmetic subgroup of $\mathcal{G}$. If G_1 and G_2 have coherently equivalent systems of maximal K-tori, then the locally symmetric spaces $\mathfrak{X}_{\Gamma_1}$ and $\mathfrak{X}_{\Gamma_2}$ are length-commensurable.*

Proof. (Cf. the proof of Corollary 8.4.) We can assume that $\Gamma_i \subset G_i(K)$ for $i = 1, 2$. Let $\gamma_1 \in \Gamma_1$ be a nontrivial semi-simple element, and let $T_1 \subset G_1$ be a maximal K-torus containing it. By our assumption, there exists an isomorphism $\varphi\colon G_1 \to G_2$ such that the restriction $\varphi|T_1$ is defined over K, hence $T_2 := \varphi(T_1)$ is a maximal K-torus of G_2. Since $\varphi(T_1 \cap \Gamma_1)$ is an arithmetic subgroup of T_2, there exists $n > 0$ such that $\gamma_2 := \varphi(\gamma_1)^n$ belongs to Γ_2. The map $\alpha \to \alpha \circ \varphi$ defines a bijection between the root systems $\Phi(G_2, T_2)$ and $\Phi(G_1, T_1)$. It follows that the sets of complex numbers

$$\{\alpha(\gamma_1^n) \mid \alpha \in \Phi(G_2, T_2)\} \quad \text{and} \quad \{\alpha(\gamma_2) \mid \alpha \in \Phi(G_1, T_1)\}$$

are identical. Using the formula (22) from Proposition 8.2(ii), we see that

$$\lambda_{\Gamma_2}(\gamma_2)/\lambda_{\Gamma_1}(\gamma_1) \in \mathbb{Q}.$$

□

9.14. We finally indicate how Theorem 9.11 can be used to construct examples of weakly commensurable arithmetic and S-arithmetic subgroups and length-commensurable locally symmetric spaces which are not commensurable. Let G be a connected absolutely simple simply connected isotropic real algebraic group of one of the following types: A_n, D_{2n+1}, $n > 1$, or E_6, and let $\mathcal{L}$ be either $\mathbb{R}$ or $\mathbb{C}$ depending on whether or not G is an inner form over $\mathbb{R}$. Fix a real quadratic extension $K/\mathbb{Q}$, and let v'_∞, v''_∞ denote its two real places. Next, pick a quadratic extension L/K so that $L \otimes_K K_{v'_\infty} = \mathcal{L}^{2/[\mathcal{L}:\mathbb{R}]}$ and $L \otimes_K K_{v''_\infty} = \mathbb{C}$, and let G_0 denote the nonsplit quasi-split K-group of the same type as G which splits over L. Since for the types under consideration, the $\mathbb{R}$-anisotropic form is an inner twist of the nonsplit quasi-split $\mathbb{R}$-group, there exist cohomology classes $\xi^{(v'_\infty)} \in H^1(K_{v'_\infty}, \overline{G}_0)$ and $\xi^{(v''_\infty)} \in H^1(K_{v''_\infty}, \overline{G}_0)$ such that the twist ${}_{\xi^{(v'_\infty)}}G_0$ is isomorphic to G and the twist ${}_{\xi^{(v''_\infty)}}G_0$ is $\mathbb{R}$-anisotropic. Then applying the construction described in Theorem 9.11 to $V = \{v'_\infty, v''_\infty\}$ and the specified cocycles, we obtain 2^t groups G_ε, $\varepsilon \in E_t$, which are pairwise nonisomorphic over K but have coherently equivalent systems of maximal K-tori as these groups are all anisotropic over $K_{v''_\infty}$. Besides, G_ε is isomorphic to G over $K_{v'_\infty} = \mathbb{R}$, for any $\varepsilon \in E_t$. Thus, torsion-free arithmetic subgroups of G_ε yield discrete torsion-free subgroups of $\mathcal{G} = G(\mathbb{R})$, and it follows from Proposition 9.13 that the resulting locally symmetric spaces are length-commensurable, but not commensurable. Finally, for any finite subset S of V^K containing

V_∞^K, the S-arithmetic subgroups of G_ε, $\varepsilon \in E_t$, are weakly commensurable, but not commensurable (cf. Example 6.5).

Remark 9.15. Most of the results of this section immediately extend to a global function field K. This applies, in particular, to Theorem 9.5, yielding a local-to-global principle for the existence of a coherent embedding, and Theorem 9.11, containing a construction of forms of a quasi-split group G_0 belonging to one of the types A_n, D_{2n+1} $(n > 1)$ or E_6, which are not K-isomorphic, but are isomorphic over K_v for all $v \in V^K$. It should be noted, however, that the construction of nonisomorphic K-groups with coherently equivalent systems of maximal K-tori, described in 9.14, extends to global function fields only for groups of type A_n. The reason is that we ensured the triviality of $Ш^2(T)$ for all maximal tori of a group under consideration by arranging that the group is anisotropic at a certain archimedean place. Over global function fields, however, any group of type different from A_n, is isotropic.

10. Isospectral locally symmetric spaces

The following theorem is known. For locally symmetric spaces of rank 1, a proof is given in [12]. However, for locally symmetric spaces of rank > 1, we have not been able to find a reference for it. For the convenience of the reader we will give below its proof which was supplied to us by Alejandro Uribe and Steve Zelditch.

Theorem 10.1. *Let M_1 and M_2 be two compact locally symmetric spaces with nonpositive sectional curvatures. Assume M_1 and M_2 are isospectral, in the sense that the spectra of their Laplace-Beltrami operators on functions are the same (their eigenvalues and their multiplicities). Then the sets*

$$L(M_j) = \{ \lambda \in \mathbb{R} \,;\, \textit{there exists a periodic geodesic in } M_j \textit{ of length } \lambda \},$$

for $j = 1, 2$, are equal.

As we will explain, this theorem is a direct consequence of theorems of Duistermaat and Guillemin, [9], and of Duistermaat, Kolk and Varadarajan, [10]. (In fact, the results of the latter paper alone imply this theorem, but it is conceptually better to use the main theorem of [9] in the proof.)

The results of [10] (cf. Proposition 5.15) include that, for M a compact locally symmetric space of non-compact type,

(i) $L(M)$ is a discrete subset of $\mathbb{R}$, and
(ii) if $\lambda \in L(M)$, the set

$$Z_\lambda := \{\overline{x} \in T^1M \,;\, \text{the geodesic through } \overline{x} \text{ is periodic of length } \lambda\}$$

is a finite union of closed submanifolds (possibly of different dimensions) of the unit tangent bundle T^1M of M.

Denote by Z_λ° the union of connected components of Z_λ of maximal dimension. It turns out that, in addition to the previous theorem, for M as above

(34) for all $\lambda \in L(M)$ dim Z_λ° and Vol Z_λ° are spectrally determined.

Here the volume is with respect to a measure naturally induced by the geodesic flow. (Equation (5.47) of [10] is a formula for this volume.)

Let us now see how one proves Theorem 10.1 and the additional statement, (34). Proposition 5.8 of [10] establishes that each Z_λ is a clean fixed-point set of the time λ map of the geodesic flow. We can therefore apply the Duistermaat-Guillemin trace formula, [9], to the square root of the Laplace-Beltrami operator on M. Specifically, pick a length λ and a Schwartz function on the real line, φ, such that its Fourier transform $\widehat{\varphi}$ is compactly supported and satisfies:

$$\widehat{\varphi}(\lambda) = 1 \quad \text{and} \quad L(M) \cap \operatorname{supp} \widehat{\varphi} = \{\lambda\}.$$

(Such a φ exists by item (i) above.) Let $0 = \mu_0 < \mu_1 \leqslant \mu_2 \leqslant \cdots$ be the square roots of the eigenvalues of the Laplacian on M, listed with their multiplicities. Then, by Theorem 4.5 of [9] one has an asymptotic expansion as $\mu \to \infty$ of the form:

$$\sum_j \varphi(\mu - \mu_j) \sim e^{i\mu\lambda} \sum_{j=0}^{\infty} c_j \, \mu^{d_\lambda - j}. \tag{35}$$

Here $d_\lambda = (\dim Z_\lambda^\circ - 1)/2$. A key point is that the leading coefficient, c_0, is not zero because the Maslov indices (the integers σ_j in equation (4.7) in [9]) of all periodic geodesics on M are zero, by Proposition 5.15 of [10]. By equation (4.8) in [9], c_0 is equal to the volume of Z_λ° times a factor that depends only on d_λ. The expansion (35) in the present context is explicitly discussed in §5.6 of [10] (see the last formula in that section which, incidentally, contains a typo: a τ is missing in the left-hand side exponent). The dimension of Z_λ° is determined spectrally by the size in μ of the left-hand side of (35), and therefore c_0 determines the volume of Z_λ°.

Theorem 10.1 and statement (34) follow from (35), the information on c_0, and the basic fact that if $L(M) \cap \operatorname{supp} \widehat{\varphi} = \emptyset$, then the left-hand side of (35) is $O(\mu^{-\infty})$. By considering all possible test functions φ as above, one can detect the set $L(M)$ from the eigenvalues of the Laplacian. □

Let $\mathcal{G}$ be a connected semi-simple real Lie group of adjoint type without compact factors, and Γ_1 and Γ_2 be two torsion-free irreducible cocompact discrete subgroups of $\mathcal{G}$. Let $\mathfrak{X}$ be the symmetric space of $\mathcal{G}$, and $M_i = \mathfrak{X}/\Gamma_i$ for $i = 1, 2$ be the corresponding local symmetric spaces. From Theorems 8.7 and 10.1 we obtain the following.

Theorem 10.2. *If M_1 and M_2 are isospectral, then Γ_1 and Γ_2 are weakly commensurable.*

We now assume that $\mathcal{G}$ is *absolutely* simple. Then using Theorem 10.2 in conjunction with Theorem F, we obtain the following result.

Theorem 10.3. *If M_1 and M_2 are isospectral, and Γ_1 is arithmetic, then so is Γ_2.*

This result combined with Theorem 8.9 yields the following theorem.

Theorem 10.4. *Any two arithmetically defined compact isospectral locally symmetric spaces of an absolutely simple real Lie group of type other than A_n $(n > 1)$, D_n $(n \geqslant 4)$, and E_6, are commensurable to each other.*

The following remark is due to Peter Sarnak.

Remark 10.5. It was proved by Hermann Weyl that any two isospectral Riemannian manifolds are of same volume (and of same dimension), see, for example, [13], Theorem 4.2.1. Now, as before, let $\mathcal{G}$ be a connected semi-simple real Lie group of adjoint type without compact factors, and $\mathfrak{X}$ be its symmetric space. If Γ is a torsion-free irreducible cocompact discrete subgroup of $\mathcal{G}$, then the set of conjugacy classes of torsion-free irreducible cocompact discrete subgroups Γ' of $\mathcal{G}$ such that $\mathfrak{X}/\Gamma'$ is isospectral to $\mathfrak{X}/\Gamma$ is finite. This follows from H.C. Wang's finiteness theorem ([30], Ch. IX) if $\mathcal{G}$ is not isomorphic to $\mathrm{PSL}_2(\mathbb{R})$, since according to a thereom of André Weil ([30], Theorem 7.63) cocompact irreducible discrete subgroups in such a $\mathcal{G}$ are locally rigid, and $\mathfrak{X}/\Gamma$ and $\mathfrak{X}/\Gamma'$, and therefore, $\mathcal{G}/\Gamma$ and $\mathcal{G}/\Gamma'$ have equal volume. On the other hand, if $\mathcal{G}$ is isomorphic to $\mathrm{PSL}_2(\mathbb{R})$, then the finiteness of the conjugacy classes of Γ's is proved in §5.3 of [18].

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