# The First Brauer-Thrall Conjecture

Claus Michael Ringel

August 31, 2007

**Abstract**

Let $\Lambda$ be an artin algebra and $M$ a $\Lambda$-module. We show that $M$ is either the direct sum of copies of a finite number of indecomposable modules of finite length, or else $M$ contains indecomposable submodules of arbitrarily large finite length. This improves the assertion of the first Brauer-Thrall conjecture as established by Roiter in 1968: Any artin algebra with infinitely many isomorphism classes of indecomposable modules of finite length has indecomposable modules of arbitrarily large finite length.

Let $\Lambda$ be an artin algebra. The modules to be considered are left $\Lambda$-modules, and not necessarily of finite length. A module $M$ is said to be *of finite type,* provided $M$ is the direct sum of (arbitrarily many) copies of a finite number of indecomposable modules of finite length.

**Theorem.** *A module $M$ which is not of finite type contains indecomposable submodules of arbitrarily large finite length.*

**Remarks: (1)** Recall that $\Lambda$ is said to be *representation-finite* provided there are only finitely many isomorphism classes of indecomposable $\Lambda$-modules of finite length, otherwise $\Lambda$ is called *representation-infinite.* The first Brauer-Thrall conjecture asserts that *a representation-infinite artin algebra has indecomposable submodules of arbitrarily large finite length.* The conjecture was solved by Roiter [7] in 1968. The theorem can be seen as a strengthening: Assume there are infinitely many isomorphism classes of indecomposable modules $M_i$; take the direct sum $M = \bigoplus M_i$. The (Krull-Remak-Schmidt-)Azumaya theorem shows that $M$ is not of finite type, thus we obtain indecomposable modules of arbitrarily large finite length as submodules of this particular module $M$.

**(2)** In order to provide a proof of the first Brauer-Thrall conjecture, it is sufficient to deal with the case where there are infinitely many isomorphism classes of indecomposable modules $M_i$ of a fixed length and consider the direct sum $M = \bigoplus M_i$. This case has been discussed in [4], Appendix A. There, we have shown that there are large indecomposable modules which are **cogenerated** by $M$ (but this means only that the direct sum of countably many copies of $M$ contains indecomposable submodules of arbitrarily large length).

Part of the present proof will consist in dealing precisely with the special case of $M$ being the direct sum of infinitely many pairwise non-isomorphic indecomposable modules of equal length.

**(3)** The theorem also yields a new proof of the following result (see [8], Corollary 9.5, or [6], and also [1], Corollary 4.8): *If an artin-algebra* $\Lambda$ *is of finite type, then any* $\Lambda$*-module is of finite type.* We should stress that the converse implication is an obvious consequence of the (Krull-Remak-Schmidt)-Azumaya theorem.

*Proof of Theorem.* We assume that the module $M$ is not of finite type and show the existence of indecomposable submodules of arbitrarily large length. Thus, assume that the indecomposable submodules of $M$ of finite length are of bounded length, thus there are only finitely many possible Gabriel-Roiter measures. Assume that the indecomposable submodules of $M$ of finite length have Gabriel-Roiter measure $\gamma_1 < \gamma_2 < \cdots < \gamma_s$. We show by induction on $s$ that $M$ is of finite type. The case $s = 1$ is well-known and easy to see: if any indecomposable submodule of $M$ of finite length is simple, then $M$ has to be semi-simple, thus of finite type.

Assume now that $s \geq 2$. Consider a submodule $M'$ of $M$ which is a direct sum of modules of Gabriel-Roiter measure $\gamma_s$, and maximal with this property. If $M'$ is of finite type, then [5], theorem 4.2 asserts that $M'$ is $\Sigma$-pure injective in $\mathcal{D}(\gamma_s)$, and of course $M'$ is a pure submodule of $M$, thus $M'$ is a direct summand of $M$, say $M = M' \oplus M''$ for some module $M''$. However, the indecomposable submodules of $M''$ of finite length have Gabriel-Roiter measure $\gamma_1, \ldots, \gamma_{s-1}$ (note that $\gamma_s$ cannot occur by the maximality of $M'$), thus by induction $M''$ is of finite type. Then also $M = M' \oplus M''$ is of finite type.

Thus we can assume that there is a submodule $M^1 = \bigoplus_{i\geq 1} M_i$ of $M$ which is an infinite direct sum of pairwise non-isomorphic indecomposable modules $M_i$ with Gabriel-Roiter measure $\gamma_s$, indexed over $\mathbb{N}$. For any $r \in \mathbb{N}$, let $M^r = \bigoplus_{i\geq r} M_i$.

The modules $M_i$ have all the same length, say length $t$. Let $\mathcal{U}_r$ be the set of isomorphism classes of indecomposable submodules of $M^r$ of length at most $t-1$.

(a) *The set $\mathcal{U}_r$ is finite for almost all $r$.* Otherwise, we chose inductively pairwise non-isomorphic submodules $U_j$ of $M^1$ of length at most $t-1$ such that $U = \sum_{j\in\mathbb{N}} U_j$ is the direct sum of the modules $U_j$. (Namely, assume we have found $U_1, \ldots, U_s$ with $U' = \bigoplus_{j=1}^{s} U_j \subseteq M^1$, then $U' \subseteq \bigoplus_{i=1}^{r-1} M_i$ for some $r$. If $\mathcal{U}_r$ is infinite, we find inside $M^r$ an indecomposable submodule $U_{s+1}$ of length at most $t-1$ which is not isomorphic to any of the $U_1, \ldots, U_s$. Since $\bigoplus_{i=1}^{r-1} M_i$ and $M^r$ intersect in zero, we see that $\sum_{j=1}^{s+1} U_j$ is a direct sum.) As a submodule of $M$, all the indecomposable submodules of $U$ of finite length have Gabriel-Roiter measure $\gamma_i$ with $1 \leq i \leq s$ and actually $\gamma_s$ does not occur as a Gabriel-Roiter measure (since such a submodule would be a direct summand of $U$, impossible). By induction, $U$ has to be of finite type — but by construction, $U = \bigoplus_{j\in\mathbb{N}} U_j$ is not of finite type.

Let $\mathcal{U} = \bigcap_r \mathcal{U}_r$. As we have seen, this is a finite set of isomorphism classes, and of course non-empty. There is some $r'$ with $\mathcal{U} = \mathcal{U}_{r'}$ and without loss of generality, we can assume that $r' = 1$ (replacing $M^1$ by $M^{r'}$). Thus we deal with the following situation: $M^1 = \bigoplus_{i\geq 1} M_i$ is an infinite direct sum of pairwise non-isomorphic indecomposable modules $M_i$ with Gabriel-Roiter measure $\gamma_s$, and any indecomposable submodule of $M^1$ of length at most $t-1$ is also a submodule of $M^r = \bigoplus_{i\geq r} M_i$ for any $r$.

(b) *Any indecomposable module of length at most $t-1$ and cogenerated by $M^1$ is isomorphic to a submodule of $M^1$.* Assume that $N$ is of length at most $t-1$ and cogenerated by $M^1$, thus there is a finite number of maps $\pi : N \to M_i$ such that the kernels of these maps intersect in zero. These maps $\pi$ cannot be surjective, since $N$ is of length at most $t-1$, whereas $M_i$ is of length $t$. If we decompose the images $\pi(N)$ of these maps, we obtain indecomposable submodules $N_j$ of $M_i$ of length at most $t-1$, and such submodules $N_j$ occur frequently inside $M^1$, namely inside $M^r$, for any $r$. This shows that $N$ is a submodule of $M^1$.

(c) In particular, we see that there are only finitely many isomorphism classes of modules which are cogenerated by $M^1$ and of length at most $t-1$.

Let $S$ be the direct sum of all the simple modules. As in [4], we consider the class $\mathcal{N}$ of all indecomposable modules cogenerated by $M^1 \oplus S$ and not isomorphic to any $M_i$. Clearly, this class is again closed under cogeneration and still finite. For any module $M_i$, let $f^{\mathcal{N}} M_i$ be the maximal factor module of $M_i$ which belongs to $\operatorname{add} \mathcal{N}$. Since $M_i$ does not belong to $\mathcal{N}$, we see that $f^{\mathcal{N}} M_i$ is a module of length at most $t-1$ and cogenerated by $M^1 \oplus S$, thus there are only finitely many possibilities. It follows that there is a module $Q$ in $\operatorname{add} N$ such that $f^{\mathcal{N}} M_i = Q$ for infinitely many $i$. Without loss of generality, we even may assume that $f^{\mathcal{N}} M_i = Q$ for all $i$ (by deleting the remaining factors). For any module $M_i$, fix a projection $q_i : M_i \to Q$ and let $K$ be the kernel of the map $(f_i)_i : M^1 \to Q$. Roiter's coamalgamation lemma (see [4]) asserts that $K$ has no direct summand isomorphic to $M_i$, thus no submodule of Gabriel-Roiter measure $\gamma_s$ (since $M^1$ belongs to $\mathcal{D}(\gamma_s)$ and $M_i$ is relative injective in $\mathcal{D}(\gamma_s)$). By induction we see that $K$ has to be of finite type. But this contradicts the Ext-Lemma [4]: for any extension of the form

$$0 \to K \to X \to Q \to 0$$

with $Q$ of finite length and $K$ of infinite length and of finite type, the modules $K$ and $X$ will have common indecomposable direct summands. For $X = M^1$, the indecomposable direct summands have Gabriel-Roiter measure $\gamma_s$, but $K$ has not even a submodule of measure $\gamma_s$. $\square$

Assume again that $M$ is a $\Lambda$-module which is not of finite type. This implies that $\Lambda$ is representation-infinite, thus according to Auslander ([2], see also [4]) there are indecomposable $\Lambda$-modules of infinite length. We have shown that $M$ contains indecomposable submodules of arbitrarily large finite length, but $M$ may not contain an indecomposable submodule of infinite length, as the following example shows:

**Example.** Here is *a module $M$ which is not of finite type, but such that all its indecomposable submodules are of finite length.* Let $\Lambda$ be the Kronecker algebra with preprojective indecomposable modules $P_i$, $i \in \mathbb{N}$. Let $M = \bigoplus_{i \in \mathbb{N}} P_i$. We show that any indecomposable submodule $U$ of $M$ is of finite length. Let $M_j = \bigoplus_{j \le i} P_i$. Assume $U$ is any submodule. If $U$ is contained in all $M_j$, then $U = 0$. Thus assume that $U$ is contained in $M_j$, but not in $M_{j+1}$. We get a non-zero map $U \to M_j/M_{j+1} = P_j$. According to [3], $U$ splits off a direct summand of the form $P_i$ with $i \le j$.

# References

[1] M. Auslander: Representation theory of artin algebras II. Communications Algebra. 1 (1974), 269-310.

[2] M. Auslander: Large modules over artin algebras. In: Algebra, Topology and Category Theory. Academic Press (1976), 1-17.

[3] C. M. Ringel: Infinite dimensional representations of finite dimensional hereditary algebras. Symposia Math. XXIII. Istituto NazÀlta Mat(1979), 321-412.

[4] C. M. Ringel: The Gabriel-Roiter measure. Bull. Sci. math. 129 (2005), 726-748.

[5] C. M. Ringel: Foundation of the Representation Theory of Artin Algebras, Using the Gabriel-Roiter Measure. In: Trends in Representation Theory of Algebras and Related Topics. Edited by de la Peña and Bautista. Contemporary Math. 406. Amer.Math.Soc. (2006), 105-135.

[6] C. M. Ringel, H. Tachikawa: QF-3 rings. J. Reine Angew. Math. 272 (1975), 49-72.

[7] A. V. Roiter: Unboundedness of the dimension of the indecomposable representations of an algebra which has infinitely many indecomposable representations. Izv. Akad. Nauk SSSR. Ser. Mat. 32 (1968), 1275-1282.

[8] H. Tachikawa: Quasi-Frobenius Rings and Generalizations. Springer Lecture Notes in Mathematics 351 (1973).

Fakultät für Mathematik, Universität Bielefeld

Germany

ringel@math.uni-bielefeld.de