Littlewood's one circle problem, revisited

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Abstract

It is known since 1994 that the answer to Littlewood's one circle problem is "no". The original construction of a counterexample is very delicate. It is simplified using a result of M. Talagrand on compact sets in the plane which are visible from one direction only.

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1 Introduction and main result

Let U be the open unit disk in $\mathbb{R}^2$ and let $\rho\colon U \to (0,1]$ denote the distance to the boundary ∂U of U, that is, $\rho(x) = 1 - |x|$. Littlewood's one circle problem is the following (see [8]). Let f be a continuous bounded function on U. Is f harmonic, if, for each $x \in U$, there exists *one* radius $r(x) \in (0, \rho(x)]$ such that

$$f(x) = \frac{1}{2\pi} \int_0^{2\pi} f(x + r(x)(\cos t, \sin t))\, dt\,? \tag{1.1}$$

As shown in [5], the answer to this question is "no". Let us note that the corresponding question in the whole plane has a positive answer (see [3]; an entirely elementary proof is given in [1]). The analogous problem in $\mathbb{R}^d$, $d \geq 3$, is still open, both for the unit ball and the whole space. In contrast to this situation, the related problems for averages on balls are solved, in any dimension $d \geq 1$ and for every open subset of $\mathbb{R}^d$ (see [4, 6, 2], cf. also [7]).

The construction of a counter-example to Littlewood's one circle problem given in [5] is very delicate. The purpose of this paper is to show that the construction becomes much simpler, if a result of M. Talagrand ([9]; see the Appendix) is used.

Let λ^2 denote Lebesgue measure on $\mathbb{R}^2$. For $x \in \mathbb{R}^2$ and $0 < s < t < \infty$, let

$$A(x,s,t) = \{y \in \mathbb{R}^2 : s < |y-x| < t\}, \quad B(x,t) := \{y \in \mathbb{R}^2 \colon |y-x| < t\}.$$

As in [5], we fix $\alpha \in (0, 1/2]$ (e.g., $\alpha = 1/2$) and prove the following.

THEOREM 1.1. *There exist continuous functions* $0 < s < t \le \alpha\rho$ *on* U *and a continuous function* $0 \le f \le 1$ *on* U *such that* f *is* not *harmonic, but*

$$f(x) = \frac{1}{\lambda^2(A(x, s(x), t(x)))} \int_{A(x,s(x),t(x))} f \, d\lambda^2 \tag{1.2}$$

for every $x \in U$. *In particular,* (1.1) *holds for some* $r(x) \in (s(x), t(x))$.

The open unit disk U will be equipped with the σ-algebra $\mathcal{B}(U)$ of all Borel measurable subsets of U. Let $\Omega := U^{\mathbb{N}\cup\{0\}}$, $X_n(\omega) := \omega_n$, and let $\mathcal{M}$ denote the σ-algebra on Ω generated by X_n, $n \in \mathbb{N} \cup \{0\}$. Given a Markov kernel P on U and $x \in U$, let P^x denote the probability measure on $(\Omega, \mathcal{M})$ such that $(\Omega, \mathcal{M}, X_n, P^x)$ is the random walk starting at x and having transition kernel P, that is, for all $n \in \mathbb{N}$ and $B_0, B_1, \dots, B_n \in \mathcal{B}(U)$,

$$\begin{aligned} &P^x[X_0 \in B_0, X_1 \in B_1, X_2 \in B_2, \dots, X_n \in B_n] \\ &\qquad = \varepsilon_x(B_0) \int_{B_1} P(x, dx_1) \int_{B_2} P(x_1, dx_2) \dots \int_{B_n} P(x_{n-1}, dx_n). \end{aligned} \tag{1.3}$$

For every $A \in \mathcal{B}(U)$, let T_A denote the first hitting time of A, that is,

$$T_A := \inf\{n \ge 1 : X_n \in A\}.$$

We shall use the following simple fact. If P and Q are Markov kernels on U such that $P(x, \cdot) = Q(x, \cdot)$ for every x outside a Borel measurable set A, then

$$P^x[X_0 \in B_0, \dots, X_n \in B_n, n \le T_A] = Q^x[X_0 \in B_0, \dots, X_n \in B_n, n \le T_A] \tag{1.4}$$

for all $x \in U \setminus A$, $n \in \mathbb{N}$, and $B_0, \dots, B_n \in \mathcal{B}(U)$.

Given Borel measurable functions $s, t \colon U \to (0, \infty)$ such that $s < t \le \alpha\rho$, let $P_{s,t}$ denote the Markov kernel defined by

$$P_{s,t}(x, B) := \frac{\lambda^2(B \cap A(x, s(x), t(x)))}{\lambda^2(A(x, s(x), t(x)))} \qquad (B \in \mathcal{B}(U)).$$

Let us fix $\delta_0 \in (0, \alpha/2)$ such that, for every harmonic function $h \colon U \to [0, 1]$,

$$|h - h(0)| < \frac{1}{4} \quad \text{on } B(0, \delta_0). \tag{1.5}$$

Then Theorem 1.1 is a consequence of the following proposition.

PROPOSITION 1.2. *There are continuous functions* $s, t \colon U \to (0, \infty)$, *a Borel measurable set* B *in* U, *and a point* $x_0 \in B(0, \delta_0) \setminus B$ *such that* $s < t \le \alpha\rho$, $0 \in B$, *and* $P := P_{s,t}$ *satisfies*

$$P^0[T_{B^c} = \infty, \lim_{n\to\infty} \rho(X_n) = 0] > \frac{3}{4}, \quad \textit{whereas} \quad P^{x_0}[T_B < \infty] < \frac{1}{4}. \tag{1.6}$$

Indeed, we then define Borel measurable functions $f_n, f \colon U \to [0, 1]$ by

$$f_n(x) := P^x[T_{B\cap\{\rho\le\alpha^n\}} < \infty], \qquad f := \inf f_n = \lim_{n\to\infty} f_n.$$

By (1.6), $f(0) \ge 3/4$, whereas $f(x_0) < 1/4$. So f is not harmonic by (1.5). If $x \in U$ and $\rho(x) \ge \alpha^{n-1}$, then $P(x, \{\rho \le \alpha^n\}) = 0$ and hence $Pf_n(x) = f_n(x)$. This implies that $Pf = f$. Thus (1.2) holds, f is continuous, since the functions s, t are continuous, and Theorem 1.1 is proven.

2 Proof of Proposition 1.2

We adopt from [5] that the random walk associated with $P_{\rho/2,\rho}$ has small probability to hit a set having small area. More precisely, we have the following result, where we could replace $1/4$ by an arbitrary $\varepsilon > 0$ (see Proposition 1 in [5]).

PROPOSITION 2.1. *There exists a decreasing sequence* (δ_n) *in* $(0, \delta_0^2)$ *such that* $\delta_{n+1} \le \delta_n/2$ *for every* $n \in \mathbb{N}$ *and* $P^x_{\rho/2,\rho}[T_B < \infty] < 1/4$ *for all* $x \in U$ *and* $B \in \mathcal{B}(U)$ *satisfying*

$$\lambda^2(B \cap \{\rho \le \alpha^{n-1}\}) < 2\delta_n \qquad (n \in \mathbb{N}).$$

In [5], the construction of a suitable set B is very delicate. Using the following consequence of [9], it will become simpler and more transparent.

PROPOSITION 2.2. *Let* L *be a finite union of closed line segments in* $\mathbb{R}^2$ *and let* $s, t \colon L \to (0, \infty)$ *be continuous,* $s < t$*. Then, for every* $\varepsilon > 0$*, there exist* $x_1, \dots, x_m \in L$ *and* $s(x_i) \le s_i < t_i \le t(x_i)$, $1 \le i \le m$*, such that*

$$L \subset \bigcup_{i=1}^m B(x_i, \varepsilon(t_i - s_i)) \quad \text{and} \quad \lambda^2\Big(\bigcup_{i=1}^m A(x_i, s_i, t_i)\Big) < \varepsilon. \tag{2.1}$$

Proof. By Corollary 3.2 (see the Appendix), there exists an upper semicontinuous real function r on L such that $s < r < t$ and the closed set

$$A \colon = \overline{\bigcup_{x \in L} \partial B(x, r(x))}$$

has zero area. So there is an open neighborhood W of A in U with $\lambda^2(W) < \varepsilon$. Let

$$\eta(x) := \min\{r(x) - s(x), t(x) - r(x), \operatorname{dist}(\partial B(x, r(x)), \partial W)\} \qquad (x \in L).$$

Since L is compact, there exist $x_1, \dots, x_m \in L$ such that L is covered by the disks $B(x_1, \varepsilon\eta(x_1)), \dots, B(x_m, \varepsilon\eta(x_m))$. Taking $s_i := r(x_i) - \eta(x_i)$ and $t_i := r(x_i) + \eta(x_i)$, $1 \le i \le m$, we have $s(x_i) \le s_i < t_i \le t(x_i)$, $t_i - s_i = 2\eta(x_i)$, $A(x_i, s_i, t_i) \subset W$. Thus (2.1) holds. □

Let $\mathcal{A}$ denote the family of all open sets A in U such that the boundary ∂A consists of finitely many line segments in U which are parallel to one of the coordinate axes. If $A, B \in \mathcal{A}$, then clearly $A \cup B, A \cap B, A \setminus \overline{B} \in \mathcal{A}$, and the closure of $A \setminus \overline{B}$ contains $A \setminus B$. Let us fix a sequence (δ_n) according to Proposition 2.1.

Lemma 2.3. *There exist sequences* $(U_n), (A_n)$ *in* $\mathcal{A}$ *and continuous real functions* s_n, t_n *on* U_n, $n \in \mathbb{N}$*, such that the following holds:*

1. $\lambda^2(U_1) < \delta_1$, $0 \in U_1$.
2. $\lambda^2(A_n) < \delta_{n+1}$ *and* $U_n \subset U_{n+1} \subset U_n \cup A_n \subset \overline{U}_{n+1} \cap \{\rho > \alpha^n\}$.
3. s_n *and* t_n *attain only finitely many values,* $0 < s_n < t_n \le \alpha\rho$ *on* U_n.

4. *For every* $x \in U_n$, $s_{n+1}(x) = s_n(x)$, $t_{n+1}(x) = t_n(x)$, *and*

$$A(x, s_n(x), t_n(x)) \subset \overline{U}_n \cup A_n,$$

where $A(x, s_n(x), t_n(x)) \subset U_n$ *or* $s_n(x) \geq \alpha\rho(x)/4$.

Proof. We first choose $\alpha/3 < \beta < \gamma < \alpha/2$ such that $\lambda^2(A(0,\beta,\gamma)) < \delta_2$. Since $\alpha \leq 1/2$, we see that $A(0,\beta,\gamma) \subset B(0,\gamma) \subset \{\rho > 1 - \alpha/2\} \subset \{\rho > \alpha\}$. So there exists $A_1 \in \mathcal{A}$ such that $A(0,\beta,\gamma) \subset A_1 \subset \{\rho > \alpha\}$ and $\lambda^2(A_1) < \delta_2$. Moreover, if $a > 0$ is sufficiently small, then the square $U_1 := (-a,a) \times (-a,a)$ is contained in $B(0,\gamma)$, satisfies $\lambda^2(U_1) < \delta_1$, and, defining functions $s_i, t_i \colon U_1 \to (\beta,\gamma)$ by

$$s_1(x) := \frac{2}{3}\beta + \frac{1}{3}\gamma, \qquad t_1(x) := \frac{1}{3}\beta + \frac{2}{3}\gamma,$$

we have $\alpha\rho/3 < s_1 < t_1 < \alpha\rho/2$ on U_1 and, for every $x \in U_1$,

$$A(x, s_1(x), t_1(x)) \subset A(0,\beta,\gamma) \subset A_1.$$

Next let us fix $n \in \mathbb{N}$ and suppose that we have constructed $U_n, A_n \in \mathcal{A}$, and functions $s_n, t_n \colon U_n \to (0,1)$ such that $U_n \cup A_n \subset \{\rho > \alpha^n\}$ and $A(x, s_n(x), t_n(x)) \subset \overline{U}_n \cup A_n$ for every $x \in U_n$. By Proposition 2.2, there are $x_1, \dots, x_m \in \partial A_n$ and $\alpha\rho(x_i)/3 < \beta_i < \gamma_i < \alpha\rho(x_i)/2$, $1 \leq i \leq m$, such that

$$\partial A_n \subset \bigcup_{i=1}^{m} B(x_i, \frac{\gamma_i - \beta_i}{6}) \quad \text{and} \quad \lambda^2\big(\bigcup_{i=1}^{m} A(x_i,\beta_i,\gamma_i)\big) < \delta_{n+2}. \tag{2.2}$$

If $x \in A(x_i,\beta_i,\gamma_i)$, $1 \leq i \leq m$, then $\rho(x) \geq \rho(x_i) - \alpha\rho(x_i)/2 \geq (1-\alpha/2)\alpha^n \geq \alpha^{n+1}$. Hence we may choose $A_{n+1} \in \mathcal{A}_{n+1}$ such that

$$\bigcup_{1\leq i\leq m} A(x_i,\beta_i,\gamma_i) \subset A_{n+1} \subset \{\rho > \alpha^{n+1}\} \quad \text{and} \quad \lambda^2(A_{n+1}) < \delta_{n+2}. \tag{2.3}$$

For every $1 \leq i \leq m$, let $a_i := (\gamma_i - \beta_i)/6$, $Q_i := (x_i - a_i, x_i + a_i) \times (x_i - a_i, x_i + a_i)$, and

$$V_i := (A_n \setminus \overline{U}_n) \cap \big(Q_i \setminus (\overline{Q}_1 \cup \dots \cup \overline{Q}_{i-1})\big).$$

Moreover, we define

$$V := (A_n \setminus \overline{U}_n) \setminus (\overline{Q}_1 \cup \dots \cup \overline{Q}_m) \quad \text{and} \quad U_{n+1} = U_n \cup V_1 \cup \dots \cup V_m \cup V.$$

Then $U_n \subset U_{n+1} \subset U_n \cup A_n \subset \overline{U}_{n+1}$ and the sets $U_n, V_1, \dots, V_m, V \in \mathcal{A}$ are pairwise disjoint.

We shall extend the functions s_n, t_n to functions s_{n+1}, t_{n+1} on U_{n+1} choosing suitable constants on each of the open sets $V_1, \dots, V_m, V$. If $x \in V_i$, $1 \leq i \leq m$, we proceed as in the case $n = 1$, define

$$s_{n+1}(x) := \frac{2}{3}\beta_i + \frac{1}{3}\gamma_i, \qquad t_{n+1}(x) := \frac{1}{3}\beta_i + \frac{2}{3}\gamma_i,$$

and obtain that, by (2.3),

$$A(x, s_{n+1}(x), t_{n+1}(x)) \subset A(x_i, \beta_i, \gamma_i) \subset A_{n+1}.$$

Moreover, since $3|x - x_i| < \gamma_i - \beta_i < \alpha\rho(x_i)/6$ and $|\rho(x) - \rho(x_i)| \le |x - x_i|$, we have

$$t_{n+1}(x) < \gamma_i < \alpha\rho(x_i)/2 < \alpha\rho(x) \quad \text{and} \quad s_{n+1}(x) > \beta_i > \alpha\rho(x_i)/3 > \alpha\rho(x)/4.$$

By (2.2), $\operatorname{dist}(V, \partial A_n) > 0$. So we may choose $\eta \in (0, \alpha^{n+1})$ such that $B(x, \eta) \subset V$ for every $x \in B$. Defining

$$s_{n+1}(x) := \frac{1}{2}\eta, \quad t_{n+1}(x) := \eta \qquad (x \in V)$$

we hence know that

$$A(x, s_{n+1}(x), t_{n+1}(x)) \subset B(x, \eta) \subset V \subset U_{n+1} \qquad \text{for every } x \in V.$$

Finally, we note that $t_{n+1} = \eta < \alpha^{n+1} < \alpha\rho$ on V, since $V \subset \{\rho > \alpha^n\}$. □

Let U_∞ denote the union of the increasing sequence (U_n). We define functions s_∞, t_∞ on U by

$$\begin{aligned} s_\infty(x) &:= s_n(x) &\quad \text{and} \quad t_\infty(x) &:= t_n(x), &\quad \text{if } n \in \mathbb{N}, x \in U_n, \\ s_\infty(x) &:= \frac{\alpha}{2}\rho(x) &\quad \text{and} \quad t_\infty(x) &:= \alpha\rho(x), &\quad \text{if } x \in U \setminus U_\infty. \end{aligned}$$

By construction, U_∞ is contained in the union of the sets $U_1, A_1, A_2, \dots$. Let us recall that $\lambda^2(U_1) < \delta_1$ and, for every $n \ge 1$, $\lambda^2(A_n) < \delta_{n+1} \le \delta_n/2$. Therefore

$$\lambda^2(U_\infty) \le \sum_{n=1}^{\infty} \delta_n \le 2\delta_1 < \pi\delta_0^2.$$

In particular, there exists a point

$$x_0 \in B(0, \delta_0) \setminus U_\infty.$$

Moreover, for every natural $k \ge 2$, $U_\infty \cap \{\rho \le \alpha^{k-1}\}$ is contained in the union of the sets $A_k, A_{k+1}, \dots$, and hence

$$\lambda^2(U_\infty \cap \{\rho \le \alpha^{k-1}\}) \le \sum_{n=k}^{\infty} \lambda^2(A_k) \le \sum_{n=k}^{\infty} \delta_{n+1} \le \delta_k.$$

Let $P_\infty := P_{s_\infty, t_\infty}$. By Lemma 2.3 and Proposition 2.1, we obtain the following.

COROLLARY 2.4. *If $x \in U \setminus U_\infty$, then $P_\infty^x[T_{U_\infty} < \infty] < 1/4$.*

If the random walk starts within U_∞, the situation is entirely different.

PROPOSITION 2.5. *If $x \in U_\infty$, then, for P_∞^x-almost every $\omega \in \Omega$, the sequence $(X_n(\omega))$ is contained in U_∞ and converges to a point in ∂U.*

Proof. Let $x \in U$ and let $p\colon \mathbb{R}^2 \to \mathbb{R}$ be the projection $(a,b) \mapsto a$. Clearly, $P_\infty p = p$, since p is harmonic. Therefore $(p \circ X_n)$ is a bounded martingale with respect to (Ω, P_∞^x) (and the natural filtration generated by (X_n)), and hence $(p \circ X_n)$ converges P_∞^x-almost surely to a random variable Y_1. Similarly, if q denotes the projection $(a,b) \mapsto b$, then $(q \circ X_n)$ converges P_∞^x-almost surely to a random variable Y_2. Thus

$$\lim_{n\to\infty} X_n = (Y_1, Y_2) =: Y \quad P^x\text{-almost surely.} \tag{2.4}$$

Let Ω_0 denote the set of all $\omega \in \Omega$ having the following properties.

(i) $\lim_{n\to\infty} X_n(\omega) = Y(\omega)$.

(ii) For all $n \in \mathbb{N}$, $X_n(\omega) \in U_\infty$ and $|X_{n+1}(\omega) - X_n(\omega)| \geq s_\infty(X_n(\omega))$.

(iii) If $m, n \in \mathbb{N}$ and $y_n := X_n(\omega) \in U_m$, then $X_{n+1}(\omega) \in U_m$ or $s_m(y_n) \geq \alpha\rho(y_n)/4$.

By (2.4) and (4) in Lemma 2.3, $P^x(\Omega_0) = 1$. Let us fix $\omega \in \Omega_0$, let $y := Y(\omega)$ and $y_n := X_n(\omega)$, $n \in \mathbb{N}$. By (i), $y = \lim_{n\to\infty} y_n \in \overline{U}$. We claim that $y \in \partial U$. Indeed, if $m \in \mathbb{N}$ and $n_m \in \mathbb{N}$ such that $y_n \in U_m$ for all $n \geq n_m$, then, by (ii),

$$|y_{n+1} - y_n| \geq \inf s_m(U_m) > 0 \qquad \text{for all } n \geq n_m.$$

This is impossible, since (y_n) converges. So there exist $n_1 < n_2 < \dots$ and $m_j \in \mathbb{N}$ such that, for every $j \in \mathbb{N}$, $y_{n_j} \in U_{m_j}$, but $y_{n_j+1} \notin U_{m_j}$. Then, by (ii) and (iii),

$$|y_{n_j} - y_{n_j+1}| \geq \frac{\alpha\rho(y_{n_j})}{4}.$$

Since $\lim_{n\to\infty} |y_n - y_{n+1}| = 0$, we see that $\lim_{j\to\infty} \rho(y_{n_j}) = 0$ and hence $y \in \partial U$. □

Lemma 2.6. *There exists a relatively closed set B in U such that $0 \in B \subset U_\infty$ and $P_\infty^0[T_{B^c} < \infty] < 1/4$.*

Proof. Let $\varepsilon := 1/4$ (or, more generally, $\varepsilon \in (0, 1/4]$). The unit disk U is the union of the compact annuli $K_m := \{\alpha^m \leq \rho \leq \alpha^{m-1}\}$, $m \in \mathbb{N}$. Let $m \in \mathbb{N}$. Since $\rho(X_n) \to 0$ P_∞^0-almost surely, there exist $n_m \in \mathbb{N}$ such that

$$P_\infty^0[X_n \in K_m \text{ for some } n > n_m] < \frac{\varepsilon}{2^{m+1}}. \tag{2.5}$$

Since $X_n \in U_\infty$ P_∞^0-almost surely, there exists closed subsets $B_{n,m}$ of $U_\infty \cap K_m$, $1 \leq n \leq n_m$, such that

$$P_\infty^0[X_n \in K_m \setminus B_{n,m}] < \frac{\varepsilon}{n_m 2^{m+1}}. \tag{2.6}$$

Let $B_m := B_{1,m} \cup \dots \cup B_{n_m,m}$. Combining (2.5) and (2.6) we see that

$$P_\infty^0[T_{K_m \setminus B_m} < \infty] = P_\infty^0[X_n \in K_m \setminus B_m \text{ for some } n \in \mathbb{N}] < \frac{\varepsilon}{2^m}.$$

We define $B := \{0\} \cup \bigcup_{m=1}^\infty B_m$. Then B is relatively closed in U, $0 \in B \subset U_\infty$, and

$$P_\infty^0[T_{B^c} < \infty] \leq \sum_{m=1}^\infty P_\infty^0[T_{K_m \setminus B_m} < \infty] < \varepsilon.$$

□

To finish the proof of Proposition 1.2 it now suffices to choose continuous functions $0 < s < t \le \alpha\rho$ on U such that $s = s_\infty$ and $t = t_\infty$ on $B \cup (U \setminus U_\infty)$, and to apply (1.4).

3 Appendix

As before, let $p\colon \mathbb{R}^2 \to \mathbb{R}$ be the projection $(x_1, x_2) \mapsto x_1$. Given $\alpha \in [-\pi/2, \pi/2]$, let $p_\alpha\colon \mathbb{R}^2 \to \mathbb{R}$ denote the projection parallel to the line L_α containing the origin and having slope $\tan\alpha$ so that $p_{-\pi/2} = p_{\pi/2} = p$. Let λ^1 denote Lebesgue measure on $\mathbb{R}$. The following result is a special case of Théorème 1 in [9]. For the convenience of the reader we give a direct proof (using a slightly different construction).

PROPOSITION 3.1. *Let $a, b \in \mathbb{R}$, $a < b$. There exists a compact subset K of $[0,1] \times [a,b]$ such that $p(K) = [0,1]$, but $\lambda^1(p_\alpha(K)) = 0$ for every $\alpha \in (-\pi/2, \pi/2)$.*

Proof. By induction, we shall construct a decreasing sequence (K_m) of finite unions of closed rectangles such that $K := \bigcap_{m=1}^\infty K_m$ has the desired properties.

To begin with let $\varepsilon_1 := \min(1, b-a)$ and $K_1 := [0,1] \times [a, a+\varepsilon_1]$. We now suppose that $m \in \mathbb{N}$, $0 < \varepsilon_m \le 1/m$, and that K_m is a union of rectangles

$$R_n := \big[\frac{n-1}{N}, \frac{n}{N}\big] \times \big[y_n, y_n + \frac{\varepsilon_m}{N}\big], \qquad 1 \le n \le N,$$

where $y_1, \dots, y_N \in \mathbb{R}$.

We fix $1 \le n \le N$, define $P_{0,1} = R_n$, and choose $k \in \mathbb{N}$ such that $k \ge 2m/\varepsilon_m$. Starting with $P_{0,1}$ we construct parallelograms $P_{i,j}$, $1 \le i \le k$, $1 \le j \le 2^i$, as indicated by Figure 1.

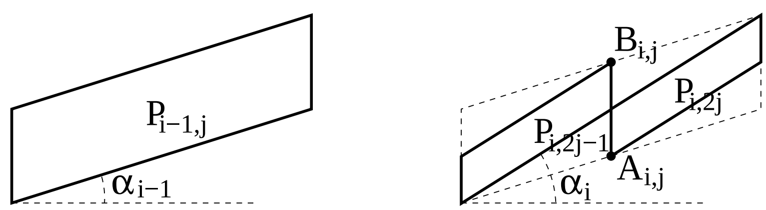

Fig. 1. From $P_{i-1,j}$ to $P_{i,2j-1}$ and $P_{i,2j}$

It follows immediately from the construction that the two intervals $p(P_{i,2j-1})$ and $p(P_{i,2j})$ have length $2^{-i}/N$ and that $|A_{i,j} - B_{i,j}| = 2^{-i}\varepsilon_m/N$. Therefore

$$\tan\alpha_i = \frac{\varepsilon_m}{2} + \tan\alpha_{i-1}. \tag{3.1}$$

Let us consider $\alpha \in [\alpha_{i-1}, \alpha_i]$. It is easily seen that lines $x + L_\alpha$, $x \in \mathbb{R}^2$, which do not intersect the segment $[A_{i,j}, B_{i,j}]$, do not intersect $P_{i,2j-1} \cup P_{i,2j}$. This shows that

$$\lambda^1(p_\alpha(P_{i,2j-1} \cup P_{i,2j})) \le \lambda^1(p_\alpha([A_{i,j}, B_{i,j}]) \le 2^{-i}\,\frac{\varepsilon_m}{N}.$$

So the union P_i of all $P_{i,j}$, $1 \le j \le 2^i$, satisfies $\lambda^1(p_\alpha(P_i)) \le \varepsilon_m/N$.

By (3.1), $\tan\alpha_k = k\varepsilon_m/2 \geq m$. Since $R_n \supset P_1 \supset P_2 \supset \cdots \supset P_k$, we obtain that

$$\lambda^1(p_\alpha(P_k)) \leq \frac{\varepsilon_m}{N} \quad \text{for all } 0 \leq \alpha < \pi/2 \text{ with } \tan\alpha \leq m. \tag{3.2}$$

Next we choose a suitable multiple N' of $2^k N$, an $\varepsilon_{m+1} \in (0, 1/(m+1))$ which is sufficiently small, and replace each of the parallelograms $P_{k,j}$, $1 \leq j \leq 2^k$, by a subset $Q_{k,j}$ which is a union of rectangles of the form $[s, s+1/N'] \times [t, t+\varepsilon_{m+1}/N']$ as indicated by Figure 2.

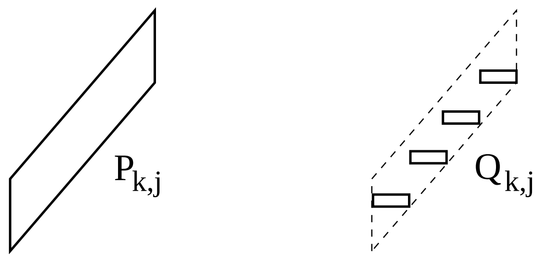

Fig. 2. From $P_{k,j}$ to $Q_{k,j}$

The union of all $Q_{k,j}$, $1 \leq j \leq 2^k$, is a subset Q_n of of P_k, hence a subset of R_n which, by (3.2), satisfies $\lambda^1(p_\alpha(Q_n)) \leq \varepsilon_m/N$. So the union $T(K_m)$ of all Q_n, $1 \leq n \leq N$, satisfies

$$\lambda^1(p_\alpha(T(K_m)) \leq N \cdot \frac{\varepsilon_m}{N} \leq \frac{1}{m} \quad \text{for all } 0 \leq \alpha < \pi/2 \text{ such that } \tan\alpha \leq m. \tag{3.3}$$

If m is odd, let $K_{m+1} := T(K_m)$. If m is even, we take $K_{m+1} := S(T(S(K_m)))$, where S denotes the reflection at the line $\{x_1 = 1/2\}$, that is, $S(x_1, x_2) = (1-x_1, x_2)$. Then (3.3) implies that $\lambda^1(p_\alpha(K_{m+1})) \leq 1/m$ for all $\alpha \in (-\pi/2, 0]$ with $\tan\alpha \geq -m$.

We may now continue our construction with K_{m+1} in place of K_m. Obviously, $K := \bigcap_{m=1}^\infty K_m$ will then be a compact set in $[0,1] \times [a,b]$ such that $p(K) = [0,1]$ and $\lambda^1(p_\alpha(K)) = 0$ for all $\alpha \in (-\pi/2, \pi/2)$. □

The following consequence of Proposition 3.1 slightly improves Proposition 2 in [9] (where no closure of the union of the circles is taken).

COROLLARY 3.2. *For all $a > 0$ and $\varepsilon > 0$, there exists an upper semicontinuous function $r \colon \mathbb{R} \to [a, a+\varepsilon]$ such that*

$$\lambda^2\Big(\overline{\bigcup_{t\in\mathbb{R}} \partial B((t,0), r(t))}\Big) = 0.$$

Proof. By scaling, we may assume without loss of generality that $a = 1$. Moreover, it suffices to consider $\varepsilon \in (0,1)$. By Proposition 3.1, there exists a compact subset K of $[0,1] \times [1, 1+\varepsilon]$ such that $p(K) = [0,1]$ and

$$\lambda^1((s,1) \cdot K) = 0 \qquad \text{for every } s \in \mathbb{R}, \tag{3.4}$$

where $\cdot$ denotes the scalar product. Let

$$A := \{(x_1, x_2) \in \mathbb{R}^2 \colon x_1^2 + x_2^2 \in (2x_1, 1) \cdot K\}.$$

Obviously, A is closed and, by Fubini's theorem and (3.4), $\lambda^2(A) = 0$.

For every $t \in [0, \varepsilon]$, let

$$\varphi(t) := \sup\{u \in [1, 1+\varepsilon] \colon (t, u) \in K\}, \qquad r_\varepsilon(t) := \sqrt{\varphi(t) + t^2}.$$

Then the graph of φ is contained in K, the functions φ and r_ε are upper semicontinuous[1], and

$$1 \le r_\varepsilon \le \sqrt{1+\varepsilon+\varepsilon^2} \le 1+\varepsilon.$$

If $t \in [0, \varepsilon]$ and $(x_1, x_2) \in \partial B((t, 0), r_\varepsilon(t))$, then

$$\begin{aligned} x_1^2 + x_2^2 &= x_1^2 + r_\varepsilon(t)^2 - (x_1 - t)^2 = \varphi(t) + 2x_1 t \\ &= (2x_1, 1) \cdot (t, \varphi(t)) \in (2x_1, 1) \cdot K. \end{aligned}$$

Thus A contains all circles $\partial B((t, 0), r_\varepsilon(t))$, $t \in [0, \varepsilon]$. To finish the proof we extend $r_\varepsilon|_{[0,\varepsilon)}$ periodically to $\mathbb{R}$, if $r_\varepsilon(0) \ge r_\varepsilon(\varepsilon)$. Otherwise, we extend $r_\varepsilon|_{(0,\varepsilon]}$. □

REMARK 3.3. Finally, let us note that Théorème 1 in [9] is much stronger than Proposition 3.1. It states that, for every upper semi-continuous real function $\varphi \ge 0$ on $[-\pi/2, \pi/2]$ with $\varphi(-\pi/2) = \varphi(\pi/2)$, there exists a compact set K in $\mathbb{R}^2$ such that $\lambda^1(p_\alpha(K)) = \varphi(\alpha)$ for every $\alpha \in [-\pi/2, \pi/2]$ (the converse is trivial).

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[1] Replacing sup by inf we would obtain a lower semicontinuous function.