GENERATING FUNCTIONS OF CAUCHY-STIELTJES TYPE FOR ORTHOGONAL POLYNOMIALS

Marek Bozejko[1] and Nizar Demni [2]

ABSTRACT. We give a free probabilistic interpretation of the multiplicative renormalization method. As a byproduct, we give a short proof of the Asai-Kubo-Kuo problem on the characterization of the family of measures for which this method applies with $h(x) = (1-x)^{-1}$ which turns out to be the free Meixner family. We also give a representation for the Voiculescu transform of all free Meixner laws (even in the non-freely infinitely divisible case). The paper closes with some comments on both classical and quantum cases as well as comments on some ways to investigate the case $h(x) = (1-x)^{-\beta}$.

1. Multiplicative renormalization method

The theory of orthogonal polynomials has seen a significant growth during the last century due to their connections with other areas such as approximation theory, quantum probability, stochastic processes (see [2], [26]). Nevertheless, for a given measure, the computation of the corresponding family of orthogonal polynomials remains a quite hard and often even impossible task by the use of Gram-Schmidt orthogonalization method (consider for instance the geometric distribution). This was the starting point and the motivation of some reserachers to find out a more subtle method to derive them. In [8], the authors propose the so-called *multiplicative renormalization* method based on generating functions. More precisely, let μ an infinitely supported measure on the real line with finite all order moments. Then, the family $(P_n)_{n\geq 0}$ of orthogonal *monic* polynomials with respect to μ satisfies the three terms-recurrence relation

$$xP_n(x) = P_{n+1}(x) + \alpha_n P_n(x) + \omega_n P_{n-1}(x)$$

where $\alpha_n \in \mathbb{R}, \omega_0 = 1, \omega_n > 0$ for all $n \geq 1$. $(\alpha_n)_n, (\omega_n)_n$ are called the Jacobi-Szegö parameters. μ is symmetric if and only if $\alpha_n = 0$ for all n. A *pre-generating function* for μ is an infinite series of the form

$$\phi(z,x) := \sum_{n\geq 0} S_n(x) z^n$$

where

- $S_n(x)$ is a polynomial of degree n for each $n \geq 0$.
- $\limsup_{n\rightarrow\infty} ||S_n||_{L^2(\mu)} < \infty$.

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[1]Instytut Matematyczny, Uniwersytet Wroclawski, Pl. Grunwaldzki 2/4, 50-384 Wroclaw, Poland.

[2]Faculty of Mathematics, Bielefeld University, Germany.

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When $(S_n)_{n\geq 0}$ are orthogonal with respect to μ, then $(z,x)\mapsto \phi(z,x)$ is said to be a *generating function*. Else, its *multilplicative renormalization* is defined by

$$\psi(z,x) := \frac{\phi(z,x)}{\mathbb{E}(\phi(z,X))} = \sum_{n\geq 0} Q_n(x) z^n$$

where X is a random variable with law μ and $\mathbb{E}_\mu$ denotes the corresponding expectation. In order to decide whether a given function ψ is a generating function for a pre-given measure, Asai-Kuo-Kubo ([6],[7],[8],[9]) derived the following criterion:

Theorem 1.1. *ψ is a generating function for μ if and only if $\mathbb{E}(\psi(z,X)\psi(v,X))$ only depends on zv.*

In that case, one writes $Q_n(x) = a_n P_n(x)$, the coefficients (a_n) being determined by

$$\lim_{z\to 0} \psi(z, \frac{x}{z}) = \sum_{n\geq 0} a_n x^n \tag{1}$$

When considering most of classical orthogonal polynomials, the authors distinguished:

$$\phi(z,x) = e^{\rho(z)x}, \quad \phi(z,x) = (1-\rho(z)x)^{-\beta},\ \beta > 0$$

for a suitable function ρ (analytic around 0, $\rho(0) = 0$ and $\rho'(0)\neq 0$). These functions of the form $h(\rho(z)x)$ for some function h so that when Theorem (1.1) holds, we say that the multiplicative renormalization method applies with a function h. In [22],[23], the authors adressed the following question: characterize the family of measures applicable with $h(x) = (1-x)^{-1}$. The Jacobi-Szegö parameters are those of free Meixner laws studied in [4] and [13]. In addition, a two-parameter measure family which include the well known Wigner and Arcsinus measures was derived as an example(see [22], [23]). However, the latter appeared many times since 1930: a particular case in Geronimus work then in [20] when dealing with random walks on finitely generated groups, in the early 80's in connection with large regular graphs ([24]) and finally in free probability as the free binomial distribution (with some rescaling , [12], [13]) and as the limiting distribution of the multivariate Beta distribution or the law of the stationary free Jacobi process (compression by a free projection, [15], [14], [16]). Moreover, computations are quite tedious and heavy. That is why we will investigate an anlogue within the scope of free probability by means of Cauchy-Stieljes and Voiculescu transforms and other related functions and continued fractions. The paper is organized as follows :
we first analyze some examples and show how these are close to free probability. Then, we prove the main result : the multiplicative renormalization method applies with $h(x) = (1-x)^{-1}$ if and only if μ is a free Meixner law. When translated to the free probability language, the second statement of the above Theorem provides an easy and short proof of the representation of the Voiculescu transforn of a free infinitely divisible Meixner law, already proved in [13]. Besides, it gives a representation for the remaining, not freely infinitely divisible, Meixner distribution. When $h(x) = e^x$, it gives a representation of the classical cumulant generating function which was already proved for classical Meixner laws ([21], [26]). We close the paper by some comments on the quantum case and some open questions concerning the case $h(x) = (1-x)^{-\beta}, \beta > 0$ and on how does this method should be interpreted in a non-analytic way.

2. Free probability and contiuned fractions

In this section, we will use continued fractions to show that both families considered in [22], [23] are only a perturbation of the first row of the continued fraction of the semi circle law with mean zero and variance 1/4. We also justify how some equations that appeared in [22], [23] are related with the G-transform of this semicircle law. The short analysis we give below not only inspired us for the results of the previous section but can be seen as a new way to explore more difficult situations (ultraspherical polynomials). To proceed, let us recall some definitions used in free probability : the Cauchy-Stieltjes transform of a measure μ is defined by

$$G_\mu(z) := \int_{\mathbb{R}} \frac{1}{z-x}\mu(dx)$$

It maps the open upper half plane $\mathbb{C}_+$ into the open lower half plane $\mathbb{C}_-$ and it is analytic there. Let $F_\mu := 1/G_\mu$, then it was shown in [10], [11] that F_μ is one-to-one in some neighborhood of infinity. The Voiculescu transform is then defined as

$$\phi_\mu(z) = F_\mu^{-1}(z) - z$$

where F_μ^{-1} is the right inverse of F_μ. Some other normalizations are used such as the *free-cumulant generating function* defined by

$$R_\mu(z) := \phi_\mu(1/z) = K_\mu(z) - \frac{1}{z}, \quad K_\mu(z) := F_\mu^{-1}(1/z),$$

Suppose that μ is compactly-supported, its continued fraction expansion is written (see [19])

$$\frac{1}{G_\mu(z)} = z - \alpha_0 - \cfrac{w_1}{z-\alpha_1 - \cfrac{\omega_2}{z-\alpha_2-\cfrac{\omega_3}{z-\dots}}} \tag{2}$$

When $\alpha_i = 0, i \geq 1$ and $\omega_i = \omega_{i+1}, i \geq 2$, this is equal to $\omega_1 G_{\nu_s}$ where ν_s is a centered semi-circle law of variance $s = \omega_2$, that is

$$\nu_s(dx) = \frac{1}{2\pi\omega_2}\sqrt{4\omega_2 - x^2}\mathbf{1}_{[-2\sqrt{\omega_2}, 2\sqrt{\omega_2}]}(x)dx$$

If besides $\omega_i = 1/4, i \geq 1$, then

$$\frac{1}{G_\mu(z)} = z - \alpha_0 - w_1 G_{\nu_{1/4}}(z), \alpha_0 \in \mathbb{R}, \, \omega_1 > 0. \tag{3}$$

Next, consider the first family of measures (Kesten law) given by

$$\mu(dx) = \frac{a\sqrt{1-x^2}}{\pi[a^2 + (1-2a)x^2]}\mathbf{1}_{[-1,1]}(x)dx, \quad 0 < |a| \leq 1.$$

corresponds to $\alpha_i = 0$ for all $i \geq 0$ and $\omega_1 = a/2, \omega_i = 1/4$ for all $i \geq 2$. The second one is more general and was considered in [23]

$$\mu(dx) = \frac{c\sqrt{1-x^2}}{\pi[b^2 + c^2 - 2b(1-c)x + (1-2c)x^2]}\mathbf{1}_{[-1,1]}(x)dx, \quad |b| < 1-c, \, 0 < c \leq 1. \tag{4}$$

Then $\alpha_0 = b, \alpha_i = 0, i \geq 1$ and $\omega_1 = c/2, \omega_i = 1/4, i \geq 2$. Hence, both measures share (3) whence we deduce that Theorem (1.1) is satisfied for some suitable $\rho(z)$. Indeed, we only have to check if (3) depends only on uv for $\nu = \nu_1$. Even more, we claim that this is true if ν_1 is replaced by ν_s. The use of $\nu_{1/4}$ is justified by the fact that μ is supported in $[-1,1]$ and the use of μ_s will provide measures supported in $[-2\sqrt{s}, 2\sqrt{s}]$. Besides, the two freedom degrees $\alpha_0 \in \mathbb{R}, \omega_1 > 0$ code the measure so that the above restrictions on b and c can be improved. To find the equation satisfied by ρ, let us first translate Theorem (1.1) using the Cauchy-Stieltjes transform :

$$\psi(z,x) = \frac{\rho(z)(1-\rho(z)x)^{-1}}{G_\mu(1/\rho(z))}, \quad (5)$$

$$\mathbb{E}(\psi(z,X)\psi(v,X)) = \frac{\rho(z)\rho(v)}{\rho(z)-\rho(v)}\left[\frac{1}{G_\mu(1/\rho(v))} - \frac{1}{G_\mu(1/\rho(z))}\right] \quad (6)$$

Moreover, (2) gives

$$\frac{1}{G_\mu(z)} = z - \alpha_0 - \omega_1 G_{\nu_{1/4}}(z)$$

From

$$G_{\nu_s}(z) = \frac{z - \sqrt{z^2-4s}}{2s}$$

then, for real z, v, the statement (6) only depends on zv is equivalent to

$$\frac{\rho(z)+\rho(v)}{|\rho(v)|\sqrt{1-\rho^2(z)} + |\rho(z)|\sqrt{1-\rho^2(v)}}$$

only depends on zv. The above expression is very similar to the one derived in [22] p. 367 and one can use similar arguments used in [8] to see that $\rho(z) = 2z/(1+z^2)$. Note also that $1/\rho(z) = K_{\nu_{1/4}}(2z)$ so that $G_{\nu_{1/4}}(1/\rho(z)) = 2z$ which will be the main ingredient for proving our main result.

3. Characterization : free Meixner laws

Using the injectivity of G_μ in some neighborhood of infinity, say $V(\infty)$, a function ρ satisfying the above-mentioned properties defines a new function in some neighborhood of zero, $V(0)$ by

$$1/g(z) := F_\mu(1/\rho(z)) \quad \Leftrightarrow \quad 1/\rho(z) = F_\mu^{-1}(1/g(z)) = K_\mu(g(z))$$

Using the relation

$$F_\mu(z) = 1 + \phi_\mu(F_\mu(z)) \quad (\text{or} \quad G_\mu(z) = \frac{1}{1+R_\mu(G_\mu(z)})$$

Theorem (1.1) then translates to

Theorem 3.1. *Let $g : V(0) \mapsto V(0)$ analytic with $g(0) = 0$ and $g'(0) \neq 0$. Then*

$$\frac{1}{g(z)(K_\mu(g(z)) - x)}$$

is a generating function of μ if and only if

$$\frac{\phi_\mu(1/g(z)) - \phi_\mu(1/g(v))}{1/g(z) - 1/g(v)}, \quad z, v \in V(0)$$

only depends on zv.

Now, let us notice that (1) implies in this case that $a_0 = 1, a_n = [g'(0)]^n, n \geq 1$. Indeed, this is easily seen when one expands $K_\mu(z) = (1/z)+$ entire function for $z \in V(0)$. As a result

$$\frac{1}{g(z)(K_\mu(g(z)) - x)} = \sum_{n\geq 0}[g'(0)]^n P_n(x)z^n$$

if and only if

$$\sum_{n\geq 0} P_n(x)z^n = \frac{1}{g(z/g'(0))(K_\mu(g(z/g'(0)) - x)} := \sum_{n\geq 0} \frac{1}{u(z)(f(z) - x)}$$

The last generating function was studied in [4] whence our main result follows:

Theorem 3.2. *The multiplicative renormalization method applies for μ with $h(x) = (1-x)^{-1}$ if and only if μ is a free Meixner law.*

Proof : this follows from Lemma 2 p. 243 in [4] and the continued fraction:

$$\frac{1}{G_\mu(z)} = z - \alpha_0 - \cfrac{w_1}{z - \alpha - \cfrac{\omega}{z - \alpha - \cfrac{\omega}{z - \dots}}}$$

for some $\alpha_0, \alpha \in \mathbb{R}$ and $\omega_1, \omega > 0$. ■

Remark 3.1. *In* [13], *authors call free Meixner family the five free Meixner laws given in* [4] *with the measure given by (4) called free Binomial distribution since it is the n-th free convolution product of freely independent free projections (free Bernoulli distributions). All of them are standard, the five first-mentioned laws are freely infinitely divisible while the latter is not. Moreover, any perturbation of the second row of the continued fraction of G_μ will provide a measure which goes beyond Theorem (1.1).*

3.1. **Representations of Voiculescu transforms.** One deduces from [4] (see the proof p. 236) that

$$u(z) := g(z/g'(0)) = \frac{z}{1 + (\alpha - \alpha_0)z + (\omega - \omega_1)z^2}, \quad g(z) = G_\mu(1/\rho(z)).$$

Let $\alpha = \alpha_0 + \sqrt{\omega_1}a, \omega = \omega_1(1+b)$, for $a \in \mathbb{R}$ and $b \geq -1$. Notice that the second assertion in Theorem (3.1) remains true when replacing g bu u. Recall also that ([13], p. 62), if $\mu_{a,b}$ is a standard free infinitely divisible Meixner distribution $(\tilde{\alpha}_0 = 1, \tilde{\omega}_1 = 1)$ of parameters a, b and $\omega_{a,b}$ is a semi-circle law of mean a and variance b, then

$$\phi_\mu(z) = \int \frac{1}{z-x}\omega_{a,b}(dx) = G_{\omega_{a,b}}(z)$$

so that the ratio in Theorem (3.1) depends only on zv. Conversely, let $\mu_{a,b}$ be a standard free Meixner distribution (not necessarily free infinitely divisible) of parameters a, b so that

$$\frac{\phi_\mu((bz^2 + az + 1)/z) - \phi_\mu((bv^2 + av + 1)/v)}{z - v}$$

depends only on zv. Taking $v = 0$, since $\phi_\mu(\infty) = R_\mu(0) = R_1 = 0$ (mean), then one writes

$$\phi_\mu((bz^2 + az + 1)/z) = Cz, \quad z \in V(0)$$

Writing $\phi(z) = R(1/z)$ and since $R_\mu(y)/y \to R_2 = 1\,(R_1 = 0)$ as $y \to 0$, then it follows by letting $z \to 0$ that $C = 1$. Set $yz = bz^2 + az + 1$, then for $b \geq 0$,

$$z = \frac{1}{y - a - bz} = \frac{1}{y - a - \dfrac{b}{y - a - bz}} = G_{\omega_{a,b}}(y).$$

Equivalently, one writes $s(z) - 1/z := 1/u(z) - 1/z = bz + a = R_{\omega_{a,b}}(z)$ so that $1/u(z) = K_{\omega_{a,b}}(z)$. When $-1 \leq b < 0$, then s maps $\mathbb{R}$ to $\mathbb{R}$ so that one can not hope a representation as the G_ξ transform of some probability measure ξ on the real line. However, one can write

$$K_\mu(u(z)) = F_\mu^{-1}(s(z)) = \phi_\mu(s(z)) + s(z) = \frac{(1+b)z^2 + az + 1}{z} = K_{\omega_{a,1+b}}(z)$$

Besides, u is a bijection in some neighborhood of 0 and it follows from the injectivity of the Cauchy-Stieltjes transform that

$$u^{-1}[G_\mu(z)] = G_{\omega_{a,1+b}}(z) \Leftrightarrow G_\mu(z) = u[G_{\omega_{a,1+b}}(z)] = u(\phi_\eta(z)),$$

in some neighborhood of ∞, where η is a Meixner law of parameters $a \in \mathbb{R}, 1 + b \in]0,1[$. Finally,

$$\frac{1}{\rho(z/g'(0))} = f(z) = K_\mu(u(z)) = \frac{(b+1)z^2 + az + 1}{z}$$

Remark 3.2. *There are similar classifications and representations of the classical cumulant generating function for generating functions of exponential type (see* [21]*). For more details see* [26].

4. Concluding remarks

The q-deformed case ($0 \leq q < 1$) was investigated in [5] where q-Meixner laws are orthogonality measures of the so-called Al-Salam- Chihara polynomilas defined in [3]. For detials see [25] for explicit expressions of q-Meixner distributions. Another interesting, however complicated, case corresponds to ultraspherical polynomials defined by

$$\sum_{n\geq 0} C_n^\beta(x) z^n = \frac{1}{(1 - 2zx + z^2)^\beta}, \quad \beta > 0. \tag{7}$$

On one hand, it fits into the multiplication method applications with $h(x) = (1 - x)^{-\beta}$ and on the other hand, it can be viewed as another deformation of the free case. However, while we were thinking that this is different from the q-deformation, we discovered that it is also a limiting case of the q-ultraspherical polynomials studied in [1] : these are defined by their generating function :

$$\sum_{n\geq 0} C_n(\cos\theta; \gamma|q) z^n = \frac{(\gamma z e^{i\theta}; q)_\infty (\gamma z e^{-i\theta}; q)_\infty}{(z e^{i\theta}; q)_\infty (z e^{-i\theta}; q)_\infty}, \, x = \cos\theta,$$

where $(a;q)_\infty = \prod_{k\geq 0}(1 - aq^{k-1})$. When $\gamma = q^\beta$ and letting $q \to 1^-$, one recovers (7). More intriguing is the fact that $q = 0$ specializes to the Kesten law. It is worth noting that the generating function of $C_n(\cos\theta; \gamma|q)$ does not cover all the free Meixner laws since the orthogonality measure is symmetric which is not the case, for instance, for the free Poisson distribution. However, this generating function is a particular case of those introduced by Fejer [18] which are of the form $|f(re^{i\theta})|^2$ for some function f around zero. From a continued fraction point of view, the G-transform of the orthogonality measure of $(C_n^\beta)_n$-called symmetric Beta-distribution-is a full perturbation of the Wigner law $\nu_{1/4}$, since

$$\omega_n = \frac{n(n+2\beta-1)}{4(n+\beta)(n+\beta-1)} = \frac{1}{4}\left[1+\beta(1-\beta)\left[\frac{1}{n+\beta-1} - \frac{1}{n+\beta}\right]\right],\ n \geq 1.$$

The last decomposition suggests the classification of measures which Jacobi-Szegö parameters ω_n, $n \geq 1$ factorizes as $c_1 + c_2(r_{n-1} - r_n)$ for some suitable sequence $(r_n)_n$. Another way of investigation of this case is Harmonic analysis, since $(C_n^\beta)_n$ are eigenfunctions of a particular Jacobi operator obtained as the radial part of the projection of the Laplace-Beltrami operator on spheres. Our last comments concerns the form of the generating function $\psi(z,x) = r(z)f(\rho(z)x)$ for some nice functions r, ρ, f. It is obvious that then $r(z) = 1/\mathbb{E}(\phi(z,X))$. One can also easily see that this function satisfies the p. d. e.

$$\psi'_z(z,x) = x\frac{\rho'(z)}{\rho(z)}\psi'_x(z,x) + \frac{r'(z)}{r(z)}\psi(z,x)$$

whence we deduce that

$$\frac{\psi'_z(z,0)}{\psi(z,0)} = \frac{r'(z)}{r(z)}, \quad z \in V(0),\ \psi(0,0) = 1$$

The first crucial question is how is this p.d.e. related to orthogonal polynomials? In orther words, under which assumptions can we start from such an equation, or may be an equivalent form of it, to write ψ as a generating function of some measure μ? This will be the topic of further future investigations. The second one is phrased as follows : we have already seen that ρ is closely related to cumulant generating functions in both classical and free case (see [21] for the former). This means that it is related to convolution structures. The problem of defining such operation in the q-deformed case is still open.

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