ORBIT CLOSURES OF DIRECTING MODULES ARE REGULAR IN CODIMENSION ONE

GRZEGORZ BOBIŃSKI

Abstract. We show that the orbit closure of a directing module is regular in codimension one. In particular, this result gives information about a distinguished irreducible component of a module variety.

Throughout the paper k is a fixed algebraically closed field. By $\mathbb{Z}$, $\mathbb{N}$, and $\mathbb{N}_+$, we denote the sets of integers, nonnegative integers, and positive integers, respectively. If $i, j \in \mathbb{Z}$, then $[i, j]$ denotes the set of all $l \in \mathbb{Z}$ such that $i \leq l \leq j$.

Introduction and the main result

Given a finite-dimensional k-algebra Λ and an element $\mathbf{d}$ of the Grothendieck group $K_0(\Lambda)$ of the category of Λ-modules, one defines the variety $\operatorname{mod}_\Lambda^{\mathbf{d}}(k)$ of Λ-modules of dimension vector $\mathbf{d}$. A product $\mathrm{GL}_{\mathbf{d}}(k)$ of general linear groups acts on $\operatorname{mod}_\Lambda^{\mathbf{d}}(k)$ in such a way that the $\mathrm{GL}_{\mathbf{d}}(k)$-orbits correspond to the isomorphism classes of Λ-modules of dimension vector $\mathbf{d}$. A study of properties of the module varieties is an interesting and important direction of research in the representation theory of algebras (for some reviews of results see [13, 17, 19] and for some more specific results see [3, 6, 8]). In particular, the author has showed [5] that if $\mathbf{d}$ is the dimension vector of a directing module and Λ is tame, then $\operatorname{mod}_\Lambda^{\mathbf{d}}(k)$ is normal if and only if it is irreducible. In general, if M is a directing module, then the closure $\overline{\mathcal{O}(M)}$ of the $\mathrm{GL}_{\mathbf{d}}(k)$-orbit $\mathcal{O}(M)$ of M is an irreducible component of $\operatorname{mod}_\Lambda^{\mathbf{d}}(k)$. Thus the above result naturally rises a question about the properties of $\overline{\mathcal{O}(M)}$. We also note that $\overline{\mathcal{O}(M)}$ coincides with the closure of the set modules of projective (injective) dimension at most 1, which is known to be an irreducible component of a module variety in many other cases (see [3, Proposition 3.1] for details).

The question about properties of $\overline{\mathcal{O}(M)}$ for a directing module M is a special case of another geometric problem investigated in representation theory of finite-dimensional algebras, namely, study of properties of orbit closures in module varieties (see for example [4, 9, 12, 25, 26]). In particular, Zwara and the author proved [10] (using, among other things, the results of [7]) that if M is an indecomposable directing module, then $\overline{\mathcal{O}(M)}$ is a normal variety. Recall that normal varieties are regular in codimension one, i.e. the set of singular points is of codimension at least two. Thus, the following main result of the paper is the first step in order to generalize the above result about the closures of the indecomposable directing modules over the tame algebras to arbitrary directing modules over arbitrary algebras.

Main Theorem. *If M is a directing module, then $\overline{\mathcal{O}(M)}$ is regular in codimension one.*

The paper is organized as follows. In Section 1 we recall definitions of quivers and their representations. We also describe properties of directing modules needed it the proof of our main result. In Section 2 we discuss interpretations of extension groups useful in geometric investigations. Next, in Section 3, we define module schemes and some schemes connected with them. Finally, in Section 4, we proof the main result of the paper.

The main idea of the proof is the following. We first observe that each minimal degeneration N of a directing module M (i.e. a module N whose orbit is maximal in $\overline{\mathcal{O}(M)} \setminus \mathcal{O}(M)$) over an algebra Λ is of the form $N = U \oplus V$ for a short exact sequence

$$\xi : 0 \to U \to M \to V \to 0.$$

Now we use a connection between the tangent space $T_N \operatorname{mod}_\Lambda^{\mathbf{d}}(k)$ to the module variety at N and the first extension group. As a consequence, it follows that $\operatorname{Ext}_\Lambda^2(V, U)$ measures a difference between $\dim \overline{\mathcal{O}(M)}$ and $\dim_k T_N \operatorname{mod}_\Lambda^{\mathbf{d}}(k)$. On the other hand, we show for a general minimal degeneration N of M, that if

$$\xi_1 : 0 \to U \to W_1 \to U \to 0 \qquad \text{and} \qquad \xi_2 : 0 \to V \to W_2 \to V \to 0$$

are short exact sequences, then (ξ_1, ξ_2) corresponds to an element of $T_N\overline{\mathcal{O}(M)}$ if only only if the sequences $\xi_1 \circ \xi$ and $\xi \circ \xi_2$ determine the same element in $\operatorname{Ext}_\Lambda^2(V, U)$. We prove that the space of such pairs of sequences is of codimension $\dim_k \operatorname{Ext}_\Lambda^2(V, U)$ in $\operatorname{Ext}_\Lambda^1(U, U) \times \operatorname{Ext}_\Lambda^1(V, V)$, and this will finish the proof.

For a basic background on the representation theory of algebras (in particular, on tilting theory) we refer to [1]. A functorial approach to schemes in algebraic geometry used in the article is explained for example in [16].

An article was written while the author was staying at University of Bielefeld as Alexander von Humboldt Foundation fellow. The author expresses his gratitude to Professor Ringel for his hospitality and helpful discussions.

1. Preliminaries on quivers and their representations

In this section we present basic facts about quivers and their representations. We also collect facts about directing modules necessary in the proof.

By a quiver Δ we mean a finite set Δ_0 of vertices and a finite set Δ_1 of arrows together with two maps $s, t : \Delta_1 \to \Delta_0$, which assign to an arrow $\alpha \in \Delta_1$ its starting vertex s_α and terminating vertex t_α, respectively. By a path of length $n \in \mathbb{N}_+$ in Δ we mean a sequence $\sigma = \alpha_1 \cdots \alpha_n$ of arrows such that $s_{\alpha_i} = t_{\alpha_{i+1}}$ for each $i \in [1, n-1]$. We write s_σ and t_σ for s_{α_n} and t_{α_1}, respectively. Additionally, for each vertex x of Δ we introduce a path x of length 0 such that $s_x = x = t_x$.

With a quiver Δ we associate its path algebra $k\Delta$, which as a k-vector space has a basis formed by all paths in Δ and whose multiplication is induced by composition of paths. If $\rho = \lambda_1\sigma_1 + \cdots + \lambda_n\sigma_n$ for scalars $\lambda_1, \ldots, \lambda_n \in k$ and paths $\sigma_1, \ldots, \sigma_n \in x(k\Delta)y$, where $x, y \in \Delta_0$, then we put $s_\rho = y$ and $t_\rho = x$. Such ρ is called a relation in Δ if the length of σ_i is at least 2 for each $i \in [1, n]$. A set $\mathfrak{R}$ of relations is called minimal if for every $\rho \in \mathfrak{R}$, ρ does not belong to the ideal $\langle \mathfrak{R} \setminus \{\rho\} \rangle$ of $k\Delta$ generated by $\mathfrak{R} \setminus \{\rho\}$. A pair $(\Delta, \mathfrak{R})$ consisting of a quiver Δ and a minimal set of relations $\mathfrak{R}$, such that there exists $n \in \mathbb{N}_+$ with the property $\sigma \in \mathfrak{R}$ for each path σ in Δ of length at least n, is called a bound quiver. If $(\Delta, \mathfrak{R})$ is a bound quiver, then the algebra $k\Delta/\langle \mathfrak{R} \rangle$ is called the path algebra of $(\Delta, \mathfrak{R})$.

Let R be a commutative k-algebra and $(\Delta, \mathfrak{R})$ a bound quiver. By an R-representation of Δ we mean a collection $M = (M_x, M_\alpha)_{x \in \Delta_0, \alpha \in \Delta_1}$ of free R-modules M_x, $x \in \Delta_0$, of finite rank and R-linear maps $M_\alpha : M_{s_\alpha} \to M_{t_\alpha}$, $\alpha \in \Delta_1$. An R-representation M of Δ is called an R-representation of $(\Delta, \mathfrak{R})$ if $M_\rho = 0$ for all $\rho \in \mathfrak{R}$, where we put $M_x = \mathrm{Id}_{M_x}$ for $x \in \Delta_0$, $M_\sigma = M_{\alpha_1} \cdots M_{\alpha_n}$ for a path $\sigma = \alpha_1 \cdots \alpha_n$ with $\alpha_1, \ldots, \alpha_n \in \Delta_1$, and

$$M_\rho = \lambda_1 M_{\sigma_1} + \cdots + \lambda_n M_{\sigma_n}$$

for $\rho = \lambda_1\sigma_1 + \cdots + \lambda_n\sigma_n$ with scalars $\lambda_1, \ldots, \lambda_n \in k$ and paths $\sigma_1, \ldots, \sigma_n \in x(k\Delta)y$, where $x, y \in \Delta_0$. By a morphism $f : M \to N$ we mean a collection $(f_x)_{x \in \Delta_0}$ of R-linear maps $f_x : M_x \to N_x$, $x \in \Delta_0$, such that $f_{t_\alpha} M_\alpha = N_\alpha f_{s_\alpha}$ for each $\alpha \in \Delta_1$. If $\Lambda = k\Delta/\langle \mathfrak{R} \rangle$, then the category of R-representations of $(\Delta, \mathfrak{R})$ is equivalent to the full subcategory $\mathrm{mod}_\Lambda(R)$ of the category of Λ-R-bimodules formed by the bimodules M such that xM is a free R-module for each $x \in \Delta_0$ (see for example [1, Theorem III.1.6] for this statement in the case $R = k$). We will identify such Λ-R-bimodules and R-representations of $(\Delta, \mathfrak{R})$. For $M, N \in \mathrm{mod}_\Lambda(R)$ we denote by $\mathrm{Hom}_\Lambda(M, N)$ the space of homomorphisms from M to N in $\mathrm{mod}_\Lambda(R)$. Moreover, if $\dim_k R < \infty$, then $[M, N]$ denotes $\dim_k \mathrm{Hom}_\Lambda(M, N)$. Similarly, for $n \in \mathbb{N}_+$ we denote by $\mathrm{Ext}^n_\Lambda(M, N)$ the n-th extension group in the category of Λ-R-bimodules and, if $\dim_k R < \infty$, by $[M,N]^n$ the dimension of $\mathrm{Ext}^n_\Lambda(M,N)$ over k. For an R-representation M of Δ its dimension vector $\mathbf{dim}\, M \in \mathbb{N}^{\Delta_0}$ is defined by $(\mathbf{dim}\, M)_x = \mathrm{rk}_R\, M_x$ for $x \in \Delta_0$.

In the rest of this section we will work in the category $\mathrm{mod}_\Lambda = \mathrm{mod}_\Lambda(k)$ of Λ-modules for the path algebra Λ of a bound quiver $(\Delta, \mathfrak{R})$. For subcategories $\mathcal{A}$ and $\mathcal{B}$ of mod_Λ we denote by $\mathcal{A} \vee \mathcal{B}$ the additive closure of their union, i.e. the full subcategory of mod_Λ whose objects are $M \oplus N$ with $M \in \mathcal{A}$ and $N \in \mathcal{B}$.

By a path in mod_Λ we mean a sequence $X_0 \to X_1 \to \cdots \to X_n$ of nonzero maps between indecomposable Λ-modules for $n \in \mathbb{N}_+$. A Λ-module M is called directing if there exists no path of the form

$$M' \to \cdots \to \tau X \to * \to X \to \cdots \to M''$$

for an indecomposable nonprojective Λ-module X and indecomposable direct summands M' and M'' of M, where τ denotes the Auslander–Reiten translation in mod_Λ. If M is indecomposable, then M is directing if and only if there is no path in mod_Λ of the form $M \to \cdots \to M$ (see [18, Section 1, Corollary]).

A Λ-module T is called tilting if $[T,T]^1 = 0$, $\mathrm{pd}_\Lambda T \leq 1$, and there exists an exact sequence of the form

$$0 \to \Lambda \to T' \to T'' \to 0$$

for $T', T'' \in \mathrm{add}\, T$, where for a Λ-module M we denote by $\mathrm{add}\, M$ the full subcategory of mod_Λ consisting of direct sums of direct summands of M. Any tilting Λ-module T determines the torsion theory $(\mathcal{T}, \mathcal{F})$, where

$$\mathcal{F} = \mathcal{F}(T) = \{N \in \mathrm{mod}_\Lambda \mid [T,N] = 0\}$$

and

$$\mathcal{T} = \mathcal{T}(T) = \{N \in \mathrm{mod}_\Lambda \mid [T,N]^1 = 0\}.$$

An algebra Λ is called tilted, if there exists a hereditary algebra Σ and a multiplicity free tilting Σ-module T such that $\Lambda \simeq \mathrm{End}_\Sigma(T)^{\mathrm{op}}$. It is known that if Λ is tilted, then $\mathrm{gl.dim}\, \Lambda \leq 2$ and there are no oriented cycles in Δ.

Let M be a directing Λ-module. Then the support of M is convex [5, Lemma 1.1], where by the support of a Λ-module N we mean the full subquiver of Δ whose vertices are all $x \in \Delta_0$ such that $N_x \neq 0$. Moreover, a full subquiver Δ' of a quiver Δ is called convex if for each path $x_0 \to \cdots \to x_n$, $n \in \mathbb{N}_+$, in Δ with $x_0, x_n \in \Delta'_0$, $x_i \in \Delta'_0$ for all $i \in [1, n-1]$. Consequently, we may assume that considered directing modules are sincere, i.e. $M_x \neq 0$ for all $x \in \Delta_0$. If this is the case, then there exists a directing tilting Λ-module T, such that $\mathrm{add}\, M \subset \mathrm{add}\, T$ [2].

Assume now that T is a directing tilting Λ-module. We have the following properties, which are either a general statements of tilting theory or were proved by Bakke [2]. First of all, Λ is a tilted algebra. Secondly, $\mathrm{mod}_\Lambda = \mathcal{F} \vee \mathcal{T}$. Moreover, $\mathrm{pd}_\Lambda N \leq 1$ for all $N \in \mathcal{F} \vee \mathrm{add}\, T$, $\mathrm{id}_\Lambda N \leq 1$ for all $N \in \mathcal{T}$, and $[N', N''] = 0$ and $[N'', N']^1 = 0$ for all $N' \in \mathcal{T}$ and $N'' \in \mathcal{F}$. Finally, modules in $\mathrm{add}\, T$ are uniquely determined by their dimension vectors and if $[X, T] \neq 0$ and $[T, X] \neq 0$ for an indecomposable Λ-module X, then $X \in \mathrm{add}\, T$.

Now assume that Λ is tilted (more generally, $\mathrm{gl.\,dim}\, \Lambda \leq 2$ and there are no oriented cycles in Δ). We define the Euler bilinear form $\langle -, - \rangle : \mathbb{Z}^{\Delta_0} \times \mathbb{Z}^{\Delta_0} \to \mathbb{Z}$ by

$$\langle \mathbf{d}', \mathbf{d}'' \rangle = \sum_{x \in \Delta_0} d'_x d''_x - \sum_{\alpha \in \Delta_1} d'_{s_\alpha} d''_{t_\alpha} + \sum_{\rho \in \mathfrak{R}} d'_{s_\rho} d''_{t_\rho}$$

for $\mathbf{d}', \mathbf{d}'' \in \mathbb{Z}^{\Delta_0}$. It is known (see [11, 2.2]), that if M and N are Λ-modules, then

$$\langle \mathbf{dim}\, M, \mathbf{dim}\, N \rangle = [M, N] - [M, N]^1 + [M, N]^2.$$

We also have the Euler characterisitic $\chi : \mathbb{Z}^{\Delta_0} \to \mathbb{Z}$ defined by $\chi(\mathbf{d}) = \langle \mathbf{d}, \mathbf{d} \rangle$.

2. Interpretations of extension groups

Throughout this section we assume that Λ is the path algebra of a bound quiver $(\Delta, \mathfrak{R})$. Moreover, R is a commutative k-algebra. Our aim is this section is to present interpretations of extension groups, which are useful in geometric considerations.

We first present a construction investigated by Bongartz [12]. For two collections $U = (U_x)_{x \in \Delta_0}$ and $V = (V_x)_{x \in \Delta_0}$ of free R-modules of finite rank we introduce the following notation:

$$\mathbb{V}^{V,U} = \prod_{x \in \Delta_0} \mathrm{Hom}_R(V_x, U_x), \qquad \mathbb{A}^{V,U} = \prod_{\alpha \in \Delta_1} \mathrm{Hom}_R(V_{s_\alpha}, U_{t_\alpha}),$$

and

$$\mathbb{R}^{V,U} = \prod_{\rho \in \mathfrak{R}} \mathrm{Hom}_R(V_{s_\rho}, U_{t_\rho}).$$

If $M \in \mathrm{mod}_\Lambda(R)$, then we denote also by M the corresponding collection $(M_x)_{x \in \Delta_0}$ of free R-modules.

Fix $U, V \in \mathrm{mod}_\Lambda(R)$. If $Z \in \mathbb{A}^{V,U}$, then we define $Z_\rho^{V,U}$ for $\rho \in x(k\Delta)y$, where x and y run through Δ_0, in the following way: $Z_x^{V,U} = 0$ for $x \in \Delta_0$,

$$Z_\sigma^{V,U} = \sum_{i \in [1,n]} U_{\alpha_1} \cdots U_{\alpha_{i-1}} Z_{\alpha_i} V_{\alpha_{i+1}} \cdots V_{\alpha_n}$$

for a path $\sigma = \alpha_1 \cdots \alpha_n$ with $\alpha_1, \ldots, \alpha_n \in \Delta_1$, and

$$Z_\rho^{V,U} = \lambda_1 Z_{\sigma_1}^{V,U} + \cdots + \lambda_n Z_{\sigma_n}^{V,U}$$

for $\rho = \lambda_1\sigma_1 + \cdots + \lambda_n\sigma_n$ with scalars $\lambda_1, \ldots, \lambda_n \in k$ and paths $\sigma_1, \ldots, \sigma_n \in x(k\Delta)y$, where $x, y \in \Delta_0$. Observe that $Z^{V,U}_{\rho_1\rho_2} = Z^{V,U}_{\rho_1} V_{\rho_2} + U_{\rho_1} Z^{V,U}_{\rho_2}$ for all possible ρ_1 and ρ_2 with $s_{\rho_1} = t_{\rho_2}$. We define $\mathbb{Z}^{V,U}$ as the set of all $Z \in \mathbb{A}^{V,U}$ such that $Z^{V,U}_\rho = 0$ for all $\rho \in \mathfrak{R}$. For $Z \in \mathbb{Z}^{V,U}$ we define an R-representation W^Z of Δ by $W^Z_x = U_x \oplus V_x$ for $x \in \Delta_0$ and

$$W^Z_\alpha = \begin{bmatrix} U_\alpha & Z_\alpha \\ 0 & V_\alpha \end{bmatrix}, \ \alpha \in \Delta_1.$$

Then $W^Z_\rho = 0$ for all $\rho \in \mathfrak{R}$, hence $W^Z \in \operatorname{mod}_\Lambda(R)$. Moreover, we have a short exact sequence

$$\xi^Z : 0 \to U \xrightarrow{f^Z} W^Z \xrightarrow{g^Z} V \to 0$$

in $\operatorname{mod}_\Lambda(R)$, where the maps f^Z and g^Z are the canonical injection and the canonical projection, respectively. On the other hand, for every short exact sequence

$$\xi : 0 \to U \to M \to V \to 0$$

of Λ-R-bimodules, there exists (nonunique) $Z \in \mathbb{Z}^{V,U}$ such that $[\xi^Z] = [\xi]$. More precisely, the map $\mathbb{Z}^{V,U} \to \operatorname{Ext}^1_\Lambda(V, U)$, $Z \mapsto [\xi^Z]$, is a surjective R-linear map. The kernel $\mathbb{B}^{V,U}$ of this map consists of $Z \in \mathbb{Z}^{V,U}$ such that the corresponding sequence splits, i.e. there exists $h \in \mathbb{V}^{V,U}$ such that $Z_\alpha = U_\alpha h_{s_\alpha} - h_{t_\alpha} V_\alpha$ for all $\alpha \in \Delta_1$. From now on we identity $\operatorname{Ext}^1_\Lambda(V, U)$ with $\mathbb{Z}^{V,U}/\mathbb{B}^{V,U}$. Observe that

$$\dim_k \mathbb{Z}^{V,U} = [V, U]^1 - [V, U] + \dim_k R \cdot \sum_{x \in \Delta_0} \operatorname{rk}_R U_x \cdot \operatorname{rk}_R V_x$$

provided $\dim_k R < \infty$.

We also present interpretations of some other homological constructions.

Proposition 2.1. *Let $U, V, M \in \operatorname{mod}_\Lambda(R)$.*

(1) *If $h \in \operatorname{Hom}_\Lambda(U, M)$, then the homomorphism*

$$\operatorname{Ext}^1_\Lambda(V, U) \to \operatorname{Ext}^1_\Lambda(V, M), \ [\xi] \mapsto [h \circ \xi],$$

is given by

$$Z + \mathbb{B}^{V,U} \mapsto (h_{t\alpha} Z_\alpha) + \mathbb{B}^{V,M}.$$

(2) *If $h \in \operatorname{Hom}_\Lambda(M, V)$, then the homomorphism*

$$\operatorname{Ext}^1_\Lambda(V, U) \to \operatorname{Ext}^1_\Lambda(M, U), \ [\xi] \mapsto [\xi \circ h],$$

is given by

$$Z + \mathbb{B}^{V,U} \mapsto (Z_\alpha h_{s\alpha}) + \mathbb{B}^{M,U}.$$

Proof. We only prove the first assertion. The proof of the second one is dual. Fix $Z \in \mathbb{Z}^{V,U}$ and let $Z'_\alpha = h_{t_\alpha} Z_\alpha$ for $\alpha \in \Delta_1$. It follows easily that $Z'^{V,M}_\rho = h_{t_\rho} Z^{V,U}_\rho = 0$ for all $\rho \in \mathfrak{R}$, hence $Z' \in \mathbb{Z}^{V,M}$. We define $h' \in \operatorname{Hom}_\Lambda(W^Z, W^{Z'})$ by

$$h'_x(u) = h_x(u),\ u \in U_x, \qquad \text{and} \qquad h'_x(v) = v,\ v \in V_x,$$

for $x \in \Delta_0$. Since $f^{Z'} h = h' f^Z$ and $g^{Z'} h' = g^Z$ (recall that $W^Z_x = U_x \oplus V_x$ and $W^{Z'}_x = M_x \oplus V_x$ for all $x \in \Delta_0$, f^Z and $f^{Z'}$ are the canonical injections, while g^Z and $g^{Z'}$ are the canonical projections), $[\xi^{Z'}] = [h \circ \xi^Z]$, and this finishes the proof. $\square$

We would like to have an analogous interpretation of second extension groups and the homomorphisms

$$\operatorname{Ext}^1_\Lambda(V,U) \to \operatorname{Ext}^2_\Lambda(V,M),\ [\xi] \mapsto [\xi_1 \circ \xi],$$

and

$$\operatorname{Ext}^1_\Lambda(V,U) \to \operatorname{Ext}^2_\Lambda(M,U),\ [\xi] \mapsto [\xi \circ \xi_2],$$

for $[\xi_1] \in \operatorname{Ext}^1_\Lambda(U,M)$, $[\xi_2] \in \operatorname{Ext}^1_\Lambda(M,V)$, and Λ-modules U, V, M.

We first present an interpretation of second extension groups. For $N \in \operatorname{mod}_\Lambda(R)$ we define R-representations P^N and Ω^N of $(\Delta, \mathfrak{R})$ and homomorphisms $f^N : \Omega^N \to P^N$ and $g^N : P^N \to N$ such that P^N is projective in $\operatorname{mod}_\Lambda(R)$ and the sequence

$$\xi^N : 0 \to \Omega^N \xrightarrow{f^N} P^N \xrightarrow{g^N} N \to 0$$

is exact, in the following way:

$$P^N_x = \bigoplus_{y \in \Delta_0} x\Lambda y \otimes N_y,\ x \in \Delta_0,$$

$$\Omega^N_x = \bigoplus_{y \in \Delta_0} x(\operatorname{rad}\Lambda)y \otimes N_y,\ x \in \Delta_0,$$

$$P^N_\alpha(\sigma \otimes n) = \alpha\sigma \otimes n,\ \alpha \in \Delta_1,\ \sigma \in s_\alpha \Lambda y,\ n \in N_y,\ y \in \Delta_0,$$

$$\Omega^N_\alpha(\sigma \otimes n) = \alpha\sigma \otimes n - \alpha \otimes N_\sigma n,$$
$$\alpha \in \Delta_1,\ \sigma \in s_\alpha(\operatorname{rad}\Lambda)y,\ n \in N_y,\ y \in \Delta_0,$$

$$f^N_x(\sigma \otimes n) = \sigma \otimes n - x \otimes N_\sigma n,\ \sigma \in x(\operatorname{rad}\Lambda)y,\ n \in N_y,\ x, y \in \Delta_0,$$

$$g^N_x(\sigma \otimes n) = N_\sigma n,\ \sigma \in x\Lambda y,\ n \in N_y,\ x, y \in \Delta_0.$$

We now assume until the end of the section that $R = k$, i.e. we are working in the category $\operatorname{mod}_\Lambda$.

Proposition 2.2. *If M and N are Λ-modules, then the map*

$$\operatorname{Ext}^1_\Lambda(\Omega^N, M) \to \operatorname{Ext}^2_\Lambda(N,M),\ [\xi] \mapsto [\xi \circ \xi^N],$$

is an isomorphism.

Proof. Since P^N is projective, the claim follows by standard homological algebra (see for example [21, Chapter III]). $\square$

From now on we identify $\mathrm{Ext}^2_\Lambda(N, M)$ with $\mathrm{Ext}^1_\Lambda(\Omega^N, M)$, hence with $\mathbb{Z}^{\Omega^N,M}/\mathbb{B}^{\Omega^N,M}$, for any Λ-modules M and N.

Proposition 2.3. *Let U, V and M be Λ-modules.*

(1) *If $Z \in \mathbb{Z}^{U,M}$, then the homomorphism*

$$\mathrm{Ext}^1_\Lambda(V, U) \to \mathrm{Ext}^2_\Lambda(V, M),\ [\xi] \mapsto [\xi^Z \circ \xi],$$

is given by $Z' + \mathbb{B}^{V,U} \mapsto Z'' + \mathbb{B}^{\Omega^V,M}$, where

$$Z''_\alpha(\sigma \otimes v) = Z_\alpha Z'^{V,U}_\sigma v,\ \alpha \in \Delta_1,\ \sigma \in s_\alpha(\mathrm{rad}\,\Lambda)y,\ v \in V_y,\ y \in \Delta_0.$$

(2) *If $Z \in \mathbb{Z}^{M,V}$, then the homomorphism*

$$\mathrm{Ext}^1_\Lambda(V, U) \to \mathrm{Ext}^2_\Lambda(M, U),\ [\xi] \mapsto [\xi \circ \xi^Z],$$

is given by $Z' + \mathbb{B}^{V,U} \mapsto Z'' + \mathbb{B}^{\Omega^M,U}$, where

$$Z''_\alpha(\sigma \otimes m) = Z'_\alpha Z^{M,V}_\sigma m,\ \alpha \in \Delta_1,\ \sigma \in s_\alpha(\mathrm{rad}\,\Lambda)y,\ m \in M_y,\ y \in \Delta_0.$$

Proof. We only prove the first assertion, the prove of the second one is analogous. Fix $Z' \in \mathbb{Z}^{V,U}$ and let $Z'' \in \mathbb{A}^{\Omega^V,M}$ be defined in the way described in the proposition. One easily checks that $Z'' \in Z^{\Omega^V,M}$. We need to show that $[\xi^Z \circ \xi^{Z'}] = [\xi^{Z''} \circ \xi^V]$. In order to do this, it is enough to construct homomorphisms $h : W^{Z''} \to W^Z$ and $h' : P^V \to W^{Z'}$ such that $f^Z = hf^{Z''}$, $f^{Z'}g^Z h = h' f^V g^{Z''}$, and $g^{Z'} h = g^V$. Such maps are defined by

$$\begin{aligned} h_x(m) &= m,\ m \in M_x, \\ h_x(\sigma \otimes v) &= Z'^{V,U}_\sigma v,\ \sigma \in x(\mathrm{rad}\,\Lambda)y,\ v \in V_y,\ y \in \Delta_0, \end{aligned}$$

and

$$h'_x(\sigma \otimes v) = Z'^{V,U}_\sigma v + V_\sigma v,\ \sigma \in x\Lambda y,\ v \in V_y,\ y \in \Delta_0,$$

for $x \in \Delta_0$. $\square$

We now construct a "smaller" model of $\mathrm{Ext}^2_\Lambda(N, M)$ for $M, N \in \mathrm{mod}_\Lambda$. Let $\mathbb{B}'^{N,M}$ be the image of the map $\mathbb{A}^{N,M} \to \mathbb{R}^{N,M}$, $Z \mapsto (Z^{N,M}_\rho)$. We define also $\Phi^{N,M} : \mathbb{Z}^{\Omega^N,M} \to \mathbb{R}^{N,M}$ by

$$\Phi^{N,M}(Z)_\rho(n) = \sum_{i \in [1,l]} \sum_{j \in [1, m_i - 1]} \lambda_i M_{\alpha_{i,1} \cdots \alpha_{i,j-1}} Z_{\alpha_{i,j}}(\alpha_{i,j+1} \cdots \alpha_{i,m_i} \otimes n)$$

for $\rho \in \mathfrak{R}$ and $n \in N_{s_\rho}$, if $\rho = \lambda_1 \sigma_1 + \cdots + \lambda_l \sigma_l$ for scalars $\lambda_1, \dots, \lambda_l \in k$ and paths $\sigma_i = \alpha_{i,1} \cdots \alpha_{i,m_i} \in t_\rho(k\Delta)s_\rho$ with $\alpha_{i,1}, \dots, \alpha_{i,m_i} \in \Delta_1$, $i \in [1, l]$.

Proposition 2.4. *Assume that Λ is tilted. If $M, N \in \operatorname{mod}_\Lambda$, then $\Phi^{N,M}$ is an epimorphism such that*

$$\operatorname{Ker} \Phi^{N,M} \subset \mathbb{B}^{\Omega^N, M} \qquad \textit{and} \qquad \Phi^{N,M}(\mathbb{B}^{\Omega^N, M}) = \mathbb{B}'^{N,M}.$$

In particular, $\Phi^{N,M}$ induces an isomorphism

$$\operatorname{Ext}^2_\Lambda(N, M) \simeq \mathbb{R}^{N,M} / \mathbb{B}'^{N,M}.$$

Proof. We first show that $\operatorname{Ker} \Phi^{N,M} \subset \mathbb{B}^{\Omega^N, M}$, i.e. for $Z \in \operatorname{Ker} \Phi^{N,M}$ we construct $h \in \mathbb{V}^{\Omega^N, M}$ such that $Z_\alpha = M_\alpha h_{s_\alpha} - h_{t_\alpha} \Omega^N_\alpha$ for all $\alpha \in \Delta_1$. The definition of such h is recursive, namely

$$h_{t_\alpha}(\alpha \otimes n) = 0, \ \alpha \in \Delta_1, \ n \in N_{s_\alpha},$$

and

$$h_{t_\alpha}(\alpha\sigma \otimes n) = M_\alpha h_{s_\alpha}(\sigma \otimes n) - Z_\alpha(\sigma \otimes n),$$
$$\alpha \in \Delta_1, \ \sigma \in s_\alpha(\operatorname{rad} \Lambda) y, \ n \in N_y, \ y \in \Delta_0.$$

In order to prove that $\Phi^{N,M}(\mathbb{B}^{\Omega^N, M}) \subset \mathbb{B}'^{N,M}$, take $h \in \mathbb{V}^{\Omega^N, M}$ and let $Z_\alpha = M_\alpha h_{s_\alpha} - h_{t_\alpha} \Omega^N_\alpha$ for $\alpha \in \Delta_1$. Direct calculations show that

$$\Phi^{N,M}(Z)_\rho(n) = \sum_{i \in [1,l]} \sum_{j \in [1, m_i - 1]} \lambda_i M_{\alpha_{i,1} \cdots \alpha_{i,j-1}} h_{t_{\alpha_{i,j}}}(\alpha_{i,j} \otimes N_{\alpha_{i,j+1} \cdots \alpha_{i,m_i}} n)$$

for ρ as above (in the definition of $\Phi^{N,M}$) and $n \in N_{s_\rho}$, i.e. $\Phi^{N,M}(Z) = (Z'^{N,M}_\rho)$ for $Z' \in \mathbb{A}^{N,M}$ defined by $Z'_\alpha(n) = h_{t_\alpha}(\alpha \otimes n)$ for $\alpha \in \Delta_1$ and $n \in N_{s\alpha}$.

We now show that $\mathbb{B}'^{N,M} \subset \Phi^{N,M}(\mathbb{B}^{\Omega^N, M})$. Take $Z \in \mathbb{A}^{N,M}$. The above calculations show that if we define $h \in \mathbb{V}^{\Omega^N, M}$ by

$$h_{t_\alpha}(\alpha \otimes n) = Z_\alpha(n), \ \alpha \in \Delta_1, \ n \in N_{s_\alpha},$$

and

$$h_{t_\alpha}(\alpha\sigma \otimes n) = 0, \ \alpha \in \Delta_1, \ \sigma \in s_\alpha(\operatorname{rad} \Lambda) y, \ n \in N_y, \ y \in \Delta_0,$$

and $Z' \in \mathbb{B}^{\Omega^N, M}$ by $Z'_\alpha = M_\alpha h_{s_\alpha} - h_{t_\alpha} \Omega^N_\alpha$ for $\alpha \in \Delta_1$, then $\Phi^{N,M}(Z') = (Z^{N,M}_\rho)$.

Finally we show that $\operatorname{Im} \Phi^{N,M} = \mathbb{R}^{N,M}$. Observe that $\dim_k \mathbb{R}^{N,M} = \sum_{\rho \in R} d''_{s_\rho} d'_{t_\rho}$, where $\mathbf{d}' = \operatorname{\mathbf{dim}} M$ and $\mathbf{d}'' = \operatorname{\mathbf{dim}} N$. On the other hand,

$$\begin{aligned}
&\dim_k \operatorname{Im} \Phi^{N,M} \\
&\quad = \dim_k \mathbb{Z}^{\Omega^N, M} - \dim_k \operatorname{Ker} \Phi^{N,M} \\
&\quad = (\dim_k \mathbb{Z}^{\Omega^N, M} - \dim_k \mathbb{B}^{\Omega^N, M}) + (\dim_k \mathbb{B}^{\Omega^N, M} - \dim_k \operatorname{Ker} \Phi^{N,M}) \\
&\quad = [N, M]^2 + \dim_k \mathbb{B}'^{N,M} \\
&\quad = [N, M]^2 + \dim_k \mathbb{A}^{N,M} - \dim_k \mathbb{Z}^{N,M} \\
&\quad = [N, M]^2 + \sum_{\alpha \in \Delta_1} d''_{s_\alpha} d'_{t_\alpha} - [N, M]^1 + [N, M] - \sum_{x \in \Delta_0} d''_x d'_x
\end{aligned}$$

$$= \langle \mathbf{d}'', \mathbf{d}' \rangle + \sum_{\alpha \in \Delta_1} d''_{s_\alpha} d'_{t_\alpha} - \sum_{x \in \Delta_0} d''_x d'_x = \sum_{\rho \in R} d''_{s_\rho} d'_{t_\rho},$$

hence the claim follows. □

We remark that we could replace the assumption Λ is tilted by a more general assumption that there are no oriented cycles in Δ and $\operatorname{gl.dim} \Lambda \leq 2$. Moreover, this assumption is necessary only for proving that $\Phi^{N,M}$ is surjective, or more generally, in order to identify $\operatorname{Im} \Phi^{N,M}$. We also mention, that the above obtained description of second extension groups is a trace of a much more general construction investigated by Butler and King [14]. In particular, using their results one could identify $\operatorname{Im} \Phi^{N,M}$ in a general case.

Using the above isomorphism we can give another formulation of Proposition 2.3. This formulation could also be deduced from [14]. Let U, V and W be Λ-modules. For $Z' \in \mathbb{Z}^{V,U}$ and $Z'' \in \mathbb{Z}^{W,V}$ we define $Z' \circ Z'' \in \mathbb{R}^{W,U}$ by

$$(Z' \circ Z'')_\rho = \sum_{i \in [1,l]} \sum_{\substack{j_1, j_2 \in [1, m_i - 1] \\ j_1 < j_2}} \lambda_i U_{\alpha_{i,1} \cdots \alpha_{i,j_1 - 1}} Z'_{\alpha_{i,j_1}} V_{\alpha_{i,j_1+1} \cdots \alpha_{i,j_2-1}} Z''_{\alpha_{i,j_2}} W_{\alpha_{i,j_2+1} \cdots \alpha_{i,m_i}}$$

for $\rho \in \mathfrak{R}$, if $\rho = \lambda_1 \sigma_1 + \cdots + \lambda_l \sigma_l$ with scalars $\lambda_1, \ldots, \lambda_l \in k$ and paths $\sigma_i = \alpha_{i,1} \cdots \alpha_{i,m_i} \in t_\rho (k\Delta) s_\rho$, where $\alpha_{i,1}, \ldots, \alpha_{i,m_i} \in \Delta_1$, $i \in [1, l]$.

Proposition 2.5. *Let U, V and M be Λ-modules.*

(1) *If $Z \in \mathbb{Z}^{U,M}$, then, under the isomorphism induced by $\Phi^{V,M}$, the homomorphism*
$$\operatorname{Ext}^1_\Lambda(V, U) \to \operatorname{Ext}^2_\Lambda(V, M), \ [\xi] \mapsto [\xi^Z \circ \xi],$$
is given by $Z' + \mathbb{B}^{V,U} \mapsto Z \circ Z' + \mathbb{B}'^{V,M}$.

(2) *If $Z \in \mathbb{Z}^{M,V}$, then, under the isomorphism induced by $\Phi^{M,U}$, the homomorphism*
$$\operatorname{Ext}^1_\Lambda(V, U) \to \operatorname{Ext}^2_\Lambda(M, U), \ [\xi] \mapsto [\xi \circ \xi^Z],$$
is given by $Z' + \mathbb{B}^{V,U} \mapsto Z' \circ Z + \mathbb{B}'^{M,U}$.

Proof. Direct calculations. □

3. Module schemes

Throughout this section Λ is the path algebra of a bound quiver $(\Delta, \mathfrak{R})$. We define in this section the schemes of Λ-modules. We also investigate subschemes of products of module schemes consisting of pairs of modules with given dimensions of the homomorphism space and the first extension group.

For $\mathbf{d} \in \mathbb{N}^{\Delta_0}$ and a commutative k-algebra R, let $R^{\mathbf{d}} = (R^{d_x})_{x \in \Delta_0}$. If $M \in \mathbb{A}^{R^{\mathbf{d}}, R^{\mathbf{d}}}$, then we can treat M as an R-representation of Δ

by taking $M_x = R^{d_x}$ for $x \in \Delta_0$. In particular, M_ρ is defined for each $\rho \in x(k\Delta)y$ with $x, y \in \Delta_0$. Obviously, if $M \in \mathbb{A}^{R^{\mathbf{d}}, R^{\mathbf{d}}}$, then $M \in \mathrm{mod}_\Lambda(R)$ if and only if $M_\rho = 0$ for all $\rho \in R$. We define $\mathrm{mod}_\Lambda^{\mathbf{d}}(R)$ as the subset of all elements of $\mathbb{A}^{R^{\mathbf{d}}, R^{\mathbf{d}}}$ for which the above condition is satisfied. Then $\mathrm{mod}_\Lambda^{\mathbf{d}}$ is a functor from the category of commutative k-algebras to the category of sets, which is an affine scheme. We call $\mathrm{mod}_\Lambda^{\mathbf{d}}$ the scheme of Λ-modules of dimension vector $\mathbf{d}$. Note that if $M \in \mathrm{mod}_\Lambda(R)$ and $\mathbf{dim}\, M = \mathbf{d}$, then there exists (usually nonunique) $M' \in \mathrm{mod}_\Lambda^{\mathbf{d}}(R)$ such that $M \simeq M'$. Consequently, we will usually treat Λ-modules as elements of $\mathrm{mod}_\Lambda^{\mathbf{d}}(k)$.

We now generalize a construction described by Zwara [24, Section 3]. For $\mathbf{d}', \mathbf{d}'' \in \mathbb{N}^{\Delta_0}$ and $d \in \mathbb{N}$ we denote by $\mathcal{H}_d^{\mathbf{d}', \mathbf{d}''}$ the subscheme of $\mathrm{mod}_\Lambda^{\mathbf{d}'} \times \mathrm{mod}_\Lambda^{\mathbf{d}''}$ such that $\mathcal{H}_d^{\mathbf{d}', \mathbf{d}''}(k)$ consists of all $(U, V) \in \mathrm{mod}_\Lambda^{\mathbf{d}'}(k) \times \mathrm{mod}_\Lambda^{\mathbf{d}''}(k)$ such that $[V, U] = d$. We briefly describe its construction.

Fix a commutative k-algebra R. Recall that for $V \in \mathrm{mod}_\Lambda^{\mathbf{d}''}(R)$ we constructed in Section 2 the exact sequence

$$0 \to \Omega^V \xrightarrow{f^V} P^V \xrightarrow{g^V} V \to 0$$

with P^V projective. Iterating this construction we obtain the exact sequence

$$P^{\Omega^V} \xrightarrow{f^V g^{\Omega^V}} P^V \xrightarrow{g^V} V \to 0.$$

In particular, for each $U \in \mathrm{mod}_\Lambda^{\mathbf{d}'}(R)$ we have the exact sequence

$$0 \to \mathrm{Hom}_\Lambda(V, U) \xrightarrow{\mathrm{Hom}_\Lambda(g^V, U)} \mathrm{Hom}_\Lambda(P^V, U) \xrightarrow{\mathrm{Hom}_\Lambda(f^V g^{\Omega^V}, U)} \mathrm{Hom}_\Lambda(P^{\Omega^V}, U).$$

Recall that

$$P^V = \bigoplus_{x \in \Delta_0} \Lambda x \otimes R^{d''_x} \qquad \text{and} \qquad P^{\Omega^V} = \bigoplus_{x, y \in \Delta_0} \Lambda x \otimes x(\mathrm{rad}\, \Lambda)y \otimes R^{d''_y}.$$

Consequently,

$$\mathrm{Hom}_\Lambda(P^V, U) \simeq \bigoplus_{x \in \Delta_0} R^{d'_x d''_x}$$

and

$$\mathrm{Hom}_\Lambda(P^{\Omega^V}, U) \simeq \bigoplus_{x, y \in \Delta_0} R^{d'_x d''_y} \otimes x(\mathrm{rad}\, \Lambda)y.$$

Moreover,

$$f_z^V g_z^{\Omega^V}(\sigma_1 \otimes \sigma_2 \otimes v) = \sigma_1 \sigma_2 \otimes v - \sigma_1 \otimes V_{\sigma_2} v$$

for all $\sigma_1 \in z\Lambda x$, $\sigma_2 \in x(\mathrm{rad}\, \Lambda)y$, $v \in R^{d''_y}$, $x, y, z \in \Delta_0$, hence if we fix bases in $x(\mathrm{rad}\, \Lambda)y$ for all $x, y \in \Delta_0$, then simple (although technical) analysis shows that there exists a morphism $\varphi : \mathrm{mod}_\Lambda^{\mathbf{d}'} \times \mathrm{mod}_\Lambda^{\mathbf{d}''} \to \mathbb{M}_{p,q}$

such that $\operatorname{Hom}_\Lambda(f^V g^{\Omega^V}, U)$ is given by $\varphi(U,V)$ for all $U \in \operatorname{mod}_\Lambda^{\mathbf{d}'}$ and $V \in \operatorname{mod}_\Lambda^{\mathbf{d}''}$, where for

$$p = \sum_{x \in \Delta_0} d'_x d''_x \qquad \text{and} \qquad q = \sum_{x,y \in \Delta_0} d'_x d''_y \dim_k(x(\operatorname{rad}\Lambda)y)$$

$\mathbb{M}_{p,q}$ denotes the scheme of $p \times q$-matrices. We define $\mathcal{H}_d^{\mathbf{d}',\mathbf{d}''}$ to be the inverse image by φ of the reduced subscheme of $\mathbb{M}_{p,q}$ whose k-rational points are the $p \times q$-matrices of rank r with coefficients in k (see [24, Subsection 3.2]). Obviously,

$$\mathcal{H}_d^{\mathbf{d}',\mathbf{d}''}(k) = \{(U,V) \in \operatorname{mod}_\Lambda^{\mathbf{d}'}(k) \times \operatorname{mod}_\Lambda^{\mathbf{d}''}(k) \mid [V,U] = d\}.$$

Let $R_0 = k[\varepsilon]/(\varepsilon^2)$ and let $\pi : R_0 \to k$ be the canonical projection. Recall, that if $\mathcal{F}$ is a scheme, then for $x \in \mathcal{F}(k)$, $\mathcal{F}(\pi)^{-1}(x)$ can be interpreted as the tangent space $T_x\mathcal{F}(k)$ to $\mathcal{F}(k)$ at x. The proof of the following proposition just repeats the arguments used in the proof of [24, Lemma 3.5], hence we omit it.

Lemma 3.1. *Let $M \in \operatorname{mod}_\Lambda^{\mathbf{d}'}(R_0)$ and $N \in \operatorname{mod}_\Lambda^{\mathbf{d}''}(R_0)$. Put $U = \operatorname{mod}_\Lambda^{\mathbf{d}'}(\pi)(M)$ and $V = \operatorname{mod}_\Lambda^{\mathbf{d}''}(\pi)(N)$. Then $(M,N) \in \mathcal{H}_d^{\mathbf{d}',\mathbf{d}''}(R_0)$ if and only if $[V,U] = d$ and $[N,M] = 2d$.*

We now give another interpretation of this result. Fix $W \in \operatorname{mod}_\Lambda^{\mathbf{d}}(k)$ for a dimension vector $\mathbf{d}$. If $L \in \operatorname{mod}_\Lambda^{\mathbf{d}}(\pi)^{-1}(W)$, then $L = W + \varepsilon\overline{L}$ for some $\overline{L} \in \mathbb{A}^{k^{\mathbf{d}},k^{\mathbf{d}}}$. Observe that

$$L_\rho = W_\rho + \varepsilon\overline{L}_\rho^{W,W}$$

for all $\rho \in x(k\Delta)y$ with $x, y \in \Delta_0$. This implies that the map

$$\operatorname{mod}_\Lambda^{\mathbf{d}}(\pi)^{-1}(W) \ni L \mapsto \overline{L} \in \mathbb{Z}^{W,W}$$

is well-defined and bijective.

Proposition 3.2. *Let $M \in \operatorname{mod}_\Lambda^{\mathbf{d}'}(R_0)$ and $N \in \operatorname{mod}_\Lambda^{\mathbf{d}''}(R_0)$. Put $U = \operatorname{mod}_\Lambda^{\mathbf{d}'}(\pi)(M)$ and $V = \operatorname{mod}_\Lambda^{\mathbf{d}''}(\pi)(N)$. Then $(M,N) \in \mathcal{H}_d^{\mathbf{d}',\mathbf{d}''}(R_0)$ if and only if $[V,U] = d$ and $[\xi^{\overline{M}} \circ f] = [f \circ \xi^{\overline{N}}]$ for all $f \in \operatorname{Hom}_\Lambda(V,U)$.*

Proof. We need to show that under the assumption $[V,U] = d$, the condition $[N,M] = 2d$ is equivalent to the latter condition in the proposition. Observe that a homomorphism $f : N \to M$ is of the form $f = f^0 + \varepsilon f^1$ for $f^0, f^1 \in \mathbb{V}^{k^{\mathbf{d}''},k^{\mathbf{d}'}}$ such that

$$U_\alpha f^0_{s_\alpha} = f^0_{t_\alpha} V_\alpha \qquad \text{and} \qquad U_\alpha f^1_{s_\alpha} + \overline{M}_\alpha f^0_{s_\alpha} = f^1_{t_\alpha} V_\alpha + f^0_{t_\alpha}\overline{N}_\alpha$$

for all $\alpha \in \Delta_1$. The first condition means that $f^0 \in \operatorname{Hom}_\Lambda(V,U)$, hence the dimension of the set of such f^0 equals d. Consequently, the condition $[N,M] = 2d$ is equivalent to the statement that for each $f \in \operatorname{Hom}_\Lambda(V,U)$ the set of $g \in \mathbb{V}^{k^{\mathbf{d}'},k^{\mathbf{d}''}}$ such that

$$\overline{M}_\alpha f_{s_\alpha} - f_{t_\alpha}\overline{N}_\alpha = g_{t_\alpha} V_\alpha - U_\alpha g_{s_\alpha}$$

for all $\alpha \in \Delta_1$, has dimension d. However linear algebra says that this is that case if and only if this system of equations has a solution, i.e. if and only if $(\overline{M}_\alpha f_{s_\alpha} - f_{t_\alpha}\overline{N}_\alpha) \in \mathbb{B}^{V,U}$, what finishes the proof according to Proposition 2.1. $\square$

We now extend the above construction to extensions. Fix $d' \in \mathbb{N}$. If $V \in \mathrm{mod}_\Lambda^{\mathbf{d}''}(k)$, then we have the exact sequence

$$P^{\Omega\Omega^V} \xrightarrow{f^{\Omega^V} g^{\Omega\Omega^V}} P^{\Omega^V} \xrightarrow{f^V g^{\Omega^V}} P^V \xrightarrow{g^V} V \to 0.$$

Repeating the previous construction (using also $\mathrm{Hom}_\Lambda(f^{\Omega^V} g^{\Omega\Omega^V}, U)$ this time) we define the subscheme $\mathcal{E}_{d,d'}^{\mathbf{d}',\mathbf{d}''}$ of $\mathcal{H}_d^{\mathbf{d}',\mathbf{d}''}$ such that $(U,V) \in \mathcal{E}_{d,d'}^{\mathbf{d}',\mathbf{d}''}(k)$ for $(U,V) \in \mathcal{H}_d^{\mathbf{d}',\mathbf{d}''}(k)$ if and only if $[V,U]^1 = d'$. Moreover, if $(M,N) \in \mathcal{H}_d^{\mathbf{d}',\mathbf{d}''}(R_0)$, $U = \mathrm{mod}_\Lambda^{\mathbf{d}'}(\pi)(M)$ and $V = \mathrm{mod}_\Lambda^{\mathbf{d}''}(\pi)(N)$, then $(M,N) \in \mathcal{E}_{d,d'}^{\mathbf{d}',\mathbf{d}''}(R_0)$ if and only if $[V,U]^1 = d'$ and $[N,M]^1 = 2d'$. An alternate description of $\mathcal{E}_{d,d'}^{\mathbf{d}',\mathbf{d}''}(R_0)$ is the following.

Proposition 3.3. *Let $M \in \mathrm{mod}_\Lambda^{\mathbf{d}'}(R_0)$ and $N \in \mathrm{mod}_\Lambda^{\mathbf{d}''}(R_0)$. Put $U = \mathrm{mod}_\Lambda^{\mathbf{d}'}(\pi)(M)$ and $V = \mathrm{mod}_\Lambda^{\mathbf{d}''}(\pi)(N)$. Then $(M,N) \in \mathcal{E}_{d,d'}^{\mathbf{d}',\mathbf{d}''}(R_0)$ if and only if $(M,N) \in \mathcal{H}_d^{\mathbf{d}',\mathbf{d}''}(R_0)$, $[V,U]^1 = d'$, and $[\xi^{\overline{M}} \circ \xi] + [\xi \circ \xi^{\overline{N}}] = 0$ for all $[\xi] \in \mathrm{Ext}_\Lambda^1(V,U)$.*

Proof. We need to show that, under the assumptions $[V,U]^1 = d'$ and $(M,N) \in \mathcal{H}_d^{\mathbf{d}',\mathbf{d}''}(R_0)$, the condition $[N,M]^1 = 2d'$ is equivalent to the condition $[\xi^{\overline{M}} \circ \xi] + [\xi \circ \xi^{\overline{N}}] = 0$ for all $[\xi] \in \mathrm{Ext}_\Lambda^1(V,U)$. Thus assume that $(M,N) \in \mathcal{H}_d^{\mathbf{d}',\mathbf{d}''}(R_0)$ and $[V,U] = d'$. Since $\mathrm{Ext}_\Lambda^1(V,U) = \mathbb{Z}^{V,U}/\mathbb{B}^{V,U}$ and $\mathrm{Ext}_\Lambda^1(N,M) = \mathbb{Z}^{N,M}/\mathbb{B}^{N,M}$, it follows that $[N,M]^1 = 2d' = 2[V,U]^1$ if and only if $\dim_k \mathbb{Z}^{N,M} = 2\dim_k \mathbb{Z}^{V,U}$.

If $Z \in \mathbb{A}^{N,M}$, then $Z = Z' + \varepsilon\overline{Z}$ for some $Z', \overline{Z} \in \mathbb{A}^{k^{\mathbf{d}''},k^{\mathbf{d}'}}$. Moreover, easy calculations show that $Z \in \mathbb{Z}^{N,M}$, i.e. $Z_\rho^{N,M} = 0$ for all $\rho \in \mathfrak{R}$, if and only if $Z' \in \mathbb{Z}^{V,U}$ and

$$\overline{M} \circ Z' + Z' \circ \overline{N} + \Phi^{V,U}(\overline{Z}) = 0.$$

Consequently, $\dim_k \mathbb{Z}^{N,M} = 2\dim_k \mathbb{Z}^{V,U}$ if and only if $Z' \in \mathbb{Z}^{V,U}$ and the set of $\overline{Z} \in \mathbb{A}^{k^{\mathbf{d}''},k^{\mathbf{d}'}}$ such that the above condition is satisfied has dimension $\dim_k \mathbb{Z}^{V,U}$. Since $\mathrm{Ker}\,\Phi^{V,U} = \mathbb{Z}^{V,U}$, linear algebra says that $\dim_k \mathbb{Z}^{N,M} = 2\dim_k \mathbb{Z}^{V,U}$ if and only if $\overline{M} \circ Z' + Z' \circ \overline{N} \in \mathrm{Im}\,\Phi^{V,U} = \mathbb{B}'^{V,U}$ for all $Z' \in \mathbb{Z}^{V,U}$, and this finishes the proof according to Proposition 2.5. $\square$

Let $\mathbf{d} \in \mathbb{N}^{\Delta_0}$. The product $\mathrm{GL}_{\mathbf{d}}(k) = \prod_{x \in \Delta_0} \mathrm{GL}_{d_x}(k)$ of general linear groups acts on $\mathrm{mod}_\Lambda^{\mathbf{d}}$ by conjugation:

$$(g \cdot M)_\alpha = g_{t_\alpha} M_\alpha g_{s_\alpha}^{-1},\ g \in \mathrm{GL}_{\mathbf{d}}(k),\ M \in \mathrm{mod}_\Lambda^{\mathbf{d}}(k),\ \alpha \in \Delta_1.$$

If $\mathcal{O}(M)$ denotes the $\mathrm{GL}_{\mathbf{d}}(k)$-orbit of $M \in \mathrm{mod}_\Lambda^{\mathbf{d}}(k)$, then $\mathcal{O}(M_1) = \mathcal{O}(M_2)$ for $M_1, M_2 \in \mathrm{mod}_\Lambda^{\mathbf{d}}(k)$ if and only if $M_1 \simeq M_2$. It is known (see for example [20, 2.2]) that

$$\dim \mathcal{O}(M) = \dim \mathrm{GL}_{\mathbf{d}}(k) - [M, M]$$

for $M \in \mathrm{mod}_\Lambda^{\mathbf{d}}(k)$. We put

$$a(\mathbf{d}) = \dim \mathbb{A}^{k^{\mathbf{d}}, k^{\mathbf{d}}} - \sum_{\rho \in \mathfrak{R}} d_{s_\rho} d_{t_\rho}.$$

Note that $a(\mathbf{d}) = \dim \mathrm{GL}_{\mathbf{d}}(k) - \chi(\mathbf{d})$. Fix $M \in \mathrm{mod}_\Lambda^{\mathbf{d}}(k)$. It follows from Voigt's result [22] that the Zariski closure $\overline{\mathcal{O}(M)}$ of $\mathcal{O}(M)$ is an irreducible component of $\mathrm{mod}_\Lambda^{\mathbf{d}}(k)$ provided $[M, M]^1 = 0$. Moreover, if $[M, M]^2 = 0$, then M is a regular point of $\mathrm{mod}_\Lambda^{\mathbf{d}}(k)$ [17], i.e. $\dim_k T_M \mathrm{mod}_\Lambda^{\mathbf{d}}(k)$ equals the maximum of the dimensions of the irreducible components of $\mathrm{mod}_\Lambda^{\mathbf{d}}(k)$ containing M. Finally, if Λ is tilted and $[M, M]^1 = 0 = [M, M]^2$, then $\dim \overline{\mathcal{O}(M)} = a(\mathbf{d})$.

Let $M, N \in \mathrm{mod}_\Lambda^{\mathbf{d}}(k)$ for $\mathbf{d} \in \mathbb{N}^{\Delta_0}$. We call N a degeneration of M and write $M \leq_{\deg} N$ if $N \in \overline{\mathcal{O}(M)}$. A deneneration $M \leq_{\deg} N$ is called minimal if $M \not\simeq N$ and either $M \simeq L$ or $N \simeq L$ for each $L \in \mathrm{mod}_\Lambda^{\mathbf{d}}(k)$ such that $M \leq_{\deg} L \leq_{\deg} N$. If $M \leq_{\deg} N$ and $M \not\simeq N$, then we write $M <_{\deg} N$.

4. Proof of the main result

Throughout this section we fix a sincere directing module M over the path algebra Λ of a bound quiver $(\Delta, \mathfrak{R})$. Recall, that this implies that Λ is a tilted algebra. We also fix a directing tilting module T such that $\mathrm{add}\, M \subset \mathrm{add}\, T$. Let $T_1, \ldots, T_n$ be the pairwise nonisomorphic indecomposable direct summands of T, $\mathcal{F} = \mathcal{F}(T)$ and $\mathcal{T} = \mathcal{T}(T)$. Upper semicontinuity of the functions $[T, -], [-, T]^1 : \mathrm{mod}_\Lambda^{\mathbf{d}}(k) \to \mathbb{Z}$ (see for example [15, Lemma 4.3]) implies that $\mathcal{F}^{\mathbf{d}}$ and $\mathcal{T}^{\mathbf{d}}$ are open subset of $\mathrm{mod}_\Lambda^{\mathbf{d}}(k)$ for each $\mathbf{d} \in \mathbb{N}^{\Delta_0}$, where $\mathcal{A}^{\mathbf{d}} = \mathcal{A} \cap \mathrm{mod}_\Lambda^{\mathbf{d}}(k)$ for a subcategory $\mathcal{A}$ of mod_Λ and $\mathbf{d} \in \mathbb{N}^{\Delta_0}$. In order to generalize the above observation we need the following lemma.

Lemma 4.1. *If $N', N'' \in \mathcal{T}$ and $[T_i, N'] = [T_i, N'']$ for all $i \in [1, n]$, then $\mathbf{dim}\, N' = \mathbf{dim}\, N''$.*

Proof. We have the isomorphism $\Phi : \mathbb{Z}^{\Delta_0} \to \mathbb{Z}^n$ induced by the assignment $\mathbf{dim}\, N \mapsto ([T_i, N] - [T_i, N]^1)$ (see [1, Theorem VI.4.3]). Since $\Phi(\mathbf{dim}\, N) = ([T_i, N])$ for all $N \in \mathcal{T}$, this implies the claim. □

For $\mathcal{U}' \subset \mathrm{mod}_\Lambda^{\mathbf{d}'}(k)$ and $\mathcal{U}'' \subset \mathrm{mod}_\Lambda^{\mathbf{d}''}(k)$, where $\mathbf{d}', \mathbf{d}'' \in \mathbb{N}^{\Delta_0}$, let $\mathcal{U}' \oplus \mathcal{U}''$ be the subset of $\mathrm{mod}_\Lambda^{\mathbf{d}'+\mathbf{d}''}(k)$ consisting of M such that $M \simeq M' \oplus M''$ for some $M' \in \mathcal{U}'$ and $M'' \in \mathcal{U}''$.

Corollary 4.2. *If $\mathbf{d}', \mathbf{d}'' \in \mathbb{N}^{\Delta_0}$, then $\mathcal{F}^{\mathbf{d}'} \oplus \mathcal{T}^{\mathbf{d}''}$ is a locally closed subset of $\mathrm{mod}_\Lambda^{\mathbf{d}'+\mathbf{d}''}(k)$.*

Proof. Let $\mathbf{d} = \mathbf{d}' + \mathbf{d}''$ and $N \in \mathrm{mod}_\Lambda^{\mathbf{d}}(k)$. If $N \simeq N' \oplus N''$ for $N' \in \mathcal{F}$ and $N'' \in \mathcal{T}$, then $[T_i, N] = [T_i, N''] = \langle \mathbf{dim}\, T_i, \mathbf{dim}\, N'' \rangle$ for each $i \in [1, n]$. Consequently, it follows from the previous lemma that $N \in \mathcal{F}^{\mathbf{d}'} \oplus \mathcal{T}^{\mathbf{d}''}$, i.e. $\mathbf{dim}\, N'' = \mathbf{d}''$, if and only if $[T_i, N] = \langle \mathbf{dim}\, T_i, \mathbf{d}'' \rangle$ for all $i \in [1, n]$, hence the claim follows. □

Lemma 4.3. *If N is a minimal degeneration of M, then there exists an exact sequence*

$$0 \to U \to M \to V \to 0$$

such that $N \simeq U \oplus V$.

Proof. If there is not such a sequence, then the minimality of the degeneration $M <_{\deg} N$ and [23, Theorem 4] imply that there exists an indecomposable direct summand X of N such that $M' <_{\deg} X$ for a direct summand M' of M. In particular,

$$[M', X] = [M', X]^1 + \chi(\mathbf{dim}\, M') > 0.$$

Similarly, $[X, M'] \neq 0$. Consequently, $X \in \mathrm{add}\, T$, hence $X \simeq M'$, a contradiction. □

Corollary 4.4. *If $U \in \mathcal{F}$ and $T \in \mathcal{T}$ are such that $U \neq 0$ and $U \oplus V$ is a minimal degeneration of M, then there exists an exact sequence*

$$0 \to U \to M \to V \to 0.$$

Proof. According to the above lemma there exists an exact sequence

$$\sigma : 0 \to U' \to M \to V' \to 0$$

such that $U' \oplus V' \simeq U \oplus V$. Since $[M, U] = 0$, U must be a direct summand of U'. Let

$$\sigma' = p \circ \sigma : 0 \to U \to M' \to V' \to 0,$$

where $p : U' \to U$ is the canonical projection. Since $U \notin \mathrm{add}\, M$, σ' does not split. Consequently,

$$M \leq_{\deg} \mathrm{Ker}\, p \oplus M' <_{\deg} \mathrm{Ker}\, p \oplus U \oplus V' \simeq U' \oplus V' \simeq U \oplus V,$$

hence $M \simeq \mathrm{Ker}\, p \oplus M'$ and $\mathrm{Ker}\, p \oplus V' \simeq V$ by the minimality of the degeneration $M <_{\deg} N$. Moreover, we have a short exact sequence

$$0 \to U \to M' \oplus \mathrm{Ker}\, p \to V' \oplus \mathrm{Ker}\, p \to 0$$

and the claim follows. □

Proof of the main result. Let $\mathcal{X}$ be an irreducible component of $\overline{\mathcal{O}(M)} \setminus \mathcal{O}(M)$. Our aim is to show that there exists an open subset of $\mathcal{X}$ such that all points of this subset are regular points of $\overline{\mathcal{O}(M)}$. Since $\mathrm{mod}_\Lambda = \mathcal{F} \vee \mathcal{T}$, it follows from Corollary 4.2 that there exist $\mathbf{d}', \mathbf{d}'' \in \mathbb{N}^{\Delta_0}$ such that $\mathcal{U} = (\mathcal{F}^{\mathbf{d}'} \oplus \mathcal{T}^{\mathbf{d}''}) \cap \mathcal{X}$ is an open subset of $\mathcal{X}$. If $\mathbf{d}' = 0$, then

$\operatorname{id}_\Lambda N \leq 1$ for all $N \in \mathcal{U}$, hence the claim follows. Thus assume that $\mathbf{d}' \neq 0$. Let $d_0 = \min\{[N, N] \mid N \in \mathcal{X}\}$, $d_1 = \min\{[N, N]^1 \mid N \in \mathcal{X}\}$, and $\mathcal{U}_0$ be the subset of $\mathcal{U}$ consisting of all $N \in \mathcal{U}$ such that $[N, N] = d_0$, $[N, N]^1 = d_1$, and N does not belong to an irreducible component of $\overline{\mathcal{O}(M)} \setminus \mathcal{O}(M)$ different from $\mathcal{X}$. Then $\mathcal{U}_0$ is an open subset of $\mathcal{X}$. Moreover, all points of $\mathcal{U}_0$ are minimal degenerations of M. We show that all points of $\mathcal{U}_0$ are regular points of $\overline{\mathcal{O}(M)}$.

We first observe that in order to prove the above claim it is enough to show that $\dim_k L(U, V) \leq a(\mathbf{d}') + a(\mathbf{d}'') - [V, U]^2$ for all $U \in \mathcal{F}^{\mathbf{d}'}$ and $V \in \mathcal{T}^{\mathbf{d}''}$ such that $U \oplus V \in \mathcal{U}_0$, where

$$L(U, V) = T_{U \oplus V}((\operatorname{mod}_\Lambda^{\mathbf{d}'}(k) \times \operatorname{mod}_\Lambda^{\mathbf{d}''}(k)) \cap \overline{\mathcal{O}(M)}),$$

for $U \in \operatorname{mod}_\Lambda^{\mathbf{d}'}(k)$ and $V \in \operatorname{mod}_\Lambda^{\mathbf{d}''}(k)$, and we identify $\operatorname{mod}_\Lambda^{\mathbf{d}'}(k) \times \operatorname{mod}_\Lambda^{\mathbf{d}''}(k)$ with

$$\left\{ \begin{bmatrix} U & 0 \\ 0 & V \end{bmatrix} \,\middle|\, U \in \operatorname{mod}_\Lambda^{\mathbf{d}'}(k),\ V \in \operatorname{mod}_\Lambda^{\mathbf{d}''}(k) \right\} \subset \operatorname{mod}_\Lambda^{\mathbf{d}}(k).$$

Indeed, if $N \in \mathcal{U}_0$, then without loss of generality we may assume that $N = U \oplus V$ for $U \in \mathcal{F}^{\mathbf{d}'}$ and $V \in \mathcal{T}^{\mathbf{d}''}$. Consequently,

$$T_N \overline{\mathcal{O}(M)} \subset L(U, V) \oplus \mathbb{Z}^{U,V} \oplus \mathbb{Z}^{V,U}$$

(here we use that $T_N \operatorname{mod}_\Lambda^{\mathbf{d}}(k) = \mathbb{Z}^{N,N}$). Since

$$\dim_k \mathbb{Z}^{U,V} = \sum_{x \in \Delta_0} d'_x d''_x - \langle \mathbf{d}', \mathbf{d}'' \rangle$$

and

$$\dim_k \mathbb{Z}^{V,U} = \sum_{x \in \Delta_0} d''_x d'_x - [V, U] + [V, U]^1,$$

the required inequality implies $\dim_k T_n \overline{\mathcal{O}(M)} \leq a(\mathbf{d}) = \dim \overline{\mathcal{O}(M)}$, and this will finish the proof.

Now we fix $U \in \mathcal{F}^{\mathbf{d}'}$ and $V \in \mathcal{T}^{\mathbf{d}''}$ such that $(U, V) \in \mathcal{U}_0$. Observe that $[U, V] = \langle \mathbf{d}', \mathbf{d}'' \rangle$, $[U, V]^1 = 0$, and $[V, U] = 0$. Recall also that $[U \oplus V, U \oplus V] = d_0$ and $[U \oplus V, U \oplus V]^1 = d_1$. Finally, $[U, U]^2 = 0$ and $[V, V]^2 = 0$. Consequently, $[V, U]^1 = d$, where

$$d = d_1 - d_0 + \chi(\mathbf{d}') + \chi(\mathbf{d}'') + \langle \mathbf{d}', \mathbf{d}'' \rangle,$$

thus $\mathcal{U}_0 \subset \mathcal{E}_{0,d}^{\mathbf{d}',\mathbf{d}''}$. Observe that $\mathcal{U}_0$ is open in $(\operatorname{mod}_\Lambda^{\mathbf{d}'}(k) \times \operatorname{mod}_\Lambda^{\mathbf{d}''}(k)) \cap \overline{\mathcal{O}(M)}$, hence $L(U, V) = T_{U \oplus V} \mathcal{U}_0 \subset T_{(U,V)} \mathcal{E}_{0,d}^{\mathbf{d}',\mathbf{d}''}$. Recall that according to Proposition 3.3 $T_{(U,V)} \mathcal{E}_{0,d}^{\mathbf{d}',\mathbf{d}''}$ consists of $(Z', Z'') \in \mathbb{Z}^{U,U} \times \mathbb{Z}^{V,V}$ such that $[\xi^{Z'} \circ \xi] + [\xi \circ \xi^{Z''}] = 0$ for all $[\xi] \in \operatorname{Ext}_\Lambda^1(V, U)$. Since $U \oplus V$ is a minimal degeneration of M, there exists a short exact sequence of the form

$$\xi : 0 \to U \to M \to V \to 0$$

by Corollary 4.4. Moreover, $\operatorname{pd}_\Lambda M \leq 1$, hence the map $[V,V]^1 \to [V,U]^2$, $[\xi'] \to [\xi \circ \xi']$, is surjective. Consequently, the map

$$\Psi : \mathbb{Z}^{U,U} \times \mathbb{Z}^{V,V} \to \operatorname{Ext}^2_\Lambda(V,U),\ (Z',Z'') \mapsto [\xi^{Z'} \circ \xi] + [\xi \circ \xi^{Z''}],$$

is surjective. Finally, using $T_{(U,V)}\mathcal{E}^{\mathbf{d}',\mathbf{d}''}_{0,d} \subset \operatorname{Ker}\Psi$, $\dim_k \mathbb{Z}^{U,U} = a(\mathbf{d}')$, and $\dim_k \mathbb{Z}^{V,V} = a(\mathbf{d}'')$, we finish the proof. $\square$

References

[1] I. Assem, D. Simson, and A. Skowroński, *Elements of the Representation Theory of Associative Algebras. Vol. 1*, London Mathematical Society Student Texts, vol. 65, Cambridge University Press, Cambridge, 2006.

[2] Ø. Bakke, *Some characterizations of tilted algebras*, Math. Scand. **63** (1988), no. 1, 43–50.

[3] M. Barot and J. Schröer, *Module varieties over canonical algebras*, J. Algebra **246** (2001), no. 1, 175–192.

[4] J. Bender and K. Bongartz, *Minimal singularities in orbit closures of matrix pencils*, Linear Algebra Appl. **365** (2003), 13–24.

[5] G. Bobiński, *Geometry of decomposable directing modules over tame algebras*, J. Math. Soc. Japan **54** (2002), no. 3, 609–620.

[6] ———, *Geometry of regular modules over canonical algebras*, Trans. Amer. Math. Soc. **360** (2008), no. 2, 717–742.

[7] G. Bobiński and A. Skowroński, *Geometry of directing modules over tame algebras*, J. Algebra **215** (1999), no. 2, 603–643.

[8] ———, *Geometry of modules over tame quasi-tilted algebras*, Colloq. Math. **79** (1999), no. 1, 85–118.

[9] G. Bobiński and G. Zwara, *Schubert varieties and representations of Dynkin quivers*, Colloq. Math. **94** (2002), no. 2, 285–309.

[10] ———, *Normality of orbit closures for directing modules over tame algebras*, J. Algebra **298** (2006), no. 1, 120–133.

[11] K. Bongartz, *Algebras and quadratic forms*, J. London Math. Soc. (2) **28** (1983), no. 3, 461–469.

[12] ———, *Minimal singularities for representations of Dynkin quivers*, Comment. Math. Helv. **69** (1994), no. 4, 575–611.

[13] ———, *Some geometric aspects of representation theory*, Algebras and Modules. I (I. Reiten, S. O. Smalø, and Ø. Solberg, eds.), CMS Conf. Proc., vol. 23, Amer. Math. Soc., Providence, RI, 1998, pp. 1–27.

[14] M. C. R. Butler and A. D. King, *Minimal resolutions of algebras*, J. Algebra **212** (1999), no. 1, 323–362.

[15] W. Crawley-Boevey and J. Schröer, *Irreducible components of varieties of modules*, J. Reine Angew. Math. **553** (2002), 201–220.

[16] D. Eisenbud and J. Harris, *The Geometry of Schemes*, Graduate Texts in Mathematics, vol. 197, Springer, New York, 2000.

[17] Ch. Geiß, *Geometric methods in representation theory of finite-dimensional algebras*, Representation Theory of Algebras and Related Topics (R. Bautista, R. Martínez-Villa, and J. A. de la Peña, eds.), CMS Conf. Proc., vol. 19, Amer. Math. Soc., Providence, RI, 1996, pp. 53–63.

[18] D. Happel and C. M. Ringel, *Directing projective modules*, Arch. Math. (Basel) **60** (1993), no. 3, 237–246.

[19] H. Kraft, *Geometric methods in representation theory*, Representations of Algebras (Maurice Auslander and Emilio Lluis, eds.), Lecture Notes in Math., vol. 944, Springer, Berlin, 1982, pp. 180–258.

[20] H. Kraft and Ch. Riedtmann, *Geometry of representations of quivers*, Representations of Algebras (P. Webb, ed.), London Math. Soc. Lecture Note Ser., vol. 116, Cambridge Univ. Press, Cambridge, 1986, pp. 109–145.
[21] S. Mac Lane, *Homology*, Classics in Mathematics, Springer, Berlin, 1995.
[22] D. Voigt, *Induzierte Darstellungen in der Theorie der endlichen, algebraischen Gruppen*, Lecture Notes in Mathematics, vol. 592, Springer, Berlin, 1977.
[23] G. Zwara, *Degenerations of finite-dimensional modules are given by extensions*, Compositio Math. **121** (2000), no. 2, 205–218.
[24] ______, *Smooth morphisms of module schemes*, Proc. London Math. Soc. (3) **84** (2002), no. 3, 539–558.
[25] ______, *Unibranch orbit closures in module varieties*, Ann. Sci. École Norm. Sup. (4) **35** (2002), no. 6, 877–895.
[26] ______, *Regularity in codimension one of orbit closures in module varieties*, J. Algebra **283** (2005), no. 2, 821–848.

Faculty of Mathematics and Computer Science, Nicolaus Copernicus University, ul. Chopina 12/18, 87-100 Toruń, Poland
E-mail address: gregbob@mat.uni.torun.pl