Area limit laws for symmetry classes of staircase polygons

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Abstract

We derive area limit laws for the various symmetry classes of staircase polygons on the square lattice, in a uniform ensemble where, for fixed perimeter, each polygon occurs with the same probability. This complements a previous study by Leroux and Rassart, where explicit expressions for the area and perimeter generating functions of these classes have been derived.

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1 Introduction

Models of planar lattice polygons have a rich history. The most challenging member is self-avoiding polygons [30], a model of interest not only in enumerative combinatorics, but also in the natural sciences such as physics and chemistry. Some solvable subclasses, obtained by imposing convexity or directedness constraints, have been understood in detail, see [5] for an overview. An important example on the square lattice are *staircase polygons*, also called parallelogram polygons. These have been enumerated with respect to various parameters such as perimeter, area and generalisations thereof, site-perimeter, and radius of gyration, see [10, 5, 39, 28] and references therein. For some of these parameters, their distribution has been asymptotically analysed. An example is the area distribution in a uniform ensemble where, for fixed perimeter, all polygons occur with the same probability. We will call this ensemble the *uniform fixed perimeter ensemble*. For a large class of planar polygon models including staircase polygons, the *Airy distribution* emerges as the limit law of area [12, 38]. There is compelling numerical evidence that the Airy distribution also appears as the limit distribution of area in self-avoiding polygons [36, 40].

When counting polygons, translated copies of a polygon are identified, but rotated or reflected versions of a polygon are often distinguished. Motivated by applications such as benzenoid counting in chemistry [44], one may be led to identify objects that are related by point symmetries. The task arises to enumerate classes of polygons up to symmetries of the underlying lattice. Two different counting problems occur. One may ask for the number of polygons which are fixed by a given subgroup of the lattice symmetries. One may also count the number of polygon orbits with respect to a such a subgroup. These two counting problems are related. In fact, it is possible to express the generating function for orbit counts in terms of those for fixed point counts, via the Lemma of Burnside. This has been studied for various classes of column-convex polygons by Leroux and co-workers [27, 26, 21], and explicit expressions for generating functions have been obtained.

Symmetry classes of staircase polygons on the square lattice are analysed in [26], by two different approaches. One of them uses a bijection between staircase polygons and Dyck paths due to Delest and Viennot [9]. By this bijection, some of the polygon symmetry classes can be identified with corresponding symmetry classes of Dyck paths. The toolbox of Dyck path analysis can then be used to solve the corresponding polygon counting problems. A second approach uses the Temperley–Bousquet-Mélou method [5] to obtain explicit expressions for the perimeter and area generating functions. In addition, it is shown in [26] that the number of polygons fixed by some non-trivial symmetry is asymptotically negligible to the total number of polygons. This implies that the number of symmetry orbits is asymptotically equal to the total number of polygons.

In this article, we derive the area limit laws for the symmetry subclasses of polygons, within the uniform fixed perimeter ensemble. Wheras the result for the full class of staircase polygons is known already [12, 38], the corresponding problem for the various symmetry subclasses has apparently not been studied before. We will use an elementary self-contained approach, which is based on a simple decomposition of staircase polygons [39]. This yields, for *every* subclass, a q-difference equation for its perimeter and area generating function. Explicit expressions could be obtained from this equation, which might differ from those of [26]. We do not focus on solving, but on manipulating the q-difference equation in order to derive the limit law, by an application of the moment method. Whereas this approach has been used previously in different contexts, see e.g. [42, 43, 17, 12, 32], we would like to stress that our alternative derivation, based on the method of *dominant balance* [38, 40], simplifies the analysis. As a consequence, corrections to the asymptotic behaviour might be mechanically obtained, compare [37].

Our results extend those of [26] to a complete symmetry analysis, and to a detailed discussion of asymptotics. We remark that the area limit laws cannot easily be obtained from the functional equations in [26], or from the explicit expressions given there. One can use the asymptotic description of Dyck paths by Brownian excursions [1, 2] to obtain part of our results. Whereas this requires some results from stochastic processes, our approach only needs basic probability theory and singularity analysis of generating functions, as e.g. described in [18].

The plan of this paper is as follows. We will discuss the various square lattice symmetry subgroups, and characterise the associated polygon classes by certain decompositions. This induces functional equations for their generating functions, from which the area limit laws are obtained, by an application of the moment method and the method of dominant balance. We will also indicate how some of our results may be obtained within a stochastic approach, and we finally discuss extensions of our results.

2 Symmetry classes and functional equations

We explain the models, introduce basic constructions, and set the notation, following [39]. We will then derive functional equations for the perimeter and area generating function of the symmetry subclasses of staircase polygons.

Consider two fully directed walks on the edges of the square lattice (i.e., walks stepping only up or right), which both start at the origin and end in the same vertex, but have no other vertex and no edge in common. The edge set of such a configuration is called a *staircase polygon*, if it is nonempty. For a given staircase polygon, consider the construction of moving the upper directed walk one unit down and one unit to the right. For each walk, remove its first and its last edge. The resulting object is a sequence of (horizontal and vertical) edges and staircase polygons, see Figure 1. The unit square yields the empty sequence. It is easy to see that this construction describes a combinatorial bijection between the set $\mathcal{P}$ of staircase polygons and the set $\mathcal{Q}$ of ordered sequences of edges and staircase polygons. Let us denote the corresponding map by $f : \mathcal{P} \to \mathcal{Q}$. Thus, for a staircase polygon $P \in \mathcal{P}$, we have $f(P) = (Q_1, \ldots, Q_n) \in \mathcal{Q}$, where Q_i is, for $i = 1, \ldots, n$, either a single edge or a staircase polygon. We denote the single horizontal edge by e_h, and the single vertical edge by e_v. The image of the unit square is the empty sequence $n = 0$, which we occasionally identify with a single point, denoted by pt. A variant of this construction will be used below, in order to derive a functional equation for the generating function of the staircase polygon symmetry classes.

The perimeter of a staircase polygon $P \in \mathcal{P}$ is defined to be the number of its edges. The half-perimeter equals the number of (negative) diagonals $n_0(P)$ plus one. The area $n_1(P)$ of a staircase polygon P is defined to be the number of its enclosed squares. It equals the sum of the

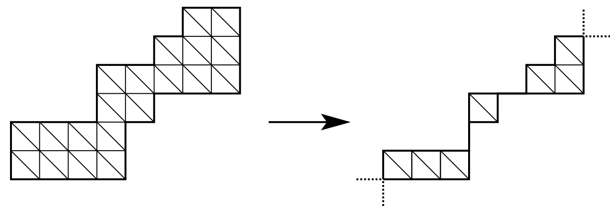

Figure 1: The set of staircase polygons is in one-to-one correspondence with the set of ordered sequences of edges and staircase polygons. A corresponding combinatorial bijection is characterised by shifting the upper walk of a staircase polygon one unit down and one unit to the right, and by then removing the first and the last edge of each walk.

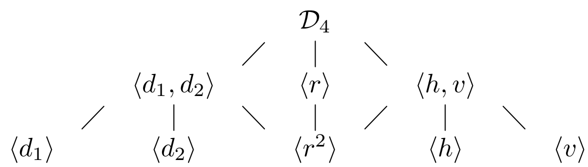

Figure 2: The lattice of subgroups of $\mathcal{D}_4$. The rotation about $\pi/2$ is denoted by r, the reflections in the positive and the negative diagonal are denoted by d_1 and d_2, and the reflections in the horizontal and vertical axes are denoted by h and v. The identity is omitted.

lengths of its (negative) diagonals. See Figure 1 for an illustration. The *weight* of a staircase polygon P is the monomial $w_P(x,q) = x^{n_0(P)+1} q^{n_1(P)}$. The half-perimeter and area generating function of a subclass $\mathcal{C} \subseteq \mathcal{P}$ of staircase polygons is the (formal) power series

$$C(x,q) := \sum_{P \in \mathcal{C}} w_P(x,q).$$

Observe that for e_h, e_v, pt, and for $P \in \mathcal{P}$ we have [39]

$$w_{f^{-1}(pt)}(x,q) = x^2 q, \qquad w_{f^{-1}(e_h)}(x,q) = x^3 q^2, \qquad w_{f^{-1}(e_v)}(x,q) = x^3 q^2,$$
$$w_{f^{-1}(P)}(x,q) = x^2 q \cdot w_P(xq, q).$$

For a polygon $P \in \mathcal{P}$, consider $f(P) = (Q_1, \ldots, Q_n)$. In order to retrieve P from $(Q_1, \ldots, Q_n)$, translate $P_i = f^{-1}(Q_i)$ in such a way that its lower left square coincides with the upper right square of P_{i-1}, for $i \in \{2, \ldots, n\}$. We say that P is the *concatenation* of $(P_1, \ldots, P_n)$, and write $P = c(P_1, \ldots, P_n)$. The weight $w_P(x,q)$ of P is retrieved from the weights of $P_1, \ldots, P_n$ via [39]

$$w_{c(P_1,\ldots,P_n)}(x,q) = \frac{1}{(x^2 q)^{n-1}} w_{P_1}(x,q) \cdot \ldots \cdot w_{P_n}(x,q).$$

Denote by $\widetilde{\mathcal{P}} \subseteq \mathcal{P}$ the subset of polygons $\widetilde{P} = f^{-1}(P)$, where $P \in \mathcal{P} \cup \{e_h, e_v\}$. We have established a combinatorial bijection between the set $\mathcal{P}$ and the set of ordered sequences from $\widetilde{\mathcal{P}}$.

The group of point symmetries of the square lattice is the dihedral group $\mathcal{D}_4$. Its non-trivial subgroups are depicted in Figure 2. Note that the above decomposition respects any subgroup of the square lattice point symmetries. This observation is the key to deriving functional equations for the generating functions of the symmetry subclasses. In the proof of the following proposition, will treat two cases in some detail, the remaining ones being handled similarly.

Proposition 2.1. *The half-perimeter and area generating functions of the staircase polygon symmetry subclasses satisfy the following functional equations.*

1. Class $\mathcal{P}$ of all staircase polygons with generating function $P(x,q)$:

$$P(x,q) = \frac{x^2 q}{1 - 2xq - P(xq,q)}. \tag{2.1}$$

2. Class $\mathcal{S}$ of $\langle r^2 \rangle$-symmetric staircase polygons with generating function $S(x,q)$:

$$S(x,q) = \frac{1}{x^2 q} \left(1 + 2xq + S(xq,q)\right) P(x^2, q^2). \tag{2.2}$$

3. Class of $\langle d_1 \rangle$-symmetric staircase polygons with generating function $D_1(x,q)$:

$$D_1(x,q) = \frac{x^2 q}{1 - D_1(xq,q)}.$$

4. Class of $\langle d_2 \rangle$-symmetric staircase polygons with generating function $D_2(x,q)$:

$$D_2(x,q) = \frac{1}{x^2 q} \left(1 + D_2(xq,q)\right) P(x^2, q^2).$$

5. Class of $\langle d_1, d_2 \rangle$-symmetric staircase polygons with generating function $D_{1,2}(x,q)$:

$$D_{1,2}(x,q) = \frac{1}{x^2 q} \left(1 + D_{1,2}(xq,q)\right) D_1(x^2, q^2).$$

6. Classes of $\langle h \rangle$-, $\langle v \rangle$-, and $\langle h, v \rangle$-symmetric staircase polygons with generating function $H(x,q)$:

$$H(x,q) = x^2 q H(xq,q) + x^2 q \frac{1 + xq}{1 - xq}.$$

7. Classes $\langle r \rangle$-symmetric staircase polygons with generating function $R(x,q)$:

$$R(x,q) = x^2 q R(xq,q) + x^2 q.$$

Proof. Denote the induced group action $\alpha : \mathcal{D}_4 \times \mathcal{P} \to \mathcal{P}$ by $\alpha(g,P) = gP$.
1. The bijection described above implies the following chain of equalities, compare also [39].

$$\begin{aligned}
P(x,q) &= \sum_{n=0}^{\infty} \sum_{(P_1,\dots,P_n)\in(\widetilde{\mathcal{P}})^n} w_{c(P_1,\dots,P_n)}(x,q) \\
&= \sum_{n=0}^{\infty} x^2 q \sum_{(P_1,\dots,P_n)\in(\widetilde{\mathcal{P}})^n} \frac{w_{P_1}(x,q)}{x^2 q} \cdot \ldots \cdot \frac{w_{P_n}(x,q)}{x^2 q} \\
&= x^2 q \sum_{n=0}^{\infty} \left(\frac{1}{x^2 q} \sum_{P \in \widetilde{\mathcal{P}}} w_P(x,q) \right)^n \\
&= x^2 q \frac{1}{1 - \frac{1}{x^2 q} \left(w_{f^{-1}(e_h)}(x,q) + w_{f^{-1}(e_v)}(x,q) + \sum_{p \in \mathcal{P}} w_{f^{-1}(P)}(x,q) \right)} \\
&= \frac{x^2 q}{1 - 2xq - P(xq,q)}.
\end{aligned} \tag{2.3}$$

2. For $P \in \mathcal{S}$, we have $f(P) = (P_1, \dots, P_n, C, r^2 P_n, \dots, r^2 P_1)$, where $C \in \mathcal{S} \cup \{e_v, e_h, pt\}$, and $P_i \in \mathcal{P} \cup \{e_v, e_h\}$ for $i = 1, \dots, n$, compare Figure 2. In analogy to the definition of $\widetilde{\mathcal{P}}$ above, define $\widetilde{\mathcal{S}} \subset \mathcal{P}$ as the pre-image of $\mathcal{S} \cup \{e_v, e_h, pt\}$ under f. Note that concatenation of $Q \in \mathcal{P}$ with

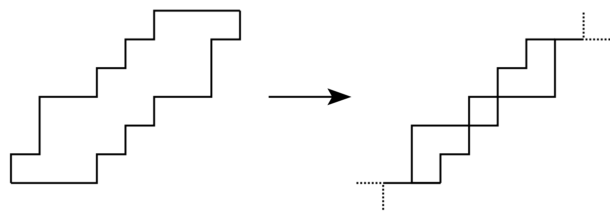

Figure 3: r^2-symmetric polygon and corresponding sequence of polygons and edges

the unit square results in Q again, and that we have $w_Q(x,q)^k = w_Q(x^k, q^k)$. With $P(x,q)$ as above, this yields

$$\begin{aligned}
S(x,q) &= \sum_{n=0}^{\infty} \sum_{(P_1,\ldots,P_n,C)\in(\widetilde{\mathcal{P}})^n\times\widetilde{\mathcal{S}}} w_{c(P_1,\ldots,P_n,C,r^2P_n,\ldots,r^2P_1)}(x,q) \\
&= \sum_{C\in\widetilde{\mathcal{S}}} w_C(x,q) \sum_{n=0}^{\infty} \sum_{(P_1,\ldots,P_n)\in(\widetilde{\mathcal{P}})^n} \frac{w_{P_1}(x,q)^2}{(x^2q)^2}\cdot\ldots\cdot\frac{w_{P_n}(x,q)^2}{(x^2q)^2} \\
&= \left(w_{f^{-1}(pt)} + w_{f^{-1}(e_h)} + w_{f^{-1}(e_v)} + \sum_{C\in\mathcal{S}} w_{f^{-1}(C)}\right) \sum_{n=0}^{\infty}\left(\sum_{Q\in\widetilde{\mathcal{P}}} \frac{w_Q(x^2,q^2)}{(x^2q)^2}\right)^n \qquad (2.4) \\
&= \left(x^2q + 2x^3q^2 + x^2qS(xq,q)\right)\frac{1}{1-(2x^2q^2+P(x^2q^2,q^2))} \\
&= \frac{1+2xq+S(xq,q)}{x^2q}\cdot\frac{x^4q^2}{1-2x^2q^2-P(x^2q^2,q^2)} \\
&= \frac{1}{x^2q}\left(1+2xq+S(xq,q)\right)P(x^2,q^2)
\end{aligned}$$

where the sum over $Q \in \widetilde{\mathcal{P}}$ in the third equation is treated as in eqn. (2.3). In the last step, we applied eqn. (2.1).

3. For a $\langle d_1\rangle$-symmetric polygon Q, we have $f(Q) = (P_1, \ldots P_n)$, with a $\langle d_1\rangle$-symmetric P_i for $i = 1, \ldots, n$. A calculation similar to that in eqn. (2.3) then yields the assertion.

4. For a $\langle d_2\rangle$-symmetric polygon Q, we have $f(Q) = (P_1, \ldots P_n, C, d_2P_n, \ldots, d_2P_1)$, where $P_i \in \mathcal{P} \cup \{e_h, e_v\}$ for $i = 1, \ldots, n$, and where C is a *pt* or $\langle d_2\rangle$-symmetric. Now a computation similar to that in eqn. (2.4) yields the assertion.

5. For a $\langle d_1, d_2\rangle$-symmetric polygon Q, we have $f(Q) = (P_1, \ldots P_n, C, d_2P_n, \ldots, d_2P_1)$, with $\langle d_1\rangle$-symmetric P_i for $i = 1, \ldots, n$, and where C is a *pt* or $\langle d_1 d_2\rangle$-symmetric. A computation similar to that of eqn. (2.4) yields the assertion.

6. Staircase polygons are also characterised by the property that they contain the lower left and the upper right corner of their smallest bounding rectangles. So the only staircase polygons with $\langle h\rangle$- or $\langle v\rangle$- symmetry are rectangles. f maps a rectangle Q either to a single rectangle, or to sequences of vertical (horizontal) edges, if the width (height) of Q is 1. This results in the above equation.

7. The only admissible polygons are squares. For a given half-perimeter, there is exactly one square. If $n > 1$, the function f maps a square of half-perimeter $2n$ to the square of half-perimeter $2n - 2$, and it maps the unit square to *pt*. We obtain the claimed equation.

□

Remark 2.2. Equations of the above form appear in different contexts. Examples are classes of directed lattice paths, counted by length and area under the path [32], or classes of simply generated trees [31], counted by number of vertices and internal path length. This is due to combinatorial bijections between these classes, which we will partly review in Section 5. In the context of polygon models, equations appear for Class 1 in [4] and for Class 6 in [37], while Class 7 is trivial. Solutions of some equations may be given in explicit form, compare [5, 34].

3 Area limit laws

In this section, we derive the limiting area laws for the various symmetry subclasses, in the uniform fixed perimeter ensemble. This will be achieved by an application of the moment method [3, Sec. 30]. Such an approach has been used previously [42, 43, 32, 12] in similar contexts, using some involved computations. We will follow a streamlined version, based on the method of dominant balance [38], which finally allows to obtain the limit distribution by a mechanical calculation. In order to give a self-contained description of the method, we will treat the two cases $\mathcal{P}$ and $\mathcal{S}$ in detail, and then indicate the analogous arguments for the remaining subclasses.

3.1 Limit law for $\mathcal{P}$

A q-difference equation for the half-perimeter and area generating function $P(x,q)$ of all staircase polygons was derived in Proposition 2.1. For $q=1$, the resulting quadratic equation describes the generating function $P(x,1)$ of staircase polygons, counted by half-perimeter. The relevant solution is

$$P(x,1)=\frac{1}{4}-\frac{1}{2}\sqrt{1-4x}+\frac{1}{4}(1-4x). \tag{3.1}$$

We are interested in the distribution of area within a uniform ensemble where, for fixed perimeter $2m$, each polygon has the same probability of occurrence. We introduce the discrete random variable X_m of area by

$$\mathbb{P}_m(X_m=n)=\frac{[x^m q^n]P(x,q)}{[x^m]P(x,1)}, \tag{3.2}$$

where $[u^k]f(u)$ denotes the coefficient of order k in the formal power series $f(u)$. In the following, we will asymptotically analyse the moments of X_m. The answer can be expressed in terms of the *Airy distribution*, see [15, 24, 25] for a discussion of its properties.

Definition 3.1. *A random variable Y is* Airy distributed *[15] if*

$$\frac{\mathbb{E}[Y^k]}{k!}=\frac{\Gamma(\gamma_0)}{\Gamma(\gamma_k)}\frac{\phi_k}{\phi_0},$$

where $\gamma_k=3k/2-1/2$, and where $\Gamma(z)$ is the Gamma function. The numbers ϕ_k satisfy for $k\in\mathbb{N}$ the quadratic recursion

$$\gamma_{k-1}\phi_{k-1}+\frac{1}{2}\sum_{l=0}^{k}\phi_l\phi_{k-l}=0,$$

with initial condition $\phi_0=-1$.

Remark 3.2. *i)* In the sequel, we shall make frequent use of *Carleman's condition*: A sequence of moments $\{M_m\}_{m\in\mathbb{N}}$ with the property $\sum_k (M_{2k})^{-1/(2k)}=\infty$ defines a unique random variable X with moments M_m, cf. [13].
ii) This implies in particular, that Y is uniquely determined by the above moment sequence. Explicit expressions can be given for its moments, its moment generating function, and its density. The name relates to the asymptotic expansion

$$\frac{\mathrm{d}}{\mathrm{d}s}\log \mathrm{Ai}(s)\sim\sum_{k\geq 0}(-1)^k\frac{\phi_k}{2^k}s^{-\gamma_k}\qquad (s\to\infty),$$

where $\mathrm{Ai}(x)=\frac{1}{\pi}\int_0^\infty \cos(t^3/3+tx)\mathrm{d}t$ is the Airy function. The Airy distribution appears in a variety of contexts. In particular, the random variable $Y/\sqrt{8}$ describes the Brownian excursion area.

We can now state the following result.

Theorem 3.3. *For staircase polygons of half-perimeter m, the area random variables X_m eqn.* (3.2)*, appropriately normalised, converge in distribution,*

$$\frac{X_m}{m^{3/2}} \xrightarrow{d} \frac{Y}{4} \qquad (m \to \infty).$$

where Y is Airy distributed. We also have moment convergence.

Remark 3.4. The previous theorem is a special case of [12, Thm. 3.1] and [38, Thm. 1.5]. In [12], a limit distribution result is stated for certain algebraic q-difference equations, together with arguments of a proof using the moment method. In [38], a general multivariate limit distribution result is proved for certain q-functional equations, using the moment method and the method of dominant balance. The corresponding argument for staircase polygons, using the moment method and the method of dominant balance, is sketched in [39].

For pedagogic purposes, we will give a proof of the above result using the moment method and the method of dominant balance. In particular, we show that the moments of X_m, appropriately normalised, converge to those of Y. Since Y is uniquely determined by its moments by Carleman's condition, this implies that the sequence $\{X_m\}_{m\in\mathbb{N}}$ converges, after normalisation, to Y in distribution and for moments, compare [6, Thm. 4.5.5].

We will analyse the asymptotic behaviour of the moments in terms of the singular behaviour of the associated *factorial moment generating functions* which are, for $(k,l) \in \mathbb{N}_0^2$, defined by

$$P_{k,l}(x) := \left.\frac{\partial^{k+l}}{\partial q^k \partial x^l} P(x,q)\right|_{q=1}.$$

In particular, $P_{0,0}(x) = P(x,1)$ is the generating function of staircase polygons, counted by half-perimeter. Note that these quantities exist as formal power series and have the same radius of convergence as $P(x,1)$. This is due to the fact that the area of a polygon is bounded by the square of its perimeter, resulting in $[x^m]\,P(x,q)$ being a *polynomial* in q. The name results from the identity

$$\mathbb{E}_m[(X_m)_k] = \frac{[x^m]P_{k,0}(x)}{[x^m]P_{0,0}(x)}, \tag{3.3}$$

where $(a)_k = a \cdot (a-1) \cdot \ldots \cdot (a-k+1)$ is the lower factorial. The factorial moment generating functions $P_{k,l}(x)$ turn out to be algebraic. Explicit expressions may be obtained recursively from the functional equation eqn. (2.1), by implicit differentiation w.r.t. x and q.

We will study the singular behaviour of the factorial moment generating functions from their defining functional equation, and will then infer the asymptotic behaviour of the moments. This will be achieved in three steps. We will first prove the existence of a certain local expansion for each factorial moment generating function about its singularity, by an application of the chain rule (or Faà di Bruno's formula). Then we will provide an explicit expression for the leading term in the expansion, by an application of the method of dominant balance. This will finally be analysed in order to obtain the asymptotic behaviour of the corresponding moment, by methods from singularity analysis of generating functions.

We note that steps one and two are usually performed simultaneously, the corresponding method being nicknamed *moment pumping* [17]. Our two-step approach uses an exponent guess, which might be obtained from an analysis of the first few factorial moment generating functions. It is then shown that the guessed exponent value is an upper bound on the true exponent, by an application of Faà di Bruno's formula. The corresponding calculation is simpler to perform than the usual asymptotic analysis of the functional equation. Once an exponent bound has been established, the method of dominant balance can be applied. It yields a recursion for the coefficients of the leading singular term in the factorial moment generating functions. If the recursion reveals non-zero coefficients, this proves that the exponent bound is actually an equality, thereby verifying the initial guess.

The first step of our method is summarised by the following lemma. For its statement, recall that a function $f(u)$ is Δ-*regular* [14] if it is analytic in the *indented disc* $\Delta = \Delta(u_c) = \{u : |u| \leq u_c + \eta, |\arg(u - u_c)| \geq \phi\}$ for some real numbers $u_c > 0$, $\eta > 0$ and ϕ, where $0 < \phi < \pi/2$. Note that $u_c \notin \Delta$, where we employ the convention $\arg(0) = 0$. The set of Δ-regular functions is closed under addition, multiplication, differentiation, and integration. Moreover, if $f(u) \neq 0$ in Δ, then $1/f(u)$ exists in Δ and is Δ-regular.

Lemma 3.5. *For $(k,l) \in \mathbb{N}_0^2$, the power series $P_{k,l}(x)$ has radius of convergence $1/4$ and is $\Delta(1/4)$-regular. It has a locally convergent expansion about $x = 1/4$, as in eqn.* (3.1) *for $(k,l) = (0,0)$, and for $(k,l) \neq (0,0)$ of the form*

$$P_{k,l}(x) = \sum_{r=0}^{\infty} \frac{d_{k,l,r}}{(1-4x)^{3k/2+l-r/2-1/2}}. \tag{3.4}$$

Remark 3.6. *i)* The exponent $3k/2$ in eqn. (3.4) might be guessed from the asymptotic behaviour of the mean area $\mathbb{E}_m[X_m] \sim Am^{3/2}$ of a random polygon. The mean area is obtained from $P_{0,0}(x)$ and $P_{1,0}(x)$, which might be easily extracted from the q-difference equation. Note that the coefficients $d_{k,l,0}$ might attain zero values at this stage. The recursion eqn. (3.10) given below however implies that all of them are non-zero.
ii) The reasoning in the following proof may be used to show that all series $P_{k,l}(x)$ are algebraic.
iii) For $\langle r^2 \rangle$-symmetric polygons, our proof below will use properties of the derivatives

$$\widetilde{P}_{k,l}(x) := \left. \frac{\partial^{k+l}}{\partial q^k \partial x^l} \left(P(x^2, q^2)\right) \right|_{q=1}.$$

These functions have all radius of convergence $1/2$, are $\Delta(1/2)$-regular, and have the same type of expansion as the functions $P_{k,l}(x)$ of the previous lemma. This may be inferred from the previous lemma by the chain rule or, more formally, by an application of Faà di Bruno's formula [8].

Proof. By the argument given above, it is seen that all functions $P_{k,l}(x)$ have the same radius of convergence. The statement of the theorem is true for $P_{0,0}(x) = P(x,1)$, as follows from the explicit expression eqn. (3.1). For the general case, we argue by induction on (k,l), using the total order $\lhd$ defined by

$$(r,s) \lhd (k,l) \Leftrightarrow r+s < k+l \vee (r+s = k+l \wedge r < k),$$

chosen to be compatible with the combinatorics of derivatives. Define

$$H(x,q) := P(x,q)(1 - 2xq - P(xq,q)) - x^2 q,$$

compare eqn. (2.1). Fix $(k,l) \rhd (0,0)$. An application of Leibniz' rule yields

$$\begin{aligned} \frac{\partial^{k+l}}{\partial q^k \partial x^l} \left(H(x,q) + x^2 q\right) = \sum_{(0,0) \unlhd (r,s) \unlhd (k,l)} \binom{k}{r}\binom{l}{s} \frac{\partial^{k+l-r-s}}{\partial q^{k-r} \partial x^{l-s}} \left(P(x,q)\right) \\ \cdot \frac{\partial^{r+s}}{\partial q^r \partial x^s} \left(1 - 2xq - P(xq,q)\right) \end{aligned} \tag{3.5}$$

In fact, terms corresponding to indices $(r,s) \unlhd (k,l)$ with $s > l$ or $r > k$ are zero. For the second derivative on the r.h.s. of eqn. (3.5), note that by the chain rule

$$\frac{\partial^r}{\partial q^r} \left(P(xq,q)\right) = \sum_{i=0}^{r} \binom{r}{i} q^{r-i} \left(\frac{\partial^r}{\partial x^{r-i} \partial q^i} P\right)(xq,q).$$

Taking further derivatives w.r.t. x, we may write

$$\begin{aligned} \frac{\partial^{r+s}}{\partial q^r \partial x^s} \left(P(xq,q)\right) &= q^{r+s} \left(\frac{\partial^{r+s}}{\partial q^r \partial x^s} P\right)(qx,q) \\ &+ \sum_{(i,j) \lhd (r,s)} \left(\frac{\partial^{i+j}}{\partial q^i \partial x^j} P\right)(xq,q) \cdot w_{i,j}(x,q), \end{aligned} \tag{3.6}$$

for polynomials $w_{i,j}(x,q)$ in x and q, which satisfy $w_{i,j}(x,q) \equiv 0$ if $i < r$. By inserting eqn. (3.6) into eqn. (3.5) and setting $q = 1$, one observes that only the $(0,0)$ and the (k,l) summand in eqn. (3.5) contribute terms with $P_{k,l}(x)$. The terms involving $P_{k,l}(x)$ sum up to

$$P_{k,l}(x)\,(1 - 2x - 2P_{0,0}(x)) = \sqrt{1-4x}\,P_{k,l}(x),$$

where we used eqn. (3.1). Now the claimed $\Delta(1/4)$-regularity of $P_{k,l}(x)$ follows from the induction hypothesis, by the closure properties of Δ-regular functions. For the particular singular expansion eqn. (3.4) note that, by induction hypothesis, each of the remaining terms in the summation in eqn. (3.5) has an expansion eqn. (3.4). Hence, the most singular exponent is bounded by

$$\left(\frac{3}{2}(k-r) + (l-s) - \frac{1}{2}\right) + \left(\frac{3}{2}r + s - \frac{1}{2}\right) = \frac{3}{2}k + l - 1.$$

We conclude that the leading singular exponent of $P_{k,l}(x)$ is at most $3k/2 + l - 1/2$, which yields the desired bound, and thus the remaining assertion of the theorem.

□

The second and third step of our method yield a proof of Theorem 3.3.

Proof of Theorem 3.3. We apply the method of dominant balance [38] in order to obtain the limit distribution of area. Its idea consists in first replacing the factorial moment generating functions, which appear in the formal expansion of $P(x,q)$ about $q = 1$, by their singular expansion of Lemma 3.5, and then in studying the equation implied by the q-difference equation eqn. (2.1). We may thus write

$$P(x,q) = \frac{1}{4} + (1-q)^{1/3}\,F\left(\frac{1-4x}{(1-q)^{2/3}}, (1-q)^{1/3}\right), \tag{3.7}$$

where $F(s,\epsilon) = \sum_r F_r(s)\epsilon^r$ is a formal power series in ϵ with coefficients $F_r(s)$ being formal Laurent series in $s^{1/2}$. The series

$$F_0(s) = F(s,0) = \sum_k \frac{d_{k,0,0}}{k!} \cdot \frac{(-1)^k}{s^{3k/2-1/2}} \tag{3.8}$$

is some generating function for the leading coefficients of $P_{k,0}(x)$. The coefficients $f_k := d_{k,0,0}/k!$, in turn, determine the asymptotic form of the factorial moments $\mathbb{E}[(X_m)_k]$ in eqn. (3.3), as we will see below. We will use the q-difference equation eqn. (2.1) to derive a defining equation for $F_0(s)$. This will lead to a simple quadratic recursion for the numbers f_k. Use the above form of $P(x,q)$ in the q-difference equation, introduce $4x = 1 - s\epsilon^2$, $q = 1 - \epsilon^3$, and expand the functional equation to second order in ϵ. This yields a Riccati equation for the generating function $F_0(s)$,

$$\frac{\mathrm{d}}{\mathrm{d}s}F_0(s) + 4F_0(s)^2 - s = 0. \tag{3.9}$$

On the level of coefficients of $F_0(s)$, we obtain the recursion

$$\gamma_{k-1} f_{k-1} + 4\sum_{l=0}^{k} f_l f_{k-l} = 0, \tag{3.10}$$

with initial condition $f_0 = d_{0,0,0} = -1/2$. We infer from the definition of the Airy distribution that $f_k = 2^{-2k-1}\phi_k$. In particular, all coefficients f_k are non-zero. Noting that the functions $P_{k,0}(x)$ are $\Delta(1/4)$-regular, we thus get by an application of the transfer lemma [16, Thm. 1] for the factorial moments of X_m the asymptotic form

$$\begin{aligned}\frac{\mathbb{E}_m[(X_m)_k]}{k!} &= \frac{1}{k!}\frac{[x^m]P_{k,0}(x)}{[x^m]P_{0,0}(x)} \sim \frac{1}{k!}\frac{[x^m]d_{k,0,0}(1-4x)^{-(3k/2-1/2)}}{[x^m]d_{0,0,0}(1-4x)^{1/2}}\\ &\sim \frac{f_k}{f_0}\frac{\Gamma(-1/2)}{\Gamma(3k/2-1/2)}m^{3k/2} = \frac{\phi_k}{\phi_0}\frac{\Gamma(\gamma_0)}{\Gamma(\gamma_k)}\left(\frac{m^{3/2}}{4}\right)^k \qquad (m \to \infty).\end{aligned}$$

The previous estimate also shows that the factorial moment $\mathbb{E}_m[(X_m)_k]$ is asymptotically equal to the ordinary moment $\mathbb{E}_m[(X_m)^k]$. It follows with [6, Thm. 4.5.5] that the sequence of random variables $\left\{4m^{-3/2}X_m\right\}_{m\in\mathbb{N}}$ converges in distribution to Y, where Y is Airy distributed. The above reasoning also implies moment convergence.

□

Remark 3.7. If we expand the functional equation to higher order in the above example, we obtain at order ϵ^{r+2} a linear differential equation for the function $F_r(s)$, which is the generating function for the numbers $d_{k,0,r}$ in the expansion eqn. (3.4), compare [37]. So we can mechanically obtain corrections to the asymptotic behaviour of the factorial moment generating functions, and hence to the moments of the limit distribution.

3.2 Limit law for $\mathcal{S}$

The above strategy can also be followed in order to study the area law for the class of $\langle r^2\rangle$-symmetric staircase polygons. The result can be expressed in terms of the distribution of area of the Brownian meander, see [43, Thms. 2,3] and the review [24].

Definition 3.8. *The random variable Z of area of the Brownian meander is given by*

$$\frac{\mathbb{E}[Z^k]}{k!} = \frac{\Gamma(\alpha_0)}{\Gamma(\alpha_k)}\frac{\omega_k}{\omega_0}\frac{1}{2^{k/2}},$$

where $\alpha_k = 3k/2 + 1/2$. The numbers ω_k satisfy for $k \in \mathbb{N}$ the quadratic recursion

$$\alpha_{k-1}\omega_{k-1} + \sum_{l=0}^{k} \phi_l 2^{-l}\omega_{k-l} = 0, \tag{3.11}$$

with initial condition $\omega_0 = 1$, where the numbers ϕ_k appear in the Airy distribution.

Remark 3.9. By Carleman's condition, the random variable Z is uniquely determined by its moments, and explicit expressions are known for the moment generating function and the distribution function.

We are particularly interested in the derivatives

$$S_{k,l}(x) := \left.\frac{\partial^{k+l}}{\partial q^k \partial x^l} S(x,q)\right|_{q=1},$$

where $(k,l) \in \mathbb{N}_0^2$. As above, these series exist as formal power series and have the same radius of convergence. We have the following lemma.

Lemma 3.10. *For $(k,l) \in \mathbb{N}_0^2$, the power series $S_{k,l}(x)$ has radius of convergence $1/2$ and is $\Delta(1/2)$-regular. It has a locally convergent expansion about $x = 1/2$ of the form*

$$S_{k,l}(x) = \sum_{r\geq 0} \frac{s_{k,l,r}}{(1-2x)^{3k/2+l-r/2+1/2}}.$$

Remark 3.11. The following proof can be used to show that all series $S_{k,l}(x)$ are algebraic.

Proof. The proof is analogous to that of Lemma 3.5. Elementary estimates show that all series $S_{k,l}(x)$ have the same radius of convergence. Setting $q = 1$ in eqn. (2.2), solving for $S_{0,0}(x)$ and expanding about $x = 1/2$ yields the assertion for $(k,l) = (0,0)$. We argue by induction on (k,l), using the total order $\lhd$. Fix $(k,l) \rhd (0,0)$. Differentiating eqn. (2.2) with Leibniz' Rule gives

$$\begin{aligned}\frac{\partial^{k+l}}{\partial q^k \partial x^l} S(x,q) = &\sum_{(0,0)\unlhd(r,s)\unlhd(k,l)} \binom{k}{r}\binom{l}{s} \frac{\partial^{r+s}}{\partial q^r \partial x^s}\left(\frac{P(x^2,q^2)}{x^2 q}\right) \\ &\cdot \frac{\partial^{k+l-r-s}}{\partial q^{k-r}\partial x^{l-s}}\left(1 + 2xq + S(xq,q)\right).\end{aligned} \tag{3.12}$$

We argue as in the proof of Lemma 3.5 that only the $(0,0)$ summand on the right hand side of eqn. (2.2) contributes (k,l) derivatives of S, and that all other derivatives of S of order (r,s) satisfy $(r,s) \lhd (k,l)$. Setting $q=1$ in (2.2) and collecting all terms involving $S_{k,l}(x)$ on the left hand side gives

$$\left(1-\frac{\widetilde{P}_{0,0}(x)}{x^2}\right) S_{k,l}(x) = \frac{\widetilde{P}_{0,0}(x)}{x^2}(1+2x) + \sum_{(r,s)\unlhd(k,l)} \frac{\partial^{r+s}}{\partial q^r \partial x^s}\left(\frac{P(x^2,q^2)}{x^2 q}\right)\Bigg|_{q=1} \cdot \left(h_{r,s}(x) + \sum_{(i,j)\lhd(k,l)} a_{i,j} S_{i,j}(x)\right), \tag{3.13}$$

where the $h_{r,s}(x)$ are (at most linear) polynomials, and the $a_{i,j}$ are some real coefficients. Note also that the terms

$$\frac{\partial^{r+s}}{\partial q^r \partial x^s}\left(\frac{P(x^2,q^2)}{x^2 q}\right)\Bigg|_{q=1} = \sum_{i,j} \binom{r}{i}\binom{s}{j} \widetilde{P}_{i,j}(x) \frac{c_{i,j}}{x^{2+r-i} q^{1+s-j}}$$

are $\Delta(1/2)$-regular, with an expansion about $x=1/2$ having the same exponents as in eqn. (3.4), see the remark following Lemma 3.5. We thus get $\Delta(1/2)$-regularity of $S_{k,l}(x)$ by induction, and by the closure properties of Δ-regular functions. For the particular expansion, note that the right hand side has a locally convergent expansion about $1/2$ with most singular exponent $3k/2+l$, as the factor $\frac{\partial^{r+s}}{\partial q^r \partial x^s}\left(\frac{P(x^2,q^2)}{x^2 q}\right)\Big|_{q=1}$ has an expansion with most singular exponent $3r/2+s-1/2$, and the inner sum has by induction an expansion with most singular exponent at most $3(k-r)/2+(l-s)+1/2$. The first factor on the left hand side has a locally convergent expansion about $1/2$ starting with

$$\left(1-\frac{\widetilde{P}_{0,0}(x)}{x^2}\right) = -2\sqrt{2}\sqrt{1-2x} + \mathcal{O}(1-2x) \qquad (x \to 1/2).$$

Solving for $S_{k,l}(x)$ yields the desired expansion. □

Theorem 3.12. *The random variables $X_m^{(sym)}$ of area of $\langle r^2\rangle$-symmetric staircase polygons of half-perimeter m, appropriately normalised, converge in distribution,*

$$\frac{X_m^{(sym)}}{m^{3/2}} \xrightarrow{d} \frac{Z}{2} \qquad (m \to \infty),$$

where Z is the meander area random variable. We also have moment convergence.

Proof. We apply the method of dominant balance. According to Lemma 3.10, the generating function $S(x,q)$ may be expressed as

$$S(x,q) = \frac{1}{(1-q)^{1/3}} G\left(\frac{1-2x}{(1-q)^{2/3}}, (1-q)^{1/3}\right), \tag{3.14}$$

where $G(s,\epsilon) = \sum G_r(s)\epsilon^r$ is a formal power series in ϵ and $s^{-1/2}$, and

$$G(s,0) = G_0(s) = \sum_{k=0}^{\infty} \frac{s_{k,0,0}}{k!} \frac{(-1)^k}{s^{3k/2+1/2}}$$

is a generating function for the leading coefficients in the singular expansions of the functions $S_{k,0}(x)$. The functional equation eqn. (2.2) induces a recursion on the numbers $g_k := s_{k,0,0}/k!$, which determines the limit distribution, as we will see below. We insert eqn. (3.14) together with eqn. (3.7) into the functional equation, introduce $q = 1-\epsilon^3$ and $2x = 1 - s\epsilon^2$, and expand the functional equation to order zero in ϵ. This gives the linear inhomogeneous first order differential equation

$$\frac{\mathrm{d}}{\mathrm{d}s}G_0(s) + 4\cdot 2^{1/3}F_0\left(2^{1/3}s\right)G_0(s) + 2 = 0, \tag{3.15}$$

where $F_0(s)$ is given by eqn. (3.8). On the level of coefficients, we have the recursion

$$\alpha_{k-1}g_{k-1} + \sum_{l=0}^{k} 2^{-l/2+5/2} f_l g_{k-l} = 0, \tag{3.16}$$

with the initial condition $g_0 = s_{0,0,0} = 2^{-1/2}$. If we set

$$g_k = \frac{\omega_k}{2^{3k/2+1/2}},$$

then the above recursion is identical to that occurring in the definition of the meander distribution. In particular, all numbers g_k are non-zero. Since the functions $S_{k,0}(x)$ are $\Delta(1/2)$-regular, we may use the transfer lemma [16, Thm. 1] to infer for the moments of $X_m^{(sym)}$ the asymptotic form

$$\begin{aligned}\frac{\mathbb{E}_m[(X_m^{(sym)})_k]}{k!} &= \frac{1}{k!}\frac{[x^m]S_{k,0}(x)}{[x^m]S_{0,0}(x)} \sim \frac{1}{k!}\frac{[x^m]s_{k,0,0}(1-2x)^{-(3k/2+1/2)}}{[x^m]s_{0,0,0}(1-2x)^{-1/2}} \\ &\sim \frac{g_k}{g_0}\frac{\Gamma(1/2)}{\Gamma(3k/2+1/2)}m^{3k/2} = \frac{1}{2^k}\frac{\omega_k}{\omega_0}\frac{\Gamma(\alpha_0)}{\Gamma(\alpha_k)}\frac{1}{2^{k/2}}m^{3k/2} \qquad (m\to\infty).\end{aligned}$$

The last term is, up to the factor $m^{3k/2}$, the k-th moment of $Z/2$, where Z is the meander area variable. The previous estimate shows that the factorial moments $\mathbb{E}_m[(X_m^{(sym)})_k]$ are asymptotically equal to the ordinary moments $\mathbb{E}_m[(X_m^{(sym)})^k]$. It follows with [6, Thm. 4.5.5] that the sequence of random variables $\left\{2m^{-3/2}X_m^{(sym)}\right\}_{m\in\mathbb{N}}$ converges in distribution to Z, where Z is distributed as the meander area. We also have moment convergence.

□

Remark 3.13. As for the full class of staircase polygons, corrections to the asymptotic behaviour of the factorial moment generating functions can be mechanically obtained also for this example, by expanding the corresonding functional equation to higher orders in ϵ.

3.3 Limit law for $\langle d_1\rangle$-symmetric polygons

These polygons always have even half-perimeter. In order to derive an area limit law, we thus restrict to area random variables of quarter-perimeter. Note that we have $\widetilde{D}_1(x,q) = D_1(x^{1/2},q)$ for the generating function of the class $\langle d_1\rangle$-symmetric polygons, counted by quarter-perimeter and area. The functional equation for $D_1(x,q)$ induces a similar one for $\widetilde{D}_1(x,q)$. Their factorial moment generating functions all have radius of convergence 1/4, and a statement as in Lemma 3.5 can be formulated and proven almost verbatim for $\widetilde{D}_1(x,q)$. The method of dominant balance then yields a generating function for the leading coefficients in the singular expansions, a defining equation similar to eqn. (3.9), and a recursion similar to eqn. (3.10). We have the following result.

Theorem 3.14. *The area random variables X_m of $\langle d_1\rangle$-symmetric staircase polygons, indexed by quarter-perimeter m and scaled by $m^{-3/2}$, converge in distribution to a random variable Y, which is Airy distributed. We also have moment convergence.*

3.4 Limit law for $\langle d_2\rangle$-symmetric polygons

In [26], a combinatorial bijection between $\langle d_2\rangle$-symmetric polygons and $\langle r^2\rangle$-symmetric polygons with even half-perimeter is described: cut a $\langle d_2\rangle$-symmetric polygon along the line of reflection, flip its upper right part, and glue the two parts together along the cut. So Theorem 3.12 translates to the $\langle d_2 \rangle$-case.

Alternatively, one may apply the methods of Section 3.2, together with modifications similar to those of Section 3.3, to the quarter-perimeter and area generating function $\widetilde{D}_2(x, q) = D_2(x^{1/2}, q)$. Lemma 3.10 holds in this case, with 1/2 replaced by 1/4, and the method of dominant balance yields results similar to eqn. (3.15) and eqn. (3.16).

Theorem 3.15. *The area random variables X_m of $\langle d_2 \rangle$-symmetric staircase polygons, indexed by quarter-perimeter m and scaled by $(2m)^{-3/2}$, converge in distribution to a random variable $Z/2$, where Z is the meander area random variable. We also have moment convergence.*

3.5 Limit law for $\langle d_1, d_2 \rangle$-symmetric polygons

In this symmetry class, every polygon has even half-perimeter. So we define $\widetilde{D}_{12}(x, q) = D_{12}(x^{1/2}, q)$ as above, and obtain from the functional equation for $D_{12}(x, q)$ one for $\widetilde{D}_{12}(x, q)$, which involves $\widetilde{D}_1(x, q)$, resembling eqn. (2.2).

It can be argued, as in the proof Lemma 3.10, that all factorial moment generating functions $\frac{\partial^k}{\partial q^k} \widetilde{D}_{12}(x, q)\Big|_{q=1}$ have radius of convergence 1/2, with singularities at $\pm 1/2$, where the leading singular behaviour of the coefficients is determined by the singularity at 1/2. We can apply the methods of Section 3.2, with the modifications of Section 3.3. This yields the following result.

Theorem 3.16. *The sequence of area random variables X_m of $\langle d_1, d_2 \rangle$-symmetric staircase polygons, indexed by quarter-perimeter m and scaled by $m^{-3/2}$, converges in distribution to $2Z$, where Z is the meander area variable. We also have moment convergence.*

3.6 Limit law for $\langle r \rangle$-symmetric polygons

The class of staircase polygons with $\langle r \rangle$-symmetry is the class of squares. These may be counted by quarter-perimeter m. Since for given m there is exactly one square, they have, after scaling by m^{-2}, a concentrated limit distribution $\delta(x - 1)$. This result can also be obtained from the q-difference equation in Proposition 2.1.

3.7 Limit law for $\langle h, v \rangle$- ($\langle h \rangle$-, $\langle v \rangle$-) symmetric polygons

The class of staircase polygons with $\langle h, v \rangle$-symmetry (or with $\langle h \rangle$- or $\langle v \rangle$-symmetry) is the class of rectangles. These have been discussed in [40]. The k-th moments of the area random variable X_m, with m half-perimeter, cf. eqn. (3.2), are given explicitly by

$$\mathbb{E}[X_m^k] = \sum_{l=1}^{m-1} (l(m-l))^k \frac{1}{m-1} \sim m^{2k} \int_0^1 (x(1-x))^k \mathrm{d}x = \frac{(k!)^2}{(2k+1)!} m^{2k} \qquad (m \to \infty),$$

where we used a Riemann sum approximation. Consider the normalised random variable

$$\widetilde{X}_m = 4X_m/m^2.$$

The moments of $\widetilde{X}_m$ converge as $m \to \infty$, and the limit sequence $M_k = \lim_{m \to \infty} \mathbb{E}_m[\widetilde{X}_m^k]$ satisfies Carleman's condition and hence defines a unique random variable with moments M_k. The corresponding distribution is the beta distribution $\beta_{1,1/2}$. We arrive at the following result.

Theorem 3.17. *The sequence $\widetilde{X}_m = 4X_m/m^2$ of area random variables of rectangles, with half-perimeter m scaled by $4/m^2$, converges in distribution to a $\beta_{1,1/2}$-distributed random variable. We also have moment convergence.*

One may also obtain this result by manipulating the associated q-difference equation, see [40]. Expansions of the factorial moment generating functions about their singularity at $x = 1$ can be derived, and bounds for their most singular exponent can be given. The method of dominant balance can then be applied to obtain the leading singular coefficient of these expansions.

4 Limit law for orbit counts

Let $\mathcal{H}$ be a subgroup of $\mathcal{D}_4$. By the Lemma of Burnside, the half-perimeter and area generating function $P_{\mathcal{H}}(x,q)$ of orbit counts w.r.t. $\mathcal{H}$ is given by [26]

$$P_{\mathcal{H}}(x,q) = \frac{1}{|\mathcal{H}|}\sum_{g\in\mathcal{H}} P_{\mathrm{Fix}(g)}(x,q) = \frac{1}{|\mathcal{H}|}\left(P(x,q)+R(x,q)\right),$$

where $|\mathcal{H}|$ denotes the cardinality of $\mathcal{H}$, and where $\mathrm{Fix}(g) \subseteq \mathcal{P}$ is the subclass of staircase polygons, which are fixed under $g \in \mathcal{D}_4$, with half-perimeter and area generating function $P_{\mathrm{Fix}(g)}(x,q)$. The series $P(x,q)$ is the full staircase polygon half-perimeter and area generating function, and $R(x,q)$ is the sum of generating functions of staircase polygons which are fixed under $g \in \mathcal{H}$, where $g \neq e$. Let $P(x,1) = \sum_m p_m x^m$ and $R(x,1) = \sum_m r_m x^m$. Due to the previous discussion, see also [26, Prop. 14], the number of polygons fixed by some non-trivial symmetry grows subexponentially w.r.t. the total number of polygons. This implies

$$\frac{m^\alpha r_m}{p_m} \to 0 \qquad (m\to\infty), \tag{4.1}$$

for any real number α. As a consequence, area limit distributions for orbit counts coincide with those for the full class of staircase polygons.

Theorem 4.1. *Let $\mathcal{H}$ be a subgroup of $\mathcal{D}_4$. Then the area limit law of the class $\mathcal{P}/\mathcal{H}$ coincides with that of $\mathcal{P}$.*

Proof. We show that both classes have asymptotically the same area moments. Since in all examples the limit distribution is uniquely determined by its moments, the claim follows.

Note that, for polygons of half-perimeter m, their area n satisfies $1 \le n \le m^2$. Let $P(x,q) = \sum p_{m,n}x^m q^n$ and $R(x,q) = \sum r_{m,n}x^m q^n$. By eqn. (4.1), we have for $k \in \mathbb{N}_0$

$$\frac{\sum_n n^k r_{m,n}}{\sum_n n^k p_{m,n}} \le \frac{m^{2k} r_m}{p_m} \to 0 \qquad (m\to\infty).$$

This implies for the coefficients of the moment generating functions the asymptotic estimate

$$\begin{aligned}[x^m]\left(q\frac{\partial}{\partial q}\right)^k P_{\mathcal{H}}(x,q)\Big|_{q=1} &= \frac{1}{|\mathcal{H}|}\sum_n n^k (p_{m,n}+r_{m,n}) \\ &\sim \frac{1}{|\mathcal{H}|}\sum_n n^k p_{m,n} = \frac{1}{|\mathcal{H}|}[x^m]\left(q\frac{\partial}{\partial q}\right)^k P(x,q)\Bigg|_{q=1} \qquad (m\to\infty).\end{aligned}$$

We conclude that both classes have asymptotically the same area moments. □

5 Staircase polygons, Dyck paths, and Brownian excursions

We briefly explain how some of our results could be alternatively obtained from the bijections described in [26], which set symmetry classes of staircase polygons in one-to-one correspondence to symmetry classes of Dyck paths. Random Dyck paths, in turn, are related to corresponding stochastic objects such as the Brownian excursion and the Brownian meander. This allows to infer area limit distributions for some polygon symmetry classes from distributions of certain Brownian excursion and meander functionals.

A combinatorial bijection between staircase polygons of perimeter $2m+2$ and Dyck paths of length $2m$ has been described by Delest and Viennot [9]. Within that bijection, the area of a staircase polygon corresponds to the sum of the peak heights of a Dyck path. In a uniform ensemble, the sequence of random Dyck paths w.r.t. half-length yields, after suitable normalisation, a sequence of stochastic processes on $(C[0,1], ||\cdot||_\infty)$ with the Borel σ-algebra, which converges in distribution to the standard Brownian excursion [2, 1]. Also, for certain continuous functionals on suitably normalised Dyck paths, including the area and the sum of peak heights, we have convergence in distribution and moment convergence to the Brownian excursion area, see [22, 20] and [11, Thm. 9]. Using stochastic techniques, the distribution of the excursion area has been initially analysed in [7, 41, 29], compare the historical remarks in [33]. Derivations by discrete methods are summarised in [15].

The above bijection, restricted to r^2-symmetric polygons, yields *symmetric* Dyck paths, which decompose in two identical discrete meanders [26]. Again, it is known that the sequence of random discrete meanders w.r.t. half-length, appropriately normalised, converges to the Brownian meander [23]. Together with the convergence theorem for continuous functionals of polynomial growth [11, Thm. 9], one may conclude that also discrete meander functionals, such as area and sum of peak heights, converge to the Brownian meander area. The distribution of the Brownian meander area was derived in [43] from a q-difference equation similar to eqn. (2.2). It can also be obtained from known results for the corresponding Brownian motion and bridges distribution, since there is a relation between the double Laplace transforms of the three distributions [35, 24]. The same considerations hold for d_2-symmetric polygons, which are in one-to-one correspondence to r^2-symmetric polygons [26].

Staircase polygons with d_1-symmetry are in bijection with pairs of identical Dyck paths, which is seen by cutting a polygon along its positive diagonal [26]. Here, the polygon area corresponds to twice the Dyck path area, hence the area limit distribution is given by that of the Brownian excursion area, by the convergence result mentioned above. A similar result holds for $\langle d_1, d_2\rangle$-symmetric polygons. Here, a combinatorial bijection between polygons and discrete meanders is known [26], where the polygon area corresponds to four times the meander area.

The classes of rectangles and squares lead to Dyck paths with an initial sequence of up steps, followed by an alternating sequence of peaks and valleys, ending in a terminal sequence of down steps. As seen above, these classes are treated by elementary methods.

6 Conclusions

We analysed the symmetry subclasses of staircase polygons on the square lattice. Exploiting a simple decomposition for staircase polygons [39], we obtained the area limit laws in the uniform fixed perimeter ensembles. This extends and completes previous results [26]. As expected, orbit counts with respect to different symmetry subgroups always lead to an Airy distribution. The enumeration of polygons fixed under a given symmetry group leads to a variety of area limit distributions, such as a concentrated distribution, the $\beta_{1,1/2}$-distribution, the Airy-distribution, or the Brownian meander area distribution.

As described in Section 5, the latter two results can also be obtained from the connection to Brownian motion and the Brownian meander. Our independent discrete approach uses only elementary methods from probability and singularity analysis of generating functions. Moreover, it may be used in order to analyse corrections to the limiting behaviour, which cannot easily be obtained by stochastic methods.

One may also study the analogous problem of *perimeter* limit laws in a uniform ensemble where, for fixed area, every polygon occurs with the same probability. For the class of staircase polygons, the associated centred and normalised random variable is asymptotically Gaussian [18, Prop. 9.11], and the same result is expected to hold for the symmetry subclasses, apart from squares. Also, limit laws in non-uniform ensembles and for other counting parameters may be studied, compare [40].

The above methods may be applied to extract limit laws related to symmetry subclasses of other polygon classes. In particular, the classes of convex polygons on the square [27] and on the hexagonal [21] lattices may be studied.

With respect to symmetry subclasses of self-avoiding polygons, exact enumeration studies may be carried out. It would be interesting to numerically analyse moments, in order to test conjectures for limit distributions, compare [36, 40].

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