

Non-Monotone Stochastic Generalized Porous Media Equations*

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Abstract

By using the Nash inequality and a monotonicity approximation argument, existence and uniqueness of strong solutions are proved for a class of non-monotone stochastic generalized porous media equations. Moreover, we prove for a large class of stochastic PDE that the solutions stay in the smaller L^2 -space provided the initial value does, so that some recent results in the literature are considerably strengthened.

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1 Introduction

Based on the classical Galerkin method of finite-dimensional approximations, a large class of nonlinear partial differential equations can be solved on a separable real Hilbert space H under certain monotonicity conditions, see e.g. [16] and the references therein for deterministic equations, and [11, 13, 5, 10, 15] and the references therein for stochastic versions. More precisely, consider for instance

$$dX_t = A(t, X_t)dt + B(t, X_t)dW_t,$$

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where W_t is a G -valued cylindrical Brownian motion on a complete filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ for some real separable Hilbert space G , $A : V \rightarrow V^*$ is a measurable map for some reflexive Banach space V and dual V^* with embeddings $V \subset H \subset V^*$ dense and continuous, and B is a progressively measurable process in the space of Hilbert-Schmidt operators from G to H . Among other conditions for existence and uniqueness of solutions for this equation, the monotonicity is expressed as

$$\boxed{\text{M}} \quad (1.1) \quad {}_{V^*}\langle A(u) - A(v), u - v \rangle_V \leq c \|u - v\|_H^2, \quad u, v \in V$$

for some constant $c > 0$.

On the other hand, however, the following stochastic porous medium equation studied in [10] is not monotone on $L^2(\mathbb{R}^d; dx)$:

$$\boxed{1.1} \quad (1.2) \quad dX_t = \Delta \{X_t |X_t|^{r-1}\} dt + B(t, X_t) dW_t,$$

where Δ is the Laplace operator on \mathbb{R}^d , $r > 1$ is a fixed number, and B and W are as above for $G = H := L^2(\mathbb{R}^d; dx)$. Indeed, for any $c > 0$, the condition

$$\langle \Delta(f|f|^{r-1} - g|g|^{r-1}), f - g \rangle \leq c \|f - g\|_2^2, \quad f, g \in C_0^\infty(\mathbb{R}^d)$$

does not hold, where $\langle \cdot, \cdot \rangle$ and $\|\cdot\|_2$ are the inner product and norm in $L^2(\mathbb{R}^d; dx)$ respectively. By combining the Sobolev inequality with Galerkin approximations, Kim [10] was able to solve this equation on $L^2(\mathbb{R}^d; dx)$ for $X_0 \in L^2(\mathbb{R}^d \times \Omega; dx \times \mathbb{P})$, and the unique solution is an adapted process on $L^2(\mathbb{R}^d; dx)$ satisfying

$$\mathbb{E} \int_0^T dt \int_{\mathbb{R}^d} |\nabla(X_t |X_t|^{r-1})|^2(x) dx < \infty.$$

The right-continuity of the solution, however, is not proved in [10].

In this paper, we show that the existence and uniqueness result for monotone equations can be extended to a class of non-monotone situations as soon as the Nash inequality holds. Indeed, our results are proved for a rather general framework in which we can also allow B to depend on the solution X . Even under the framework of Kim [10] where B is independent of X (“additive noise”), we allow B to be Hilbert-Schmidt from $L^2(\mathbb{R}^d; dx)$ to H^{-1} , where H^{-1} is the dual of $H^1(\mathbb{R}^d) :=$ classical Sobolev space of order 1 in $L^2(\mathbb{R}^d; dx)$, and allow X_0 to be any H^{-1} -valued \mathcal{F}_0 -measurable random variable. Since H^{-1} is much larger than $L^2(\mathbb{R}^d; dx)$ and the norm in H^{-1} is much smaller than that in $L^2(\mathbb{R}^d; dx)$, our assumptions are considerably weaker than Kim’s in [10]. If furthermore B_t is a Hilbert-Schmidt operator on $L^2(\mathbb{R}^d; dx)$, then our results also generalize Kim’s, namely, the solution with $\mathbb{E}\|X_0\|_2^2 < \infty$ satisfies

$$\mathbb{E} \sup_{t \in [0, T]} \|X_t\|_2^2 < \infty \quad \text{and} \quad |X|^{r-1} X \in L^2([0, T] \times \Omega \rightarrow \mathcal{F}_e; dt \times \mathbb{P}), \quad T > 0,$$

where \mathcal{F}_e is the completion of $C_0^\infty(\mathbb{R}^d; dx)$ under the inner product $\langle f, g \rangle_{\mathcal{F}_e} := \int_{\mathbb{R}^d} \langle \nabla f, \nabla g \rangle dx$. Some other properties are also derived (cf. Theorem 1.2 below). Our result, in fact, hold

for a large class of (not necessarily differential) operators L replacing the Laplacian. The appropriate class are operators which are associated to Dirichlet forms satisfying a Nash-type irregularity. The reader unfamiliar with Dirichlet forms should think e.g. of L being a globally elliptic differential operator of order 2 on \mathbb{R}^d , $d \geq 3$.

Let us introduce our framework in detail. Let $(E, \mathcal{B}, \mathbf{m})$ be a σ -finite separable measure space and $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ a symmetric Dirichlet form on $L^2(\mathbf{m})$ (cf. [9]). Assume that the following Nash inequality

$$\boxed{1.2} \quad (1.3) \quad \|f\|_2^2 \leq C \mathcal{E}(f, f)^{d/(d+2)}, \quad f \in \mathcal{D}(\mathcal{E}), \mathbf{m}(|f|) = 1,$$

holds for some constant $C > 0$, where $\|\cdot\|_p$ is the norm in $L^p(\mathbf{m})$ for $p \geq 1$. This inequality is equivalent to the classical Sobolev inequality with dimension d if $d > 2$ (cf. [6, Theorems 2.4.2 and 2.4.6]) i.e. there exists $C_d \in (0, \infty)$ such that

$$\boxed{1.3a} \quad (1.4) \quad \|f\|_{\frac{2s}{d-2}} \leq C_d \mathcal{E}(f, f)^{1/2}, \quad f \in \mathcal{D}(\mathcal{E}).$$

In particular, it holds for the classical Dirichlet form generated by the Laplacian on \mathbb{R}^d , $d \geq 3$. We adopt the above formulation (1.3) here to include also examples with dimension ≤ 2 . In particular, this inequality holds for the Dirichlet Laplace operator on bounded domains in a Riemannian manifold and on the whole Riemannian manifold provided the injectivity radius is infinite (see [3]). Moreover, (1.3) also holds for Dirichlet forms associated with stable-like processes, since according to Theorem 1.3 in [2] the Nash inequality holds for fractional Dirichlet forms with parameter $d > 0$. Let $(L, \mathcal{D}(L))$ be the associated Dirichlet operator, which is thus a negative definite self-adjoint operator on $L^2(\mathbf{m})$. We shall use $\langle \cdot, \cdot \rangle$ for the inner product in $L^2(\mathbf{m})$ and $\|\cdot\|_2$ for its norm. More generally, we set $\langle f, g \rangle := \mathbf{m}(fg) := \int fg d\mathbf{m}$ for any two measurable functions f, g such that $fg \in L^1(\mathbf{m})$. Let $\mathcal{D}(\mathcal{E})$ be equipped with the inner product $\mathcal{E}_1 := \mathcal{E} + \langle \cdot, \cdot \rangle$ and H its dual space. H is then a separable Hilbert space equipped with the induced inner product $\langle \cdot, \cdot \rangle_H$ and norm $\|\cdot\|_H := \langle \cdot, \cdot \rangle_H^{1/2}$. For $a > 0$ we shall also consider the inner products $\mathcal{E}_a := a\mathcal{E} + \langle \cdot, \cdot \rangle$ on $\mathcal{D}(\mathcal{E})$ and their dual inner products $\langle \cdot, \cdot \rangle_{H_a}$ on H with corresponding norms $\|\cdot\|_{H_a}$ (see Section 2 below for details). If H is equipped with $\langle \cdot, \cdot \rangle_{H_a}$ (and $\|\cdot\|_{H_a}$) we denote it by H_a , hence $H_1 = H$. By continuity $1 - L$ (and hence L) extends from $\mathcal{D}(L)$ to an operator from $\mathcal{D}(\mathcal{E})$ to H , denoted by the same symbol. Finally, let \mathcal{F}_e be the completion of $\mathcal{D}(\mathcal{E})$ under the inner product $\langle f, g \rangle_{\mathcal{F}_e} := \mathcal{E}(f, g)$, which is called the extended domain of the Dirichlet form (see [9]). If $d > 2$, (1.4) (hence (1.3)) immediately implies that $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is transient in the sense of [9], that is, there exists $g \in L^1(\mathbf{m}) \cap L^\infty(\mathbf{m})$, such that $\mathcal{F}_e \subset L^1(g \cdot \mathbf{m})$ continuously. We denote the extension of \mathcal{E} from $\mathcal{D}(\mathcal{E})$ to \mathcal{F}_e by $\bar{\mathcal{E}}$, and denote the dual space of \mathcal{F}_e by \mathcal{F}_e^* . Since $\mathcal{D}(\mathcal{E}) \subset \mathcal{F}_e$ densely and continuously, also $\mathcal{F}_e^* \subset H$ densely and continuously. But in general $\mathcal{F}_e^* \neq H$. We equip \mathcal{F}_e^* with the inner product $\langle \cdot, \cdot \rangle_{\mathcal{F}_e^*}$ and corresponding norm $\|\cdot\|_{\mathcal{F}_e^*}$, induced by the Riesz map $\mathcal{F}_e \ni u \mapsto \bar{\mathcal{E}}(\cdot, u) \in \mathcal{F}_e^*$. We recall that if $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is transient, then $\mathcal{F}_e \cap L^2(\mathbf{m}) = \mathcal{D}(\mathcal{E})$ (cf. [9]). If $\mathbf{m}(E) < \infty$, then (1.3) implies that $\inf \sigma(-L) > 0$ and thus that $\mathcal{D}(\mathcal{E}) = \mathcal{F}_e$, hence $H = \mathcal{F}_e^*$ and $(\mathcal{E}, (\mathcal{D}(\mathcal{E})))$ is transient in this case.

Let $r_2 > r_1 > 1$ be two constants and ν a probability measure on $[r_1, r_2]$. We consider the following stochastic partial differential equation on H :

$$\boxed{1.3} \quad (1.5) \quad dX_t = \left\{ \bar{L} \int_{r_1}^{r_2} \xi(t, r) |X_t|^{r-1} X_t \nu(dr) + \eta_t X_t \right\} dt + B(t, X_t) dW_t,$$

where W is a cylindrical Brownian motion on $L^2(\mathbf{m})$, ξ, η and B are specified in the following assumptions and \bar{L} in Definition 2.3 below. For two Hilbert spaces H_1 and H_2 , let $\mathcal{L}_{HS}(H_1; H_2)$ denote the Hilbert space of all Hilbert-Schmidt operators from H_1 to H_2 , equipped with the usual Hilbert-Schmidt inner product. Consider the following conditions:

(H1) $\xi : [0, \infty) \times [r_1, r_2] \times \Omega \rightarrow [0, \infty)$ is progressively measurable and for any $T > 0$, there exists a locally bounded function $R : [0, \infty) \rightarrow [1, \infty)$ such that $\frac{1}{R(t)} \leq \xi(t, \cdot) \leq R(t)$ holds on $[r_1, r_2] \times \Omega$ for all $t \in [0, T]$.

(H2) η is a real-valued locally bounded progressively measurable process (i.e. $\sup_{\substack{s \in [0, T], \\ \omega \in \Omega}} |\eta_s(\omega)| < \infty$ for every $T > 0$).

(H3) For every $T > 0$ the map $B : [0, T] \times V \times \Omega \rightarrow \mathcal{L}_{HS}(L^2(\mathbf{m}); H)$ is progressively measurable such that

(i) there exists $C \in (0, \infty)$ such that for all $a \in (0, \infty)$

$$\|B(\cdot, u) - B(\cdot, v)\|_{\mathcal{L}_{HS}(L^2(\mathbf{m}), H_a)} \leq C \|u - v\|_{H_a}^2 \quad \text{on } [0, T] \times \Omega \text{ for all } u, v \in V;$$

(ii)

$$\int_0^T \|B(s, 0)\|_{\mathcal{L}_{HS}(L^2(\mathbf{m}); H)}^2 ds \in L^{r_2}(\mathbb{P}).$$

We give examples where condition (H.3(i)) holds in Remark 2.9 at the end of Section 2 below. Obviously, when $\xi = 1, \eta = 0$ and $\nu = \delta_r$ (the Dirac measure at r), equation (1.5) reduces to (1.2). The following definition of a solution is taken from [15] (see also [11]).

First, however, we need to introduce auxiliary spaces V and V^* :

It is easy to see that $N(s) := \int_{r_1}^{r_2} |s|^{r+1} \nu(dr)$, $s \in \mathbb{R}$, is a Δ_2 -regular Young function so that the corresponding Orlicz space $L_N(\mathbf{m})$ is a reflexive separable Banach space (see [14]). By [15, Propostion 3.1] applied to $L - 1$ instead of L the embedding $V := H \cap L_N(\mathbf{m}) \subset H$ is dense and continuous. Furthermore, V is reflexive (see [15]). Let V^* be the dual of V and N^* the dual Young function to N^* (cf. Section 2 below for details).

def:1.1

Definition 1.1. A continuous adapted process $\{X_t\}_{t \geq 0}$ on H is called a solution to (1.5), if for any $T > 0$, $X \in L^2([0, T] \times \Omega \rightarrow H, dt \times \mathbb{P})$ with

$$\boxed{01} \quad (1.6) \quad \int_0^T \int_{r_1}^{r_2} \|X_t\|_{r+1}^{r+1} \nu(dr) dt < \infty \quad \mathbb{P} - \text{a.s.}$$

such that \mathbb{P} -a.s.

$$\boxed{02} \quad (1.7) \quad X_t = X_0 + \bar{L} \left[\int_0^t \left(\int_{r_1}^{r_2} \xi(s, r) |X_s|^{r-1} X_s \nu(dr) \right) ds \right] \\ + \int_0^t \eta_s X_s ds + \int_0^t B(s, X_s) dW_s, \text{ for all } t \geq 0$$

holds in H , where the first integral in (1.7) is an L_{N^*} -valued Bochner integral which takes values in $\mathcal{D}(\bar{L})$ \mathbb{P} -a.s. $\forall t \geq 0$ and $\bar{L} : D(\bar{L}) \subset L_{N^*} \rightarrow V^*$ is a natural extension of $L : D(\mathcal{E}) \cap L_{N^*} \rightarrow V^*$ defined in Definition 2.3 below.

T1.1 **Theorem 1.2.** *Assume (1.3), (H1), (H2) and (H3).*

- (1) *For any \mathcal{F}_0 -measurable H -valued random variable X_0 , (1.5) has a unique solution in the sense of Definition 1.1. This solution is a Markov process provided ξ, η and B are constant (i.e. independent of t and ω).*
- (2) *Let $\{X^{(n)}\}$ be a sequence of solutions to (1.5). If $X_0^{(n)} \rightarrow X_0$ in H in probability as $n \rightarrow \infty$, then for any $t > 0$,*

$$X_t^{(n)} \rightarrow X_t \text{ in } H \quad \text{and} \quad \int_0^t \int_{r_1}^{r_2} \|X_s^{(n)} - X_s\|_{r+1}^{r+1} \nu(dr) ds \rightarrow 0$$

in probability as $n \rightarrow \infty$. Consequently, if ξ, η and B are independent of t and ω , then the transition semigroup of the solution is a Feller semigroup.

- (3) *For all $p \in [2, \infty)$, $T > 0$, and some constant $c(p, T)$*

$$\mathbb{E} \sup_{t \in [0, T]} \|X_t\|_H^p \leq c(p, T) \left[\mathbb{E} \|X_0\|_H^p + \mathbb{E} \left(\int_0^T \|B(s, 0)\|_{\mathcal{L}_{HS}(L^2(\mathbf{m}); H)}^2 ds \right)^{\frac{p}{2}} \right]$$

which is finite provided $p \leq 2r_2$ and $\mathbb{E} \|X_0\|_H^p < \infty$. In the latter case we have

$$\mathbb{E} \left[\int_0^T \int_{r_1}^{r_2} \|X_t\|_{r+1}^{r+1} \nu(dr) dt \right]^{p/(r_2+1)} < \infty, \quad \text{provided } p \geq r_2 + 1.$$

- (4) *In addition, assume that $B(\cdot, 0) \in L^2([0, T] \times \Omega \rightarrow \mathcal{L}_{HS}(L^2(\mathbf{m}); L^2(\mathbf{m})), dt \times \mathbb{P})$. If $X_0 \in L^2(\mathbf{m})$ a.s. then X_t is a right-continuous process in $L^2(\mathbf{m})$ (“ $L^2(\mathbf{m})$ -invariance”). If moreover $\mathbb{E} \|X_0\|_2^2 < \infty$, then $\mathbb{E} \sup_{t \in [0, T]} \|X_t\|_2^2 < \infty$. If, in addition, E is a Lusin space, then $\zeta(X_t) := \int_{r_1}^{r_2} |X_t|^{(r-1)/2} X_t \nu(dr) \in \mathcal{D}(\mathcal{E})$ $dt \times \mathbb{P}$ -a.e. with*

$$\boxed{LL1} \quad (1.8) \quad \mathbb{E} \int_0^T \mathcal{E}(\zeta(X_t), \zeta(X_t)) dt < \infty.$$

Consequently, if $\mathbb{E}(\|X_0\|_2^2 + \|X_0\|_H^{r_2+1}) < \infty$ then $\zeta(X) \in L^2([0, T] \times \Omega \rightarrow \mathcal{D}(\mathcal{E}); dt \times \mathbb{P})$ for any $T > 0$.

The uniqueness and the Markov property can be proved in a standard way as in [11, 5, 15] by using the Itô formula for the square of the norm. So, the main point is to prove the existence. Since in general the map (cf. Section 3 in [15])

$$V \ni x \mapsto A(t, x) := L \int_{r_1}^{r_2} \xi(t, r) |x|^{r-1} x \nu(dr) + \eta_t x \in V^*$$

is not monotone in H , known results concerning monotone stochastic SPDEs do not work directly. To make the equation monotone, in [15] we replaced H by \mathcal{F}_e^* , the dual space of the extended Dirichlet space \mathcal{F}_e , but had to assume that $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is transient. In general, the embedding $\mathcal{F}_e^* \subset H$ is dense and continuous, but \mathcal{F}_e^* and $L^2(\mathbf{m})$ are incomparable except $\inf \sigma(-L) > 0$, where $\sigma(-L)$ is the spectrum of $(-L)$. Under a stronger condition than (H3), namely that B is in $L^2([0, T] \times \Omega \rightarrow \mathcal{L}_{HS}(L^2(\mathbf{m}); \mathcal{F}_e^*), dt \times \mathbb{P})$, in [15] existence and uniqueness of the solution to (1.5) was proved for all $X_0 \in L^2(\Omega \rightarrow \mathcal{F}_e^*; \mathcal{F}_0, \mathbb{P})$. Since \mathcal{F}_e^* and $L^2(\mathbf{m})$ are generally incomparable, the solutions constructed in [15] do not automatically provide solutions starting from points in $L^2(\mathbf{m}) \setminus \mathcal{F}_e^*$. So, in this paper we first construct solutions in H , which is larger than $L^2(\mathbf{m})$, then prove that the the solution will be in $L^2(\mathbf{m})$ for $t \geq 0$ provided the initial value is so and B is as in Theorem 1.2(4).

To construct solutions starting from all \mathcal{F}_0 -measurable H -valued random variables, we develop an approximation argument by first considering the equation (1.5) for $L - \varepsilon$ in place of L to make the equation monotone on H , then taking the limit $\varepsilon \rightarrow 0$ we obtain a solution for the original equation. To realize this approximation procedure, the Nash inequality (1.3) will play a crucial role.

In Section 2 we first briefly recall some general results obtained in [15] concerning monotone stochastic equations, prove some technical auxiliary results and then prove a criterion for the $L^2(\mathbf{m})$ -invariance of solutions. Some a priori estimates are presented in Section 3 by using the Nash inequality, which will be used in Section 4 to construct the solution to (1.5) for H -valued X_0 satisfying a moment condition. Finally, the complete proof of Theorem 1.2 is contained in Section 5.

From now on we fix $(E, \mathcal{B}, \mathbf{m})$ and $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ as above.

2 Some known results and $L^2(\mathbf{m})$ -invariance

2.1 Review of known results

In this subsection we recall some results obtained recently in [15] which will be used in the sequel for constructing solutions to (1.5). In all of this subsection we assume that $\inf \sigma(-L) > 0$, hence $H = \mathcal{F}_e^*$. But at least initially we shall consider the inner product $\langle \cdot, \cdot \rangle_{\mathcal{F}_e^*}$ on H and only later $\langle \cdot, \cdot \rangle_H$.

Let $N \in C(\mathbb{R})$ be a Young function, i.e. a nonnegative, continuous, convex and even function such that $N(s) = 0$ if and only if $s = 0$, and

$$\lim_{s \rightarrow 0} \frac{N(s)}{s} = 0, \quad \lim_{s \rightarrow \infty} \frac{N(s)}{s} = \infty.$$

For any measurable function f on E with $\mathbf{m}(N(\alpha f)) < \infty$ for some $\alpha > 0$, define

$$\|f\|_N := \inf\{\lambda \geq 0 : \mathbf{m}(N(f/\lambda)) \leq 1\}.$$

Then the space

$$L_N(\mathbf{m}) := \{f : \|f\|_N < \infty\}$$

is a real separable Banach space, which is called the Orlicz space induced by the Young function N (cf. [14, Proposition 1.2.4]). There is an equivalent norm defined by using the dual function:

$$N^*(s) := \sup\{r|s| - N(r) : r \geq 0\}, \quad s \in \mathbb{R},$$

which is once again a Young function. More precisely, letting

$$\|f\|_{(N)} := \sup\{\langle f, g \rangle : \mathbf{m}(N^*(g)) \leq 1\},$$

one has (see [14, Theorem 1.2.8 (ii)])

$$\boxed{*0} \quad (2.1) \quad \|\cdot\|_N \leq \|\cdot\|_{(N)} \leq 2\|\cdot\|_N.$$

The function N is called Δ_2 -regular, if there exists a constant $c > 0$ such that

$$N(2s) \leq c(N(s) + 1_{\{\mathbf{m}(E) < \infty\}}), \quad s \in \mathbb{R}.$$

We assume that N and N^* are Δ_2 -regular. By [14, Proposition 1.2.11(iii) and Theorem 1.2.13], $L_N(\mathbf{m})$ and $L_{N^*}(\mathbf{m})$ are dual spaces of each other, and hence are reflexive. By the Δ_2 -regularity, $f \in L_N(\mathbf{m})$ if and only if $\mathbf{m}(N(f)) < \infty$. For simplicity, we sometimes use L_N and L_{N^*} instead of $L_N(\mathbf{m})$, $L_{N^*}(\mathbf{m})$ respectively.

Let $V := H \cap L_N(\mathbf{m})$ with $\|\cdot\|_V := \|\cdot\|_N + \|\cdot\|_H$. More precisely,

$$V = \{v \in L_N(\mathbf{m}) \mid \mathcal{D}(\mathcal{E}) \cap L_{N^*}(\mathbf{m}) \ni u \mapsto \mathbf{m}(uv) \text{ is in } H\}.$$

Since by [15, Proposition 3.1 and its proof] $\mathcal{D}(\mathcal{E}) \cap L_{N^*}$ is dense in $\mathcal{D}(\mathcal{E})$, V is indeed embedded into H . Furthermore, V is complete, by [15, Proposition 3.1], reflexive and dense in H and L_N . Let

$$\Psi : [0, \infty) \times \mathbb{R} \times \Omega \rightarrow \mathbb{R}$$

be progressively measurable, i.e. for any $t \geq 0$, Ψ restricted to $[0, t] \times \mathbb{R} \times \Omega$ is measurable w.r.t. $\mathcal{B}([0, t]) \times \mathcal{B}(\mathbb{R}) \times \mathcal{F}_t$. We assume that for any $(t, \omega) \in [0, \infty) \times \Omega$, $\Psi(t, \cdot)(\omega)$ is continuous.

Finally, let $B : [0, \infty) \times V \times \Omega \rightarrow \mathcal{L}_{HS}(L^2(\mathbf{m}); H)$ be progressively measurable as in the last section. We shall make use of the following assumptions:

- (B) For any $T > 0$, $\|B(\cdot, 0)\|_{\mathcal{L}_{HS}(L^2(\mathbf{m}); H)} \in L^2([0, T] \times \Omega; dt \times \mathbb{P})$ and there exists a constant $c \geq 0$ such that $\|B(\cdot, u) - B(\cdot, v)\|_{\mathcal{L}_{HS}(L^2(\mathbf{m}); H)}^2 \leq c\|u - v\|_H^2$ holds on $[0, T] \times \Omega$ for all $u, v \in V$.

(Ψ) For any $T > 0$, there exist a nonnegative \mathcal{F}_t -adapted process $f \in L^1([0, T] \times \Omega; dt \times \mathbb{P})$ and a constant $c \geq 1$, such that for all $s, s_1, s_2 \in \mathbb{R}$ on $[0, T] \times \Omega$

$$(\Psi 1) \quad (s_2 - s_1)(\Psi(\cdot, s_2) - \Psi(\cdot, s_1)) \geq 0.$$

$$(\Psi 2) \quad c^{-1}N(s) - 1_{\{\mathbf{m}(E) < \infty\}}f \leq s\Psi(\cdot, s) \leq cN(s) + 1_{\{\mathbf{m}(E) < \infty\}}f.$$

$$(\Psi 3) \quad N^*(\Psi(\cdot, 0)) 1_{\{\mathbf{m}(E) < \infty\}} \in L^1([0, T] \times \Omega; dt \times \mathbb{P}).$$

Let

$$K := L_N([0, T] \times E \times \Omega; dt \times \mathbf{m} \times \mathbb{P}) \cap L^2([0, T] \times \Omega \rightarrow H; dt \times \mathbb{P})$$

with norm

$$\|\cdot\|_K := \|\cdot\|_{L_N([0, T] \times E \times \Omega; dt \times \mathbf{m} \times \mathbb{P})} + \|\cdot\|_{L^2([0, T] \times \Omega \rightarrow H; dt \times \mathbb{P})}.$$

Then, $K \subset L^1([0, T] \times \Omega \rightarrow V; dt \times \mathbb{P})$ continuously and densely (cf. [15, Lemmas 3.7 and 3.5]). Let K^* be the dual of K . Then by [15, Lemma 2.5] K^* is the completion of $L^\infty([0, T] \times \Omega \rightarrow V^*; dt \times \mathbb{P})$ w.r.t.

$$\|z^*\|_{K^*} := \sup_{\|z\|_K \leq 1} \mathbb{E} \int_0^T V^* \langle z_t^*, z_t \rangle_V dt.$$

Furthermore, $K^* \subset L^1([0, T] \times \Omega \rightarrow V^*; dt \times \mathbb{P})$ and we recall that by (Ψ) and [15, Lemma 3.6(i)] for all $u \in L_N$

$$\Psi(\cdot, u) \in L^1([0, T] \times \Omega \rightarrow L_{N^*}; dt \times \mathbb{P}).$$

We want to apply the existence and uniqueness result [15, Theorem 3.9] in this case. We recall that in [15], $H = \mathcal{F}_e^*$ was identified with its dual $H^* = \mathcal{D}(\mathcal{E}) = \mathcal{F}_e$ using the Riesz map coming from the inner product $\langle \cdot, \cdot \rangle_{\mathcal{F}_e^*}$ defined in the introduction. The reason is that only in this inner product we have monotonicity for our drift coefficient. Since below we want to consider other inner products on H (generating, however, equivalent norms) and to avoid confusion we are going to recall the main existence and uniqueness result from [15] in a version not based on this specific identification of H and H^* . First, we fix some notation and conventions: for a Banach space B we denote its dual by B^* and use ${}_B \langle \cdot, \cdot \rangle_B$ for their dualization. We always consider B^* with the standard dual norm $\|l\|_{B^*} := \sup_{\|v\|_B=1} l(v)$, $l \in B^*$. If B is reflexive, then $B^{**} = B$ canonically and by convention we use this below without further mentioning it. By [15, Lemma 3.4(i)] and since $\inf \sigma(-L) > 0$, the map

$$\boxed{\text{eq:2.1a}} \quad (2.2) \quad \mathcal{D}(\mathcal{E}) \ni v \mapsto -\mathcal{E}(v, \cdot) \in H$$

(i.e. the Riesz isomorphism on $(\mathcal{D}(\mathcal{E}), \mathcal{E})$ multiplied by (-1)) is the unique continuous linear extension of the map

$$\mathcal{D}(L) \ni v \mapsto \langle Lv, \cdot \rangle \in H.$$

Here, as above, $\mathcal{D}(\mathcal{E})$ is equipped with the norm $\mathcal{E}^{1/2}(u) := \mathcal{E}(u, u)^{1/2}$, $u \in D(\mathcal{E})$, which is equivalent to the norm $\mathcal{E}_1^{1/2}(u) := (\mathcal{E}(u, u) + \langle u, u \rangle)^{1/2}$, $u \in \mathcal{D}(\mathcal{E})$, since $\inf \sigma(-L) > 0$. Let us denote the map in (2.2) again by L . Let $i : h \mapsto \langle \cdot, h \rangle_{\mathcal{F}_e^*}$ be the Riesz map on

$(H, \langle \cdot, \cdot \rangle_{\mathcal{F}_e^*})$. Then clearly, $i = (-L)^{-1} : H \rightarrow H^* = \mathcal{D}(\mathcal{E})$ and by [15, Lemma 3.4(iii)] (and since $\inf \sigma(-L) > 0$)

$$-1 = i \circ L : \mathcal{D}(\mathcal{E}) \cap L_{N^*} \rightarrow H^* \subset V^*$$

uniquely extends to a continuous linear map

$$\boxed{\text{eq:2.1b}} \quad (2.3) \quad \overline{i \circ L} : L_{N^*} \rightarrow V^*.$$

The map $\overline{i \circ L}$ is of course nothing but (-1) times the natural embedding $L_{N^*} \subset V^*$ induced by the continuous and dense embedding $V \subset L_N$. So, below we always replace $\overline{i \circ L}(u)$ by $-u$ for $u \in L_{N^*}$. Now we can formulate the existence and uniqueness result [15, Theorem 3.9] in our situation:

$\boxed{\text{T2.1}}$ **Theorem 2.1.** *Let the Young function N and its dual function N^* be Δ_2 -regular, and let $\inf \sigma(-L) > 0$. Assume (H2), (B) and (Ψ). Then for any $X_0 \in L^2(\Omega \rightarrow H; \mathcal{F}_0; \mathbb{P})$, the equation*

$$dX_t = (L\Psi(t, X_t) + \eta_t X_t)dt + B(t, X_t)dW_t$$

has a unique solution in the sense that X_t is a continuous adapted process in H such that $X \in K$, $-\Psi(\cdot, X) + \eta i(X)$ is a progressively measurable process in K^* for any $T > 0$, and \mathbb{P} -a.s.

$$\boxed{\text{T2.1-Xt}} \quad (2.4) \quad i(X_t) = i(X_0) + \int_0^t \{ -\Psi(s, X_s) + \eta_s i(X_s) \} ds + i \left(\int_0^t B(s, X_s) dW_s \right), \quad t \geq 0,$$

holds in $i(H) = H^* = \mathcal{D}(\mathcal{E})$ (where the first integral in (2.4) is an $L_{N^*}(\subset V^*)$ -valued Bochner integral, which a posteriori is in $\mathcal{D}(\mathcal{E})$ \mathbb{P} -a.e. $\forall t \geq 0$) or equivalently,

$$\boxed{\text{eq:2.4b}} \quad (2.5) \quad X_t = X_0 + L \left(\int_0^t \Psi(s, X_s) ds \right) + \int_0^t \eta_s X_s ds + \int_0^t B(s, X_s) dW_s, \quad t \geq 0,$$

holds in H . Furthermore, $\mathbb{E} \sup_{t \in [0, T]} \|X_t\|_{\mathcal{F}_e^*}^2 < \infty$ for $T > 0$ and \mathbb{P} -a.s.

$$\begin{aligned} \|X_t\|_{\mathcal{F}_e^*}^2 &= \|X_0\|_{\mathcal{F}_e^*}^2 + \int_0^t [2 {}_{V^*} \langle -\Psi(s, X_s) + \eta_s i(X_s), X_s \rangle_V + \|B(s, X_s)\|_{\mathcal{L}_{HS}(L^2(\mathbf{m}); \mathcal{F}_e^*)}^2] ds \\ &\quad + 2 \int_0^t \langle X_s, B(s, X_s) dW_s \rangle_{\mathcal{F}_e^*}, \quad t \geq 0. \end{aligned}$$

We note that since by (2.4) we have that $\int_0^t \Psi(s, X_s) ds \in \mathcal{D}(\mathcal{E}) \cap L_{N^*}$, we can replace L by \bar{L} in (2.5). So, (2.5) means that X is indeed a solution in the sense of Definition 1.1. We also emphasize that the existence result in [16] is considerably more general. In particular, we do not need that $\inf \sigma(-L) > 0$, but only that $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is transient. Below, however, we shall only use the weaker version formulated in Theorem 2.1 above.

The above Itô formula for the square of the norm was proved in the Appendix of [15], generalizing the version proved in the fundamental work [11] for a special case where $K := L^p([0, T] \times \Omega \rightarrow V; dt \times \mathbb{P}) \cap L^2([0, T] \times \Omega \rightarrow H; dt \times P)$ for some $p > 1$. Below, however, we shall apply this formula to other, but equivalent norms $\|\cdot\|_{H_a}$ on H which for $a \searrow 0$ increase to $\|\cdot\|_2$ and come from inner products $\langle \cdot, \cdot \rangle_{H_a}$ on H_a which are defined in the next subsection in which we drop the assumption that $\inf \sigma(-L) > 0$.

2.2 Some technical lemmas and change of norms

In this subsection we do neither assume $\inf \sigma(-L) > 0$ nor (1.3), unless explicitly stated. Let $a > 0$ and define the following inner product on $\mathcal{D}(\mathcal{E})$ by

$$\mathcal{E}_a(u, v) := a\mathcal{E}(u, v) + \langle u, v \rangle; \quad u, v \in \mathcal{D}(\mathcal{E}).$$

Let $\langle \cdot, \cdot \rangle_{H_a}$ be its dual inner product on H_a , i.e. the inner product induced on H by the Riesz map on $(\mathcal{D}(\mathcal{E}), \mathcal{E}_a)$ which is given by

$$\boxed{\text{eq:2.4}} \quad (2.6) \quad \mathcal{D}(\mathcal{E}) \ni u \mapsto a\mathcal{E}(u, \cdot) + \langle u, \cdot \rangle \in H$$

and which is the unique continuous linear extension of

$$(1 - aL) : \mathcal{D}(L) \subset \mathcal{D}(\mathcal{E}) \rightarrow H,$$

hence we denote it by the same symbol $1 - aL$. Then $i_a := (1 - aL)^{-1}$ is just the Riesz map on $(H, \langle \cdot, \cdot \rangle_{H_a})$. In particular, we have

$$\boxed{\text{2.1c}} \quad (2.7) \quad {}_H \langle i_a^{-1}u, v \rangle_{\mathcal{D}(\mathcal{E})} = \mathcal{E}_a(u, v), \quad u, v \in \mathcal{D}(\mathcal{E}).$$

As usual we set

$$\mathcal{E}_a^{1/2}(u) := (a\mathcal{E}(u, u) + \langle u, u \rangle)^{1/2}, \quad u \in D(\mathcal{E}).$$

If $a \leq a'$, then $\mathcal{E}_a^{1/2} \leq \mathcal{E}_{a'}^{1/2} \leq \sqrt{\frac{a'}{a}} \mathcal{E}_a^{1/2}$, so $\|\cdot\|_{H_a} \geq \|\cdot\|_{H_{a'}} \geq \sqrt{\frac{a}{a'}} \|\cdot\|_{H_a}$, where $\|\cdot\|_{H_a} := \langle \cdot, \cdot \rangle_{H_a}^{1/2}$.

We emphasize that for different inner products $\langle \cdot, \cdot \rangle_{H_a}$, $a > 0$, on H the corresponding Riesz isomorphisms $i_a : H \rightarrow H^*$, $h \mapsto \langle \cdot, h \rangle_{H_a}$ depend on $a > 0$. To avoid confusion, we shall therefore always distinguish between a Hilbert space and its dual, except for $L^2(\mathbf{m})$, which we canonically identify with its dual. So, we have

$$\boxed{\text{2.1a}} \quad (2.8) \quad V \subset H \xrightarrow{i_a} H^* \subset V^*$$

and

$$\mathcal{D}(\mathcal{E}) \subset L^2(\mathbf{m}) \equiv L^2(\mathbf{m})^* \subset H.$$

In order to apply the Itô formula from [15] to $\|X_t\|_{H_a}^2$, $t \geq 0$, we have to find the stochastic equation satisfied by $i_a(X_t)$, $t \geq 0$. To this end we first have to define and calculate the unique continuous extension

$$\overline{i_a \circ L} : L_{N^*} \rightarrow V^*$$

of

$$i_a \circ L : \mathcal{D}(\mathcal{E}) \cap L_{N^*} \rightarrow H \xrightarrow{i_a} H^* \subset V^*.$$

lem-2.2 **Lemma 2.2.** *Let $a > 0$. Then the map*

$$i_a \circ L : \mathcal{D}(\mathcal{E}) \cap L_{N^*} \rightarrow V^*$$

extends continuously to L_{N^} , and for its extension $\overline{i_a \circ L} : L_{N^*} \rightarrow V^*$ we have*

$$\overline{i_a \circ L} u = \frac{1}{a} \left(\overline{(1 - aL)^{-1} L_{N^*}} - 1 \right) u \in L_{N^*}$$

for all $u \in L_{N^}$, $v \in V$, where as usual 1 denotes the identity map and*

$$\overline{(1 - aL)^{-1} L_{N^*}} : L_{N^*} \rightarrow L_{N^*}$$

denotes the continuous extension of $(1 - aL)^{-1} : \mathcal{D}(\mathcal{E}) \cap L_{N^} \rightarrow L_{N^*}$ to all of L_{N^*} (which exists by a simple application of Jensen's inequality). In particular, $\overline{i_a \circ L}(L_{N^*}) \subset L_{N^*}$ and $\overline{i_a \circ L} : L_{N^*} \rightarrow L_{N^*}$ is continuous.*

Altogether, we have the following diagram:

$$\begin{array}{ccccc}
 H \cap L_N =: V & \subset & H & \xrightarrow{i_a} & H^* = \mathcal{D}(\mathcal{E}) \subset V^* \\
 & & \uparrow L & & \nearrow \overline{i_a \circ L} \\
 & & \mathcal{D}(\mathcal{E}) & & \\
 & & \cup & & \\
 & & \mathcal{D}(\mathcal{E}) \cap L_{N^*} & & \\
 & & \cap & & \\
 & & L_{N^*} & &
 \end{array}$$

where by [15, Proposition 3.1] (applied to the operator $-(1 - \alpha L)$ instead of L) all inclusions are dense and continuous.

Proof. Let $\varepsilon > 0$. Then for all $u \in \mathcal{D}(\mathcal{E}) \cap L_{N^*}$, $v \in V$,

$$\begin{aligned}
 V^* \langle (i_a \circ L)u, v \rangle_V &= \mathcal{D}(\mathcal{E}) \langle (i_a \circ L)u, v \rangle_H = \mathcal{D}(\mathcal{E}) \langle (1 - aL)^{-1} Lu, v \rangle_H \\
 &= \frac{1}{a} \left(-\mathcal{D}(\mathcal{E}) \langle u, v \rangle_H + \mathcal{D}(\mathcal{E}) \langle (1 - aL)^{-1} u, v \rangle_H \right) \\
 &= \frac{1}{a} \left(-\mathbf{m}(uv) + \mathbf{m}([(1 - aL)^{-1} u]v) \right) \\
 &= \frac{1}{a} \cdot \mathbf{m} \left([((1 - aL)^{-1} - 1)u] \cdot v \right),
 \end{aligned}$$

where we used the identification of $L^2(\mathbf{m})$ with its dual (so $\mathcal{D}(\mathcal{E}) \subset L^2(\mathbf{m}) \subset H$). Using the fact that by Jensen's inequality $(1 - aL)^{-1}$ with initial domain $\mathcal{D}(\mathcal{E}) \cap L_{N^*}$ is a bounded linear operator on L_{N^*} , and since by [15, Proposition 3.1] (applied to \mathcal{E}_1 replacing \mathcal{E}) $\mathcal{D}(\mathcal{E}) \cap L_{N^*}$ is dense in L_{N^*} , the assertion follows. \square

Now let us define the operator $\bar{L} : \mathcal{D}(\bar{L}) \subset L_{N^*} \rightarrow H$ appearing in Definition 1.1.

def:2.3 **Definition 2.3.** Let

$$\begin{aligned} \mathcal{D}(\bar{L}) := & \{u \in L_{N^*} \mid \exists u_n \in \mathcal{D}(\mathcal{E}) \cap L_{N^*} \text{ and a sequence } \varepsilon_n \rightarrow 0 \\ & \text{such that } \lim_{n \rightarrow \infty} u_n = u \text{ in } L_{N^*} \\ & \text{and } \lim_{n \rightarrow \infty} (Lu_n - \varepsilon_n u_n) \text{ exists in } H\}, \end{aligned}$$

and for $u \in \mathcal{D}(\bar{L})$ let

$$\bar{L}u := \lim_{n \rightarrow \infty} (Lu_n - \varepsilon_n u_n) (\in H).$$

The following lemma implies that $(\bar{L}, \mathcal{D}(\bar{L}))$ is well-defined. Below we add prefixes $\mathcal{D}(\mathcal{E})$, V^* , L_{N^*} in front of “lim” to indicate in which spaces the respective limit is taken.

lem:2.3 **Lemma 2.4.** Let $u \in L_{N^*}$ and $u_n, \varepsilon_n, n \in \mathbb{N}$, as in the definition of $\mathcal{D}(\bar{L})$. Then for all $a > 0$

$$i_a(H\text{-}\lim_{n \rightarrow \infty} (Lu_n - \varepsilon_n u_n)) = \overline{i_a \circ L}u.$$

In particular, $(\bar{L}, \mathcal{D}(\bar{L}))$ is a well-defined operator from L_{N^*} to H and $\overline{i_a \circ L}u \in \mathcal{D}(\mathcal{E})$ and $i_a \circ \bar{L} = \overline{i_a \circ L}$ on $\mathcal{D}(\bar{L})$.

Proof. We have

$$\begin{aligned} i_a(H\text{-}\lim_{n \rightarrow \infty} (Lu_n - \varepsilon_n u_n)) &= \mathcal{D}(\mathcal{E})\text{-}\lim_{n \rightarrow \infty} (i_a(L - \varepsilon_n)u_n) \\ &= V^*\text{-}\lim_{n \rightarrow \infty} \frac{1}{a}(1 - a\varepsilon_n)i_a u_n - u_n \\ &= L_{N^*}\text{-}\lim_{n \rightarrow \infty} \frac{1}{a} \left(\overline{(1 - aL)^{L_{N^*}}} u - u \right) \\ &= \overline{i_a \circ L}u \end{aligned}$$

by Lemma 2.2. □

cor:2.5 **Corollary 2.5.** Let $T > 0$ and $Z \in L^1([0, T] \rightarrow L_{N^*}, dt)$. Let $t \in [0, T]$ such that

$$\int_0^t Z_s ds \in \mathcal{D}(\bar{L})$$

and let $a > 0$. Then

$$i_a \circ \bar{L} \left(\int_0^t Z_s ds \right) = \int_0^t \overline{i_a \circ L}(Z_s) ds.$$

Proof. The assertion is an immediate consequence of Lemma 2.4 and the last part of Lemma 2.2. □

Now we can state and prove the Itô formula for the norms $\|\cdot\|_{H_a}$, $a > 0$.

T2.2 **Theorem 2.6.** *Let X be the solution from Theorem 2.1 or, assuming (1.3), (H1)–(H3), as in Definition 1.1 (where in the latter case below we set $\Psi(t, s) := \int_{r_1}^{r_2} \xi(t, r) |s|^{r-1} s \nu(dr)$, $s \in \mathbb{R}$, $t \geq 0$), and let $a > 0$. Then $\overline{i_a \circ L}(\Psi(\cdot, X)) + \eta i_a(X)$ is a progressively measurable process in K^* for any $T > 0$, and \mathbb{P} -a.s.*

eq:2.5b (2.9)
$$i_a(X_t) = i_a(X_0) + \int_0^t [\overline{i_a \circ L}(\Psi(s, X_s)) + \eta_s i_a(X_s)] ds + i_a \left(\int_0^t B(s, X_s) dW_s \right), \quad t \geq 0.$$

Furthermore, \mathbb{P} -a.s.

eq:2.6 (2.10)
$$\begin{aligned} \|X_t\|_{H_a}^2 &= \|X_0\|_{H_a}^2 + \int_0^t [2 \, {}_{V^*} \langle \overline{i_a \circ L}(\Psi(s, X_s)) + \eta_s i_a(X_s), X_s \rangle_V \\ &\quad + \|B(s, X_s)\|_{\mathcal{L}_{HS}(L^2(\mathbf{m}); H_a)}^2] ds + 2 \int_0^t \langle X_s, B(s, X_s) dW_s \rangle_{H_a}, \quad t \geq 0. \end{aligned}$$

Proof. Applying i_a to (2.5) and (1.7) respectively, (2.9) follows from Corollary 2.5. (2.10) follows immediatley from (2.9) and the Itô formula in [15, Theorem 4.2] applied to the Hilbert space $(H_a, \langle \cdot, \cdot \rangle_{H_a})$. \square

lem:2.7 **Lemma 2.7.** *Let $a > 0$.*

(i) *Let $v \in V$. Then $(1 - aL)^{-1}v \in V$ and, in particular,*

$$L(1 - aL)^{-1}v = -\frac{1}{a}(v - (1 - aL)^{-1}v) \in V.$$

(ii) *Let $u \in L_{N^*}$, $v \in V$. Then*

$${}_{V^*} \langle \overline{i_a \circ L}u, v \rangle_V = {}_{V^*} \langle u, L(1 - aL)^{-1}v \rangle_V.$$

(iii) *$(1 - aL)^{-1} : V \rightarrow V$ is continuous. Furthermore, its dual operator $((1 - aL)^{-1})^* : V^* \rightarrow V^*$ is the continuous extension of both $\overline{(1 - aL)^{-1}}^{L_{N^*}} : L_{N^*} \rightarrow L_{N^*}$ defined in Lemma 2.2 and of $(1 - aL)^{-1}|_{\mathcal{D}(\mathcal{E})} : \mathcal{D}(\mathcal{E}) \rightarrow \mathcal{D}(\mathcal{E})$. (Here we recall that both $\mathcal{D}(\mathcal{E}) \subset V^*$ and $L_N^* \subset V^*$ continuously and densely.)*

Proof. (i) We first note that since $v \in H$, $(1 - aL)^{-1}v$ is a well-defined element in $\mathcal{D}(\mathcal{E})$ and since $i_a = (1 - aL)^{-1}$, we have by (2.7) for $u \in D(\mathcal{E}) \cap L_{N^*}$

eq:2.9a (2.11)
$$\begin{aligned} \langle u, (1 - aL)^{-1}v \rangle &= {}_H \langle u, (1 - aL)^{-1}v \rangle_{\mathcal{D}(\mathcal{E})} \\ &= \langle u, v \rangle_{H_a} \\ &= {}_{\mathcal{D}(\mathcal{E})} \langle (1 - aL)^{-1}u, v \rangle_H \\ &= \langle (1 - aL)^{-1}u, v \rangle \\ &= \langle \overline{(1 - aL)^{-1}}^{L_{N^*}} u, v \rangle. \end{aligned}$$

(cf. the proof and statement of Lemma 2.2). Since $\mathcal{D}(\mathcal{E}) \cap L_{N^*}$ is dense in L_{N^*} it follows that for fixed v the right hand side uniquely determines a continuous linear functional on L_{N^*} , since $v \in L_N$. Hence so does its left hand side. Therefore,

$$(1 - aL)^{-1}v \in L_N,$$

because $L_N = (L_{N^*})^*$.

(ii) Let $u_n \in \mathcal{D}(\mathcal{E}) \cap L_{N^*}$, $n \in \mathbb{N}$, such that $\lim_{n \rightarrow \infty} u_n = u$ in L_{N^*} . Then by Lemma 2.2

$$\begin{aligned} v^* \langle \overline{i_a \circ Lu}, v \rangle_V &= \frac{1}{a} \langle [(\overline{(1 - aL)^{-1}L_{N^*}} - 1]u, v \rangle \\ &= \lim_{n \rightarrow \infty} \frac{1}{a} \langle (1 - aL)^{-1}u_n - (1 - aL)(1 - aL)^{-1}u_n, v \rangle \\ &= \lim_{n \rightarrow \infty} \langle L(1 - aL)^{-1}u_n, v \rangle. \end{aligned}$$

Let $v_m \in \mathcal{D}(\mathcal{E}) \subset L^2(\mathbf{m}) \subset H$, $m \in \mathbb{N}$, such that $\lim_{m \rightarrow \infty} v_m = v$ in H . Then for all $n \in \mathbb{N}$, since $L(1 - aL)^{-1}u_n = \frac{1}{a}[(1 - aL)^{-1}u_n - u_n] \in \mathcal{D}(\mathcal{E}) \cap L_{N^*}$

$$\begin{aligned} \langle L(1 - aL)^{-1}u_n, v \rangle &= \mathcal{D}(\mathcal{E}) \langle L(1 - aL)^{-1}u_n, v \rangle_H \\ &= \lim_{m \rightarrow \infty} \langle L(1 - aL)^{-1}u_n, v_m \rangle \\ &= - \lim_{m \rightarrow \infty} \mathcal{E}((1 - aL)^{-1}u_n, v_m) \\ &= - \lim_{m \rightarrow \infty} \mathcal{E}(u_n, (1 - aL)^{-1}v_m) \\ &= -\mathcal{E}(u_n, (1 - aL)^{-1}v) \\ &= -\frac{1}{a} \mathcal{E}_a(u_n, i_a v) + \frac{1}{a} \langle u_n, (1 - aL)^{-1}v \rangle_H \\ &= -\frac{1}{a} \mathcal{D}(\mathcal{E}) \langle u_n, v \rangle_H + \frac{1}{a} \mathcal{D}(\mathcal{E}) \langle u_n, (1 - aL)^{-1}v \rangle_H \\ &= \mathcal{D}(\mathcal{E}) \langle u_n, L(1 - aL)^{-1}v \rangle_H \\ &= v^* \langle u_n, L(1 - aL)^{-1}v \rangle_V \end{aligned}$$

by (i) But again by (i) and since $L_{N^*} \subset V^*$ continuously, the latter converges to $v^* \langle u, L(1 - aL)^{-1}v \rangle_V$ as $n \rightarrow \infty$.

(iii) Since by (i)

$$(1 - aL)^{-1}(V) \subset V$$

and since $(1 - aL)^{-1} : H \rightarrow \mathcal{D}(\mathcal{E}) \subset L^2(\mathbf{m}) \subset H$ is continuous, the continuity of $(1 - aL)^{-1}$ on V follows from the closed graph theorem, since the topology on V is stronger than that on H . Since $L_{N^*} \subset V^*$ continuously and densely, the second statement follows from (ii).

To prove the last assertion let $u \in \mathcal{D}(\mathcal{E})$, $v \in V$. Then

$$\begin{aligned}
{}_{V^*} \langle ((1 - aL)^{-1})^* u, v \rangle_V &= {}_{V^*} \langle u, (1 - aL)^{-1} v \rangle_V \\
&= \mathcal{D}(\mathcal{E}) \langle u, (1 - aL)^{-1} v \rangle_H \\
&= \langle u, v \rangle_{H_a} \\
&= \mathcal{D}(\mathcal{E}) \langle (1 - aL)^{-1} u, v \rangle_H \\
&= {}_{V^*} \langle (1 - aL)^{-1} u, v \rangle_V.
\end{aligned}$$

□

2.3 $L^2(\mathbf{m})$ -invariance

T2.3 **Theorem 2.8.** *Consider the situation of Theorem 2.6. Assume that $\mathbb{E} \|X_0\|_2^2 < \infty$, that there exist a progressively measurable $b \in L^2([0, T] \times \Omega \rightarrow \mathbb{R}, dt \times \mathbb{P})$ and $c_0 \in (0, \infty)$ such that for all $n \in \mathbb{N}$, $v \in V$*

2.11b (2.12) $\|B(\cdot, v)\|_{\mathcal{L}_{HS}(L^2(\mathbf{m}); H_{\frac{1}{n}})}^2 \leq c_0 \|v\|_{H_{\frac{1}{n}}}^2 + b^2 \quad dt \times \mathbb{P} - a.s. \text{ on } [0, T] \times \Omega$

(where we note that by assumption (B) the $dt \times \mathbb{P}$ -zero set is independent of $v \in V$). If there exists a constant $c > 0$ such that for all $a \in (0, 1)$

****** (2.13) $2 {}_{V^*} \langle \overline{i_a \circ L}(\Psi(s, X_s)) + \eta_s i_a(X_s), X_s \rangle_V \leq c \|X_s\|_{H_a}^2, \quad \mathbb{P}\text{-a.s. for } ds\text{-a.e. } s \in [0, T],$

then

2.3' (2.14) $\mathbb{E} \sup_{t \in [0, T]} \|X_t\|_2^2 < \infty$

and, in particular, $(X_t)_{t \in [0, T]}$ is weakly continuous in $L^2(\mathbf{m})$. Furthermore, $(X_t)_{t \in [0, T]}$ is right-continuous in $L^2(\mathbf{m})$.

Proof. By (2.13), the condition on B and Theorem 2.6, we have for $0 \leq r < t \leq T$ and $n \in \mathbb{N}$

2.6 (2.15) $e^{-ct} \|X_t\|_{H_{1/n}}^2 \leq e^{-cr} \|X_r\|_{H_{1/n}}^2 + \int_r^t \|B(s, X_s)\|_{\mathcal{L}_{HS}(L^2(\mathbf{m}); H_{1/n})}^2 e^{-cs} ds + 2 \int_r^t e^{-cs} dM_s^{(n)},$

where $M_t^{(n)} := \int_0^t \langle X_s, B(s, X_s) dW_s \rangle_{H_{1/n}}$, $t \in [0, T]$, is a local real martingale. Therefore, setting $r = 0$ in (2.15), it follows for every stopping time $\tau \leq T$

2.6a (2.16)
$$\begin{aligned}
&\mathbb{E} \sup_{t \in [0, \tau]} \left(\|X_t\|_{H_{1/n}}^2 e^{-ct} \right) \\
&\leq \mathbb{E} \|X_0\|_2^2 + \mathbb{E} \int_0^\tau (c_0 \|X_s\|_{H_{\frac{1}{n}}}^2 + b_s^2) e^{-cs} ds + 2 \mathbb{E} \sup_{t \in [0, \tau]} \left| \int_0^t e^{-cs} dM_s^{(n)} \right|.
\end{aligned}$$

But by the Burkholder-Davis-Gundy inequality (for $p = 1$)

$$\begin{aligned}
& \mathbb{E} \sup_{t \in [0, \tau]} \left| \int_0^t e^{-cs} dM_s \right| \leq 3 \mathbb{E} \left(\int_0^\tau \|B^*(s, X_s) X_s\|_{L^2(\mathbf{m})}^2 e^{-2cs} ds \right)^{1/2} \\
& \leq 3 \mathbb{E} \left(\int_0^\tau \|X_s\|_{H_{1/n}}^2 \|B(s, X_s)\|_{\mathcal{L}_{HS}(L^2(\mathbf{m}); H_{1/n})}^2 e^{-2cs} ds \right)^{1/2} \\
\boxed{2.6b} \quad (2.17) \quad & \leq 3 \left(\mathbb{E} \sup_{t \in [0, \tau]} \|X_t\|_{H_{1/n}}^2 e^{-ct} \right)^{1/2} \cdot \left(\mathbb{E} \int_0^\tau (c_0 \|X_s\|_{H_{\frac{1}{n}}}^2 + b_s^2) e^{-cs} ds \right)^{1/2}.
\end{aligned}$$

By Grownwall's lemma (2.16) and (2.17) imply that

$$\boxed{\text{eq:2.12}} \quad (2.18) \quad \mathbb{E} \sup_{t \in [0, T]} \|X_t\|_2^2 = \sup_{n \in \mathbb{N}} \mathbb{E} \sup_{t \in [0, T]} \|X_t\|_{H_{1/n}}^2 < \infty,$$

since $\|\cdot\|_2 = \sup_n \|\cdot\|_{H_{1/n}} = \lim_{n \rightarrow \infty} \|\cdot\|_{H_{1/n}}$, so we can apply monotone convergence. In particular, X_t is weakly continuous in $L^2(\mathbf{m})$, since it is continuous in H .

Next, letting $n \rightarrow \infty$ in (2.12) by (2.18) and the Burkholder-Davis-Gundy inequality (for $p = 1$) we obtain

$$\begin{aligned}
& \limsup_{n \rightarrow \infty} \left\{ \mathbb{E} \sup_{t \in [0, T]} \left| \int_0^t (\langle X_s, B(s, X_s) dW_s \rangle_{H_{1/n}} - \langle X_s, B(s, X_s) dW_s \rangle) \right| \right\} \\
& \leq 3 \limsup_{n \rightarrow \infty} \mathbb{E} \left(\int_0^T \|(1 - n^{-1}L)^{-1} X_s - X_s\|_2^2 \|B(s, X_s)\|_{\mathcal{L}_{HS}(L^2(\mathbf{m}); L^2(\mathbf{m}))}^2 ds \right)^{1/2} \\
& \leq 3 \lim_{n \rightarrow \infty} \mathbb{E} \left(\int_0^T \|(1 - n^{-1}L)^{-1} X_s - X_s\|_2 (c_0 \|X_s\|_2^2 + b_s^2) ds \right)^{1/2} = 0, \quad T > 0.
\end{aligned}$$

Thus, up to a subsequence, \mathbb{P} -a.s.

$$\lim_{n \rightarrow \infty} \int_0^t \langle X_s, B(s, X_s) dW_s \rangle_{H_{1/n}} = \int_0^t \langle X_s, B(s, X_s) dW_s \rangle, \quad t \geq 0,$$

which is a real valued continuous martingale. Hence in (2.15) we can let first $n \rightarrow \infty$ and then $t \downarrow r$, to obtain

$$\limsup_{t \downarrow r} \|X_t\|_2 \leq \|X_r\|_2.$$

On the other hand, by the $L^2(\mathbf{m})$ -weak continuity of X_t we have $\liminf_{t \rightarrow r} \|X_t\|_2 \geq \|X_r\|_2$. So $\|X_t\|_2$ is right-continuous and hence, X_t is right-continuous in $L^2(\mathbf{m})$ again due to the $L^2(\mathbf{m})$ -weak continuity. \square

rem-2.7 Remark 2.9. (i) We emphasize that Theorem 2.8 applies to solutions as in Theorem 2.1 without the assumption $\inf \sigma(-L) > 0$. We just need an Itô formula as in (2.10).

(ii) Obviously, (H3 (i)) implies (2.12) provided

$$\int_0^T \|B(s, 0)\|_{\mathcal{L}_{HS}(L^2(\mathbf{m}); L^2(\mathbf{m}))}^2 ds < \infty.$$

(iii) Now we want to describe examples in which (H3 (i)) holds with B non-constant in $v \in V$. The easiest is to take $B_0 : [0, T] \times \Omega \rightarrow \mathcal{L}_{HS}(L^2(\mathbf{m}), H)$ progressively measurable, $u_0 \in L^2(\mathbf{m})$ and $f : [0, T] \times \Omega \rightarrow \mathbb{R}$ progressively measurable and bounded. Then

$$B(t, v) := f(t)\langle \cdot, u_0 \rangle u + B_0$$

is easily checked to satisfy (H3 (i)). Further examples one obtains as follows:

(M) Let $N \in \mathbb{N} \cup \{+\infty\}$ and $e_k \in L^2(\mathbf{m}) \cap L^\infty(\mathbf{m})$, $1 \leq k \leq N$, be an orthonormal system in $L^2(\mathbf{m})$ such that for every $1 \leq k \leq N$ there exists $\xi_k \in (0, \infty)$ such that for all $a \in (0, \infty)$

$$|{}_H\langle x, e_k u \rangle_{\mathcal{D}(\mathcal{E})}| \leq \xi_k \|x\|_{H_a} \mathcal{E}_a(u, u)^{1/2} \quad \text{for all } u \in \mathcal{D}(\mathcal{E}).$$

(M) just means that each e_k is a multiplier on H_a with norm independent of $a > 0$. Choose $\mu_k \in (0, \infty)$ such that

$$\boxed{2.17b} \quad (2.19) \quad \sum_{k=1}^{\infty} \xi_k^2 \mu_k^2 < \infty,$$

and define for $x \in H$, $B(x) \in \mathcal{L}_{HS}(L^2(\mathbf{m}); H)$ by

$$B(x)h := \sum_{k=1}^{\infty} \mu_k \langle e_k, h \rangle x \cdot e_k, \quad h \in L^2(\mathbf{m}).$$

Indeed, (extending $\{e_k | k \in \mathbb{N}\}$ to an orthonormal basis of $L^2(\mathbf{m})$) by (M) we have for $x \in H$, $a \in (0, \infty)$

$$\begin{aligned} \|B(x)\|_{\mathcal{L}_{HS}(L^2(\mathbf{m}); H_a)}^2 &= \sum_{k=1}^{\infty} \|B(x)e_k\|_{H_a}^2 \\ &= \sum_{k=1}^{\infty} \mu_k^2 \|xe_k\|_{H_a}^2 \\ &\leq \sum_{k=1}^{\infty} \mu_k^2 \xi_k^2 \|x\|_{H_a}^2 \end{aligned}$$

and since $x \mapsto B(x)$ is linear and $V \subset H$, condition (H3(i)) follows.

Now let us describe a large class of Dirichlet forms $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ for which (M) holds. Let us assume that (1.3) holds, and define the square field operator of L by

$$\Gamma(u, v) := \frac{1}{2}(L(uv) - uLv - vLu), \quad u, v \in \mathcal{A},$$

where $\{e_k | k \in \mathbb{N}\} \subset \mathcal{A} \subset \mathcal{D}(L)$ and \mathcal{A} is an algebra of bounded functions which is dense in $\mathcal{D}(\mathcal{E})$ with respect to \mathcal{E}_1 . Γ is symmetric in u, v . Suppose that there exist $\chi_n \in \mathcal{D}(L)$, $\chi_n \geq 0$, $\chi_n \rightarrow 1$ in $L^2(\mathbf{m})$ as $n \rightarrow \infty$. Then clearly

$$\mathcal{E}(u, v) = \int \Gamma(u, v) d\mathbf{m} \quad \text{for all } u, v \in \mathcal{D}(\mathcal{E}).$$

Assume further that for all $u_1, u_2, v \in \mathcal{A}$

$$\Gamma(u_1 u_2, v) = u_1 \Gamma(u_2, v) + u_2 \Gamma(u_1, v)$$

which is e.g. the case if (L, \mathcal{A}) is a diffusion operator in the sense of [8, Appendix B, Definition 1.5], like e.g. a partial differential operator of order 2. Assume $d > 2$ and that $\Gamma(e_k, e_k) \in L^{d/2}(\mathbf{m})$. Then by (1.4) we obtain for $u \in \mathcal{A}$ and $1 \leq k \leq N$

$$\begin{aligned} \mathcal{E}_a(e_k u, e_k u) &\leq 2a \int (u^2 \Gamma(e_k, e_k) + e_k^2 \Gamma(u, u)) d\mathbf{m} + \int e_k^2 u^2 d\mathbf{m} \\ &\leq 2a \left(\|\Gamma(e_k, e_k)\|_{\frac{d}{2}} \|u\|_{\frac{2d}{d-2}}^2 + \|e_k\|_{\infty}^2 \mathcal{E}(u, u) \right) + \|e_k\|_{\infty}^2 \|u\|_2^2 \\ &\leq 2a \left(C_d^2 \|\Gamma(e_k, e_k)\|_{\frac{d}{2}} + \|e_k\|_{\infty}^2 \right) \mathcal{E}(u, u) + \|e_k\|_{\infty}^2 \|u\|_2^2. \end{aligned}$$

Hence (M) holds in this case with

$$\boxed{2.20a} \quad (2.20) \quad \xi_k := \sqrt{2(C_d^2 \|\Gamma(e_k, e_k)\|_{\frac{d}{2}} + \|e_k\|_{\infty}^2)}.$$

If one wants to choose μ_k in (2.19) in a somewhat optimal way, one needs bounds on ξ_k . To this end let us assume that e_k , $1 \leq k \leq N := \infty$, is an eigenbasis of L , with corresponding eigenvalues $-\lambda_k$, $k \in \mathbb{N}$. Then one can get estimates on ξ_k in terms of merely e_k (not $\Gamma(e_k, e_k)$) and λ_k or even λ_k alone, for which the asymptotics is precisely known in a large number of cases. Note first that (1.3) then implies that $\lambda_k > 0$, $k \in \mathbb{N}$. In what follows we do not need that $d > 2$. In the present situation it is then easy to check that for all $u \in \mathcal{A}$, $k \in \mathbb{N}$,

$$\boxed{2.18} \quad (2.21) \quad \mathcal{E}(e_k u, e_k u) = \int \Gamma(e_k u, e_k u) d\mathbf{m} = \int (\lambda_k u^2 + \Gamma(u, u)) e_k^2 d\mathbf{m}.$$

We consider two cases:

Case 1: $d > 2$.

Then by (1.4), (2.21) and Hölder's inequality for all $u \in \mathcal{A}$, $k \in \mathbb{N}$,

$$\mathcal{E}_a(e_k u, e_k u) \leq \|e_k\|_{\infty}^2 \|u\|_2^2 + a(C_d \lambda_k \|e_k\|_d^2 + \|e_k\|_{\infty}^2) \mathcal{E}(u, u) \leq \xi_k^2 \mathcal{E}_a(u, u),$$

with

$$\xi_k := \sqrt{C_d \lambda_k \|e_k\|_d^2 + \|e_k\|_{\infty}^2}.$$

It is worth noting that if $d \leq 4$, hence $d \leq \frac{2d}{d-2}$, and if $\mathbf{m}(E) < \infty$, applying Hölder's inequality and (2.21) with $u := e_k$ we obtain that up to a constant $\|e_k\|_d^2$ is bounded by $\mathcal{E}(e_k, e_k) = \langle -Le_k, e_k \rangle = \lambda_k$, hence

$$\boxed{2.20} \quad (2.22) \quad \xi_k \leq \text{const} \cdot (\max(\lambda_k, \|e_k\|_\infty) + 1)$$

in this case.

Case 2: $d = 1, 2$, $E \subset \mathbb{R}^d$, E open, bounded, and $L = \Delta$ with Dirichlet boundary conditions on ∂E , $\mathbf{m} = dx = \text{Lebesgue measure}$.

In this case it is well known that for $p = \infty$, if $d = 1$, and $p \in [1, \infty)$, if $d = 2$, there exists $C_p \in (0, \infty)$ such that for all $u \in \mathcal{D}(\mathcal{E})$

$$\|u\|_p \leq C_p \mathcal{E}(u, u)^{1/2},$$

hence $\|e_k\|_p \leq C_p \lambda_k^{1/2}$, $k \in \mathbb{N}$, and by Sobolev's embedding for all $k \in \mathbb{N}$

$$\boxed{2.21} \quad (2.23) \quad \|e_k\|_\infty \leq \text{const} \cdot \lambda_k.$$

Hence by (2.21) for all $a \in (0, \infty)$, $u \in \mathcal{A}$

$$\begin{aligned} \mathcal{E}_a(e_k u, e_k u) &\leq C \lambda_k^2 \|u\|_2^2 + a (\lambda_k \|u\|_4^2 \|e_k\|_4^2 + \lambda_k^2) \mathcal{E}(u, u) \\ &\leq \tilde{C} \lambda_k^2 \|u\|_2^2 + a (\lambda_k^3 C_4^4 + \lambda_k^2) \mathcal{E}(u, u) \\ &\leq \xi_k^2 \cdot \mathcal{E}_a(u, u) \end{aligned}$$

with

$$\xi_k := \tilde{C} \cdot (\lambda_k^{3/2} + 1),$$

and the constant \tilde{C} is independent of a, k, u .

We also note that if we consider Case 2 for $d = 3$, then (2.23) still holds (see e.g. [1]). In fact for nice domains E even $\sup_{k \in \mathbb{N}} \|e_k\|_\infty < \infty$ for all $d \in \mathbb{N}$. Hence by (2.22) we get

$$\xi_k \leq \text{const} \cdot (\lambda_k + 1), \quad k \in \mathbb{N}.$$

3 Some estimates

Let $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ be as in the introduction satisfying (1.3). In this section we first present some estimates on the operator $(\varepsilon - L)^{-1/2}$ which will be used in the next section for constructing solutions of (1.5), where $(L, \mathcal{D}(L))$ is the Dirichlet operator associated with $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ (see Section 1).

L2.2 Lemma 3.1. *Assume (1.3). For any $p \in (2, 2d/(d-2)^+)$, there exists $\alpha_p \in (0, 1/2)$ and $c_p \geq 1$, both continuous in p , such that*

$$\|(\varepsilon - L)^{-1/2}\|_{2 \rightarrow p} \leq c_p \varepsilon^{-\alpha_p}, \quad \varepsilon \in (0, 1).$$

Proof. Let $P_t := e^{tL}$ and $\{E_\lambda : \lambda \geq 0\}$ the spectral family of $-L$. By the spectral representation theorem we have

$$\begin{aligned} \text{A} \quad (3.1) \quad & \int_0^\infty \frac{e^{-\varepsilon t}}{\sqrt{t}} P_t dt = \int_0^\infty dE_\lambda \int_0^\infty \frac{e^{-(\varepsilon+\lambda)t}}{\sqrt{t}} dt \\ & = 2 \int_0^\infty dE_\lambda \int_0^\infty e^{-(\varepsilon+\lambda)t^2} dt = \sqrt{\pi} \int_0^\infty \frac{1}{\sqrt{\varepsilon+\lambda}} dE_\lambda = \sqrt{\pi}(\varepsilon - L)^{-1/2} \end{aligned}$$

for all $\varepsilon > 0$. By the Nash inequality (1.3), there exists $c \geq 1$ such that (cf. [6])

$$\|P_t\|_{2 \rightarrow \infty} \leq ct^{-d/4}, \quad t > 0.$$

But $\|P_t\|_{2 \rightarrow 2} \leq 1$. By the Riesz-Thorin interpolation theorem, we obtain

$$\text{B} \quad (3.2) \quad \|P_t\|_{2 \rightarrow p} \leq ct^{-d(p-2)/4p}, \quad t > 0.$$

Taking $\delta_p := \frac{1}{2} + \frac{d(p-2)}{4p}$, we have $\delta_p \in (1/2, 1)$ since $p \in (2, 2d/(d-2)^+)$. Let $\delta'_p := \frac{1}{2} + \frac{1}{4(1-\delta_p)}$, so that $\alpha_p := \delta'_p(1-\delta_p) \in (0, \frac{1}{2})$. Then by (3.1) and (3.2), there exists $c_1 > 0$ such that for all $\varepsilon \in (0, 1)$

$$\begin{aligned} \|(\varepsilon - L)^{-1/2}\|_{2 \rightarrow p} & \leq c_1 \int_0^\infty e^{-\varepsilon t} t^{-\delta_p} dt \leq c_1 \int_0^{\varepsilon^{-\delta'_p}} t^{-\delta_p} dt + c_1 \int_{\varepsilon^{-\delta'_p}}^\infty e^{-\varepsilon t} dt \\ & \leq \frac{c_1 \varepsilon^{-\alpha_p}}{1-\delta_p} + \frac{c_1}{\varepsilon} \exp[-\varepsilon^{-(\delta'_p-1)}]. \end{aligned}$$

Since $\delta'_p > 1$, the last term is bounded w.r.t. $\varepsilon \in (0, 1)$, so that the desired assertion holds for some $c_p \geq 1$ continuous in $p \in (2, 2d/(d-2)^+)$ and all $\varepsilon \in (0, 1)$. \square

L2.3 **Lemma 3.2.** *Let (1.3) hold and let ε, p, c_p and α_p be as in Lemma 3.1. Then for any $r > p-1$ and any $x \in L^2(\mathbf{m}) \cap L^{r+1}(\mathbf{m})$,*

$$\|(\varepsilon - L)^{-1/2}x\|_{r+1} \leq c_p \varepsilon^{-(\frac{1}{2} - (1-2\alpha_p)(p-2)/2(r-1))} \|x\|_2^{(p-2)/(r-1)} \|x\|_{r+1}^{(r+1-p)/(r-1)}.$$

Consequently, for any $\delta \in (0, 1 \wedge \frac{4}{(d-2)^+(r_2-1)})$, there exist $c > 0$ and $\alpha \in (0, 1/2)$ such that

$$\|(\varepsilon - L)^{-1/2}x\|_{r+1} \leq c\varepsilon^{-\alpha} \|x\|_2^\theta \|x\|_{r+1}^{1-\theta},$$

for $r \in [r_1, r_2]$, $x \in L^2(\mathbf{m}) \cap L^{r_2+1}(\mathbf{m})$, $\theta \in [\delta, 1 \wedge \frac{4}{(d-2)^+(r_2-1)} - \delta]$.

Proof. Since $s := (r-1)/(r+1)$ satisfies

$$\frac{s}{\infty} + \frac{1-s}{2} = \frac{1}{r+1}, \quad \frac{s}{\infty} + \frac{1-s}{p} = \frac{1}{p(r+1)/2},$$

by the interpolation theorem

$$\|(\varepsilon - L)^{-1/2}\|_{r+1 \rightarrow p(r+1)/2} \leq \|(\varepsilon - L)^{-1/2}\|_{\infty \rightarrow \infty}^s \|(\varepsilon - L)^{-1/2}\|_{2 \rightarrow p}^{1-s}.$$

Moreover, (3.1) implies

$$\|(\varepsilon - L)^{-1/2}\|_{\infty \rightarrow \infty} \leq \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-\varepsilon t}}{\sqrt{t}} dt \leq \varepsilon^{-1/2}.$$

So, combining the above with Lemma 3.1, we obtain

$$\boxed{3.0} \quad (3.3) \quad \|(\varepsilon - L)^{-1/2}\|_{r+1 \rightarrow p(r+1)/2} \leq c_p \varepsilon^{-(4\alpha_p + r - 1)/2(r+1)} \leq c_p \varepsilon^{-1/2}, \quad \varepsilon \in (0, 1).$$

Let $t := (r + 1)(p - 2)/(r - 1)$. By Hölder inequality we obtain

$$\begin{aligned} \mathbf{m}(|(\varepsilon - L)^{-1/2}x|^{r+1}) &= \mathbf{m}(|(\varepsilon - L)^{-1/2}x|^t \cdot |(\varepsilon - L)^{-1/2}x|^{r+1-t}) \\ &\leq \mathbf{m}(|(\varepsilon - L)^{-1/2}x|^p)^{t/p} \mathbf{m}(|(\varepsilon - L)^{-1/2}x|^{(r+1-t)p/(p-t)})^{(p-t)/p} \\ &= \|(\varepsilon - L)^{-1/2}x\|_p^{(r+1)(p-2)/(r-1)} \|(\varepsilon - L)^{-1/2}x\|_{(r+1)p/2}^{(r+1)(r+1-p)/(r-1)}. \end{aligned}$$

Combining this with (3.3) and Lemma 3.1 we prove the first assertion. Finally, for fixed $\theta \in (0, 1 \wedge \frac{4}{(r_2-1)(d-2)^+})$, the second assertion follows from the first by taking $p_{r,\theta} := 2 + \theta(r - 1)$ so that $c_{p_{r,\theta}}$ is bounded for $r \in [r_1, r_2]$ and $\theta \in [\delta, 1 \wedge \frac{4}{(d-2)^+(r_2-1)} - \delta]$. \square

Now assume that (H1) – (H3) hold. Our next aim is to apply Theorem 2.1 with $L - \varepsilon$ instead of L , i.e., we fix $\varepsilon \in (0, 1)$ and consider the equation

$$\boxed{3.1} \quad (3.4) \quad dX_t^\varepsilon = [(L - \varepsilon)\Psi(t, X_t^\varepsilon) + \eta_t X_t^\varepsilon] dt + B_t dW_t, \quad X_0^\varepsilon = X_0,$$

where

$$\Psi(t, s) := \int_{r_1}^{r_2} \xi(t, r) |s|^{r-1} s \nu(dr), \quad s \in \mathbb{R}, t \geq 0.$$

Define

$$N(s) := \int_{r_1}^{r_2} |s|^{r+1} \nu(dr), \quad s \in \mathbb{R}.$$

It is trivial to see that both N and $N^*(s) := \inf_{r \geq 0} \{|sr| - N(r)\}$ are Δ_2 -regular, which follows directly from the calculation in [15, Example 3.5] where $\nu := \sum_{i=1}^n c_i \delta_{r_i}$ for $c_i > 0$ and $r_i > 1$. Then (Ψ) follows from (H1) and (B) from (H3).

By Theorem 2.6 (applied to $L - \varepsilon$ replacing L) for any $a \in (0, \varepsilon^{-1})$ we have that \mathbb{P} -a.s.

$$\boxed{3.4a} \quad (3.5) \quad i_a(X_t) = i_a(X_0) + \int_0^t [\overline{i_a \circ (L - \varepsilon)}(\Psi(s, X_s)) + \eta_s i_a(X_s)] ds + i_a \left(\int_0^t B_s dW_s \right), \quad t \geq 0,$$

where we used that

$$\boxed{3.4b} \quad (3.6) \quad i_a = (1 - aL)^{-1} = \frac{1}{1 - a\varepsilon} \left(1 - \frac{a}{1 - a\varepsilon} (L - \varepsilon) \right)^{-1} \quad \text{for } a \in (0, \varepsilon^{-1}).$$

Furthermore, applying Lemma 2.2 with $L - \varepsilon$ replacing L and using (3.6) we obtain for all $u \in L_{N^*}$, $v \in V$, $a \in (0, \varepsilon^{-1})$

$$\boxed{3.6b} \quad (3.7) \quad {}_{V^*} \left\langle \overline{i_a \circ (L - \varepsilon)u}, v \right\rangle_V = \frac{1 - a\varepsilon}{a} \left\langle \overline{(1 - aL)^{-1}L_{N^*} u}, v \right\rangle - \frac{1}{a} \langle u, v \rangle,$$

which by an easy approximation argument is equal to

$$\frac{1 - a\varepsilon}{a} \langle u, \overline{(1 - aL)^{-1}L_N v} \rangle - \frac{1}{a} \langle u, v \rangle,$$

where $\overline{(1 - aL)^{-1}L_N}$ is the unique continuous extension of $(1 - aL)^{-1} : L^1(\mathbf{m}) \cap L^\infty(\mathbf{m}) \rightarrow L_N$ to all of L_N . It, however, follows immediately from (2.11) that

$$\boxed{eq:3.4b} \quad (3.8) \quad \overline{(1 - aL)^{-1}L_N} v = (1 - aL)^{-1}v \quad \text{for all } v \in V (= H \cap L_N),$$

where we recall that the right hand side is by definition the Riesz map $(1 - aL)^{-1} : (H, \langle \cdot, \cdot \rangle_{H_a}) \rightarrow (\mathcal{D}(\mathcal{E}), \mathcal{E}_a)$ applied to v as an element in H . Therefore, we do not distinguish $\overline{(1 - aL)^{-1}L_N}$ and $(1 - aL)^{-1}$ below. So, altogether we obtain

$$\boxed{eq:3.4c} \quad (3.9) \quad {}_{V^*} \left\langle \overline{i_a \circ (L - \varepsilon)u}, v \right\rangle_V = \frac{1 - a\varepsilon}{a} \langle u, (1 - aL)^{-1}v \rangle - \frac{1}{a} \langle u, v \rangle, \quad \text{for all } u \in L_{N^*}, v \in V, a \in (0, \frac{1}{\varepsilon}).$$

Therefore, by Theorem 2.1, applied to $L - \varepsilon$ in place of L , if $\mathbb{E}\|X_0\|_H^2 < \infty$ then (3.4) has a unique solution X^ε which is a continuous adapted process in H and $X^\varepsilon \in L_N([0, T] \times E \times \Omega; dt \times \mathbf{m} \times \mathbb{P}) \cap L^2([0, T] \times \Omega \rightarrow H; dt \times \mathbb{P})$.

L3.1 Lemma 3.3. *Assume that (H1)–(H3) and (1.3) hold. Let $X_0 : \Omega \rightarrow H$ be \mathcal{F}_0 -measurable such that $\mathbb{E}\|X_0\|_H^2 < \infty$. Let $T > 0$ be fixed. Then for any $q \geq 1$ there exists a constant $c(q) > 0$ such that for any $\varepsilon \in (0, 1)$,*

$$\boxed{3.2} \quad (3.10) \quad \mathbb{E} \sup_{t \in [0, T]} \|X_t^\varepsilon\|_H^{q+1} \leq c(q) \left(\mathbb{E}\|X_0\|_H^{q+1} + \mathbb{E} \left(\int_0^T \|B(s, 0)\|_{\mathcal{L}_{HS}(L^2(\mathbf{m}); H)}^2 ds \right)^{\frac{q+1}{2}} \right)$$

and

$$\boxed{3.3} \quad (3.11) \quad \mathbb{E} \left(\int_0^T dt \int_{r_1}^{r_2} \|X_t^\varepsilon\|_{r+1}^{r+1} \nu(dr) \right)^q \leq c(q) \left(1 + \mathbb{E}\|X_0\|_H^{(r_2+1)q} + \mathbb{E} \left(\int_0^T \|B(s, 0)\|_{\mathcal{L}_{HS}(L^2(\mathbf{m}); H)}^2 ds \right)^{\frac{(r_2+1)q}{2}} \right).$$

Proof. We may assume that the right hand sides of (3.10) and (3.11) are finite. We recall that $\langle \cdot, \cdot \rangle_H = \langle \cdot, \cdot \rangle_{H_1}$, $\| \cdot \|_H = \| \cdot \|_{H_1}$.

(a) By assumptions (H1) – (H3) and using the Itô formula in Theorem 2.6 and (3.9) for $a = 1$, we have

$$\begin{aligned}
& d\|X_t^\varepsilon\|_H^2 \\
&= 2 \int_{r_1}^{r_2} \langle \overline{i_1 \circ (L - \varepsilon)} \Psi(t, X_t^\varepsilon) + \eta_t i_1(X_t^\varepsilon), X_t^\varepsilon \rangle_V dt \\
\boxed{3.4} \quad (3.12) \quad & + \|B(t, X_t^\varepsilon)\|_{\mathcal{L}_{HS}(L^2(\mathbf{m}); H)}^2 dt + 2\langle X_t^\varepsilon, B(t, X_t^\varepsilon) dW_t \rangle_H \\
& \leq (c\|X_t^\varepsilon\|_H^2 + \|B(t, 0)\|_{\mathcal{L}_{HS}(L^2(\mathbf{m}); H)}^2) dt - 2\langle X_t^\varepsilon, \Psi(t, X_t^\varepsilon) \rangle dt \\
& + 2(1 - \varepsilon) \langle (1 - L)^{-1} X_t^\varepsilon, \Psi(t, X_t^\varepsilon) \rangle dt + 2\langle X_t^\varepsilon, B(t, X_t^\varepsilon) dW_t \rangle_H
\end{aligned}$$

for some constant $c > 0$. Since

$$\begin{aligned}
& - 2\langle X_t^\varepsilon, \Psi(t, X_t^\varepsilon) \rangle + 2(1 - \varepsilon) \langle (1 - L)^{-1} X_t^\varepsilon, \Psi(t, X_t^\varepsilon) \rangle \\
& \leq -2 \int_{r_1}^{r_2} \xi(t, r) \|X_t^\varepsilon\|_{r+1}^{r+1} \nu(dr) + 2(1 - \varepsilon) \int_{r_1}^{r_2} \xi(t, r) \|(1 - L)^{-1} X_t^\varepsilon\|_{r+1} \|X_t^\varepsilon\|_{r+1}^r \nu(dr) \\
& \leq 0,
\end{aligned}$$

(3.12) implies

$$d\|X_t^\varepsilon\|_H^2 \leq (c\|X_t^\varepsilon\|_H^2 + \|B(t, 0)\|_{\mathcal{L}_{HS}(L^2(\mathbf{m}), H)}^2) dt + 2\langle X_t^\varepsilon, B(t, X_t^\varepsilon) dW_t \rangle_H.$$

By Itô's formula, applied to the real valued semimartingale $Z_t := \|X_t^\varepsilon\|_H^2$, $t \in [0, t]$, for any $q \geq 1$ there exists $c_1(q) > 0$ such that

$$\begin{aligned}
\boxed{3.5} \quad (3.13) \quad & d\|X_t^\varepsilon\|_H^{q+1} \\
& \leq c_1(q) (\|X_t^\varepsilon\|_H^{q+1} + \|X_t^\varepsilon\|_H^{q-1} \|B(t, 0)\|_{\mathcal{L}_{HS}(L^2(\mathbf{m}); H)}^2) dt + (q+1) \|X_t^\varepsilon\|_H^{q-1} \langle X_t^\varepsilon, B(t, X_t^\varepsilon) dW_t \rangle_H.
\end{aligned}$$

Thus, any stopping time $\tau \leq T$, applying first Itô's product rule, then the Burkholder-Davis-

Gundy inequality for $p = 1$, and using (H3) we obtain

$$\begin{aligned}
& \mathbb{E} \sup_{t \in [0, \tau]} \|X_t^\varepsilon\|_H^{q+1} e^{-c_1(q)t} \\
& \leq \mathbb{E} \|X_0^\varepsilon\|_H^{q+1} + c_1(q) \mathbb{E} \int_0^\tau \|X_s^\varepsilon\|_H^{q-1} \|B(s, 0)\|_{\mathcal{L}_{HS}(L^2(\mathbf{m}); H)}^2 e^{-c_1(q)s} ds \\
& \quad + (q+1) \mathbb{E} \sup_{t \in [0, \tau]} \left| \int_0^t \|X_s^\varepsilon\|_H^{q-1} e^{-c_1(q)s} \langle X_s^\varepsilon, B(s, X_s^\varepsilon) dW_s \rangle_H \right| \\
& \leq \mathbb{E} \|X_0^\varepsilon\|_H^{q+1} + c_1(q) \mathbb{E} \sup_{t \in [0, \tau]} (\|X_t^\varepsilon\|_H^{q-1} e^{-c_1(q)t}) \int_0^t \|B(s, 0)\|_{\mathcal{L}_{HS}(L^2(\mathbf{m}); H)}^2 ds \\
& \quad + 3(q+1) \mathbb{E} \left(\int_0^\tau \|X_s^\varepsilon\|_H^{2q} \|B(s, X_s^\varepsilon)\|_{\mathcal{L}_{HS}(L^2(\mathbf{m}); H)}^2 e^{-2c_1(q)s} ds \right)^{\frac{1}{2}} \\
& \leq \mathbb{E} \|X_0^\varepsilon\|_H^{q+1} + c_1(q) \left[\mathbb{E} \sup_{t \in [0, \tau]} (\|X_t^\varepsilon\|_H^{q+1} e^{-c_1(q)t}) \right]^{\frac{q-1}{q+1}} \left[\mathbb{E} \left(\int_0^T \|B(s, 0)\|_{\mathcal{L}_{HS}(L^2(\mathbf{m}); H)}^2 ds \right)^{\frac{q+1}{2}} \right]^{\frac{2}{q+1}} \\
& \quad + 3(q+1)c \left[\mathbb{E} \sup_{t \in [0, \tau]} (\|X_t^\varepsilon\|_H^{q+1} e^{-c_1(q)t}) \right]^{\frac{1}{2}} \left[\mathbb{E} \int_0^\tau \|X_s^\varepsilon\|_H^{q+1} e^{-c_1(q)s} ds \right]^{\frac{1}{2}} \\
& \quad + 3(q+1) \left[\mathbb{E} \sup_{s \in [0, \tau]} (\|X_s^\varepsilon\|_H^{q+1} e^{-c_1(q)s}) \right]^{\frac{q}{q+1}} \left[\mathbb{E} \left(\int_0^T \|B(s, 0)\|_{\mathcal{L}_{HS}(L^2(\mathbf{m}); H)}^2 ds \right)^{\frac{q+1}{2}} \right]^{\frac{1}{q+1}} \\
& \leq \mathbb{E} \|X_0^\varepsilon\|_H^{q+1} + \frac{1}{2} \mathbb{E} \sup_{t \in [0, \tau]} (\|X_t^\varepsilon\|_H^{q+1} e^{-c_1(q)t}) \\
& \quad + \tilde{C}(q) \left(\mathbb{E} \left[\int_0^T \|B(s, 0)\|_{\mathcal{L}_{HS}(L^2(\mathbf{m}); H)}^2 ds \right]^{\frac{q+1}{2}} + \mathbb{E} \int_0^\tau \|X_s^\varepsilon\|_H^{q+1} e^{-c_1(q)s} ds \right)
\end{aligned}$$

for some constant $\tilde{C}(q) > 0$, where we used Young's inequality in the last step.

By Gronwall's Lemma this implies (3.10) for some $c(q) > 0$ (independent of ε).

(b) By (3.12), assumptions (H1), (H3), and Lemma 3.2 with $\varepsilon = 1$, there exist $\delta_1, \delta_2, \delta_3 > 0$ (independent of ε) such that

$$\begin{aligned}
d\|X_t^\varepsilon\|_H^2 & \leq (c\|X_t^\varepsilon\|_H^2 + \|B(t, 0)\|_{\mathcal{L}_{HS}(L^2(\mathbf{m}); H)}^2) dt - 2\delta_1 \int_{r_1}^{r_2} \|X_t^\varepsilon\|_{r+1}^{r+1} \nu(dr) dt \\
& \quad + \delta_2 \int_{r_1}^{r_2} \|X_t^\varepsilon\|_H^\theta \|X_t^\varepsilon\|_{r+1}^{r+1-\theta} \nu(dr) + 2\langle X_t^\varepsilon, B(t, X_t^\varepsilon) dW_t \rangle_H \\
& \leq \delta_3 (1 + \|X_t^\varepsilon\|_H^{r_2+1} + \|B(t, 0)\|_{\mathcal{L}_{HS}(L^2(\mathbf{m}); H)}^2) dt \\
& \quad - \delta_1 \int_{r_1}^{r_2} \|X_t^\varepsilon\|_{r+1}^{r+1} \nu(dr) dt + 2\langle X_t^\varepsilon, B(t, X_t^\varepsilon) dW_t \rangle_H,
\end{aligned}$$

where the last step follows from the fact that

$$a^\theta b^{r+1-\theta} \leq \frac{\delta_1}{\delta_2} b^{r+1} + c_0 a^{r+1}$$

holds for some constant $c_0 > 0$ and all $a, b \geq 0, r \in [r_1, r_2]$. This implies

$$\begin{aligned} & \delta_1 \int_0^T dt \int_{r_1}^{r_2} \|X_t^\varepsilon\|_{r+1}^{r+1} \nu(dr) \\ & \leq \|X_0\|_H^2 + \delta_3 \int_0^T (1 + \|X_t^\varepsilon\|_H^{r_2+1} + \|B(t, 0)\|_{\mathcal{L}_{HS}(L^2(\mathbf{m}); H)}^2) dt + 2 \int_0^T \langle X_t^\varepsilon, B(t, X_t^\varepsilon) dW_t \rangle_H. \end{aligned}$$

Therefore, (3.11) follows from (3.10) by similar arguments as above. \square

4 Existence of solutions for special initial conditions

P4.1 **Proposition 4.1.** *Consider the situation of Theorem 1.2. If $\|X_0\|_H \in L^{2r_2}(\mathbb{P})$ then (1.5) has a unique solution, and the solution satisfies*

$$(4.1) \quad \mathbb{E} \sup_{t \in [0, T]} \|X_t\|_H^{2r_2} + \mathbb{E} \left(\int_0^T \int_{r_1}^{r_2} \|X_t\|_{r+1}^{r+1} \nu(dr) dt \right)^{\frac{2r_2}{r_2+1}} < \infty, \quad \forall T > 0.$$

Proof. (a) Existence: Let $0 < \varepsilon' < \varepsilon < 1$. Then by (2.5) \mathbb{P} -a.s. for all $t \geq 0$

$$\begin{aligned} X_t^\varepsilon - X_t^{\varepsilon'} &= (L - \varepsilon) \int_0^t \Psi(s, X_s^\varepsilon) ds - (L - \varepsilon') \int_0^t \Psi(s, X_s^{\varepsilon'}) ds + \int_0^t \eta_s (X_s^\varepsilon - X_s^{\varepsilon'}) ds \\ &= \varepsilon \left(\frac{1}{\varepsilon} L - 1 \right) \int_0^t \left(\Psi(s, X_s^\varepsilon) - \Psi(s, X_s^{\varepsilon'}) \right) ds + \int_0^t \eta_s (X_s^\varepsilon - X_s^{\varepsilon'}) ds \\ &\quad + (\varepsilon' - \varepsilon) \int_0^t \Psi(s, X_s^{\varepsilon'}) ds + \int_0^t (B(s, X_s^\varepsilon) - B(s, X_s^{\varepsilon'})) dW_s. \end{aligned}$$

Therefore,

$$\begin{aligned} & i_{\frac{1}{\varepsilon}} (X_t^\varepsilon - X_t^{\varepsilon'}) \\ &= -\varepsilon \int_0^t (\Psi(s, X_s^\varepsilon) - \Psi(s, X_s^{\varepsilon'})) ds + \int_0^t \eta_s i_{\frac{1}{\varepsilon}} (X_s^\varepsilon - X_s^{\varepsilon'}) ds \\ &\quad + (\varepsilon' - \varepsilon) \int_0^t \left((1 - \frac{1}{\varepsilon} L)^{-1} \right)^* \Psi(s, X_s^{\varepsilon'}) ds + i_{\frac{1}{\varepsilon}} \left(\int_0^t (B(s, X_s^\varepsilon) - B(s, X_s^{\varepsilon'})) dW_s \right). \end{aligned}$$

where for the last term we used Lemma 2.7 (iii) and that the involved integrals are Bochner integrals in V^* .

Now we can use the Itô formula in [15, Theorem 4.2] applied to the Hilbert space $H_{\frac{1}{\varepsilon}}$ and obtain for

$$M_t^{\varepsilon, \varepsilon'} := 2 \int_0^t \langle X_s^\varepsilon - X_s^{\varepsilon'}, (B(s, X_s^\varepsilon) - B(s, X_s^{\varepsilon'})) dW_s \rangle_{H_{\frac{1}{\varepsilon}}}$$

by (H3(i)) that for $t \in [0, T]$, $T > 0$ fixed,

$$\begin{aligned}
& \|X_t^\varepsilon - X_t^{\varepsilon'}\|_{H_{\frac{1}{\varepsilon}}}^2 \\
&= -2\varepsilon \int_0^t \langle \Psi(s, X_s^\varepsilon) - \Psi(s, X_s^{\varepsilon'}), X_s^\varepsilon - X_s^{\varepsilon'} \rangle ds \\
&\quad + 2 \int_0^t \eta_s \|X_s^\varepsilon - X_s^{\varepsilon'}\|_{H_{\frac{1}{\varepsilon}}}^2 ds \\
&\quad + 2(\varepsilon' - \varepsilon) \int_0^t \langle \Psi(s, X_s^{\varepsilon'}), (1 - \frac{1}{\varepsilon}L)^{-1}(X_s^\varepsilon - X_s^{\varepsilon'}) \rangle ds \\
&\quad + \int_0^t \|B(s, X_s^\varepsilon) - B(s, X_s^{\varepsilon'})\|_{\mathcal{L}_{HS}(L^2(\mathbf{m}); H_{\frac{1}{\varepsilon}})}^2 ds + M_t^{\varepsilon, \varepsilon'} \\
\text{eq:4.3} \quad (4.2) \quad &\leq -2\varepsilon \int_0^t \int_{r_1}^{r_2} \xi(s, r) \langle X_s^\varepsilon |X_s^\varepsilon|^{r-1} - X_s^{\varepsilon'} |X_s^{\varepsilon'}|^{r-1}, X_s^\varepsilon - X_s^{\varepsilon'} \rangle \nu(dr) ds \\
&\quad + \int_0^t (2\eta_s + c^2) \|X_s^\varepsilon - X_s^{\varepsilon'}\|_{H_{\frac{1}{\varepsilon}}}^2 ds \\
&\quad + 2(\varepsilon' - \varepsilon) \int_0^t \int_{r_1}^{r_2} \xi(s, r) \langle X_s^{\varepsilon'} |X_s^{\varepsilon'}|^{r-1}, (1 - \frac{1}{\varepsilon}L)^{-1}(X_s^\varepsilon - X_s^{\varepsilon'}) \rangle \nu(dr) ds + M_t^{\varepsilon, \varepsilon'} \\
&\leq -\varepsilon \delta \int_0^t \int_{r_1}^{r_2} \|X_s^\varepsilon - X_s^{\varepsilon'}\|_{r+1}^{r+1} \nu(dr) ds \\
&\quad + c_1 \int_0^t \|X_s^\varepsilon - X_s^{\varepsilon'}\|_{H_{\frac{1}{\varepsilon}}}^2 ds + c_1 I_t^{\varepsilon, \varepsilon'} + M_t^{\varepsilon, \varepsilon'},
\end{aligned}$$

where we used the elementary estimate that $(x|x|^{r-1} - y|y|^{r-1})(x - y) \geq 2^{-r+1}|x - y|^{r+1}$ for all $r \in (1, \infty)$, $x, y \in \mathbb{R}$, we set $\delta := 2^{-r_2+2} \inf \xi$, $c_1 := 2 \sup \eta \vee \sup \xi + c^2$ and where

$$I_t^{\varepsilon, \varepsilon'} := \varepsilon^{\frac{3}{2}} \int_0^t \int_{r_1}^{r_2} \|(\varepsilon - L)^{-\frac{1}{2}}(1 - \frac{1}{\varepsilon}L)^{-\frac{1}{2}}(X_s^\varepsilon - X_s^{\varepsilon'})\|_{r+1} \|X_s^{\varepsilon'}\|_{r+1}^r \nu(dr) ds.$$

We note that $(1 - \frac{1}{\varepsilon}L)^{-\frac{1}{2}}$ is a contraction on $L^{r+1}(\mathbf{m})$ and that $X_s^\varepsilon - X_s^{\varepsilon'} \in L^{r+1}(\mathbf{m}) \mathbb{P} \otimes ds \otimes \nu$ -a.e. on $\Omega \times [0, t] \times [r_1, r_2]$. Hence by Lemma 3.2 for any given continuous function $[r_1, r_2] \ni r \mapsto \theta_r \in (0, 1 \wedge \frac{4}{(d-2)^+(r_2-1)})$ there exist $c > 0$ and $\alpha \in (0, \frac{1}{2})$ such that

$$I_t^{\varepsilon, \varepsilon'} \leq c \varepsilon^{\frac{3}{2}-\alpha} \int_0^t \int_{r_1}^{r_2} \|X_s^\varepsilon - X_s^{\varepsilon'}\|_{H_{\frac{1}{\varepsilon}}}^{\theta_r} \|X_s^\varepsilon - X_s^{\varepsilon'}\|_{r+1}^{1-\theta_r} \|X_s^{\varepsilon'}\|_{r+1}^r \nu(dr) ds,$$

which by Young's inequality is dominated by

$$\begin{aligned}
& \frac{\delta}{2} \varepsilon \int_0^t \int_{r_1}^{r_2} \|X_s^\varepsilon - X_s^{\varepsilon'}\|_{r+1}^{r+1} \nu(dr) ds \\
\text{eq:4.4} \quad (4.3) \quad & + C_\delta \varepsilon^{\frac{3}{2}-\alpha} \int_0^t \int_{r_1}^{r_2} \|X_s^\varepsilon - X_s^{\varepsilon'}\|_{H_{\frac{1}{\varepsilon}}}^{\theta_r(r+1)/(r+\theta_r)} \|X_s^{\varepsilon'}\|_{r+1}^{r(r+1)/(r+\theta_r)} \nu(dr) ds,
\end{aligned}$$

where $C_\delta > 0$ is a large enough constant (which is independent of $\varepsilon, \varepsilon'$ and by the continuity of $r \mapsto \theta_r$ can indeed be chosen independently of r). Now define the increasing continuous function

$$\theta_r := \frac{\theta \cdot r}{r + 1 - \theta}, \quad r \in [r_1, r_2],$$

where $\theta \in (0, 1)$ is chosen so small that $(\theta_r \leq) \theta_{r_2} \in (0, 1 \wedge \frac{4}{(d-2)^+(r_2-1)})$. Then $\theta = \frac{\theta_r(r+1)}{r+\theta_r}$ for all $r \in [r_1, r_2]$ and by (4.2) and (4.3) we hence obtain

$$\begin{aligned} \|X_t^\varepsilon - X_t^{\varepsilon'}\|_{H_{\frac{1}{\varepsilon}}}^2 &\leq -\varepsilon \frac{\delta}{2} \int_0^t \int_{r_1}^{r_2} \|X_s^\varepsilon - X_s^{\varepsilon'}\|_{r+1}^{r+1} \nu(dr) ds \\ &\quad + c_1 \int_0^t \|X_s^\varepsilon - X_s^{\varepsilon'}\|_{H_{\frac{1}{\varepsilon}}}^2 ds \\ &\quad + c_1 C_\delta \varepsilon^{\frac{3}{2}-\alpha} \int_0^t \|X_s^\varepsilon - X_s^{\varepsilon'}\|_{H_{\frac{1}{\varepsilon}}}^\theta \int_{r_1}^{r_2} \|X_s^{\varepsilon'}\|_{r+1}^{r+1-\theta} \nu(dr) ds + M_t^{\varepsilon, \varepsilon'} \end{aligned}$$

which for $\tilde{C}_\delta := c_1 C_\delta$ in turn implies for $t \leq T$

$$\begin{aligned} &e^{-c_1 t} \|X_t^\varepsilon - X_t^{\varepsilon'}\|_{H_{\frac{1}{\varepsilon}}}^2 \\ &\leq \tilde{C}_\delta \varepsilon^{\frac{3}{2}-\alpha} \sup_{s \in [0, t]} \|X_s^\varepsilon - X_s^{\varepsilon'}\|_{H_{\frac{1}{\varepsilon}}}^\theta \int_0^t \int_{r_1}^{r_2} \|X_s^{\varepsilon'}\|_{r+1}^{r+1-\theta} \nu(dr) ds \\ \text{eq: 4.5} \quad (4.4) \quad &- \varepsilon \frac{\delta}{2} e^{-c_1 T} \int_0^t \int_{r_1}^{r_2} \|X_s^\varepsilon - X_s^{\varepsilon'}\|_{r+1}^{r+1} \nu(dr) ds \\ &+ 2 \int_0^t e^{c_1 s} \langle X_s^\varepsilon - X_s^{\varepsilon'}, (B(s, X_s^\varepsilon) - B(s, X_s^{\varepsilon'})) dW_s \rangle_{H_{\frac{1}{\varepsilon}}}. \end{aligned}$$

So, for any fixed $T > 0$ by (H3(i)) and by the Hölder and Burkholder-Davies-Gundy inequalities we have for all $t \in [0, T]$

$$\begin{aligned} \text{4.4b} \quad (4.5) \quad &E \sup_{s \in [0, t]} \|X_s^\varepsilon - X_s^{\varepsilon'}\|_{H_{\frac{1}{\varepsilon}}}^2 + \varepsilon \frac{\delta}{2} e^{-c_1 T} \int_0^t \int_{r_1}^{r_2} \|X_s^\varepsilon - X_s^{\varepsilon'}\|_{r+1}^{r+1} \nu(dr) ds \\ &\leq \tilde{C}_\delta \varepsilon^{\frac{3}{2}-\alpha} e^{c_1 T} \left[\mathbb{E} \sup_{s \in [0, t]} \|X_s^\varepsilon - X_s^{\varepsilon'}\|_{H_{\frac{1}{\varepsilon}}}^2 \right]^{\theta/2} \left[\mathbb{E} \left(\int_0^T \int_{r_1}^{r_2} \|X_s^{\varepsilon'}\|_{r+1}^{r+1-\theta} \nu(dr) ds \right)^{\frac{2}{2-\theta}} \right]^{\frac{2-\theta}{2}} \\ &\quad + 2c \left[E \sup_{s \in [0, t]} \|X_s^\varepsilon - X_s^{\varepsilon'}\|_{H_{\frac{1}{\varepsilon}}}^2 \right]^{\theta/2} \left[\mathbb{E} \left(\int_0^t \|X_s^\varepsilon - X_s^{\varepsilon'}\|_{H_{\frac{1}{\varepsilon}}}^{2(2-\theta)} ds \right)^{\frac{1}{2-\theta}} \right]^{\frac{2-\theta}{2}}. \end{aligned}$$

Dropping the integral on the left hand side for $t \in [0, T]$ this yields

$$\begin{aligned} & \mathbb{E} \sup_{s \in [0, t]} \|X_s^\varepsilon - X_s^{\varepsilon'}\|_{H_{\frac{1}{\varepsilon}}}^2 \\ & \leq 2^{\frac{\theta}{2-\theta}} \left(\tilde{C}_\delta e^{\frac{3}{2}-\alpha} e^{c_1 T} \right)^{\frac{2}{2-\theta}} \mathbb{E} \left(\int_0^T \int_{r_1}^{r_2} \|X_s^{\varepsilon'}\|_{r+1}^{r+1-\theta} \nu(dr) ds \right)^{\frac{2}{2-\theta}} \\ & \quad + 2^{\frac{2+\theta}{2-\theta}} c^{\frac{2}{2-\theta}} \mathbb{E} \left(\int_0^t \|X_s^\varepsilon - X_s^{\varepsilon'}\|_{H_{\frac{1}{\varepsilon}}}^{2(2-\theta)} ds \right)^{\frac{1}{2-\theta}}. \end{aligned}$$

But the last term is dominated by

$$\begin{aligned} & 2^{\frac{2+\theta}{2-\theta}} c^{\frac{2}{2-\theta}} \left[\mathbb{E} \sup_{s \leq t} \|X_s^\varepsilon - X_s^{\varepsilon'}\|_{H_{\frac{1}{\varepsilon}}}^2 \right]^{1/2} \left[\mathbb{E} \left(\int_0^t \|X_s^\varepsilon - X_s^{\varepsilon'}\|_{H_{\frac{1}{\varepsilon}}}^{2-\theta} ds \right)^{\frac{2}{2-\theta}} \right]^{1/2} \\ & \leq \frac{1}{2} \mathbb{E} \sup_{s \leq t} \|X_s^\varepsilon - X_s^{\varepsilon'}\|_{H_{\frac{1}{\varepsilon}}}^2 + C_{T, \theta} \mathbb{E} \int_0^t \|X_s^\varepsilon - X_s^{\varepsilon'}\|_{H_{\frac{1}{\varepsilon}}}^2 ds, \end{aligned}$$

where $C_{T, \theta}$ is a constant (independent of $\varepsilon, \varepsilon'$). Hence by Gronwall's Lemma

$$\boxed{\text{eq:4.6}} \quad (4.6) \quad \mathbb{E} \sup_{s \in [0, t]} \|X_s^\varepsilon - X_s^{\varepsilon'}\|_{H_{\frac{1}{\varepsilon}}}^2 \leq \left(\varepsilon^{\frac{3}{2}-\alpha} \tilde{C}_\delta e^{c_1 T} \right)^{\frac{2}{2-\theta}} \mathbb{E} \left(\int_0^T \int_{r_1}^{r_2} \|X_s^{\varepsilon'}\|_{r+1}^{r+1-\theta} \nu(dr) ds \right)^{\frac{2}{2-\theta}}.$$

Since $\|\cdot\|_{H_1}^2 \leq \frac{1}{\varepsilon} \|\cdot\|_{H_{\frac{1}{\varepsilon}}}^2$, by (3.10) applied with $q := \frac{2r_2}{r_2+1}$ and the assumption that $\|X_0\|_{H_1} \in L^{2r_2}(\mathbb{P})$, we conclude that

$$\boxed{\text{eq:4.7}} \quad (4.7) \quad \mathbb{E} \sup_{t \in [0, T]} \|X_t^\varepsilon - X_t^{\varepsilon'}\|_{H_1}^2 \leq \varepsilon^{\frac{1+\theta-2\alpha}{2-\theta}} C$$

for some constant C (independent of $\varepsilon, \varepsilon'$). Here we applied Hölder's inequality to the right hand side of (4.6) and used that $\frac{r+1-\theta}{r+1} \frac{2}{2-\theta} \leq \frac{2r_2}{r_2+1}$ for all $\theta \in (0, 1)$ and all $r \in [r_1, r_2]$. Since $\|\cdot\|_{H_{\frac{1}{\varepsilon}}}^2 \leq \|\cdot\|_{H_1}^2$, analogously one deduces from (4.5) that for some constant $C > 0$ (independent of $\varepsilon, \varepsilon'$)

$$\begin{aligned} \boxed{\text{eq:4.8}} \quad (4.8) \quad & \mathbb{E} \int_0^T \int_{r_1}^{r_2} \|X_s^\varepsilon - X_s^{\varepsilon'}\|_{r+1}^{r+1} \nu(dr) ds \\ & \leq C \varepsilon^{\frac{1}{2}-\alpha} \left(\mathbb{E} \sup_{t \in [0, T]} \|X_t^\varepsilon - X_t^{\varepsilon'}\|_{H_1}^2 \right)^{\frac{\theta}{2}} \cdot \left(\mathbb{E} \left(\int_0^T \int_{r_1}^{r_2} \|X_s^{\varepsilon'}\|_{r+1}^{r+1} \nu(dr) ds \right)^{\frac{2(r+1-\theta)}{(r+1)(2-\theta)}} \right)^{\frac{2-\theta}{2}} \\ & \quad + 2cT^{1/2} \mathbb{E} \sup_{t \in [0, T]} \|X_t^\varepsilon - X_t^{\varepsilon'}\|_{H_{\frac{1}{\varepsilon}}}^2 \end{aligned}$$

So, as above by (3.11) (with q as above), (4.8) together with (4.7) imply that there exists an adapted continuous process X in $H(=H_1)$ such that for $\varepsilon_n \rightarrow 0$

$$\boxed{\text{eq:4.9}} \quad (4.9) \quad \lim_{n \rightarrow \infty} \mathbb{E} \left(\sup_{t \in [0, T]} \|X_t^{\varepsilon_n} - X_t\|_H^2 + \int_0^T \int_{r_1}^{r_2} \|X_t^{\varepsilon_n} - X_t\|_{r+1}^{r+1} \nu(dr) dt \right) = 0.$$

By Fatou's lemma and Lemma 3.3 applied with $p := q + 1$ in (3.10) and $q := \frac{2r_2}{r_1+1}$ in (3.11) we obtain (4.1), so in particular X satisfies (1.6). Now let us show that it also satisfies (1.7).

Claim: There exists a sequence $\varepsilon_n \rightarrow 0$ such that \mathbb{P} -a.s.

$$\int_0^t \Psi(s, X_s^{\varepsilon_n}) ds \rightarrow \int_0^t \Psi(s, X_s) ds \quad \text{as } n \rightarrow \infty \text{ in } L_{N^*} \text{ for all } t \geq 0.$$

To prove the claim let $v \in L_N$. Then by (H1) for some $C \in (0, \infty)$

$$\begin{aligned} & \left| \mathbf{m} \left(\int_0^t (\Psi(s, X_s^\varepsilon) - \Psi(s, X_s)) ds \cdot v \right) \right| \\ \boxed{4.9} \quad (4.10) \quad & \leq C \cdot \int_0^t \int_{r_1}^{r_2} \mathbf{m} (||X_s^\varepsilon|^{r-1} X_s^\varepsilon - |X_s|^{r-1} X_s||v|) \nu(dr) ds \\ & \leq r_2 C \int_0^t \int_{r_1}^{r_2} \mathbf{m} (|X_s^\varepsilon - X_s| (|X_s^\varepsilon| \vee |X_s|)^{r-1} |v|) \nu(dr) ds \end{aligned}$$

where we used the elementary estimate $||x|^{r-1}x - |y|^{r-1}y| \leq r|x-y|(|x| \vee |y|)^{r-1}$; $x, y \in \mathbb{R}$. Applying Hölder's and Young's inequalities the above up to a constant can be estimated from above by

$$\begin{aligned} & \int_0^t \int_{r_1}^{r_2} ||X_s^\varepsilon - X_s| (|X_s^\varepsilon| \vee |X_s|)^{r-1} ||\frac{r+1}{r}||v||_{r+1} \nu(dr) ds \\ & \leq C(\delta) \int_0^t \int_{r_1}^{r_2} ||X_s^\varepsilon - X_s| (|X_s^\varepsilon| \vee |X_s|)^{r-1} ||\frac{r+1}{r}||\frac{r+1}{r}||\nu(dr) ds \\ & \quad + \delta \int_{r_1}^{r_2} ||v||_{r+1}^{r+1} \nu(dr) \\ & \leq \tilde{C}(\delta) \int_0^t \int_{r_1}^{r_2} ||X_s^\varepsilon - X_s||_{r+1}^{r+1} \nu(dr) ds + \delta \int_0^t \int_{r_1}^{r_2} (||X_s^\varepsilon||_{r+1}^{r+1} + ||X_s||_{r+1}^{r+1}) \nu(dr) ds \\ & \quad + \delta \cdot \mathbf{m}(N(v)) \end{aligned}$$

for any $\delta > 0$ and some constants $C(\delta), \tilde{C}(\delta) > 0$ (only depending on δ, r_1, r_2). But by (4.9) for some sequence $\varepsilon_n \rightarrow 0$ the first term of the right hand side \mathbb{P} -a.s. converges to zero for all $t \geq 0$ and the second is \mathbb{P} -a.s. bounded by a random number $c(t)$ times δ . Hence first taking $n \rightarrow \infty$ and then $\delta \rightarrow 0$ we see that the left hand side of (4.10)

converges to zero \mathbb{P} -a.s. for all $t \geq 0$ uniformly for all $v \in L_N$ such that $\mathbf{m}(N(v)) \leq 1$. Hence by the equivalence of the norms $\|\cdot\|_{(N^*)}$ and $\|\cdot\|_{N^*}$ on L_{N^*} (see (2.1)) the claim follows.

We have \mathbb{P} -a.s.

$$\boxed{4.10} \quad (4.11) \quad X_t^{\varepsilon_n} = X_0 + (L - \varepsilon_n) \int_0^t \Psi(s, X_s^{\varepsilon_n}) ds + \int_0^t \eta_s X_s^{\varepsilon_n} ds + \int_0^t B(s, X_s^{\varepsilon_n}) dW_s, \quad t \geq 0.$$

Obviously, by (H3(i)) and 4.7

$$\lim_{\varepsilon \rightarrow 0} \mathbb{E} \sup_{t \in [0, T]} \int_0^t \|B(s, X_s^\varepsilon) - B(s, X_s)\|_{\mathcal{L}_{HS}(L^2(\mathbf{m}); H)}^2 ds = 0.$$

Hence by (4.9) all terms in (4.11) except for the second on the right converge in H . But hence also this term must converge in H . By Claim 1 it follows that \mathbb{P} -a.s.

$$\int_0^t \Psi(s, X_s) ds \in \mathcal{D}(\bar{L}) \quad \forall t \geq 0$$

and

$$(L - \varepsilon_n) \int_0^t \Psi(s, X_s^{\varepsilon_n}) ds \rightarrow \bar{L} \int_0^t \Psi(s, X_s) ds \quad \text{as } n \rightarrow \infty \text{ in } H \quad \forall t \geq 0.$$

Consequently, X satisfies (1.5).

Since by Theorem 2.6 we have an Itô formula for any solution of (1.5), by exactly the same arguments as in the proof of Lemma 3.3 and choosing q as we did for our solution X constructed above, we obtain that any solution Y of (1.5) with $\|Y_0\|_H \in L^{2r_2}(\mathbb{P})$ satisfies (4.1).

It remains to prove uniqueness. So, let X, Y be two solutions of (1.5) such that $X_0 = Y_0$ and $\|X_0\|_H \in L^{2r_2}(\mathbb{P})$. Let $T > 0$ and $\varepsilon \in (0, 1)$. Then by Theorem 2.6 we have \mathbb{P} -a.s.

$$\begin{aligned} i_{\frac{1}{\varepsilon}}(X_t - Y_t) &= \int_0^t \left[\overline{i_{\frac{1}{\varepsilon}} \circ \bar{L}}(\Psi(s, X_s) - \Psi(s, Y_s)) + \eta_s i_{\frac{1}{\varepsilon}}(X_s - Y_s) \right] ds \\ &\quad + i_{\frac{1}{\varepsilon}} \int_0^t (B(s, X_s) - B(s, Y_s)) dW_s, \quad t \geq 0. \end{aligned}$$

So, applying the Itô formula in [15, Theorem 4.2] we obtain (as in Theorem 2.6) \mathbb{P} -a.s.

for all $t \in [0, T]$

4.11 (4.12)

$$\begin{aligned}
\|X_t - Y_t\|_{H_{\frac{1}{\varepsilon}}}^2 &= \int_0^t 2 \left\langle \overline{i_{\frac{1}{\varepsilon}} \circ L}(\Psi(s, X_s) - \Psi(s, Y_s)), X_s - Y_s \right\rangle_V ds \\
&\quad + \int_0^t \left[2\eta_s \|X_s - Y_s\|_{H_{\frac{1}{\varepsilon}}}^2 + \|B(s, X_s) - B(s, Y_s)\|_{\mathcal{L}_{HS}(L^2(\mathbf{m}); H_{\frac{1}{\varepsilon}})}^2 \right] ds \\
&\quad + M_t^\varepsilon \\
&\leq -2\varepsilon \int_0^t \langle \Psi(s, X_s) - \Psi(s, Y_s), X_s - Y_s \rangle ds \\
&\quad + 2\varepsilon \int_0^t \langle \overline{(1 - \varepsilon^{-1}L)^{-1} N^*}(\Psi(s, X_s) - \Psi(s, Y_s)), X_s - Y_s \rangle ds \\
&\quad + c_1 \int_0^t \|X_s - Y_s\|_{H_{\frac{1}{\varepsilon}}}^2 ds + M_t^\varepsilon,
\end{aligned}$$

for some constant $c_1 > 0$ and

$$M_t^\varepsilon := 2 \int_0^t \langle X_s - Y_s, (B(s, X_s) - B(s, Y_s)) dW_s \rangle_{H_{\frac{1}{\varepsilon}}}, \quad t \geq 0.$$

Here we used Lemma 2.2 and the assumed Lipschitz continuity of B for the last inequality. Using the same arguments that led to (4.2) we deduce from (4.12) that

$$\begin{aligned}
\|X_t - Y_t\|_{H_{\frac{1}{\varepsilon}}}^2 &\leq -\varepsilon\delta \int_0^t \int_{r_1}^{r_2} \|X_s - Y_s\|_{r+1}^{r+1} \nu(dr) ds \\
&\quad + c_1 \int_0^t \|X_s - Y_s\|_{H_{\frac{1}{\varepsilon}}}^2 ds + c_1 I_t^\varepsilon + M_t^\varepsilon
\end{aligned}$$

with δ, c_1 as in (4.2) and

$$I_t^\varepsilon := \varepsilon^{\frac{3}{2}} \int_0^t \int_{r_1}^{r_2} \|(\varepsilon - L)^{-\frac{1}{2}} (1 - \frac{1}{\varepsilon} L)^{-\frac{1}{2}} (X_s - Y_s)\|_{r+1} \|X_s - Y_s\|_{r+1}^r \nu(dr) ds.$$

Now, since $\|\cdot\|_H^2 \leq \frac{1}{\varepsilon} \|\cdot\|_{H_{\frac{1}{\varepsilon}}}^2$, and proceeding in exactly the same way as in the proof of (4.6) and (4.7) we obtain that for some constant $C > 0$

$$\mathbb{E} \sup_{s \in [0, T]} \|X_s - Y_s\|_H^2 \leq C \varepsilon^{\frac{1+\theta-2\alpha}{2-\theta}} \mathbb{E} \left(\int_0^T \int_{r_1}^{r_2} (\|X_t\|_{r+1}^{r+1} + \|Y_t\|_{r+1}^{r+1}) \nu(dr) dt \right)^{\frac{2r_2}{r_2+1}}$$

with α, θ as in (4.6), (4.7). Letting $\varepsilon \rightarrow 0$ shows $X_t = Y_t$ for all $t \in [0, T]$. □

5 Proof of Theorem 1.2

Proof of Theorem 1.2(1) and (3). For any $n \geq 1$, by Proposition 4.1 we let $X^{(n)}$ be the unique solution of (1.5) with $X_0^{(n)} := X_0 1_{\{n-1 \leq \|X_0\|_H < n\}}$. Then

$$\begin{aligned} X_t^{(n)} &= X_0 1_{\{n-1 \leq \|X_0\|_H < n\}} + \bar{L} \int_0^t \Psi(s, X_s^{(n)}) ds \\ &+ \int_0^t \eta_s X_s^{(n)} ds + \int_0^t B(s, X_s^{(n)}) dW_s, \quad n \geq 1, \end{aligned} \tag{5.1}$$

holds in H . Letting $X_t := \sum_{n=1}^{\infty} X_t^{(n)} 1_{\{n-1 \leq \|X_0\|_H < n\}}$, we obtain from (5.1) that

$$X_t = X_0 + \bar{L} \int_0^t \Psi(s, X_s) ds + \int_0^t \eta_s X_s ds + \int_0^t B(s, X_s) dW_s, \quad t \geq 0,$$

holds on $\{n-1 \leq \|X_0\|_H < n\}$ for all $n \geq 1$. Therefore, X_t is a solution of (1.5) in the sense of Definition (1.1). Since by Theorem 2.6 we have an Itô formula for the solution X above, by exactly the same arguments as in the proof of Lemma 3.3 we obtain assertion (3).

Let Y_t be another solution with the same initial values X_0 . Then both $X_t 1_{\{\|X_0\|_H \leq n\}}$ and $Y_t 1_{\{\|X_0\|_H \leq n\}}$ solve (1.5) for $B 1_{\{\|X_0\|_H \leq n\}}$ in place of B . By the uniqueness stated in Proposition 4.1 we have $X 1_{\{\|X_0\|_H \leq n\}} = Y 1_{\{\|X_0\|_H \leq n\}}$. Therefore $X = Y$ since $n \geq 1$ was arbitrary. \square

Proof of Theorem 1.2(2). If $\{X_0^{(n)} : n \geq 1\}$ is uniformly bounded in H , then the desired assertion follows from Theorem 2.6 as in the proof of Proposition 4.1. In general, the proof can be completed as above by restricting the solutions first on $\{\sup_{n \geq 1} \|X_0^{(n)}\|_H \leq m\}$ then letting $m \rightarrow \infty$. For instance, if $X_t^{(n)} \rightarrow X_t$ does not hold in probability, then there exist $\varepsilon_0, \varepsilon_1 > 0$ and a subsequence $n_k \rightarrow \infty$ such that

$$\mathbb{P}(\|X_t^{(n_k)} - X_t\|_H \geq \varepsilon_0) \geq \varepsilon_1, \quad k \geq 1. \tag{5.2}$$

Moreover, since $X_0^{(n)} \rightarrow X_0$ in probability, we may assume further that

$$\sum_{k=1}^{\infty} \mathbb{P}(\|X_0^{(n_k)} - X_0\|_H \geq \varepsilon_0) < \infty.$$

Then, by the Borel-Cantelli lemma we obtain $\sup_{k \geq 1} \|X_0^{n_k}\|_H < \infty$ \mathbb{P} -a.s., hence

$$\lim_{m \rightarrow \infty} \mathbb{P}(\sup_{k \geq 1} \|X_0^{n_k}\|_H > m) = 0.$$

Therefore, letting $\Omega_m := \{\sup_{k \geq 1} \|X_0^{n_k}\|_H \leq m\}$, by the assertion with uniformly bounded initial values we obtain (recall that $1_{\Omega_m} X$ solves (1.5) with B replaced by $1_{\Omega_m} B$ for any solution X)

$$\lim_{k \rightarrow \infty} \mathbb{P}(\|X_t^{(n_k)} - X_t\|_H \geq \varepsilon_0) \leq \mathbb{P}(\Omega_m^c) + \lim_{k \rightarrow \infty} \mathbb{P}(\|X_t^{(n_k)} - X_t\|_H 1_{\Omega_m} \geq \varepsilon_0) = \mathbb{P}(\Omega_m^c)$$

which is smaller than ε_1 for large m , and hence is contradictive to (5.2). \square

Proof of Theorem 1.2(4). (a) We first assume that $\mathbb{E}\|X_0\|_2^2 < \infty$. Let $\varepsilon \in (0, 1)$. Since $(1 - \varepsilon L)^{-1}$ is contractive in $L^p(\mathbf{m})$ for $p \geq 1$ we have

$$\langle \Psi(t, v), v - (1 - \varepsilon L)^{-1}v \rangle = \int_{r_1}^{r_2} \xi(t, r) \mathbf{m}(|v|^{r+1} - |v|^{r-1}v(1 - \varepsilon L)^{-1}v) \nu(dr) \geq 0 \quad \forall v \in V.$$

This and Lemma 2.7 (i), (ii) imply that for all $v \in V$

$$\begin{aligned} \boxed{5.3} \quad (5.3) \quad v^* \langle \overline{i_\varepsilon \circ L} \Psi(t, v), v \rangle_V &= v^* \langle \Psi(t, v), L(1 - \varepsilon L)^{-1}v \rangle_V \\ &= -\frac{1}{\varepsilon} \langle \Psi(t, v), v - (1 - \varepsilon L)^{-1}v \rangle \\ &\leq 0. \end{aligned}$$

Then Theorem 2.8 implies that X is right-continuous in $L^2(\mathbf{m})$ and $\mathbb{E} \sup_{t \in [0, T]} \|X_t\|_2^2 < \infty$. In general, letting $X^{(n)}$ be the solution with initial value $X_0 1_{\{\|X_0\|_2 \leq n\}}$, we have $X = X^{(n)}$ on $\{\|X_0\|_2 \leq n\}$, and hence X is right-continuous in $L^2(\mathbf{m})$ according to the results for $X_0 \in L^2(\mathbf{m})$ and the arbitrariness of n .

(b) We first prove (1.8). Let $T > 0$. We first note that by the left hand side of (5.3) and (H3) we have that for some constant $C > 0$ independent of $\varepsilon \in (0, 1)$

$$\begin{aligned} \boxed{5.4} \quad (5.4) \quad \mathbb{E} \int_0^T \frac{1}{\varepsilon} \langle \Psi(t, X_t), X_t - (1 - \varepsilon L)^{-1}X_t \rangle dt &\leq -\mathbb{E} \int_0^T v^* \langle \overline{i_\varepsilon \circ L} \Psi(t, X_t), X_t \rangle_V dt \\ &\leq \mathbb{E} \|X_0\|_{H_\varepsilon}^2 + C(1 + \mathbb{E} \sup_{t \in [0, T]} \|X_t\|_{H_\varepsilon}^2) \\ &\leq \mathbb{E} \|X_0\|_2^2 + C(1 + \mathbb{E} \sup_{t \in [0, T]} \|X_t\|_2^2) \\ &< \infty \end{aligned}$$

where we used the Itô formula from Theorem 2.6 in the second step. Define

$$\zeta(s) := \int_{r_1}^{r_2} |s|^{(r-1)/2} s \nu(dr), \quad s \in \mathbb{R}.$$

By (H1) and the Schwartz inequality,

$$\begin{aligned}
(\Psi(t, s) - \Psi(t, s'))(s - s') &= \int_{r_1}^{r_2} \xi(t, r) (|s|^{r-1}s - |s'|^{r-1}s')(s - s') \nu(dr) \\
&= \int_{r_1}^{r_2} \xi(t, r) (s - s') \int_{s'}^s |u|^{r-1} du \nu(dr) \\
&\geq \frac{(\int_{r_1}^{r_2} \xi(t, r) \int_{s'}^s |u|^{(r-1)/2} du \nu(dr))^2}{\int_{r_1}^{r_2} \xi(t, r) \nu(dr)} \\
&\geq c_2 |\zeta(s) - \zeta(s')|^2, \quad t \in [0, T], s, s' \in \mathbb{R},
\end{aligned}$$

5.5 (5.5)

holds for some constant $c_2 > 0$, where we applied the fundamental theorem of calculus to ζ . In particular, since $\Psi(t, 0) = 0$ and $\zeta(0) = 0$, it follows that

$$5.6 \quad (5.6) \quad \Psi(t, s)s \geq c_2 \zeta(s)^2$$

By Lemma 5.1 below with p being the kernel corresponding to $P := (1 - \varepsilon L)^{-1}$ defined there, (5.5) and (5.6) imply

$$\begin{aligned}
&\frac{1}{\varepsilon} \langle \Psi(t, X_t), X_t - (1 - \varepsilon L)^{-1} X_t \rangle \\
&= \frac{1}{2\varepsilon} \int_E \int_E [\Psi(t, X_t(\tilde{\xi})) - \Psi(t, X_t(\xi))] [X_t(\tilde{\xi}) - X_t(\xi)] p(\xi, d\tilde{\xi}) \mathbf{m}(d\xi) \\
&\quad + \frac{1}{\varepsilon} \int_E (1 - (1 - \varepsilon L)^{-1}) \Psi(t, X_t) X_t \, d\mathbf{m} \\
&\geq c_2 \frac{1}{2\varepsilon} \int_E \int_E (\zeta(X_t(\tilde{\xi})) - \zeta(X_t(\xi)))^2 p(\xi, d\tilde{\xi}) \mathbf{m}(d\xi) \\
&\quad + \frac{1}{\varepsilon} \int_E (1 - (1 - \varepsilon L)^{-1}) |\zeta(X_t)|^2 \, d\mathbf{m} \\
&= c_2 \frac{1}{\varepsilon} \langle \zeta(X_t), \zeta(X_t) - (1 - \varepsilon L)^{-1} \zeta(X_t) \rangle = c_2 \mathcal{E}^{(\varepsilon)}(\zeta(X_t), \zeta(X_t)),
\end{aligned}$$

where for $f \in L^2(\mathbf{m})$

$$(5.7) \quad \mathcal{E}^{(\varepsilon)}(f, f) := \frac{1}{\varepsilon} \langle f, f - (1 - \varepsilon L)^{-1} f \rangle.$$

Combining this with (5.4), we obtain

$$\mathbb{E} \int_0^T \sup_{\varepsilon > 0} \mathcal{E}^{(\varepsilon)}(\zeta(X_t), \zeta(X_t)) dt < \infty.$$

Here we recall that $\mathcal{E}^{(\varepsilon)}(f, f) \nearrow \infty$ as $\varepsilon \searrow 0$ and that

$$f \in \mathcal{D}(\mathcal{E}) \Leftrightarrow \sup_{\varepsilon > 0} \mathcal{E}^{(\varepsilon)}(f, f) < \infty, f \in L^2(\mathbf{m})$$

in which case $\mathcal{E}(f, f) = \sup_{\varepsilon > 0} \mathcal{E}^{(\varepsilon)}(f, f)$ (cf. [12, Chap. I, Theorem 2.13] or [10, Subsection 1.5]). We also note that by (1.6) and Jensen's inequality indeed $\zeta(X_t) \in L^2(\mathbf{m}) \, dt \times \mathbb{P}$ -a.e. Hence $\zeta(X_t) \in \mathcal{D}(\mathcal{E}) \, dt \times \mathbb{P}$ -a.e. and (1.8) holds.

Finally, if $\mathbb{E}\|X_0\|_H^{r_2+1} < \infty$, then Theorem 1.2(3) implies that

$$\zeta(X) = \int_{r_1}^{r_2} |X|^{r-1} X \, dr \in L^2([0, T] \times \Omega \rightarrow L^2(\mathbf{m}); \, dt \times \mathbb{P})$$

and hence also the last part of assertion (4) is proved. □

lem-5.1 **Lemma 5.1.** *Let E be a Lusin space. Let P be a symmetric contraction on $L^2(\mathbf{m})$ which is sub-Markovian (i.e. $0 \leq Pf \leq 1$ if $f \in L^2(\mathbf{m})$, $0 \leq f \leq 1$).*

(i) *There exists a probability kernel p on (E, \mathcal{B}) such that for all \mathcal{B} -measurable $f : E \rightarrow \mathbb{R}$ whose \mathbf{m} -class \bar{f} is in $L^2(\mathbf{m})$ $P\bar{f}$ is the \mathbf{m} -class corresponding to pf where*

$$pf(\xi) := \int_E f(\tilde{\xi}) p(\xi, d\tilde{\xi}), \quad \xi \in E.$$

(ii) *Let $f \in L_{N^*}$, $g \in L_N$. Then*

$$E \ni \xi \mapsto p((f - f(\xi))(g - g(\xi)))(\xi)$$

is \mathbf{m} -integrable and

$$\mathbf{m}(f(g - Pg)) = \frac{1}{2} \int \int (f(\tilde{\xi}) - f(\xi))(g(\tilde{\xi}) - g(\xi)) p(\xi, d\tilde{\xi}) \mathbf{m}(d\xi) + \int_E (1 - p1) f g \, d\mathbf{m}.$$

Proof. (i) follows from [7, Chapter IX.11], since E is Lusin.

(ii) We first note that by Jensen's inequality and symmetry P extends to a contraction on $L^p(\mathbf{m})$ for all $p \in [1, \infty)$ and that for $\xi \in E$

5.5a (5.8)

$$p((f - f(\xi))(g - g(\xi)))(\xi) = p(fg)(\xi) - f(\xi)pg(\xi) - g(\xi)pf(\xi) + f(\xi)g(\xi)p1(\xi).$$

Since by Jensen's inequality p leaves both L_N and L_{N^*} invariant, $fg \in L^1(\mathbf{m})$ and $p1$ is bounded, it follows that the function in (5.8) is in $L^1(\mathbf{m})$. Hence integrating with respect to \mathbf{m} and using the symmetry of P the assertion follows. □

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