Bessis-Moussa-Villani conjecture and generalized Gaussian random variables

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Abstract

In this paper we give the solution of Bessis-Moussa-Villani conjecture (BMV) conjecture for the generalized Gaussian random variables

$$G(f) = a(f) + a^*(f) \quad ,$$

where f is in the real Hilbert space $\mathcal{H}$.
The main examples of generalized Gaussian random variables are q-Gaussian random variables , $(-1 \leq q \leq 1)$, related to $q - CCR$ relation and others commutation relations. We will prove that (BMV) conjecture is true for all operators $A = G(f), B = G(g)$; i.e. we will show that the function

$$F(x) = tr(exp(A + ixB))$$

is positive definite function on the real line. The case $q = 0$,i.e. when $G(f)$ are the free Gaussian (Wigner) random variables and the operators A and B are free with respect to the vacuum trace was proved by M.Fannes and D.Petz [23].

1 Generalized Gaussian Random Variable.

Generalized Gaussian random variables, $G(f)$ were introduced in our paper with R.Speicher [16], where the main example was coming from the q-CCR relation for $q \in [-1, 1]$:

$$a(f)a^*(g) - qa^*(g)a(f) = < f, g > I,$$

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here f, g are in a real Hilbert space $\mathcal{H}$ and

$$G(f) = a(f) + a^*(f).$$

The others examples of generalized Gaussian random variables were constructed by L.Accardi and M.Bozejko[1],M.Bozejko and M.Guta[9], M.Bozejko and J.Wysoczanski[17, 18], M.Guta and H.Maassen[25, 26], M.Bozejko and H.Yoshida[19], A.Buchholz[20], M.Bozejko M.,A.Krystek and L.Wojakowski[10] and recently M.Bozejko M.[8] constructed q-Gaussian random variables for $|q| > 1$.

Let $\mathcal{H}$ be a real Hilbert space. A family of self-adjoint operators $G(f) = G(f)^*, f \in \mathcal{H}$ is called Generalized Gaussian random variables or Generalized Brownian Motion (GBM) , if there exists a state ε on the von Neumann algebra generated by $G(f), f \in \mathcal{H}$ and a complex valued function

$$t : \bigcup_{n=1}^{\infty} \mathcal{P}_2(2n) \to \mathcal{C},$$

(here - $\mathcal{P}_2(2n)$ is the set of 2-partitions of the set $\{1, 2, \ldots, 2n\}$),
such that the following generalized Wick formula holds:

$$\varepsilon(G(f_1)...G(f_k)) = \begin{cases} 0 & \text{if k is odd} \\ \sum_{\mathcal{V} \in \mathcal{P}_2(2n)} t(\mathcal{V}) \prod_{(i,j) \in \mathcal{V}} < f_i, f_j > & \text{if k =2n .} \end{cases}$$

If the dimension of a Hilbert space $\mathcal{H}$ is infinite, then the above definition is equivalent to the following (see F.Lehner-II,[32]):

for each orthogonal linear map $O : \mathcal{H} \to \mathcal{H}$ *and* $f_i \in \mathcal{H}$:

$$\varepsilon(G(f_1)...G(f_k)) = \varepsilon(G(O(f_1))...G(O(f_k))),$$

Typical examples of (GBM) was obtained by R.Speicher and myself[13] in 1991, using q-CCR relations for $-1 \leq q \leq 1$, then putting $G(f) = a(f) + a^*(f)$ and knowing that $a(f)\Omega = 0$ we obtain the following Wick formula:

$$< G(f_1)...G(f_{2n})\Omega, \Omega >= \sum_{\mathcal{V} \in \mathcal{P}_2(2n)} q^{cr(\mathcal{V})} \prod_{(i,j) \in \mathcal{V}} < f_i, f_j > .$$

Here $cr(\mathcal{V})$ is the number of crossing, which is given by the number of pairs of blocks of $\mathcal{V}$ which will cross. To obtain the above Wick formula we need a deformed Fock space $\mathcal{F}_q(\mathcal{H}_C)$ constructed by the completion of the free Fock space

$$\mathcal{F}(\mathcal{H}_C) = \mathcal{C}\Omega \oplus \mathcal{H}_C \oplus \ldots$$

by introducing a new scalar product on $\mathcal{H}_C^{\otimes n}$ as follows:
For $\xi, \eta \in \mathcal{H}_C^{\otimes n}$ we define a q-deformed scalar product,

$$< \xi, \eta >_q = < P_q^{(n)} \xi, \eta >,$$

where

$$P_q^{(n)} = \sum_{\pi \in \mathrm{S}(n)} q^{cr(\pi)} \pi,$$

and where for a permutation $\pi \in \mathrm{S}(n)$,

$$cr(\pi) = \sharp \ \{(i,j) : 1 \leq i \leq j \leq n, \quad and \qquad \pi(i) > \pi(j)\}.$$

In the construction of $\mathcal{F}_q(\mathcal{H}_C)$ we need the positivity of the operator $P_q^{(n)}$ for $-1 \leq q \leq 1$,which was done by Bozejko and Speicher in the papers [11,12,13] .

The very large class of Gaussian random variables were obtained in the very important paper of M.Guta and H.Maassen[25]. Many others examples of Generalized Gaussian random variables we can find in the related papers of L.Accardi-M.Bozejko, M.Anshelevich, Ph.Biane, M.Bozejko , M.Bozejko-M.Guta, M.Bozejko-A.Krystek-L.Wojakowski, M.Bozejko-M.Leinert-R.Speicher, M.Bozejko-R.Speicher, M.Bozejko-J.Wysoczanski, M.Bozejko-H.Yoshida, A.Buchholz, M.Guta-H.Maassen, F.Hiai, I.Krolak, A.Krystek-H.Yoshida, F.Lehner, W.Mlotkowski, L.Wojakowski and H.Yoshida.

2 Generalized Bessis-Moussa-Villani conjecture.

Let $(\mathcal{A}, \tau)$ be a von Neumann algebra $\mathcal{A}$ with a finite trace τ. We say that generalized (BMV) conjecture holds, if for all $a = a^*$, $b = b^*$ in $\mathcal{A}$, the function

$$F_{a,b}(x) = \tau(exp(a + ixb))$$

is positive definite on the real line,
i.e. there exists a positive , bounded Borel measure μ on the real line such that

$$F_{a,b}(x) = \int_{-\infty}^{\infty} e^{ixs} \mu(ds).$$

In the case of the algebra of all complex nxn matrices $\mathcal{A} = M_n(\mathcal{C})$ and the trace τ is the classical trace,
(BMV) conjecture is called $(BMV)_n$.

From the paper of Lieb-Seiringer[33] , we know that for
a fixed natural number n ,
$(BMV)_n$-condition is equivalent to the following statement:

$(LS)_n$: For each positive definite matrices $A, B \in M_n(\mathcal{C})$, for all natural m and complex z , the polynomial

$$L_{A,B,m}(z) = Tr[(A + zB)^m]$$

has only non-negative coefficients.

From the last condition it is easy to show that $(LS)_n$ is true for $n = 2$.

The hint for that result is the following: for two positive definite 2x2-complex matrices there exists a basis , in which that matrices have only non-negative entries.
The case $(BMV)_3$ for 3x3 matrices is STILL OPEN!

See more in the recent paper of Ch.Hillar[**?**].
In this note we show the generalized(BMV) conjecture for all generalized Gaussian random variables by reducing to the case
q-Gaussian, where $q = -1$, i.e. the classical canonical anticommutation relation-(CAR) , in which we have representation of (CAR) relations using Dirac-Pauli matrices.

3 Pauli-Dirac matrices.

We are looking for 2x2-complex self-adjoint matrices Q_1, Q_2
satisfying the following conditions:

$$(i) \quad Q_1^2 = Q_2^2 = I, Q_k = Q_k^*, k = 1, 2.$$

$$(ii) \quad Q_1 Q_2 + Q_2 Q_1 = O$$

and

$$(iii) \quad Tr(Q_{j_1} Q_{j_2} ... Q_{j_{2n}}) = \sum_{\mathcal{V} \in \mathcal{P}_2(2n)} (-1)^{cr(\mathcal{V})} \prod_{(l,m) \in \mathcal{V}} \delta_{j_l, j_m}.$$

One can see that the following matrices satisfying (i)-(iii).

$$Q_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and

$$Q_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$$

Hence we have the following well know fact(folklore):

Proposition 3.1. For all real s, x , the function

$$\varphi_s(x) = Tr(exp(s(Q_1 + ixQ_2))$$

is positive definite on the real line as a function of variable x.

Moreover

$$\varphi_s(x) = \sum_{n=0}^{\infty} \frac{(1-x^2)^n s^{2n}}{(2n)!} = cosh(s\sqrt{1-x^2}).$$

Proof.

$$Let\ Z(x) = Q_1 + ixQ_2 = \begin{pmatrix} 0 & 1-x \\ 1+x & 0 \end{pmatrix}.$$

Hence

$$Z(x)^{2n} = (1-x^2)^n I \ \ , \ Tr(Z(x)^{2n+1}) = 0$$

and

$$Tr(Z(x)^{2n}) = (1-x^2)^n \ \ ,$$

where Tr is the normalized trace.
By the result of Lieb-Seiringer [33], we obtain that our function φ_s is positive definite.

This finishes the proof.

Remark. One can also show that the function φ_s is of the following form:

$$\varphi_s(x) = \int_{-\infty}^{\infty} e^{ixs} \mu_s(dx),$$

where the measure

$$\mu_s(dx) = \frac{1}{2}(\delta(s) + \delta(-s)) + f_s(x)dx \ \ .$$

The density

$$f_s(x) = \frac{I_1(\sqrt{s^2 - x^2})}{\sqrt{s^2 - x^2}}.$$

Here the function

$$I_1(x) = \sum_{k=0}^{\infty} \frac{x^k}{2^k k!(k+1)!}$$

is the modified Bessel function.

Now we are in position to state and proof our main result:

Theorem 3.2. If f_1, f_2 are in the real Hilbert space $\mathcal{H}$ and $G(f_1), G(f_2)$ are generalized Gaussian random variables with respect to the state ε, then the function

$$F_G(x) = \varepsilon(exp(G(f_1) + ixG(f_2))$$

is positive definite on the real line.

Moreover, if $< f_1, f_2 >= 0$ and $\|f_i\| = 1$, then

$$F_G(x) = \int_{-\infty}^{\infty} \varphi_s(x) d\nu_G(ds),$$

where ν_G is the probability distribution of the operator $G(f)$ $, \|f\| = 1$ with respect to the state ε,
i.e.

$$\varepsilon(G(f)^k) = \int_{-\infty}^{\infty} x^k d\nu_G(dx),$$

for all $k = 0, 1, 2 \ldots$.

Proof. In the first part of the proof, we can assume that

$$\|f_i\| = 1 \;, and \quad < f_1, f_2 >= 0 \;.$$

Let us consider the following function on the complex plane $\mathbf{C}$:

$$\delta_n(z) = \varepsilon((G(f_1) + zG(f_2))^n) \;\; , z \in \mathbf{C}.$$

By the very definition we have that $\delta_{2n+1}(z) = 0$ and $\delta_{2n}(z)$ is a polynomial of the degree 2n.

But for real x we have $G(f_1) + xG(f_2) = G(f_1 + xf_2)$.

That last fact for the q-Gaussian case follows from the construction, but for generalized Gaussian field it follows from the Theorem of Guta-Maassen, [26], that each generalized Gaussian field is of the following form $G(f) = a(f) + a^*(f)$, where a creation and an annihilation operators are $\mathbf{R} - linear$.

Therefore for real x we have:

$$\delta_{2n}(x) = \varepsilon((G(f_1 + xf_2))^{2n}) = \|f_1 + xf_2\|^{2n} \sum_{\mathcal{V}\in\mathcal{P}_2(2n)} t(\mathcal{V}) =$$

$$= (1 + x^2)^n m_{2n}(d\nu_G),$$

Since

$$m_{2n}(d\nu_G) = \sum_{\mathcal{V}\in\mathcal{P}_2(2n)} t(\mathcal{V}) = \int_{-\infty}^{\infty} x^k d\nu_G(dx).$$

By the analytic property of $\delta_{2n}(z)$, we have that for real x ,

$$\delta_{2n}(ix) = (1 - x^2)^n m_{2n}(\nu_G).$$

Therefore

$$F_G(x) = \sum_{n=0}^{n=\infty} \frac{(1-x^2)^n}{(2n)!} m_{2n}(\nu_G) = \int_{-\infty}^{\infty} \sum_{n=0}^{n=\infty} \frac{(1-x^2)^n s^{2n}}{(2n)!} d\nu_G(ds) =$$

$$= \int_{-\infty}^{\infty} cosh(s\sqrt{1-x^2}) d\nu_G(ds) = \int_{-\infty}^{\infty} \varphi_s(x) d\nu_G(ds).$$

In more general case, if

$$\|f_i\| = 1 \; , and \quad < f_1, f_2 >= \alpha \; .$$

then as before we get

$$F_G(x) = \int_{-\infty}^{\infty} cosh(s\sqrt{1 - x^2 - 2ix\alpha})d\nu_G(ds),$$

and this function is positive definite on the real line.
This can be proved as before by reducing to the CAR relation. We are looking for 2x2 complex, self-adjoint matrices

$$R_1 = aQ_1 + bQ_2, R_2 = cQ_1 + dQ_2,$$

a, b, c and d are real and $a^2 + b^2 = 1, c^2 + d^2 = 1$, and also $ac + bd = \alpha$.
By the same calculations as before we get:

$$Tr(exp(R_1 + ixR_2)) = \sum_{n=0}^{\infty} \frac{(1 - x^2 - 2ix\alpha)^n}{(2n)!} = cosh(\sqrt{1 - x^2 - 2ix\alpha}).$$

So that function is positive definite on the real line by Lieb-Seiringer [33]result.
This finishes the proof of Theorem 3.2.

Problem 3.3. The following problem seems to be interesting:

For real s and $-1 \leq \alpha \leq 1$, find explicit the positive measure $\mu_{s,\alpha}$ on the real line such that

$$cosh(s\sqrt{1 - x^2 - 2ix\alpha}) = \int_{-\infty}^{\infty} e^{ixy}\mu_{s,\alpha}(dy) \quad ?$$

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