POINTED SPHERICAL TILINGS AND HYPERBOLIC VIRTUAL POLYTOPES

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ABSTRACT. The paper presents an introduction to the theory of hyperbolic virtual polytopes from the viewpoint of combinatorial rigidity theory. Namely, we give a shortcut to the notion of hyperbolic polytopes for a reader who is acquainted with the notions of Laman graphs, 3D liftings and pointed tilings.

From this viewpoint, a hyperbolic virtual polytope is a stressed pointed spherically embedded graph.

The advantage of such a presentation is that it gives an alternative and the most convincing proof of existence of hyperbolic polytopes (and therefore, counterexamples to A.D. Alexandrov's conjecture).

1. Introduction

In the paper we give an alternative presentation of the theory of hyperbolic virtual polytopes.

This theory arose arose originally as a tool for constructing counterexamples (see [10], [11], [4], [19]; see also the very first counterexample constructed without hyperbolic polytopes [8]) to the following uniqueness conjecture, proven by A.D. Alexandrov (see [1]) for analytic surfaces.

Uniqueness conjecture for smooth convex surfaces.
Let $K \subset \mathbb{R}^3$ be a smooth convex body. If for a constant C, at every point of ∂K, we have $R_1 \leq C \leq R_2$, then K is a ball. (R_1 and R_2 stand for the principal curvature radii of ∂K).

A convex polytope is the convex hull of some finite set of points. Denote by $\mathcal{P}$ the set of all convex polytopes in $\mathbb{R}^3$. Equipped with the Minkowski addition $\otimes$, the set $\mathcal{P}$ is a commutative semigroup with the unit element $\{O\}$. The set of all formal Minkowski differences $\mathcal{P}^* = \{K \otimes L^{-1} \mid K, L \in \mathcal{P}\}$ is a group which is called the *group of virtual polytopes*.

In this group we have the same cancelation law as for rational fractions, that is, we identify the elements of type $K \otimes L^{-1}$ and $(K \otimes M) \otimes (L \otimes M)^{-1}$.

The elements of $\mathcal{P}^*$, which are called *virtual polytopes* are not mere formal expressions. They can be interpreted geometrically.

The first geometrical interpretation appeared in the paper [5]. From its viewpoint, a virtual polytope is a polytopal function (a piecewise constant

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function,in the terminology of [5], a convex chain) with some specific properties.

Paper [10] presents a virtual polytope as a pair (a closed piecewise linear surface in $\mathbb{R}^3$, an associated fan).

From the viewpoint of the present paper, a virtual polytope is a stressed spherically embedded graph. Together with the addition operation (defined in Section 3) the set of all such graphs form a group which is shown (Theorem 3.7) to be canonicaly isomorphic to the group of virtual polytopes.

Among the virtual polytopes we single out the class of hyperbolic virtual polytopes (for short, hyperbolic polytopes).

Very roughly, we have the following situation. Hyperbolic polytopes are defined to be as non-convex as possible. By definition, the graph of the support function of a hyperbolic polytope is a saddle surface (in contrast to convex polytopes, for which the graph of the support function is a convex surface).

The crucial link to the pointed tilings is the following: if a spherically embedded stressed graph is pointed, then the corresponding virtual polytope is hyperbolic.

The theory of hyperbolic polytopes has the following curious feature. The most interesting and important fact is the existence and diversity of hyperbolic polytopes. In other words, it took a lot of efforts to construct different examples of hyperbolic polytopes (see [19] for some 3D images).

The advantage of the approach of the paper is that it gives an alternative and the most convincing proof of existence of hyperbolic polytopes.

The paper pulls the theory of planar pointed tilings to the sphere S^2. Necessary facts of graphs rigidity are transfered onto the sphere due to some simple adjustments of Section 2 and the papers [2] and [17]. The only difference with the planar case (which changes the situation very much) is the existence of pseudo-di-gons. Namely, each planar polygon has at least three convex angles, whereas on the sphere there exist polygons with just two convex angles (see Fig. 5).

This fact changes Laman-type counts for pointed tilings. As a consequence, there exist pointed spherically embedded Laman-plus-one (and even Laman-plus-k graphs (see principally different Example 4.5 and Example 4.6). They possess a non-trivial saddle 3D lifting. By definition, this is nothing but a hyperbolic polytope.

Thus a hard problem of constructing hyperbolic polytopes (which originally were 3D objects) is reduced to construction of a spherically embedded pointed graph. We get a simple proof of existence of hyperbolic polytopes.

Using this technique, the author obtained also a refinement of A.D Alexandrov theorem on 3D polytopes with mutually non-insertable faces [13].

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2. Graphs on the sphere. Space of equilibrium stresses.

A *graph* is a pair $G = (V, E)$ where
$V = \{1, 2, ..., n\}$ is a finite set,
E is a set of unordered pairs (i, j) such that $i, j = 1, ..., n$, and $i \neq j$. The elements of V and E are called *vertices* and *edges* respectively.

By a *graph embedded in* R^3 we mean a triple $G = (V, E, p)$ where
V and E are as above, and
p is an injective mapping $p : V \to \mathbb{R}^3$.
The points $p(i)$ are denoted for short by p_i and are called *vertices* of the graph. The segments p_ip_j for $(i, j) \in E$ are called the edges of the graph and are assumed to be non-crossing.

Denote by $S^2 \subset \mathbb{R}^3$ the unit sphere centered at O. Its points we identify with their radius vectors.
By a *spherically embedded graph* we mean a quadruple $G = (V, E, p, l)$ where
V and E are as above,
p is an injective mapping $p : V \to S^2$. The points $p_i = p(i)$ are called *vertices* of the graph.
The function l defined on the set E maps each pair $(i, j) \in E$ to some geodesic segment with endpoints at p_i and p_j. The segments $l(i, j)$ are denoted for short by l_{ij} and are called *edges* of the graph. We don't claim that l_{ij} is the shortest geodesic segment connecting p_i and p_j, so there are two possible edges with fixed endpoints (or even infinitely many possible edges for antipodal points).
We assume that the edges l_{ij} are non-crossing.

Besides, in the section we assume that all embeddings are *generic* [3]. In its general stating this means that the vertex coordinates are algebraicaly independent. In particular this means that for a spherically embedded graph, there are no antipodal vertices.

Example 2.1. *It is convenient to consider a great semicircle on S^2 as an embedded graph (with no vertices and a single closed edge) as well. We call it an exotic graph EG.*

We will used a slightly modified (in comparison with [3]) definition of an equilibrium stress of a graph G embedded in R^3 and its natural version for a spherically embedded graph.

Definition 2.2. A mapping $s : E \to R$ is called an *equilibrium stress*(or shortly a *stress*) of a graph $G = (V, E, p)$ embedded in $\mathbb{R}^3$ if
$\sum_{(ij)\in E} s(i, j)\mathbf{u}_{ij} = 0$ holds for each i.
Here $\mathbf{u}_{i,j} = \frac{\overrightarrow{p_ip_j}}{|p_ip_j|}$.
A stress is called *non-trivial* if it is not identically zero.

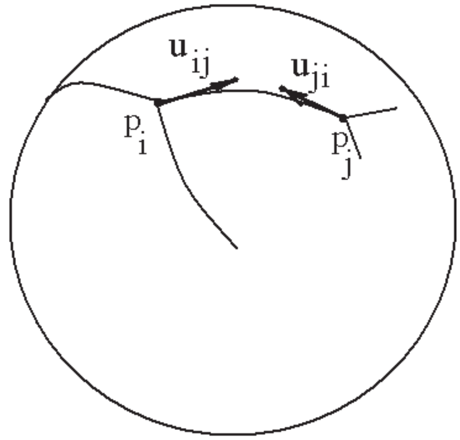

FIGURE 1

A stress is called *non-zero* if it is non-zero on each edge.

The space of all stresses of G we denote by $\mathcal{S}(G)$.

Definition 2.3. A mapping $s : E \to R$ is called an *equilibrium stress* (or shortly, a *stress*) of a spherically embedded graph $G = (V, E, p, l)$ if

$\sum_{(ij)\in E} s(i,j)\tilde{\mathbf{u}}_{ij} = 0$ holds for each i. Here $\tilde{\mathbf{u}}_{ij}$ are the unit vectors tangent to $l_{i,j}$ at the point p_i.

Their direction are choosen as is depicted in Fig. 1.

The space of all stresses of G we denote by $\mathcal{S}(G)$.

Definition 2.4. We assume that the exotic graph EG possesses a stress. It is any real number assigned to its only edge.

The following construction reduces the stress of a spherically embedded graph G to a stress of some graph embedded in R^3. The ideas are borrowed from [2] and [17].

Given a graph G embedded in S^2, add the point $p_{n+1} = O$ as a new vertex. Replace the edges of G by corresponding line segments. Finally, add the edges $(i, n+1)$ for $i = 1, ..., n$ as new edges. The embedded graph obtained denote by $\overline{G} = (\overline{V}, \overline{E}, \overline{p})$.

Proposition 2.5. *The spaces of stresses of $\mathcal{S}(G)$ and $\mathcal{S}(\overline{G})$ are canonically isomorphic.*

Proof. Let s be a stress of G. Define the stress $\overline{s}$ of $\overline{G}$ as follows. For $i, j < n+1$ let

$\alpha_{i,j}$ be the angle between $\overrightarrow{p_ip_j}$ and $\mathbf{u}_{ij}$.

Put $\overline{s}(i,j) = s(i,j)/\cos\alpha_{ij}$.

Put also $\overline{s}(i, n+1) = -\sum_{j=1}^{n} s(i,j)\tan\alpha_{ij}$. This mapping is an isomorphism between $\mathcal{S}(G)$ and $\mathcal{S}(\overline{G})$.We check first that $\overline{s}$ is a self-stress of $(\overline{G})$. The condition

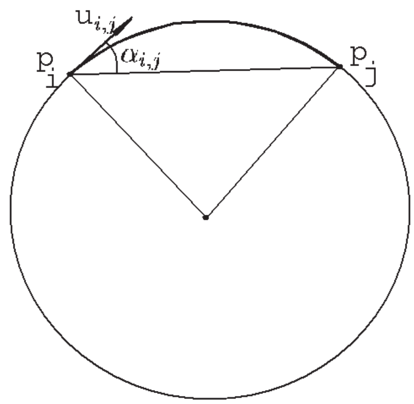

FIGURE 2

$\sum_{(ij)\in E} s(i,j)\mathbf{u}_{ij} = 0$ at a vertex p_i for $i \leq n$ is valid by construction. The mapping is obviously invertable.

Besides, the sum of all vectors $\overline{s}(i,j)\mathbf{u}_{ij}$ equals zero. Therefore, the condition $\sum_{(ij)\in E} s(i,j)\mathbf{u}_{ij} = 0$ for the remained vertex $p_{n+1} = O$ is also valid. □

Definition 2.6. ([3]) A graph $G = (V, E)$ with n vertices and m edges is a Laman graph if

- $m = 2n - 3$,
- each subset V' of k vertices spans not more than $2k - 3$ edges. (An edge $(i, j) \in E$ is spanned by V' if $i, j \in V'$.)

Definition 2.7. ([6]) A Laman graph with one extra edge is a *Laman-plus-one graph.*

Equivalently, a Laman graph with k extra edges is a *Laman-plus-k graph.*

Definition 2.8. ([6]) A graph G is a *rigidity cirquit* if the removal of any of its edge yields a Laman graph. Equivalently, G is a rigidity circuit if it is a Laman-plus-one graph and has no Laman-plus-one subgraph (except for G).

The below theorem adjust some classical theorems for spherically embedded graphs. This fact lies very close to the ideas of [2] and [17], but still we present it here for completeness.

Proposition 2.9. *Let G be a generic spherically embedded graph.*

(1) *If G is a Laman graph, then G is infinitesimally rigid.*
(2) *If G is a Laman-plus-one graph, then G possesses a non-trivial (not identically zero) stress.*
(3) *If G is a rigidity circuit, then G possesses an everywhere non-zero stress.*

Proof. 1. The rigidity of generic Laman graphs is valid for graphs embedded in the plane [3]. The paper [17] proves that it is also valid for spherically embedded graphs. More precisely, it is proven that infinitesimal motions of a spherically embedded graph are in a one-to-one correspondence with the infinitesimal motions of its projection on the plane.

The paper [17] treats only only those spherical embedding that fit an open hemisphere. Still the general case is easily reduced to this one via the following trick.

Fix a hemisphere S^+. For a spherically embedded graph $G = (V, E, p, l)$ construct the new graph $G^+ = (V, E, p^+, l^+)$ such that $p_i^+ \in S^+$, and $p_i^+ = \pm p_i$ depending on which point p_i or $-p_i$ belongs to S^+. Besides, l_{ij} is defined to be the segment lying also in S^+.

It is easy to see that this mapping preserves rigidity.

2. Denote by n the number of vertices of G and by m the number of its edges. In [17] it is proven that G is infinitesimally rigid. Together with from Corollary 2.3.1 from [3] applied to the graph $\overline{G}$ this directly implies that

$6 = 3(n+1) - (m+n) + dim(\mathcal{S}(\overline{G}))$.

Therefore, $dim(\mathcal{S}(\overline{G})) = 1$.

3. This is true by the below usual reason. Suppose the contrary, i.e., that G has a non-trivial stress which admits zero values. Therefore G has a proper subgraph with a non-trivial stress. It is at least a Laman-plus-one graph. A contradiction. □

3. 3D liftings for graphs on the sphere

A (spherical) polygon on S^2 is a domain of $S^2 \subset \mathbb{R}^3$ bounded by a closed non-crossing broken line (its edges are assumed to be geodesic segments).

Given a spherical polygon A, put $C(A) = \{\lambda x \in \mathbb{R}^3 \mid \lambda \in R^+,\ x \in A\}$. Thus $C(A)$ is a cone with the apex at O based on the polygon A.

A spherically embedded graph G generates a tiling $\mathcal{ST}(G)$ of S^2. In turn, $\mathcal{ST}(G)$ yields a tiling of the 3D space $\mathbf{R}^3$ into the union of cones:

$\mathcal{CT}(G) = \{C(A) \mid A \in \mathcal{ST}(G)\}$.

Definition 3.1. A 3D lifting of a spherically embedded graph G is a function $h : \mathbf{R}^3 \to R$ possessing the four properties:

(1) h is continuous;
(2) $h(0) = 0$;
(3) h is piecewise linear;
(4) h is linear on each cone of $\mathcal{CT}(G)$.

A 3D lift is *non-trivial* if it is not a globally linear function.

A 3D lift is *tight* if it is not a 3D lift of some proper subgraph of G.

Obviously, the set of all 3D lifts of a graph is a linear space.

An important example. Let $K \subset \mathbf{R}^3$ be a convex polytope. Remind that its support function $h_K : \mathbf{R}^3 \to \mathbb{R}$ is defined by $h_K(\mathbf{x}) = max_{y \in K}(\mathbf{x}, \mathbf{y})$. It is known to satisfy the above four properties with respect to some spherical tiling Σ_K which is called *outer normal fan* of K [18]. Its 1-skeleton is some spherically embedded graph G_K.

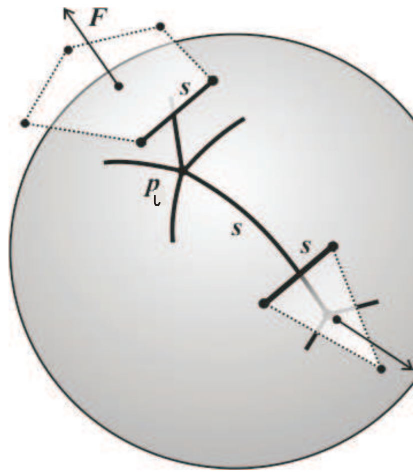

FIGURE 3

The polytope K and its fan Σ_K are combinatorially dual. In particular, this means that the edges of G_K are in one-to-one correspondence with the edges of K.

Proposition 3.2. *Let $K \subset \mathbf{R}^3$ be a convex polytope. In the above notation, we have*

(1) *h_K is a tight 3D lifting of G_K.*
(2) *For each plane $e \subset \mathbf{R}^3$, the restriction $h_K \mid_e$ is a convex function. Equivalently, the graph of $h_K \mid_e$ is concave down.*
(3) *The function s_K which maps each edge of the graph G_K to the length of the corresponding edge of K is a non-zero positive stress of G_K.*
(4) *Vice versa, given a spherically embedded graph G with a positive non-zero stress s, there exists a unique (up to a translation) convex polytope $K \subset \mathbf{R}^3$ such that $G = G_K$ and $s = s_K$.*

Proof. The proposition is a mere reformulation of some classical facts on convex polytopes for which we refer the reader to [18] and to [2] for advanced details. (1) reformulates the definitions of the outer normal fan and support function. (2) means just the convexity of h.

The statement (3) is obvious. Indeed, let p_i be a vertex of G_K. By duality, it corresponds to a face F of K such that the outer normal of F equals p_i. The edges of F correspond by duality to the edges of G_K (see Fig. 3). The condition from the Definition 2.3 means that the sum of edge vectors of the polygon F equals zero.

Prove (4). By the above reason, a stressed graph G yields a collection of convex polytopes (for each vertex p_i we have a polytope) which can be patched together to form a convex polytope (see Fig. 3). □

Example 3.3. *In the framework of Proposition 3.2 (4), the positively stressed exotic graph EG generates a line segment. Its length equals the value of the stress.*

Denote by $\mathcal{SG}$ the set of pairs of type
(a spherically embedded graph G; its non-zero stress s).
Exotic graphs and the empty graph are also included. The following definition turns this set to a group which is called the *group of stressed graphs*.

Definition 3.4. The sum $(G_1; s_1) + (G_2; s_2)$ is defined via the following procedure.

Take the tilings $\mathcal{T}(G_1)$ and $\mathcal{T}(G_2)$ generated by the graphs. Taken together, they generate their common refinement. There appear new vertices, and some of the edges get split. The 1-skeleton of the refinement is some embedded graph G.

The sum of s_1 and s_2 is a self-stress s of G. (More precisely, let l be an edge of G. If it lies on some edge of G_1 and on no edge of G_2, then we assign to l the stress inherited from s_1. If it lies on an edge of G_1 and on an edge of G_2, we take the sum of inherited stresses.)

Next, we remove all the edges of G which are zero stressed.

After this, there appear redundant vertices, namely those possessing just two adjacent edges. In this case the edges form the angle π and have the equal stress. Finally, we remove all such redundant vertices (patching together the adjacent edges).

Remark 3.5. Exotic graphs and the empty graph fit nicely in this scheme. Moreover, without them we would fail to get a group.

Proposition 3.6. *Each stressed graph $(G, s) \in \mathcal{SG}$ is the difference of some two positively stressed graphs from $\mathcal{SG}$.*

Proof. For each edge of (G, s) with a negative stress s, we add to (G, s) the corresponding positively stressed exotic graph (the stress should be greater or equal than $-s$). This makes the sum positively stressed. □

Summarizing, we get the following theorem.

Theorem 3.7.
(1) *The group of stressed graphs $\mathcal{SG}$ is generated by $\{(G_K, s_K)\}$, where K ranges over the set of convex polytopes in $\mathbb{R}^3$.*
(2) *The group $\mathcal{SG}$ is canonically isomorphic to the group of virtual polytopes $\mathcal{P}$.*
(3) *Therefore, we arrive at the same group of virtual polytopes as it was defined by A. Pukhlikov and A. Khovanskii in* [5]. □

Definition 3.8. Keeping in mind the canonical isomorphism from Theorem 3.7, we will call the elements of the group of stressed graphs *virtual polytopes.*

Theorem 3.9.
(1) *Given a spherically embedded graph G, the space of its stresses is canonically isomorphic to the space of its 3D lifts.*
(2) *A generic spherically embedded Laman-plus-k graph has a non-trivial 3D lift for any $k = 1, 2, ...$*
(3) *A spherically embedded rigidity circuit has a tigth 3D lift.*

Proof. The assertion (1) is already proven for the case described in Proposition 3.2, i.e., for the graph generated by a convex polytope. Then we get the general statement by linearity and Proposition 3.6.

(2) and (3) follow from Theorem 3.7 and Proposition 2.9. □

Definition 3.10. The 3D lift $h = h(G, s)$ of a virtual polytope (G, s) is called the *support function* of (G, s).

This definition is consistent with the definition of the support function of a convex polytope K.

4. Pointed graphs and hyperbolic virtual polytopes

Among virtual polytopes we single out the class of *hyperbolic* (in other words, saddle) virtual polytopes. The reader should not confuse them with polytopes lying in a hyperbolic space. In the context of the paper, the term "hyperbolic" means "saddle". In some sense hyperbolic polytopes are opposite to convex polytopes by their convexity property.

Here are the definitions.

Definition 4.1. A surface $F \subset \mathbb{R}^3$ is called *a saddle surface* if there is no plane cutting a bounded connected component off F.

Equivalently, a surface F is *saddle* if no plane intersects F locally at just one point.

Definition 4.2. A function $h : \mathbb{R}^3 \to \mathbb{R}$ is called *hyperbolic* if the graph of its restriction $h \mid e$ to a plane e is a saddle surface for every e.

A virtual polytope (G, s) is called *hyperbolic*, if the induced 3D lifting h is hyperbolic.

A spherically embedded graph is called *pointed* (see Fig. 4) if each of its vertices is incident to an angle larger than π.

Hyperbolic polytopes and pointed graphs are closely related due to the following simple fact.

Lemma 4.3. [10] *Given $(G, s) \in \mathcal{SG}$, if G is pointed, then (G, s) is hyperbolic.* □

A spherical polygone is called a *pseudo-triangle* (respectively, *pseudo-di-gon*) if it has exactly three (respectively, exactly two) angles smaller than π.

We borrow the definitions and the following proposition (including its proof) from the theory of planar pointed pseudo-triangulations (see [15], [16]).

Proposition 4.4. *Let G be a spherically embedded graph with n vertices and m edges. Suppose that each tile of $\mathcal{T}(G)$ is either a pseudo-triangle or a pseudo-di-gon. Then $m = 2n - 6 + d$, where d is the number of di-gons in $\mathcal{T}(G)$.*

Proof. Denote by c the total number of convex angles (i.e., the angles smaller than π) of all tiles from $\mathcal{T}(G)$. Denote by t the number of pseudo-triangles. We have

$n - m + d + t = 2$ (Euler formula),

FIGURE 4. A pointed graph

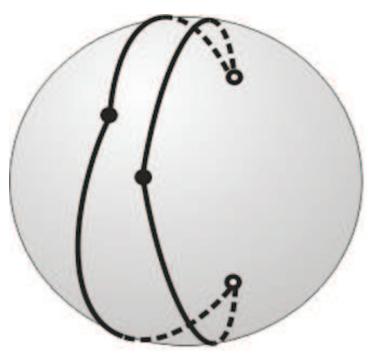

FIGURE 5. Pseudo-di-gon

$c = 2d + 3t$ (first count of convex angles), and
$c = 2n - m$ (second count of convex angles),
which imply together the required. □

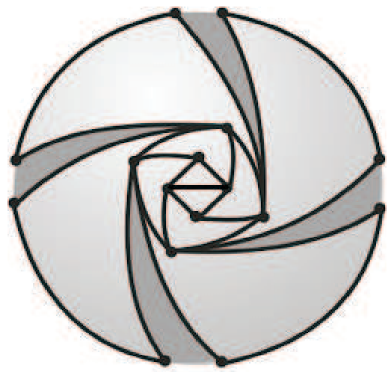

FIGURE 6. A pointed rigidity circuit

We are interested in stressed pointed embedded graphs.

Recall (see [16]) that a planar pointed graph never has a non-zero stress. We repeat here the proof which appeals to the theory of saddle surface.

If a pointed graph has an equilibrium stress, it has a 3D lifting. Its graph would be a piecewise linear surface which is saddle (due to the pointed property) and which coincides with the plane everywhere except for a bounded set. The latter is impossible.

The crucial property of pointed spherically embedded graphs is that some of them (actually, many of them) have a non-trivial 3D lifting. This means that there are many hyperbolic virtual polytopes.

Example 4.5. *Figure 6 presents a spherically embedded rigidity circuit. It has 24 vertices and 46 edges. The graph generates a tiling which has four pseudo-di-gons (marked grey). Due to Proposition 2.9, it has a tight 3D lifting. The figure depicts one side of the sphere, the other side looks analogously.*

In the framework of the above theory it is quite easy to construct a pointed rigidity circuit G. Indeed, we know in advance that the tiling $\mathcal{T}(G)$ should contain four pseudo-di-gons. So one has place on the sphere the pseudo-di-gons and after that add a pointed pseudo triangulation of their complement. This is not tricky at all. It should be mentioned how much efforts were involved to construct the first examples of hyperbolic polytopes (see [10], [11], [8]).

Example 4.6. *Fig. 7 presents a procedure which leads to a pointed embedded Laman-plus-k graph (on the left). Its space of stresses is k-dimensional.*

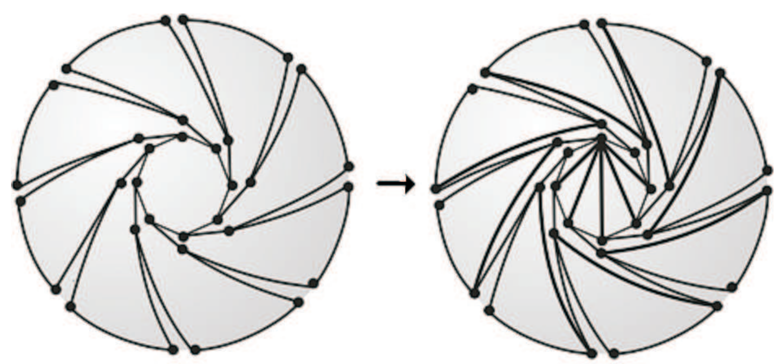

FIGURE 7. A pointed Laman plus 5 graph

Example 4.7. *Figure 8 presents another spherically embedded rigidity circuit.*

Similarly to the Example 4.5, the graph generates a tiling which has four pseudo-di-gons, but this time the pseudo-di-gons lie in a different position in the following sense.

It is easy to observe that each pseudo-di-gon contains a great semicircle. Given a pointed embedding of a rigidity circuit G, fix a great semicircle for each of the pseudo-di-gons of $\mathcal{T}(G)$. This yields a configuration of great semicircles.

The Example 4.5 and Example 4.7 give configurations from Fig. 9. (the first one and the second one respectively). These configurations are known to be non-isotopic [13]: there is no continuous motion which brings one of them to another avoiding crossings.

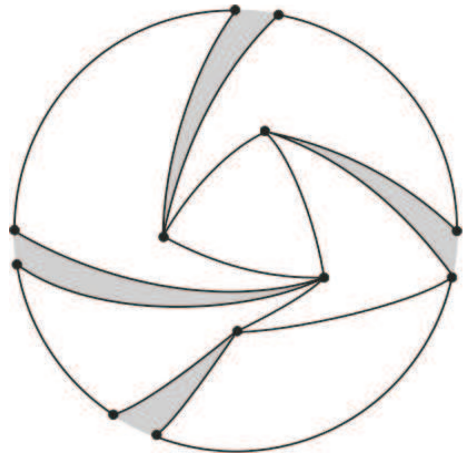

FIGURE 8. Another pointed rigidity circuit

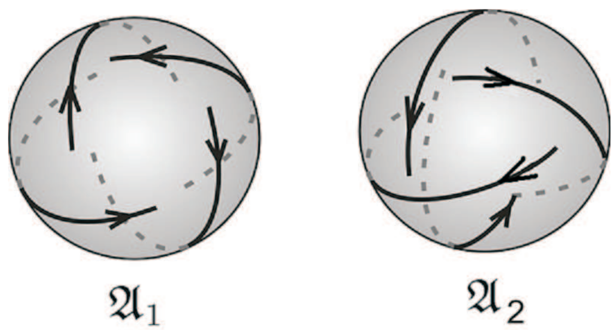

FIGURE 9. Two non-isotopic configurations of great circles

These different examples have yielded examples of non-isotopic hyperbolic hérissons (see [12], [4]). We recall that the existence of just one such surface was an open problem for a long time. It was also a surprise the existence of the second isotopy type, which is constructed easily in the framework of the present paper.

5. Maxwell reciprocal diagrams

In some sense Maxwell reciprocal diagrams of spherically embedded sels-stressed graphs are the virtual polytopes as they appeared originally in [5]. We will not repeat the definitions from [5], but an alternative one which is consistent with the approach of the paper. Proposition 3.2 says that given a positively stressed graph (G, s), for each its vertex p_i, we have a convex closed broken line (a convex polygon).The polygon lies in the plane $e(p_i)$ with the normal vector p_i. Parallel copies of all these polygons are the faces of some convex polytope which is an analog of Maxwell reciprocal for (G, s).

For an arbitrary stressed graph we can apply the same procedure (with the lost of convexity, of course). Namely, for each its vertex, we have a closed broken line (with possible self-intersections and self-overlappings) lying in $e(P_i)$ constructed according to the same rule of Maxwell reciprocal diagram (see [15]). Their parallel copies can be patched together to form an analog of Maxwell reciprocal for (G, s). We give here no further comments and proofs, referring the reader to [5], [10], and [9].

A warning. In such a scheme, different stressed graphs can yield one and the same reciprocal diagram. A funny example (due to Vlad Scherbina, see [4], see also http://club.pdmi.ras.ru/ panina/hyperbolicpolytopes.html) shows that 52 (sic!) different graphs can have one and the same reciprocal diagram which is a tetrahedron. To distinguish them, one needs some additional structure (encoded in the values of some polytopal function), for which we refer the reader to [5]).

Maxwell reciprocal diagrams of hyperbolic polytopes are interesting complicated piecewise linear surfaces.

An increasing collection of 3D pictures is presented at [19].

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