Weak existence of the squared Bessel process, CIR model, and Longstaff model, with skew reflection on a deterministic time dependent curve [1]

Gerald TRUTNAU

Universität Bielefeld, Fakultät für Mathematik, Postfach 100131 D-33501 Bielefeld, Germany (e-mail: trutnau@math.uni-bielefeld.de)

Summary: Let $\sigma > 0, \delta \geq 1, b \geq 0$, $0 < p < 1$. Let λ be a continuous and positive function in $H^{1,2}_{loc}(\mathbb{R}^+)$. Using the technique of moving domains (see [9]), and classical direct stochastic calculus, we construct a pair of continuous semimartingales $(R, \sqrt{R})$ solving weakly

$$dR_t = \sigma\sqrt{R_t}dW_t + \frac{\sigma^2}{4}(\delta - bR_t)dt + (2p-1)d\ell^0_t(R-\lambda^2),$$

and

$$d\sqrt{R_t} \;=\; \frac{\sigma}{2}dW_t + \frac{\sigma^2}{8}\left(\frac{\delta-1}{\sqrt{R_t}} - b\sqrt{R_t}\right)dt + (2p-1)d\ell^0_t(\sqrt{R}-\lambda) + \frac{\mathbb{I}_{\{\delta=1\}}}{2}\ell^{0+}_t(\sqrt{R}),$$

where the symmetric local times $\ell^0(R-\lambda^2), \ell^0(\sqrt{R}-\lambda)$, of the respective semimartingales are related through the formula

$$2\sqrt{R}d\ell^0(\sqrt{R}-\lambda) = d\ell^0(R-\lambda^2).$$

We only consider positive initial conditions. In particular, the pair $(R, \sqrt{R})$ provides another typical example for diffusions with discontinuous local time (see Remark 2.7). Well-known special cases are the (squared) Bessel processes (choose $\sigma = 2$, $b = 0$, and $\lambda^2 \equiv 0$, or equivalently $p = \frac{1}{2}$), and the Cox-Ingersoll-Ross process (i.e. R, with $\lambda^2 \equiv 0$, or equivalently $p = \frac{1}{2}$). The case $0 < \delta < 1$ can also be handled, but is different, see Remark 2.9. We also explain how a generalized Longstaff's model, known as the double square-root model, is obtained (see Remark 2.5).

2000 Mathematics Subject Classification: Primary: 60H10, 60J60, 60J55, 35K20,

[1] Supported by the SFB-701 and the BIBOS-Research Center.

91B28; Secondary: 31C25, 31C15.

Key words: Primary: Stochastic ordinary differential equations, Diffusion processes, Local time and additive functionals. Boundary value problems for second-order, parabolic equations, Finance, portfolios, investment; Secondary: Dirichlet spaces, Potentials and capacities.

1 Introduction and motivation

For parameters $\sigma > 0, \delta, b \geq 0$, consider the Cox-Ingersoll-Ross process, i.e. the unique solution (in any probabilistic sense) of the 1-dimensional SDE

$$dR_t = \sigma\sqrt{|R_t|}dW_t + \frac{\sigma^2}{4}(\delta - bR_t)dt,$$

and denote by $(\ell_t^{a+}(R))_{(t,a)\in\mathbb{R}^+\times\mathbb{R}}$ its associated right-continuous family of local times (upper local times), i.e. $(t,a) \mapsto \ell^{a+}(R)_t$ is a.s. continuous in t and càdlàg in a. Applying the occupation time formula (see e.g. [4, VI. (1.6) Corollary, (1.15) Exercise]) we obtain

$$\int_{\mathbb{R}} \frac{\mathbb{I}_{\{a\neq 0\}}}{|a|}\ell_t^{a+}(R)da = \int_0^t \frac{\mathbb{I}_{\{R_s\neq 0\}}}{|R_s|}d\langle R, R\rangle_s \leq \sigma^2 \int_0^t \mathbb{I}_{\{R_s\neq 0\}}ds \leq \sigma^2 t,$$

so that by non-integrability of $a \mapsto \frac{1}{a}$ in any neighborhood of zero, the upper (resp. lower) local time at zero must vanish, i.e. $\ell^{0^+}(R) \equiv 0$ (resp. $\ell^{0^-}(R) \equiv 0$). Accordingly, the symmetric local time

$$\ell^0(R) = \frac{\ell^{0^+}(R) + \ell^{0^-}(R)}{2},$$

vanishes. In short, the lower (resp. upper, symmetric) local time corresponds to the right-continuous (resp. left-continuous, point-symmetric) derivative of $r \mapsto |r|$ in Tanaka's formula for $|R|$. If we consider a continuous, and positive function $\lambda : \mathbb{R}^+ \to \mathbb{R}^+$, which is locally of bounded variation, and if $\lambda \not\equiv 0$, then the symmetric local time at zero $\ell_t^0(R - \lambda^2)$ (here $\lambda^2(t) = \lambda(t)\cdot\lambda(t)$) of the continuous semimatingale $R - \lambda^2$, where now R is a solution to (1) below, doesn't vanish. In fact at least for $\lambda \in H^{1,2}_{loc}(\mathbb{R}^+)$, and $\delta \geq 1$, its associated smooth measure is not identically zero (cf. Proposition 2.3 and subsequent transformations). It is therefore natural to look at a solution to

$$dR_t = \sigma\sqrt{|R_t|}dW_t + \frac{\sigma^2}{4}(\delta - bR_t)dt + (2p-1)d\ell_t^0(R - \lambda^2), \tag{1}$$

where $p \in (0,1)$ and

$$\int_0^t \mathbb{I}_{\{\lambda^2(s)=0\}}d\ell_s^0(R - \lambda^2) = 0 \quad \text{a.s. for any } t \geq 0. \tag{2}$$

In particular a.s.

$$\ell_t^0(R-\lambda^2) = \lim_{\varepsilon\downarrow 0}\frac{\sigma^2}{2\varepsilon}\int_0^t \mathbb{I}_{(-\varepsilon,\varepsilon)}(R_s-\lambda^2(s))|R_s|ds.$$

The reason to work with symmetric local times is given in Remark 2.7(ii). Applying Tanaka's formula for R^- (cf. Lemma 3.2), it is easy to see that a solution to (1) always stays positive when started with positive initial condition. One can hence in that case discard the absolute value under the square root in (1) as in the classical situation where $\lambda^2\equiv 0$. Assuming that a solution to (1) is unique in a certain sense we shall call it p-skew squared Bessel process on λ^2 if $b=0$, $\sigma=2$, and p-skew CIR model on λ^2 if $b>0$. Questions of uniqueness are handled in [10].
Let $\delta\ge 1$. Assuming that it is a semimartingale, one can similarly consider a solution $Y, Y_0\ge 0$ to

$$dY_t = \frac{\sigma}{2}dW_t + \frac{\sigma^2}{8}\left(\frac{\delta-1}{Y_t}-bY_t\right)dt + (2p-1)\mathbb{I}_{\{\lambda>0\}}d\ell_t^0(Y-\lambda) + \frac{\mathbb{I}_{\{\delta=1\}}}{2}d\ell_t^{0+}(Y). \quad (3)$$

Applying Itô's formula we observe that $R_t := Y_t^2$, $R_0\ge 0$, solves

$$dR_t = \sigma\sqrt{R_t}dW_t + \frac{\sigma^2}{4}(\delta - bR_t)dt + (2p-1)2\sqrt{R_t}d\ell_t^0(\sqrt{R}-\lambda).$$

Thus, if we want that R solves (1) then we must have

$$2\sqrt{R_t}d\ell_t^0(\sqrt{R}-\lambda) = d\ell_t^0(R-\lambda^2). \quad (4)$$

This relation reflects (2), and can be shown probabilistically using a product formula for local times (see [12], and also [2]). In fact, we have $R-\lambda^2 = (\sqrt{R}+\lambda)(\sqrt{R}-\lambda)$, but see also Remark 2.8(i).

In this work we will use new bilinear form techniques and classical stochastic calculus in order to construct a pair of positive processes $(R,\sqrt{R})$ which solve (1), (3), respectively, and which are related by (4). The fact that $\sqrt{R}$ is a semimartingale if $\delta\ge 1$ is confirmed in Remark 2.4(i). Various properties of $(R,\sqrt{R})$ are discussed, in particular, we solve a martingale problem related to R on a nice class of test functions (see Proposition 3.5, and Remark 3.6). In Remark 2.7(i) we show that if $|p|>1$, then there is no solution to (1). The construction will take place for arbitrary $\lambda\in H^{1,2}_{loc}(\mathbb{R}^+)$, if $2p-1>0$, and for increasing λ, if $2p-1<0$. (cf. below (5), and Remark 2.4, 2.8(ii) for more general λ). Thus, if e.g. λ^2 is a constant, then we obtain a solution for every $p\in(0,1)$. The construction of $(R,\sqrt{R})$ is in the sense of equivalence of additive functionals of Markov processes (see Remark 2.6). The case $0\le\delta<1$ can also be handled, but is different, see Remark 2.9. In Remark 2.5 we explain how a generalized Longstaff's model, known as the double square-root model, is obtained.

2 Construction of the skew reflected process

Throughout this article $\mathbb{I}_A$ will denote the indicator function of a set A. Let $E := \mathbb{R}^+ \times \mathbb{R}^+$, where $\mathbb{R}^+ := \{x \in \mathbb{R} | \, x \geq 0\}$. Let $C_0^1(\mathbb{R}^+) = \{f : \mathbb{R}^+ \to \mathbb{R} | \, \exists u \in C_0^1(\mathbb{R}) \text{ with } u\mathbb{I}_{\mathbb{R}^+} = f\}$, and $C_0^1(\mathbb{R})$ denotes the continuously differentiable functions with compact support in $\mathbb{R}$. Let $H^{1,2}(\mathbb{R}^+)$ be the Sobolev space of order one in $L^2(\mathbb{R}^+)$, that is the completion of $C_0^1(\mathbb{R}^+)$ w.r.t. $|\phi|_{H^{1,2}(\mathbb{R})} = (\int_{\mathbb{R}^+} |\partial_u \phi|^2 + |\phi|^2 du)^{\frac{1}{2}}$. When considering an element of $H^{1,2}(\mathbb{R}^+)$, we always assume that is it continuous by choosing such a version. Later we will use the notions ∂_u, du for the space variable (notation $u = x$) as well as for the time variable (notation $u = t$, or $u = s$). For the space-time variable we use y, e.g. $y = (s, x)$, $y = (t, x)$. Let $H^{1,2}_{loc}(\mathbb{R}^+)$ denote the space of all continuous $\phi : \mathbb{R}^+ \to \mathbb{R}$ such that $\phi f \in H^{1,2}(\mathbb{R}^+)$ for any $f \in C_0^1(\mathbb{R}^+)$.

If $\lambda \in H^{1,2}_{loc}(\mathbb{R}^+)$, then it has a uniquely determined continuous version w.r.t. the Lebesgue measure dt. We will always assume that λ is continuous. Furthermore we assume that λ is positive, i.e. $\lambda \geq 0$. In particular $\lambda = \beta + \gamma$ (so that $\beta \geq -\gamma$), where $\beta, \gamma \in H^{1,2}_{loc}(\mathbb{R}^+)$, and β is decreasing, γ is increasing. Indeed, since $\partial_t \lambda \in L^2_{loc}(\mathbb{R}^+)$ we may consider its positive part $(\partial_t \lambda)^+$, and its negative part $(\partial_t \lambda)^-$, and fix from now on

$$\beta(t) := -\int_0^t (\partial_t \lambda)^-(s) ds + \lambda(0), \qquad \gamma(t) := \int_0^t (\partial_t \lambda)^+(s) ds.$$

Consider the following moving domain

$$E := \{(t, x) \in \mathbb{R}^+ \times \mathbb{R} | x \geq -\gamma(t)\}.$$

Observe, that its t-section $E_t = \{x \in \mathbb{R} | (t, x) \in E\} = [-\gamma(t), \infty)$ is increasing in t since $-\gamma(t)$ decreases in t. In particular $E = \cup_{t \geq 0} \{t\} \times E_t$.
Let $\delta \geq 1$, $b \in \mathbb{R}^+$ (for the case $\delta \in (0, 1)$ see Remark 2.9). As reference measure on E we take

$$m(dy) = m(dxdt) := \rho(t, x) dxdt,$$

where

$$\rho(t, x) := \left((1 - p)\mathbb{I}_{[-\gamma(t)), \beta(t))}(x) + p\mathbb{I}_{[\beta(t), \infty)}(x)\right) |x + \gamma(t)|^{\delta - 1} e^{-\frac{bx^2}{2}}$$

is assumed to be increasing in t, that is

$$\rho(s, x) \leq \rho(t, x) \quad \forall\, 0 \leq s \leq t, \; x \in E_s. \tag{5}$$

For instance, if $p \in (\frac{1}{2}, 1)$, so that $1 - p < p$, then $\rho(\cdot, x)$ always increases, or if $\beta = const$, then $\rho(\cdot, x)$ increases for any $p \in (0, 1)$.
Due to the monotonicity properties of ρ we are in the framework of [9]. More precisely, there is a time-dependent generalized Dirichlet form $\mathcal{E}$ with domain $\mathcal{F} \times \mathcal{V} \cup \mathcal{V} \times \hat{\mathcal{F}}$ on $\mathcal{H} := L^2(E, m)$ which we determine right below. For $q \geq 1$ let

$$C_0^q(E) := \{f : E \to \mathbb{R} | \, \exists u \in C_0^q(\mathbb{R}^2) \text{ with } u\mathbb{I}_E = f\},$$

and $C_0^q(\mathbb{R}^2)$ denotes the q-times continuously differentiable functions with compact support in $\mathbb{R}^2$. Let $0 < \sigma \in \mathbb{R}$

$$\mathcal{A}(F,G) := \frac{\sigma^2}{8}\int_0^\infty \int_{-\gamma(s)}^\infty \partial_x F(s,x)\,\partial_x G(s,x)\rho(s,x)dxds; \quad F,G \in C_0^1(E), \tag{6}$$

with closure $(\mathcal{A},\mathcal{V})$ in $\mathcal{H}$. The closability easily follows since ρ satisfies a Hamza type condition (see [9, Lemma 1.1]). Let $\mathcal{A}_\alpha(F,G) := \mathcal{A}(F,F) + \alpha(F,F)$, $\alpha > 0$, where $(\cdot,\cdot)$ is the inner product in $\mathcal{H}$. For $K \subset E$ compact, the capacity related to $\mathcal{A}$ is defined by

$$\mathrm{Cap}^{\mathcal{A}}(K) = \inf\{\mathcal{A}_1(F,F); F \in C_{0,K}^1(E)\}, \tag{7}$$

where $C_{0,K}^1(E) = \{F \in C_0^1(E) | F(s,x) \geq 1, \forall (s,x) \in K\}$. For general $A \subset E$ it is extended by inner regularity. Define

$$U_t F(s,x) := F(s+t,x); \quad F \in C_0^1(E).$$

It then follows from results in [9] that $(U_t)_{t\geq 0}$ can be extended to a C_0-semigroup of contractions on $\mathcal{H}$ which can be restricted to a C_0-semigroup on $\mathcal{V}$. For the corresponding generator $(\partial_t, D(\partial_t,\mathcal{H})$ on $\mathcal{H}$ it follows that

$$\partial_t : D(\partial_t, \mathcal{H}\cap\mathcal{V}) \to \mathcal{V}'$$

is closable as operator from $\mathcal{V}$ to its dual $\mathcal{V}'$ (see [5, I.Lemma 2.3.]). Let $(\partial_t,\mathcal{F})$ be the closure. $\mathcal{F}$ is a real Hilbert space with norm

$$|F|_{\mathcal{F}} := \sqrt{|F|_{\mathcal{V}}^2 + |\partial_t F|_{\mathcal{V}'}^2}.$$

The adjoint semigroup $(\widehat{U}_t)_{t\geq 0}$ of $(U_t)_{t\geq 0}$ in $\mathcal{H}$ can be extended to a C_0-semigroup on $\mathcal{V}'$. The corresponding generator $(\hat{\Lambda}, D(\hat{\Lambda},\mathcal{V}'))$ is the dual operator of $(\partial_t, D(\partial_t,\mathcal{V}))$. $\widehat{\mathcal{F}} := D(\hat{\Lambda},\mathcal{V}')\cap\mathcal{V}$ is a real Hilbert space with norm

$$|F|_{\widehat{\mathcal{F}}} := \sqrt{|F|_{\mathcal{V}}^2 + |\hat{\Lambda} F|_{\mathcal{V}'}^2}.$$

Let $\langle\cdot,\cdot\rangle$ be the dualization between $\mathcal{V}'$ and $\mathcal{V}$. The *time-dependent generalized Dirichlet form* is now given through

$$\mathcal{E}(F,G) := \begin{cases} \mathcal{A}(F,G) - \langle \partial_t F, G\rangle & \text{for } F\in\mathcal{F},\ G\in\mathcal{V} \\ \mathcal{A}(F,G) - \langle \hat{\Lambda} G, F\rangle & \text{for } G\in\widehat{\mathcal{F}},\ F\in\mathcal{V}. \end{cases}$$

Note that $\langle\cdot,\cdot\rangle$ when restricted to $\mathcal{H}\times\mathcal{V}$ coincides with the inner product $(\cdot,\cdot)$ in $\mathcal{H}$. In particular when $F \in C_0^1(E)$, $G\in\mathcal{V}$, then

$$\begin{aligned}\mathcal{E}(F,G) &= \frac{\sigma^2}{8}\int_0^\infty\int_{-\gamma(s)}^\infty \partial_x F(s,x)\,\partial_x G(s,x)\rho(s,x)dxds \\ &\quad - \int_0^\infty\int_{-\gamma(s)}^\infty \partial_t F(s,x) G(s,x)\rho(s,x)dxds. \end{aligned} \tag{8}$$

For all corresponding objects to $\mathcal{E}$ which might not rigorously be defined here we refer to [9]. We also point out that the monotonicity assumption on E_t as well as on the density ρ in time is crucial for the construction of $\mathcal{E}$.
By [9, Lemma 1.6, Lemma 1.7] the resolvent $(G_\alpha)_{\alpha>0}$ and the coresolvent $(\widehat{G}_\alpha)_{\alpha>0}$ associated with $\mathcal{E}$ are sub-Markovian and $C_0^1(E)\subset\mathcal{F}$ dense. Let $\mathcal{E}_\alpha(F,G):=\mathcal{E}(F,G)+\alpha(F,G)$ for $\alpha>0$. Then

$$\mathcal{E}_\alpha(G_\alpha F,G)=(F,G)_{\mathcal{H}}=\mathcal{E}_\alpha(F,\widehat{G}_\alpha G)\quad F,G\in\mathcal{V}.$$

Proposition 2.1 *$(G_\alpha)_{\alpha>0}$ is Markovian, i.e. $G_1\mathbb{I}_E=\mathbb{I}_E$ m-a.e.*

Proof We start this proof with a general observation. In order to prove the conservativity of $\mathcal{E}$ it is enough to show that for one $F\in\mathcal{H}\cap L^1(E,m)$, $F>0$ m-a.e., there exists $(W_n)_{n\geq1}\subset\mathcal{F}$, $0\leq W_n\leq\mathbb{I}_E$, $n\geq1$, $W_n\uparrow\mathbb{I}_E$ as $n\to\infty$, such that

$$\lim_{n\to\infty}\mathcal{E}(W_n,\widehat{G}_1F)=0.$$

Indeed, if this is the case then

$$0=\lim_{n\to\infty}\mathcal{E}(W_n,\widehat{G}_1F)=\lim_{n\to\infty}\int_E(W_n-G_1W_n)F\rho dxds=\int_E(\mathbb{I}_E-G_1\mathbb{I}_E)F\rho dxds,$$

and $G_1\mathbb{I}_E=\mathbb{I}_E$ as desired. We now fix F as above, and determine below $(W_n)_{n\geq1}$.
Let $g_n\in C_0^1(\mathbb{R}^+)$, $u_n\in C_0^2(\mathbb{R})$, $n\geq1$, such that $0\leq g_n,u_n\leq1$, $|\partial_t g_n|_\infty,|\partial_x u_n|_\infty\leq L\cdot n^{-1}$, $|\partial_{xx}u_n|_\infty\leq L\cdot n^{-2}$, where L is some positive constant, and

$$g_n(s)=\begin{cases}1 & \text{if } s\in[0,n]\\ 0 & \text{if } s\in[2n,\infty),\end{cases}$$

and

$$u_n(x)=\begin{cases}1 & \text{if } [-\gamma(2n)]\leq x\leq[\lambda(0)+1]+n\\ 0 & \text{if } x\geq[\lambda(0)+1]+2n,\end{cases}$$

where $[x]:=\sup\{k\in\mathbb{Z}|k\leq x\}$. Then $W_n:=g_nu_n\mathbb{I}_E\in\mathcal{F}$, $n\geq1$, satisfies $W_n\uparrow\mathbb{I}_E$ as $n\to\infty$, and since $\partial_xW_n(s,\beta(s))=\partial_xW_n(s,-\gamma(s))=0$ for all s, we easily find

$$\mathcal{E}(W_n,\widehat{G}_1F)=-\frac{\sigma^2}{8}\int_0^\infty\int_{-\gamma(s)}^\infty\left(\partial_{xx}W_n+\left(\frac{\delta-1}{(x+\gamma(s))}-bx\right)\partial_xW_n+\frac{8}{\sigma^2}\partial_tW_n\right)\widehat{G}_1F\rho dxds,$$

so that $|\mathcal{E}(W_n,\widehat{G}_1F)|$ is dominated by

$$\frac{L\cdot\sigma^2}{8}\int_0^{2n}\int_{[\lambda(0)+1]+n}^{[\lambda(0)+1]+2n}\left(\frac{1}{n^2}+\frac{\delta-1}{n([\lambda(0)+1]+n+\gamma(0))}+\frac{b([\lambda(0)+1]+2n)}{n}\right)\widehat{G}_1F\rho dxds$$
$$+\int_0^{2n}\int_{-\gamma(s)}^\infty\frac{L}{n}\widehat{G}_1F\rho dxds.$$

Noting that $\widehat{G}_1 F\rho dxds$ is a finite measure, we just apply Lebesgue's theorem, and the last sum is easily seen to converge to zero as $n \to \infty$. This concludes the proof.

$\square$

Let us define the strict capacity corresponding to $\mathcal{E}$. We fix $\Phi \in L^1(E,m)$, $0 < \Phi \leq 1$. Let $(kG_1\Phi \wedge 1)_U$ be the 1-reduced function of $kG_1\Phi \wedge 1 := min(kG_1\Phi, 1)$ on U, and let

$$\mathrm{Cap}_{1,\widehat{G}_1\Phi}(U) = \lim_{k\to\infty} \int_E (kG_1\Phi \wedge 1)_U \Phi dm \;\text{ if }\; U \subset E \text{ is open.}$$

If $A \subset E$ arbitrary then

$$\mathrm{Cap}_{1,\widehat{G}_1\Phi}(A) = \inf\{\mathrm{Cap}_{1,\widehat{G}_1\Phi}(U) | U \supset A, U \text{ open}\}.$$

We adjoin an extra point Δ to E and let $E_\Delta := E \cup \{\Delta\}$ be the one point compactification of E. As usual any function defined on E is extended to E_Δ putting $f(\Delta) = 0$. Given an increasing sequence $(F_k)_{k\in\mathbb{N}}$ of closed subsets of E, we define

$$C_\infty(\{F_k\}) = \{f : A \to \mathbb{R} \mid \bigcup_{k\geq 1} F_k \subset A \subset E,\ f_{|F_k \cup\{\Delta\}} \ \textit{is continuous}\ \forall k\}.$$

A subset $N \subset E$ is called strictly $\mathcal{E}$-exceptional if $\mathrm{Cap}_{1,\widehat{G}_1\Phi}(N) = 0$. An increasing sequence $(F_k)_{k\in\mathbb{N}}$ of closed subsets of E is called a strict $\mathcal{E}$-nest if $\mathrm{Cap}_{1,\widehat{G}_1\Phi}(F_k^c) \downarrow 0$ as $k \to \infty$. A property of points in E holds strictly $\mathcal{E}$-quasi-everywhere (s.$\mathcal{E}$-q.e.) if the property holds outside some strictly $\mathcal{E}$-exceptional set. A function f defined up to some strictly $\mathcal{E}$-exceptional set $N \subset E$ is called strictly $\mathcal{E}$-quasi-continuous (s.$\mathcal{E}$-q.c.) if there exists a strict $\mathcal{E}$-nest $(F_k)_{k\in\mathbb{N}}$, such that $f \in C_\infty(\{F_k\})$.
For a subset $A \subset E_\Delta$ let $\sigma_A := \inf\{t > 0 \mid \overline{Y}_t \in A\}$ (resp. $D_A = \inf\{t \geq 0 | \overline{Y}_t \in A\}$) be the *first hitting time* (resp. *first entry time*) w.r.t. $\mathbb{M}$. For a Borel measure ν on E and a Borel set B let $P_\nu(B) := \int_E P_y(B)\nu(dy)$ and E_ν be the expectation w.r.t. P_ν. As usual we denote by E_y the expectation w.r.t. P_y. If $U \subset E$ is open, then

$$\mathrm{Cap}_{1,\widehat{G}_1\Phi}(U) = \int_E E_y[e^{-\sigma_U}]\Phi(y)m(dy). \tag{9}$$

If $B \subset E$ is an arbitrary Borel measurable set, then

$$\mathrm{Cap}_{1,\widehat{G}_1\Phi}(B) = \int_E E_y[e^{-D_B}]\Phi(y)m(dy).$$

Both follows from [8, Lemma 0.8].
By strict quasi-regularity every element in $\mathcal{F}$ admits a strictly $\mathcal{E}$-q.c. m-version (see [8, Proposition 0.9]). For a subset $\mathcal{D} \subset \mathcal{H}$ denote by $\widetilde{\mathcal{D}}$ all the s.$\mathcal{E}$-q.c. m-versions of elements in $\mathcal{D}$. In particular $\widetilde{\mathcal{P}}_{\mathcal{F}}$ denotes the set of all s.$\mathcal{E}$-q.c. ρdy-versions of 1-excessive elements in $\mathcal{V}$ which are dominated by elements of $\mathcal{F}$. We have an analogy, namely [8, Theorem 0.16],

to [6, Theorem 2.3]. That is: Let $\hat{u} \in \widehat{\mathcal{P}}_{\widehat{\mathcal{F}}}$. Then there exists a unique σ-finite and positive measure $\mu_{\hat{u}}$ on $(E, \mathcal{B}(E))$ charging no strictly $\mathcal{E}$-exceptional set, such that

$$\int_E \widetilde{f}\, d\mu_{\hat{u}} = \lim_{\alpha \to \infty} \mathcal{E}_1(f, \alpha \widehat{G}_{\alpha+1}\hat{u}) \quad \forall \widetilde{f} \in \widetilde{\mathcal{P}}_{\mathcal{F}} - \widetilde{\mathcal{P}}_{\mathcal{F}} \ .$$

Also in analogy to [6] we introduce the following class of measures

$$\widehat{S}_{00} := \{\mu_{\hat{u}} \mid \ \hat{u} \in \widehat{\mathcal{P}}_{\widehat{G}_1 \mathcal{H}_b^+} \text{ and } \mu_{\hat{u}}(E) < \infty\}$$

where $\widehat{G}_1 \mathcal{H}_b^+ := \{\widehat{G}_1 h \mid h \in \mathcal{H}_b^+\}$.
For $B \in \mathcal{B}(E)$ the following is known from [8, Theorem 0.17]: B is strictly $\mathcal{E}$-exceptional if, and only if $\mu(B) = 0$ for all μ in $\widehat{S}_{00}$.
Since $(\mathcal{E}, \mathcal{F})$ is regular, i.e. $C_0(E) \cap \mathcal{F}$ is dense in $C_0(E)$ w.r.t. the uniform norm as well as in $\mathcal{F}$, it follows that $(\mathcal{E}, \mathcal{F})$ is a (strictly) quasi-regular generalized Dirichlet form on E. On the other hand we can find a dense algebra of functions, namely $C_0^1(E)$, in $\mathcal{F}$. These two facts imply the existence of a Hunt process associated to $\mathcal{E}$. Applying additionally Proposition 2.1, and [9, Theorem 1.9] we have:

Theorem 2.2 *There exists a Hunt process $\mathbb{M} = (\Omega, (\mathcal{F}_t)_{t \geq 0}, (\overline{Y}_t)_{t \geq 0}, (P_y)_{y=(s,x) \in E_\Delta})$ with state space E, and infinite life time, such that $R_\alpha F(s,x) := \int_0^\infty \int_\Omega e^{-\alpha t} F(\overline{Y}_t(\omega)) P_{(s,x)}(d\omega) dt$ is a $\mathcal{E}$-q.c. m-version of $G_\alpha F$ for any $\alpha > 0$ and any $F \in \mathcal{H}_b$. Moreover there exists a $\mathcal{E}$-exceptional set $N \subset E$ such that*

$$P_{(s,x)}\big(t \mapsto \overline{Y}_t \text{ is continuous on } [0, \infty)\big) = 1 \text{ for every } (s,x) \in E \setminus N.$$

We want to identify $\overline{Y}$. Let us first recall some basic definitions and facts about additive functionals related to generalized Dirichlet forms.

A family $(A_t)_{t \geq 0}$ of extended real valued functions on Ω is called an *additive functional* (abbreviated AF) of $\mathbb{M} = (\Omega, (\mathcal{F}_t)_{t \geq 0}, (\overline{Y}_t)_{t \geq 0}, (P_y)_{y \in E_\Delta})$ (w.r.t. $\mathrm{Cap}_{1, \widehat{G}_1 \Phi}$), if:

(i) $A_t(\cdot)$ is $\mathcal{F}_t$-measurable for all $t \geq 0$.

(ii) There exists a *defining* set $\Lambda \in \mathcal{F}_\infty$ and a strictly $\mathcal{E}$-exceptional set $N \subset E$, such that $P_y(\Lambda) = 1$ for all $y \in E \setminus N$, $\theta_t(\Lambda) \subset \Lambda$ for all $t > 0$ and for each $\omega \in \Lambda$, $t \mapsto A_t(\omega)$ is right continuous on $[0, \infty)$ and has left limits on $(0, \zeta(\omega))$, $A_0(\omega) = 0$, $|A_t(\omega)| < \infty$ for $t < \zeta(\omega)$, $A_t(\omega) = A_\zeta(\omega)$ for $t \geq \zeta(\omega)$ and $A_{t+s}(\omega) = A_t(\omega) + A_s(\theta_t \omega)$ for $s, t \geq 0$.

An AF A is called a *continuous additive functional* (abbreviated CAF), if $t \mapsto A_t(\omega)$ is continuous on $[0, \infty)$, a *positive, continuous additive functional* (abbreviated PCAF) if $A_t(\omega) \geq 0$ and a finite AF, if $\mid A_t(\omega) \mid < \infty$ for all $t \geq 0, \omega \in \Lambda$. Two AF's A,B are said to be equivalent (in notation $A = B$) if for each $t > 0$ $P_y(A_t = B_t) = 1$ for strictly $\mathcal{E}$-q.e. $y \in E$. The *energy* of an AF A of $\mathbb{M}$ is defined by

$$e(A) \;=\; \lim_{\alpha \to \infty} \frac{1}{2} \alpha^2 E_{\rho dy} \left[\int_0^\infty e^{-\alpha t} A_t^2 dt \right], \qquad (10)$$

whenever this limit exists in $[0,\infty]$. We will set $\overline{e}(A)$ for the same expression but with $\overline{\lim}$ instead of lim.

Let $\widetilde{F}$ be a strictly $\mathcal{E}$-q.c. ρdy-version of some element in $\mathcal{H}$. The additive functional

$$A^{[F]} := (\widetilde{F}(\overline{Y}_t) - \widetilde{F}(\overline{Y}_0))_{t\geq 0}$$

is independent of the choice of $\widetilde{F}$ (i.e. defines the same equivalence class of AF's for any strictly $\mathcal{E}$-q.c. ρdy-version $\widetilde{F}$ of F). The sub-Markovianity of $(\widehat{G}_\alpha)_{\alpha>0}$ implies

$$\begin{aligned}\overline{e}(A^{[F]}) &= \overline{\lim_{\alpha\to\infty}} \left(\alpha(F-\alpha G_\alpha F, F)_{\mathcal{H}} - \frac{\alpha}{2}\int_E (F^2 - \alpha G_\alpha F^2)\rho dy\right) \\ &\leq \overline{\lim_{\alpha\to\infty}}\ \alpha(F-\alpha G_\alpha F, F)_{\mathcal{H}}.\end{aligned}$$

Since $\mathcal{F} \subset \mathcal{V}^{\mathcal{F}}$ (cf. e.g. proof of [7, Lemma 3.1]) it follows $\lim_{\alpha\to\infty}\alpha\widehat{G}_\alpha F = F$ weakly in $\mathcal{V}$. Hence $\lim_{\alpha\to\infty}\alpha(F-\alpha G_\alpha F, F)_{\mathcal{H}} = \lim_{\alpha\to\infty}\mathcal{E}(F, \alpha\widehat{G}_\alpha F) = \mathcal{E}(F,F)$ whenever $F\in\mathcal{F}$. In particular

$$\overline{e}(A^{[F]}) \leq 2|F|^2_{\mathcal{F}} \text{ for any } F\in\mathcal{F}. \tag{11}$$

Define

$$\begin{aligned}\mathcal{M} &= \{M|M \text{ is a finite AF}, E_y[M_t^2]<\infty, E_y[M_t]=0 \\ &\qquad \text{for strictly } \mathcal{E}\text{-q.e } y\in E \text{ and all } t\geq 0\}.\end{aligned}$$

$M\in\mathcal{M}$ is called a *martingale additive functional* (MAF). Furthermore define

$$\overset{\circ}{\mathcal{M}} = \{M\in\mathcal{M}|\ e(M)<\infty\}.$$

The elements of $\overset{\circ}{\mathcal{M}}$ are called MAF's of finite energy.

Let A be a PCAF of $\mathbb{M}$. Its Revuz measure μ_A (see [6, Theorem 3.1]) is defined by

$$\int_E G(y)\mu_A(dy) = \lim_{\alpha\to\infty}\alpha E_{\rho dy}\left[\int_0^\infty e^{-\alpha t}G(\overline{Y}_t)dA_t\right] \text{ for all } G\in\mathcal{B}^+. \tag{12}$$

The dual predictable projection $\langle M\rangle$ of the square bracket of $M\in\overset{\circ}{\mathcal{M}}$ is a PCAF of $\mathbb{M}$. It then follows from (10), (12), that one half of the total mass of the Revuz measure $\mu_{\langle M\rangle}$ is equal to the energy of M, i.e.

$$e(M) = \frac{1}{2}\int_E \mu_{\langle M\rangle}(dy). \tag{13}$$

Therefore $\mu_{\langle M\rangle}$ is also called the energy measure of M. For M, $L\in\overset{\circ}{\mathcal{M}}$ let

$$\langle M, L\rangle := \frac{1}{2}\left(\langle M+L\rangle - \langle M\rangle - \langle L\rangle\right).$$

Then $(\langle M, L\rangle_t)_{t\geq 0}$ is a CAF of bounded variation on each finite interval. Furthermore the finite signed measure $\mu_{\langle M,L\rangle}$ defined by $\mu_{\langle M,L\rangle} := \frac{1}{2}(\mu_{\langle M+L\rangle} - \mu_{\langle M\rangle} - \mu_{\langle L\rangle})$ is related to $\langle M, L\rangle$ in the sense of (12). If $G \in \mathcal{B}_b^+$, then $\int_E G d\mu_{\langle\cdot,\cdot\rangle}$ is symmetric, bilinear and positive on $\overset{\circ}{\mathcal{M}} \times \overset{\circ}{\mathcal{M}}$.

Define

$$\begin{aligned}\mathcal{N}_c &= \{N | N \text{ is a finite CAF}, e(N) = 0, E_y[|N_t|] < \infty \\ &\qquad \text{for strictly } \mathcal{E}\text{-q.e. } y \in E \text{ and all } t \geq 0\}.\end{aligned}$$

For $F \in \mathcal{F}$, $A^{[F]}$ can uniquely be decomposed (see [6, Theorem 4.5.(i)], [8, Remark 0.17]) as

$$A^{[F]} = M^{[F]} + N^{[F]}, \quad M^{[F]} \in \overset{\circ}{\mathcal{M}}, \quad N^{[F]} \in \mathcal{N}_c. \tag{14}$$

The identity (14) means that both sides are equivalent as additive functionals w.r.t. $\text{Cap}_{1,\widehat{G}_1\Phi}$. The uniqueness of (14) implies $aM^{[F]} + bM^{[G]} = M^{[aF+bG]}$, $aN^{[F]} + bN^{[G]} = N^{[aF+bG]}$, for any $a, b \in \mathbb{R}$, $F, G \in \mathcal{F}$.
From Lemma 2.1 in [9] we know that for $F \in \mathcal{F}$

$$\mu_{\langle M^{[F]}\rangle}(dxds) = \frac{\sigma^2}{4}(\partial_x F)^2 \rho dxds,$$

and moreover, if F is constant ρdy-a.e. on a Borel set B. Then

$$\mu_{\langle M^{[F]}\rangle}(B) = 0.$$

Now let us come back to the identification of $\overline{Y}$. In order to identify the drift part we might proceed as follows. Denote by δ_x the Dirac measure in $x \in \mathbb{R}$. If $\beta(s) > -\gamma(s)$ a.e. s, then integrating by parts in (6) we obtain that the generator of the diffusion is given informally in the sense of distributions by

$$LF(s,x) = \frac{\sigma^2}{8}\partial_{xx}F(s,x) + \frac{\sigma^2}{8}\frac{\delta - 1}{(x+\gamma(s))}\partial_x F(s,x) + \partial_t F(s,x) + \nu_F(dx)ds$$

where the boundary term ν_F is given by

$$\begin{aligned}\nu_F(dx)ds &= \left\{p\partial_x^+ F(s,x) - (1-p)\partial_x^- F(s,x)\right\}\frac{\sigma^2}{8}|x+\gamma(s)|^{\delta-1}e^{-\frac{bx^2}{2}}\delta_{\beta(s)}(dx)ds \\ &\quad -(1-p)\partial_x^+ F(s,x)\frac{\sigma^2}{8}|x+\gamma(s)|^{\delta-1}e^{-\frac{bx^2}{2}}\delta_{-\gamma(s)}(dx)ds,\end{aligned}$$

and where as usually ∂_x^+, resp. ∂_x^-, denote the right hand, resp. the left hand derivative in space.

If we can show that $|x+\gamma(s)|^{\delta-1}e^{-\frac{bx^2}{2}}\delta_{\kappa(s)}(dx)ds$, $\kappa:\mathbb{R}^+\to\mathbb{R}^+$ locally bounded and measurable, is a smooth measure w.r.t. $\mathcal{A}$, then there is a unique PCAF representing this measure by Theorem 2.2 in [9]. Theorem 2.3 in [9] then allows to identify the drift part. We will identify the corresponding diffusion when $\delta\geq 1$.
Let $\mathbb{R}^+\times\mathbb{R}=\bigcup_{n\geq 1}K_n$, where $(K_n)_{n\geq 1}$ be an increasing sequence of compact subsets of $\mathbb{R}^+\times\mathbb{R}$. Let $\overline{E}_n := K_n\cap E$, $n\geq 1$. Since $C_0^1(E)\subset\mathcal{F}$ dense, it follows from [5, III.Remark 2.11] that $(\overline{E}_n)_{n\geq 1}$ is an $\mathcal{E}$-nest in the sense of [5, III.Definition 2.3(i)]. Consequently, $P_y(\lim_{n\to\infty}\sigma_{\overline{E}_n^c}<\infty)=0$ for $\mathcal{E}$-q.e. $y\in E$, hence in particular for ρdy-a.e. $y\in E$ (see [5, IV. Lemma 3.10]). We obtain that $(\overline{E}_n)_{n\geq 1}$ is an strict $\mathcal{E}$-nest by (9). [8, Lemma 0.8(ii)] now implies $P_y(\lim_{n\to\infty}\sigma_{\overline{E}_n^c}<\infty)=0$ for strictly $\mathcal{E}$-q.e. $y\in E$. We may without loss of generality assume that $\overline{E}_n\subset[0,n]\times\mathbb{R}\cap E$, $n\geq 1$, and that $\overline{E}_n$ is contained in the interior of $\overline{E}_{n+1}$ for any $n\geq 1$. From now on we will fix such a strict $\mathcal{E}$-nest $(\overline{E}_n)_{n\geq 1}$.

Proposition 2.3 *Let $\delta\geq 1$, $\kappa:\mathbb{R}^+\to\mathbb{R}$ be measurable and locally bounded, such that $(s,\kappa(s))\in E_s$ for each $s\geq 0$. The measure*

$$\mathbb{I}_{\overline{E}_N}(s,x)|x+\gamma(s)|^{\delta-1}e^{-\frac{bx^2}{2}}\delta_{\kappa(s)}(dx)ds,\ N\geq 1,$$

is smooth w.r.t. $(\mathcal{A},\mathcal{V})$.

Proof We only show the statement for $\delta>1$. The proof for $\delta=1$ works in the same manner and is even easier since the derivative of $y\mapsto|y|^{\delta-1}$ disappears as it is constant, so there are less additional terms (cf. below). Let $N\geq 1$. Let $F\in C_0^1(E)$, $\psi\in C_0^\infty(\mathbb{R})$, $0\leq\psi\leq 1$, $|\partial_x\psi|_\infty\leq 2$. Since κ is locally bounded, the following two values

$$\kappa_{Nmin}:=\inf\{\kappa(t)|t\in[0,N]\},\qquad \kappa_{Nmax}:=\sup\{\kappa(t)|t\in[0,N]\},$$

are finite. Let $\psi=1$ on $[\kappa_{Nmin},\kappa_{Nmax}]$, $\psi=0$ on $[\kappa_{Nmax}+1,\infty[$. For $s\in[0,N]$ we have

$$F(s,\kappa(s))|\kappa(s)+\gamma(s)|^{\delta-1}e^{-\frac{b\kappa(s)^2}{2}}=-\int_{\kappa(s)}^{\kappa_{Nmax}+1}\partial_x\left(\psi(x)F(s,x)|x+\gamma(s)|^{\delta-1}e^{-\frac{bx^2}{2}}\right)dx$$

and thus

$$\begin{aligned}
&\int_{\overline{E}_N}|F|(s,x)|x+\gamma(s)|^{\delta-1}e^{-\frac{bx^2}{2}}\delta_{\kappa(s)}(dx)ds\\
&\leq \int_0^N\left|\int_{\kappa(s)}^{\kappa_{Nmax}+1}\partial_x\left(\psi(x)F(s,x)|x+\gamma(s)|^{\delta-1}e^{-\frac{bx^2}{2}}\right)dx\right|ds\\
&\leq 2\int_0^N\int_{\kappa(s)}^{\kappa_{Nmax}+1}\left(|\partial_xF|+|F|\right)|x+\gamma(s)|^{\delta-1}e^{-\frac{bx^2}{2}}dxds\\
&\qquad\qquad+\int_0^N\int_{\kappa(s)}^{\kappa_{Nmax}+1}|F|\left((\delta-1)|x+\gamma(s)|^{\delta-2}-bx|x+\gamma(s)|^{\delta-1}\right)e^{-\frac{bx^2}{2}}dxds\\
&\leq C_N\sqrt{\mathcal{A}_{\frac{\sigma^2}{4}}(F,F)}+I(F),
\end{aligned}\tag{15}$$

with

$$I(F) := \int_0^N \int_{\kappa(s)}^{\kappa_{Nmax}+1} |F| \left((\delta-1)|x+\gamma(s)|^{\delta-2} - bx|x+\gamma(s)|^{\delta-1} \right) e^{-\frac{bx^2}{2}} dxds,$$

and $C_N = \frac{8}{\sigma}\sqrt{\int_0^N \int_{\kappa(s)}^{\kappa_{Nmax}+1} |x+\gamma(s)|^{\delta-1} e^{-\frac{bx^2}{2}} dxds}$.
Let $K \subset E$ be compact, and $\mathrm{Cap}^{\mathcal{A}}(K) = 0$. By (7)

$$\mathrm{Cap}^{\mathcal{A}}(K) = \inf\{\mathcal{A}_1(F,F); F \in C^1_{0,K}(E)\},$$

where $C^1_{0,K}(E) = \{F \in C^1_0(E) | F(s,x) \geq 1, \forall (s,x) \in K\}$. Hence, there exists $(F_n)_{n\in\mathbb{N}} \subset C^1_0(E)$, $F_n(s,x) \geq 1$, for every $n \in \mathbb{N}$, $(s,x) \in K$, such that $|F_n|_{\mathcal{V}} \to 0$ as $n \to \infty$. Since normal contractions operate on $\mathcal{V}$ we may assume that $\sup_{n\in\mathbb{N}} \sup_{(s,x)\in K} |F_n(s,x)| \leq C$. Selecting a subsequence if necessary we may also assume that $\lim_{n\to\infty} |F_n| = 0$ $\rho(s,x)dxds$-a.e, hence $dxds$-a.e. Consequently, using Lebesgue's theorem we obtain

$$I(F_n) \to 0 \text{ as } n \to \infty.$$

Therefore by (15)

$$\begin{aligned}
& \int_{\overline{E}_N} 1_K(s,x)|x+\gamma(s)|^{\delta-1} e^{-\frac{bx^2}{2}} \delta_{\kappa(s)}(dx)ds \\
\leq\ & \limsup_{n\to\infty} \int_{\overline{E}_N} |F_n|(s,x)|x+\gamma(s)|^{\delta-1} e^{-\frac{bx^2}{2}} \delta_{\kappa(s)}(dx)ds \\
\leq\ & \limsup_{n\to\infty} \left\{ C_N \sqrt{\mathcal{A}_{\frac{\sigma^2}{4}}(F_n,F_n)} + I(F_n) \right\} = 0
\end{aligned}$$

Since $1_{\overline{E}_N}(s,x)|x+\gamma(s)|^{\delta-1} e^{-\frac{bx^2}{2}} \delta_{\kappa(s)}(dx)ds$, as well as $\mathrm{Cap}^{\mathcal{A}}$ are inner regular we obtain that the measure $1_{\overline{E}_N}(s,x)|x+\gamma(s)|^{\delta-1} e^{-\frac{bx^2}{2}} \delta_{\kappa(s)}(dx)ds$ is smooth w.r.t. $(\mathcal{A},\mathcal{V})$.

□

Let us choose $(J_M)_{M\geq1}$, $(H_M)_{M\geq1} \subset C^2_0(E)$, with

$$H_M(s,x) := \begin{cases} x & \text{for } (s,x) \in \overline{E}_M \\ 0 & \text{for } (s,x) \in \overline{E}^c_{M+1}, \end{cases}$$

$M \geq 1$, and

$$J_M(s,x) := \begin{cases} s & \text{for } (s,x) \in \overline{E}_M \\ 0 & \text{for } (s,x) \in \overline{E}^c_{M+1}, \end{cases}$$

$M \geq 1$. Let further

$$H(s,x) := x,$$

and

$$J(s,x) := s.$$

$(H_M)_{M\geq 1}$ (resp. $(J_M)_{M\geq 1}$) is a localizing sequence for H (resp. J). Obviously

$$A^{[H_K]}_{t\wedge\sigma_{\overline{E}^c_M}} = A^{[H_L]}_{t\wedge\sigma_{\overline{E}^c_M}} \text{ for any } K\geq L\geq M.$$

We claim that

$$M^{[H_K]}_{t\wedge\sigma_{\overline{E}^c_M}} = M^{[H_L]}_{t\wedge\sigma_{\overline{E}^c_M}} \text{ for any } K\geq L\geq M.$$

Indeed, for strictly $\mathcal{E}$-q.e $y\in E$, and any $t\geq 0$,

$$\begin{aligned} E_y\left[\langle M^{[H_K-H_L]}\rangle_{t\wedge\sigma_{\overline{E}^c_M}}\right] &= E_y\left[\int_0^{t\wedge\sigma_{\overline{E}^c_M}} 1_{\overline{E}_M}(\overline{Y}_s)\, d\langle M^{[H_K-H_L]}\rangle_s\right] \\ &\leq E_y\left[\int_0^t 1_{\overline{E}_M}(\overline{Y}_s)\, d\langle M^{[H_K-H_L]}\rangle_s\right]. \end{aligned}$$

By Lemma 2.1(ii) in [9] $\mu_{\int_0^\cdot 1_{\overline{E}_M}(\overline{Y}_s)d\langle M^{[H_K-H_L]}\rangle_s} = \mu_{\langle M^{[H_K-H_L]}\rangle}(\overline{E}_M) = 0$. Thus by injectivity of the Revuz-correspondence (see [6, Remark 5.2(ii)]) $E_y\left[\int_0^t 1_{\overline{E}_M}(\overline{Y}_s)\, d\langle M^{[H_K-H_L]}\rangle_s\right] = 0$ strictly $\mathcal{E}$-q.e. $y\in E$. Hence the same is true for $E_y\left[\langle M^{[H_K-H_L]}\rangle_{t\wedge\sigma_{\overline{E}^c_M}}\right]$. We know that $\left((M_t^{[H_K-H_L]})^2 - \langle M^{[H_K-H_L]}\rangle_t\right)_{t\geq 0}$ is a martingale w.r.t. P_y for strictly $\mathcal{E}$-q.e. $y\in E$. The optional sampling theorem then implies

$$E_y\left[(M^{[H_K-H_L]}_{t\wedge\sigma_{\overline{E}^c_M}})^2\right] = E_y\left[\langle M^{[H_K-H_L]}\rangle_{t\wedge\sigma_{\overline{E}^c_M}}\right] = 0$$

for strictly $\mathcal{E}$-q.e. $y\in E$ and the claim is shown. The analogous statements hold for $A^{[J_K]}$, $M^{[J_K]}$. Thus we may set

$$M_t^{[H]} := \lim_{M\to\infty} M_t^{[H_M]} \qquad\qquad N_t^{[H]} := A_t^{[H]} - M_t^{[H]},$$

and

$$M_t^{[J]} := \lim_{M\to\infty} M_t^{[J_M]} \qquad\qquad N_t^{[J]} := A_t^{[J]} - M_t^{[J]},$$

in order to obtain

$$A_t^{[H]} = M_t^{[H]} + N_t^{[H]}, \qquad\qquad A_t^{[J]} = M_t^{[J]} + N_t^{[J]}.$$

Note that $N_t^{[H]} = \lim_{M\to\infty} N_t^{[H_M]}$, $N_t^{[J]} = \lim_{M\to\infty} M_t^{[J_M]}$. We want to find the explicit expressions for $M^{[J]}$, $N^{[J]}$, $M^{[H]}$, $N^{[H]}$. Let $F\in C_0^2(E)$. Integrating by parts we obtain for any $G \in C_0^1(E)$

$$\begin{aligned}
-\mathcal{E}(F,G) &= \int_E \left(\frac{\sigma^2}{8} \left(\partial_{xx} F + \left(\frac{\delta-1}{(x+\gamma(s))} - bx \right) \partial_x F \right) + \partial_t F \right) G \rho dx ds \\
&\quad -(1-p)\frac{\sigma^2}{8} \int_0^\infty \int_{-\gamma(s)}^{\beta(s)} \partial_x \left(\partial_x F\, G |x+\gamma(s)|^{\delta-1} e^{-\frac{bx^2}{2}} \right) dx ds \\
&\quad -p\frac{\sigma^2}{8} \int_0^\infty \int_{\beta(s)}^{\infty} \partial_x \left(\partial_x F\, G |x+\gamma(s)|^{\delta-1} e^{-\frac{bx^2}{2}} \right) dx ds \\
&= \int_E \left(\frac{\sigma^2}{8} \left(\partial_{xx} F + \left(\frac{\delta-1}{(H+\gamma\circ J)} - bH \right) \partial_x F \right) + \partial_t F \right) G \rho dx ds \\
&+\frac{\sigma^2}{8} \int_E G \partial_x F \left\{ (1-p)\, \mathbb{I}_{\{\beta(s) > -\gamma(s)\}} + p \mathbb{I}_{\{\beta(s) = -\gamma(s)\}} \right\} |x+\gamma(s)|^{\delta-1} e^{-\frac{bx^2}{2}} \delta_{-\gamma(s)}(dx) ds \\
&\quad +\frac{\sigma^2}{8} \int_E G \partial_x F \,(2p-1) \mathbb{I}_{\{\beta(s) > -\gamma(s)\}} |x+\gamma(s)|^{\delta-1} e^{-\frac{bx^2}{2}} \delta_{\beta(s)}(dx) ds. \qquad (16)
\end{aligned}$$

Obviously, (16) extends to $G \in \mathcal{V}_b$. By Proposition 2.3, the measure

$$\mathbb{I}_{\overline{E}_M}(s,x) \frac{\sigma^2}{8} |x+\gamma(s)|^{\delta-1} e^{-\frac{bx^2}{2}} \delta_{\kappa(s)}(dx) ds, \;\; M \geq 1, \;\; \delta \geq 1,$$

$\kappa = -\gamma, \beta$, is smooth w.r.t. $(\mathcal{A}, \mathcal{V})$. Let ℓ_t^κ denote the unique positive continuous additive functional (PCAF) of $\overline{Y}$ associated to $\frac{\sigma^2}{8}|x+\gamma(s)|^{\delta-1} e^{-\frac{bx^2}{2}} \delta_{\kappa(s)}(dx) ds$ (see Theorem 2.2 in [9]). Then $\int_0^t G(\overline{Y}_s) d\ell_s^\kappa$ is associated to $G(s,x)\frac{\sigma^2}{8}|x+\gamma(s)|^{\delta-1} e^{-\frac{bx^2}{2}} \delta_{\kappa(s)}(dx) ds$ for any $G \in \mathcal{B}_b(E)$. In particular, $\ell_t^{-\gamma}$ vanishes, if $\delta \neq 1$. We obtain

$$\begin{aligned}
N_t^{[F]} &= \int_0^t \left(\frac{\sigma^2}{8} \left(\partial_{xx} F + \left(\frac{\delta-1}{H+\gamma\circ J} - bH \right) \partial_x F \right) + \partial_t F \right) (\overline{Y}_s) ds \\
&\quad +(2p-1) \int_0^t \partial_x F \, \mathbb{I}_{\{\beta\circ J > -\gamma\circ J\}} (\overline{Y}_s) d\ell_s^\beta \\
&\quad +\mathbb{I}_{\{\delta=1\}} \int_0^t \partial_x F \left\{ (1-p) \mathbb{I}_{\{\beta\circ J > -\gamma\circ J\}} + p \mathbb{I}_{\{\beta\circ J = -\gamma\circ J\}} \right\} (\overline{Y}_s) d\ell_s^{-\gamma}. \qquad (17)
\end{aligned}$$

Indeed, if we denote the r.h.s. of (17) by A_t then in particular by (16) $-\mathcal{E}(F, \widehat{G}_1 W) = \lim_{\alpha\to\infty} \alpha^2 E_{\widehat{G}_1 W \rho dy} \left[\int_0^\infty e^{-\alpha t} A_t dt \right]$ for all $W \in \mathcal{H}_b$. Hence $N_t^{[F]} = A_t$ by Theorem 2.3 in [9]. On the other hand, by Lemma 2.1(i) in [9] the Revuz measure $\mu_{\langle M^{[F]} \rangle}$ is equal to $\frac{\sigma^2}{4}(\partial_x F)^2 \rho dy$. A simple calculation shows that the Revuz measure of $\frac{\sigma^2}{4} \int_0^t (\partial_x F)^2 (\overline{Y}_s) ds$ is also equal to $\frac{\sigma^2}{4}(\partial_x F)^2 \rho dy$. Consequently, we have $\langle M^{[F]} \rangle_t = \frac{\sigma^2}{4} \int_0^t (\partial_x F)^2 (\overline{Y}_s) ds$ (see [6, Remark 5.2(ii)]) and therefore we may assume that

$$M_t^{[F]} = \frac{\sigma}{2} \int_0^t \partial_x F(\overline{Y}_s) dW_s \qquad (18)$$

with $((W_t)_{t\geq 0}, P_y, (\mathcal{F}_t)_{t\geq 0})$ being a Brownian motion starting at zero for strictly $\mathcal{E}$-q.e. $y \in E$.
By (16), (17), applied to J_M, letting $M \to \infty$, we obtain

$$J(\overline{Y}_t) = J(\overline{Y}_0) + t.$$

We put

$$X_t := H(\overline{Y}_t), \; t \geq 0,$$

so that

$$\overline{Y}_t = (J(\overline{Y}_0) + t, X_t).$$

Applying again (16), (17), but this time to H_M, letting $M \to \infty$, we obtain

$$\begin{aligned} X_t &= X_0 + \frac{\sigma}{2} W_t + \frac{\sigma^2}{8} \int_0^t \frac{\delta - 1}{X_s + \gamma(J(\overline{Y}_s))} - bX_s ds + (2p-1) \int_0^t \mathbb{I}_{\{\beta\circ J > -\gamma\circ J\}}(\overline{Y}_s) d\ell_s^{\beta} \\ &\quad + \mathbb{I}_{\{\delta=1\}} \int_0^t \left\{(1-p)\mathbb{I}_{\{\beta\circ J > -\gamma\circ J\}} + p\mathbb{I}_{\{\beta\circ J = -\gamma\circ J\}}\right\}(\overline{Y}_s) d\ell_s^{-\gamma}. \end{aligned} \tag{19}$$

(19) holds P_y-a.s for $\mathcal{E}$-q.e. $y \in E$.

Remark 2.4 *(i) If $\delta > 1$, then $(x + \gamma(s))^{-1}\rho(s,x)dxds$ is a smooth measure, since $(x+\gamma(s))^{-1} \in L^1_{loc}(E, \rho dy)$, and therefore X is a semimartingale. If $\delta = 1$, $\int_0^t \frac{\delta-1}{X_s+\gamma(J(\overline{Y}_s))} ds$ disappears, and X is again a semimartingale.*
If we had admitted $\delta < 1$, then clearly $(x+\gamma(s))^{-1} \notin L^1_{loc}(E, \rho dy)$, hence X would not be a semimartingale, and we would have to work with principal values in (19). Further Proposition 2.3 wouldn't apply. Nonetheless, it is clearly possible to consider the case $\delta < 1$, but we didn't do it because we want to work in the framework of semimartingale local times where the representations are clearer and less involved.
Finally, since all subsequent transformations applied to X ($\delta \geq 1$!) are keeping the class of semimartingales invariant, the processes Y, Z, constructed below, and renamed as $R, \sqrt{R}$, in the introduction, will remain semimartingales.
(ii) If κ is regular enough, e.g. $\kappa \in H^{1,2}_{loc}(\mathbb{R}^+)$, then ℓ_t^{κ} (restricted to its support) is a constant multiple of a classical semimartingale local time (see e.g. [4, chapter VI]). We will determine these constants for $\kappa = \beta$, $\kappa = -\gamma$, but we remark that everything would have worked exactly in the same way, if we only assumed β, $-\gamma$, to be measurable and decreasing. Except for the later Girsanov transformation, for which only $\gamma \in H^{1,2}_{loc}(\mathbb{R}^+)$ is needed (see also Remark 2.8(ii)).

In order to determine the constants mentioned in Remark 2.4(ii), we will compare (19) with Tanaka's formula (20) below. We consider the point-symmetric derivative

$$sgn(x) := \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0, \end{cases}$$

the left continuous derivative

$$\overline{sgn}(x) := \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x \le 0, \end{cases}$$

and the right continuous derivative

$$\underline{sgn}(x) := \begin{cases} 1 & \text{if } x \ge 0 \\ -1 & \text{if } x < 0, \end{cases}$$

of $|x|$. Let κ be continuous, and locally of bounded variation. Since $X - \kappa$ is a continuous semimartingale, we may apply Tanaka's formula

$$|X_t - \kappa(t)| = |X_0 - \kappa(0)| + \int_0^t f(X_s - \kappa(s))dX_s + \ell_t^{0f}(X - \kappa), \tag{20}$$

(cf. e.g. [4, VI.(1.2) Theorem, (1.25) Exercise]), where

$$\ell_t^{0f}(X - \kappa) = \begin{cases} \ell_t^0(X - \kappa) & \text{if } f = sgn \\ \ell_t^{0+}(X - \kappa) & \text{if } f = \overline{sgn} \\ \ell_t^{0-}(X - \kappa) & \text{if } f = \underline{sgn}, \end{cases}$$

and $\ell_t^0(X - \kappa)$ (resp. $\ell_t^{0+}(X - \kappa)$, $\ell_t^{0-}(X - \kappa)$), is called the *symmetric local time* (resp. *upper local time*, *lower local time*) in zero of the continuous semimartingale $X - \kappa$. In the book [4] they authors decided to work with the left continuous derivative of $x \mapsto |x|$, and they use the expression L_t^0 for our ℓ_t^{0+}. For our framework it is more intuitive to work with the point symmetric derivative (see Remark 2.7 below).
By [4, VI.(1.9) Corollary, (1.25) Exercise] we have P_y-a.s. for $\mathcal{E}$-q.e. $y \in E$

$$\ell_t^{0+}(X - \kappa) = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_0^t \mathbb{I}_{[0,\varepsilon)}(X_s - \kappa(s))d\langle X, X\rangle_s,$$

$$\ell_t^{0-}(X - \kappa) = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_0^t \mathbb{I}_{(\varepsilon,0]}(X_s - \kappa(s))d\langle X, X\rangle_s,$$

and

$$\ell_t^0(X - \kappa) = \frac{\ell_t^{0+}(X - \kappa) + \ell_t^{0-}(X - \kappa)}{2}.$$

Lemma 3.4 below implies that $\{s \ge 0 | X_s - \kappa(s) = 0\}$ is P_y-a.s. of Lebesgue measure zero for $\mathcal{E}$-q.e. $y \in E$.

Let $(F_M)_{M \ge 1} \subset C_0^2(E)$, such that

$$F_M(s, x) := \begin{cases} 1 & \text{for } (s, x) \in \overline{E}_M \\ 0 & \text{for } (s, x) \in \overline{E}_{M+1}^c, \end{cases}$$

$M \geq 1$, and let

$$\overline{\kappa}(s,x) := |x-\kappa(s)|, \quad \kappa \in H^{1,2}_{loc}(\mathbb{R}^+).$$

It is easy to see that $\overline{\kappa} F_M \in \mathcal{F}$ for any $M \geq 1$. We will use the same localization procedure as before. Thus, if $M^{[\overline{\kappa}]}_t := \lim_{M\to\infty} M^{[\overline{\kappa} F_M]}_t$, and $N^{[\overline{\kappa}]}_t := \lim_{M\to\infty} N^{[\overline{\kappa} F_M]}_t$, then $A^{[\overline{\kappa}]}_t = M^{[\overline{\kappa}]}_t + N^{[\overline{\kappa}]}_t$. If $G \in C^1_0(E)$, then

$$\begin{aligned}
-\mathcal{E}(\overline{\beta} F_M, G) &= \frac{\sigma^2}{8}\int_E \overline{\beta}\partial_{xx} F_M G\rho dx ds + \frac{\sigma^2}{8}\int_E 2sgn(\overline{\beta})\partial_x F_M G \rho dx ds \\
&+\frac{\sigma^2}{8}\int_E \left(\frac{\delta-1}{\overline{\gamma}} - bH\right)\left(\overline{\beta}\partial_x F_M + sgn(\overline{\beta})F_M\right) G\rho dx ds \\
&-(1-p)\int_E (\overline{\beta}\partial_x F_M - F_M) G \mathbb{I}_{\{\beta(s)>-\gamma(s)\}} \frac{\sigma^2}{8}|x+\gamma(s)|^{\delta-1} e^{-\frac{bx^2}{2}}\delta_{\beta(s)}(dx)ds \\
&+p\int_E (\overline{\beta}\partial_x F_M + F_M) G \frac{\sigma^2}{8}|x+\gamma(s)|^{\delta-1} e^{-\frac{bx^2}{2}}\delta_{\beta(s)}(dx)ds \\
&-(1-p)\int_E (\overline{\beta}\partial_x F_M - F_M) sgn(\overline{\beta}) G \mathbb{I}_{\{\beta(s)>-\gamma(s)\}} \frac{\sigma^2}{8}|x+\gamma(s)|^{\delta-1} e^{-\frac{bx^2}{2}}\delta_{-\gamma(s)}(dx)ds \\
&+\int_E \left(|x-\beta(s)|\partial_t F_M - F_M sgn\left(x-\beta(s)\right)\beta'(s)\right) G\rho dy.
\end{aligned}$$

Obviously, the last equation extends to $G \in \mathcal{V}_b$. Thus, letting $M \to \infty$,

$$\begin{aligned}
N^{[\overline{\beta}]}_t &= \frac{\sigma^2}{8}\int_0^t \left(\frac{\delta-1}{H+\gamma\circ J} - bH\right) sgn(\overline{\beta})(\overline{Y}_s)ds - \int_0^t sgn(\overline{\beta})(\overline{Y}_s)d\beta(J(\overline{Y}_s)) \\
&+(1-p)\int_0^t \mathbb{I}_{\{\beta\circ J>-\gamma\circ J\}}(\overline{Y}_s)d\ell^\beta_s + p\ell^\beta_t \\
&+(1-p)\int_0^t sgn(\overline{\beta})(\overline{Y}_s)\mathbb{I}_{\{\beta\circ J>-\gamma\circ J\}}(\overline{Y}_s)d\ell^{-\gamma}_s. \qquad (21)
\end{aligned}$$

On the other hand by Lemma 2.1(i) in [9]

$$\begin{aligned}
\mu_{\langle M^{[\overline{\beta} F_M]}\rangle} &= \frac{\sigma^2}{4}\partial_x\left(|x-\beta(s)|F_M\right)^2\rho dy \\
&= \frac{\sigma^2}{4}\left(\left(|x-\beta(s)|\partial_x F_M\right)^2 + 2|x-\beta(s)|\partial_x F_M sgn\left(x-\beta(s)\right)F_M\right)\rho dy \\
&+\frac{\sigma^2}{4}\left(F_M sgn\left(x-\beta(s)\right)\right)^2\rho dy.
\end{aligned}$$

We obtain $\langle M^{[\overline{\beta}]}\rangle_t = \frac{\sigma^2}{4}\int_0^t sgn(\overline{\beta})(\overline{Y}_s)^2 ds$. Consequently, we may assume that

$$M^{[\overline{\beta}]}_t = \frac{\sigma}{2}\int_0^t sgn(\overline{\beta})(\overline{Y}_s)dW_s.$$

Note that $\int_0^t sgn(\overline{\beta})(\overline{Y}_s)d\ell_s^\beta = 0$, because for its associated signed smooth measure we have $sgn(x-\beta(s))\delta_{\beta(s)}(dx)ds = 0$, since $sgn(0) = 0$. Therefore

$$\begin{aligned}|X_t - \beta(J(\overline{Y}_t))| &= |X_0 - \beta(J(\overline{Y}_0))| + \int_0^t sgn(X_s - \beta(J(\overline{Y}_s)))d(X_s - \beta(J(\overline{Y}_s))) \\ &\quad + p\ell_t^\beta + (1-p)\int_0^t \mathbb{I}_{\{\beta\circ J > -\gamma\circ J\}}(\overline{Y}_s)d\ell_s^\beta. \end{aligned} \tag{22}$$

For $\kappa : \mathbb{R}^+ \to \mathbb{R}$, and $u \in \mathbb{R}^+$, define

$$\kappa_u(t) := \kappa(u+t).$$

Recall that

$$P_y(J(\overline{Y}_t) = J(y) + t) = 1,$$

so that

$$\kappa_{J(y)}(t) = \kappa(J(\overline{Y}_t)) \quad P_y\text{-a.s.}$$

Comparing (22) with Tanaka's formula (20) we see that

$$\ell_t^0(X - \beta_{J(y)}) = p\ell_t^\beta + (1-p)\int_0^t \mathbb{I}_{\{\beta_{J(y)}(s) > -\gamma_{J(y)}(s)\}}d\ell_s^\beta \qquad P_y\text{-a.s} \tag{23}$$

for $\mathcal{E}$-q.e. $y \in E$. In a similar way one can see that we have

$$\begin{aligned}\ell_t^0(X + \gamma_{J(y)}) &= \frac{1}{2}\ell_t^{0+}(X + \gamma_{J(y)}) \\ &= \int_0^t p\mathbb{I}_{\{\beta_{J(y)}(s) = -\gamma_{J(y)}(s)\}} + (1-p)\mathbb{I}_{\{\beta_{J(y)}(s) > -\gamma_{J(y)}(s)\}}d\ell_s^{-\gamma},\end{aligned} \tag{24}$$

P_y-a.s. for $\mathcal{E}$-q.e. $y \in E$. Therefore, (19) rewrites P_y-a.s. as

$$\begin{aligned}X_t &= X_0 + \frac{\sigma}{2}W_t + \frac{\sigma^2}{8}\int_0^t \frac{\delta - 1}{X_s + \gamma_{J(y)}(s)} - bX_s ds \\ &\quad + (2p-1)\int_0^t \mathbb{I}_{\{\beta_{J(y)}(s) > -\gamma_{J(y)}(s)\}}d\ell_s^0(X - \beta_{J(y)}) + \frac{\mathbb{I}_{\{\delta=1\}}}{2}\ell_t^{0+}(X + \gamma_{J(y)}).\end{aligned} \tag{25}$$

Let $b \in \mathbb{R}^+$. We define

$$B_t := W_t + \frac{1}{4\sigma}\int_0^t 8\gamma'_{J(\overline{Y}_0)}(s) + \sigma^2 b\gamma_{J(\overline{Y}_0)}(s)ds,$$

and

$$dQ_y = e^{-\frac{1}{4\sigma}\int_0^t 8\gamma'_{J(y)}(s) + \sigma^2 b\gamma_{J(y)}(s)dW_s - \frac{1}{32\sigma^2}\int_0^t |8\gamma'_{J(y)}(s) + \sigma^2 b\gamma_{J(y)}(s)|^2 ds}dP_y \text{ on } \mathcal{F}_t.$$

Obviously, Novikov's condition is satisfied since $\gamma \in H^{1,2}_{loc}(\mathbb{R}^+)$, hence B_t is a Brownian motion under the equivalent measure Q_y. Put

$$Y_t := X_t + \gamma_{J(Y_0)}(t).$$

Then

$$\ell^0_t(X - \beta_{J(y)}) = \ell^0_t(X + \gamma_{J(y)} - (\beta_{J(y)} + \gamma_{J(y)})) = \ell^0_t(Y - \lambda_{J(y)}) \quad Q_y\text{-a.s},$$

since Q_y is equivalent to P_y. Analogously, $\ell^{0+}_t(X + \gamma_{J(y)}) = \ell^{0+}_t(Y)$ holds Q_y-a.s. Thus, under Q_y, $Y_t \geq 0$ (since $X_t \geq -\gamma(t)$), and

$$\begin{aligned} Y_t &= Y_0 + \frac{\sigma}{2}B_t + \frac{\sigma^2}{8}\int_0^t \frac{\delta - 1}{Y_s} - bY_s ds \\ &+ (2p-1)\int_0^t \mathbb{I}_{\{\lambda_{J(y)}(s)>0\}} d\ell^0_s(Y - \lambda_{J(y)}) \ + \ \frac{\mathbb{I}_{\{\delta=1\}}}{2}\ell^{0+}_t(Y), \end{aligned} \tag{26}$$

where $Q_y(Y_0 = H(y) + \gamma(J(y))) = 1$. Moreover, under Q_y, $Z_t := Y_t^2$, satisfies

$$\begin{aligned} Z_t &= Z_0 + \sigma\int_0^t \sqrt{Z_s}dB_s + \frac{\sigma^2}{4}\int_0^t (\delta - bZ_s)ds \\ &\quad +(2p-1)\int_0^t \mathbb{I}_{\{\lambda_{J(y)}(s)>0\}} 2\sqrt{Z_s}\, d\ell^0_s(\sqrt{Z} - \lambda_{J(y)}), \end{aligned} \tag{27}$$

and $Z_0 = (H(y) + \gamma(J(y)))^2$ Q_y-a.s.

Remark 2.5 *Setting $b = 0$, and replacing $\sigma^2 b\gamma_{J(y)}(s)$ by $\sigma^2 c$, $c \in \mathbb{R}^+$, in the expression for Q_y, we construct instead of (27) a positive solution $Z = Y^2$ to*

$$\begin{aligned} Z_t &= Z_0 + \sigma\int_0^t \sqrt{Z_s}dB_s + \frac{\sigma^2}{4}\int_0^t (\delta - c\sqrt{Z_s})ds \\ &\quad +(2p-1)\int_0^t \mathbb{I}_{\{\lambda_{J(y)}(s)>0\}} 2\sqrt{Z_s}\, d\ell^0_s(\sqrt{Z} - \lambda_{J(y)}). \end{aligned} \tag{28}$$

For $p = \frac{1}{2}$, (28) is well-known as the double square-root (DSR) model of Longstaff in financial mathematics. For $p = \frac{1}{2}$, (27) is the well-known Cox-Ingersoll-Ross model (CIR).

Remark 2.6 *The equations (25), (27), (28), are in the sense of equivalence of additive functionals. This means that they hold for initial conditions outside some exceptional set. If λ^2 is constant, say $\lambda^2 = c > 0$, we are in the symmetric case, and it is clear that the parabolic capacity is comparable with the elliptic one, so that we can start from every $X_0, Y_0, Z_0 = x \geq 0$, if $\delta < 2$, and for every $X_0, Y_0, Z_0 = x > 0$, if $\delta \geq 2$. Indeed Y is e.g. associated to the Dirichlet form, which is uniquely determined as the closure of*

$$\mathcal{E}^p(f,g) := \int_0^\infty \frac{\sigma^2}{2} x f'(x) g'(x) x^{\frac{\delta}{2}-1} e^{-\frac{bx}{2}} \rho(x)dx; \quad f,g \in C_0^\infty([0,\infty))$$

in $L^2([0,\infty), x^{\frac{\delta}{2}}e^{-\frac{bx}{2}}\rho(x)dx)$*, where* $\rho(x) = (1-p)\mathbb{I}_{\{x<c\}} + p\mathbb{I}_{\{x\geq c\}}$*, and it is clear that the capacities are all equivalent for* $p \in (0,1)$*.* $p = \frac{1}{2}$ *corresponds to the classical case.*
The regularity of the equations (25), (27), (28), i.e. the question whether we can start pointwise in the parabolic case, i.e. if $\lambda^2 \neq const$ *may be subject of forthcoming work. Note however, that the structure of (25), (27), (28), is not influenced by these questions of regularity.*

Remark 2.7 *(i) Rewriting (22) with* $\overline{sgn}, \underline{sgn}$*, one can easily see that* $\ell_t^{0+}(X-\beta) = p\ell_t^{\beta}$*, and* $\ell_t^{0-}(X-\beta) = (1-p)\int_0^t \mathbb{I}_{\{\beta(s)>-\gamma(s)\}}d\ell_s^{\beta}$*. This implies the following relations for* $\ell_t^0(R-\lambda^2)$ *in (1):*

$$\ell_t^0(R-\lambda^2) = \frac{1}{2p}\ell_t^{0+}(R-\lambda^2) = \frac{1}{2(1-p)}\ell_t^{0-}(R-\lambda^2).$$

One observes immediately the discontinuity of the local times in the space variable, thus we provide another example of diffusion with discontinuous local time (see e.g. [11]). Moreover, if $|p| > 1$*, then any of these local times is identically zero. Consequently, the associated process is the CIR process, for which uniqueness in any sense is known to hold. Regarding the time dependent Dirichlet form* $\mathcal{E}$ *(see [9]) correponding to the time dependent CIR process* (t, R_t)*, then*

$$\mathcal{E}(F,G) := \int_0^\infty\int_0^\infty \frac{\sigma^2}{2}|x|\partial_x F(t,x)\partial_x G(t,x)|x|^{\frac{\delta}{2}-1}e^{-\frac{bx}{2}}dxdt$$
$$-\int_0^\infty\int_0^\infty \partial_t F(t,x)G(t,x)|x|^{\frac{\delta}{2}-1}e^{-\frac{bx}{2}}dxdt;$$

we can see that the local time $\ell^0(R-\lambda^2)$ *is uniquely associated to the measure*

$$\frac{\sigma^2}{2}|x|^{\frac{\delta}{2}}e^{-\frac{bx}{2}}\delta_{\lambda^2(t)}(dx)dt$$

which doesn't vanish if λ^2 *is different from zero on a set of positive Lebesgue measure. Therefore* $\ell^0(R-\lambda^2)$ *cannot vanish identically.*
(ii) One reason to work with the point symmetric derivative, is that it just simply better works out the intuitive structure of the skew reflection, namely $2p-1 = p-(1-p)$*, so upper reflection with probability* p*, and lower reflection with probability* $1-p$*. This is here so far of course only intuitive, but at least in the classical case (*$\delta = 1, \sigma = 2, b = 0, \lambda^2 \equiv 0$*) it is rigorously described for the squareroot process (the skew BM) through excursion theory (see e.g. [1], [11]). The other reason is that symmetric local times correspond to symmetric derivatives, which are used in distribution theory, and therefore correspond to our analytic construction of the Markov process generator.*

Remark 2.8 *(i) The relation (4) can easily be derived by writing down Fukushima's extended decomposition (14) in localized form for* $|Z_t - \lambda^2(t)| = |(X+\gamma(t))^2 - \lambda^2(t)|$ *(Z* *as in (27)) and then comparing it with the symmetric Tanaka formula. More precisely, we obtain*

$$2\sqrt{Z_s}\,d\ell_s^0(\sqrt{Z}-\lambda_{J(y)}) = d\ell_s^0(Z-\lambda_{J(y)}^2)$$

by doing so.
(ii) Coming back to Remark 2.4(ii), suppose that we had assumed $\lambda = \beta + \gamma$, where only $\gamma \in H_{loc}^{1,2}(\mathbb{R}^+)$, and β not necessarily continuous, but decreasing. Then, we would have obtained (26), (27), except that $\ell^0(\sqrt{Z}-\lambda_{J(y)})$ has to be replaced by ℓ^λ, where ℓ^λ is a positive continuous additive functional of Y, which only grows when $Y = \lambda$, or equivalently $Z = \lambda^2$.

Remark 2.9 (The case $\delta \in (0,1)$) *Since we are no longer in the semimartingale case we can no longer make use of the Girsanov formula with γ. Therefore one puts $\gamma \equiv 0$. Then one may still start as before with (6) and the following*

$$\rho(t,x) = \left((1-p)\mathbb{I}_{[0,\lambda(t))}(x) + p\mathbb{I}_{[\lambda(t),\infty)}(x)\right)|x|^{\delta-1}e^{-\frac{bx^2}{2}}.$$

Note that $\rho(\cdot,x)$ is still assumed to be increasing in t, and that Proposition 2.3 could no longer be available.
One may preferably directly start with the squared process since the technique of changing the measure will not be used. Thus one could proceed with the following (cf. (8)) time dependent form

$$\begin{aligned}\mathcal{E}(F,G) &= \int_0^\infty\int_0^\infty \frac{\sigma^2}{2}|x|\partial_x F(s,x)\,\partial_x G(s,x)\rho(s,x)dxds \\ &\quad - \int_0^\infty\int_0^\infty \partial_t F(s,x)G(s,x)\rho(s,x)dxds,\end{aligned}$$

with

$$\rho(t,x) = \left((1-p)\mathbb{I}_{[0,\lambda^2(t))}(x) + p\mathbb{I}_{[\lambda^2(t),\infty)}(x)\right)|x|^{\frac{\delta}{2}-1}e^{-\frac{bx}{2}},$$

and where $p \in (0,1)$, and $\lambda^2 \in H_{loc}^{1,1}(\mathbb{R}^+)$, are chosen, such that ρ is increasing in t. In order to convince the reader, we just line out the argument for the existence of the corresponding local time on λ^2. In fact one only has to show that

$$\frac{\sigma^2}{2}|x|^{\frac{\delta}{2}}e^{-\frac{bx}{2}}\delta_{\lambda^2(t)}(dx)dt$$

is smooth with respect to the symmetric part of $\mathcal{E}$. This can be done analogously to Proposition 2.3. Of course, if λ^2 is a constant, we consider the symmetric Dirichlet form of Remark 2.6.

3 Some conclusions and the martingale problem

Let $\sigma > 0$, $\delta, b \geq 0$, and $\lambda^2 : \mathbb{R}^+ \to \mathbb{R}^+$ be continuous and locally of bounded variation. Throughout this section suppose that we are given a weak solution

$$R_t = R_0 + \int_0^t \sigma\sqrt{|R_s|}dW_s + \int_0^t \frac{\sigma^2}{4}(\delta - bR_s)ds + (2p-1)d\ell_t^0(R-\lambda^2), \qquad (29)$$

w.r.t. to some filtered probability space $(\Omega, (\mathcal{F}_t)_{t\geq 0}, P)$, where $p \in (0,1)$.

Lemma 3.1 *We have*

$$\int_0^t \mathbb{I}_{\{\lambda^2(s)=0\}} d\ell_s^0(R-\lambda^2) = 0 \quad P\text{-a.s. for any } t \geq 0. \tag{30}$$

In particular

$$supp(d\ell_s^0(R-\lambda^2)) \subset \overline{\{\lambda^2 > 0\}}$$

Proof By an extension of the occupation time formula (cf. [4, VI. (1.15) Exercise]) we have

$$\int_{\mathbb{R}} \frac{\mathbb{I}_{\{a\neq 0\}}}{|a|} \int_0^t \mathbb{I}_{\{\lambda^2(s)=0\}} d\ell_s^a(R-\lambda^2) da \;=\; \int_0^t \frac{\mathbb{I}_{\{R_s-\lambda^2(s)\neq 0\}}}{|R_s-\lambda^2(s)|} \mathbb{I}_{\{\lambda^2(s)=0\}} \sigma^2 |R_s| ds \leq \sigma^2 t.$$

Since $\frac{1}{|a|}$ is not integrable in any neighborhood of zero the statement holds for $\ell^{0+}(R-\lambda^2)$, $\ell^{0-}(R-\lambda^2)$, thus also for $\ell^0(R-\lambda^2)$.

□

Lemma 3.2 (Justification to discard $|\cdot|$ in (29)) *If $R_0 \geq 0$ P-a.s, then a solution to (1) is always positive.*

Proof Recall that as a direct consequence of the occupation time formula $\ell_t^{0+}(R) \equiv 0$. Then, using (30), applying Tanaka's formula (cf. e.g. [4, VI. (1.2) Theorem]), and taking expectations (note that we may assume that all integrals exist, since otherwise we can regard everything up to the exit time of balls of radius n and then let $n \to \infty$)

$$\begin{aligned} E[R_t^-] &= E[R_0^-] - E[\int_0^t \mathbb{I}_{\{R_s\leq 0\}} \frac{\sigma^2}{4}(\delta - bR_s)ds] \\ &\quad -(2p-1)E[\int_0^t \mathbb{I}_{\{R_s\leq 0\}} \mathbb{I}_{\{\lambda^2(s)>0\}} d\ell_s^0(R-\lambda^2)] + \frac{1}{2}E[\ell_t^{0+}(R)] \\ &\leq -E[\int_0^t \mathbb{I}_{\{R_s\leq 0\}} \frac{\sigma^2}{4}(\delta - bR_s)ds] \leq 0, \end{aligned}$$

It follows that R_t is P-a.s. equal to its positive part R_t^+. This concludes the proof.

□

Lemma 3.3 *The time of R spent at zero has Lebesgue measure zero, i.e. $\int_0^t \mathbb{I}_{\{R_s=0\}} ds = 0$ P-a.s.*

Proof Due to the presence of the squareroot in the diffusion part, we have $\ell_t^{0+}(R), \ell_t^{0-}(R) \equiv 0$. Using [4, VI. (1.7) Theorem], and (30), it follows P-a.s.

$$\begin{aligned} 0 \;=\; \ell_t^{0+}(R) - \ell_t^{0-}(R) &= \int_0^t \mathbb{I}_{\{R_s=0\}} \left\{ \frac{\sigma^2}{4}(\delta - bR_s)ds + (2p-1)d\ell_s(R-\lambda^2)) \right\} \\ &= \frac{\sigma^2\delta}{4} \int_0^t \mathbb{I}_{\{R_s=0\}} ds \end{aligned}$$

□

Lemma 3.4 *The time of R spent on λ^2 has Lebesgue measure zero, i.e.* $\int_0^t \mathbb{I}_{\{R_s=\lambda^2(s)\}}ds = 0$ *P-a.s.*

Proof By [4, VI. (1.16) Exercise 2°)], it follows P-a.s.

$$\int_0^t \mathbb{I}_{\{R_s=\lambda^2(s)\}}\sigma\sqrt{|R_s|}dW_s = 0.$$

This together with Lemma 3.3 gives

$$\int_0^t \mathbb{I}_{\{R_s=\lambda^2(s)\}}\sigma^2|R_s|\mathbb{I}_{\{R_s\neq 0\}}ds = 0.$$

But P-a.s. $\sigma^2|R_s|\mathbb{I}_{\{R_s\neq 0\}} > 0$ ds-a.e. and the assertion follows.

□

Let

$$\Gamma(\lambda^2) := \{(s,x) \in \mathbb{R}^+ \times \mathbb{R}^+ | x = \lambda^2(s)\}.$$

Consider the following linear operator

$$\mathcal{L}F(t,x) = \frac{\sigma^2}{2}|x|\partial_{xx}F(t,x) + \frac{\sigma^2}{4}(\delta - bx)\partial_x F(t,x) + \partial_t F(t,x)$$

acting pointwise on $C^{1,2}(\mathbb{R}^+ \times \mathbb{R})$.

Proposition 3.5 *Define*

$$\begin{aligned} D(\mathcal{L}) \;:=\; & C_0(\mathbb{R}^+ \times \mathbb{R}) \cap \{F \in C^{1,2}(\mathbb{R}^+ \times \mathbb{R} \setminus \Gamma(\lambda^2)) | (1-p)\partial_x F(t,\lambda^2(t)+) = p\partial_x F(t,\lambda^2(t)-); \\ & \partial_t F(t,\lambda^2(t)\pm), \partial_{xx}F(t,\lambda^2(t)\pm) \text{ is bounded}\}. \end{aligned}$$

Let $F \in D(\mathcal{L})$. Then

$$\left(F(t,R_t) - F(0,R_0) - \int_0^t \mathcal{L}F(s,R_s)ds\right)_{t\geq 0}$$

is a P-martingale.

Proof First observe that $\int_0^t \mathcal{L}F(s,R_s)ds$, $F \in D(\mathcal{L})$, is well-defined by Lemma 3.4. By [3, Theorem 2.1], we have the following Itô-formula for F:

$$\begin{aligned} F(t,R_t) &= F(t,R_0) + \int_0^t \frac{1}{2}(\partial_t F(s,R_s+) + \partial_t F(s,R_s-))ds \\ &\quad + \int_0^t \frac{1}{2}(\partial_x F(s,R_s+) + \partial_x F(s,R_s-))dR_s \\ &\quad + \frac{1}{2}\int_0^t \partial_{xx}F(s,R_s)\mathbb{I}_{\{R_s\neq\lambda^2(s)\}}d[R]_s \\ &\quad + \frac{1}{2}\int_0^t (\partial_x F(s,R_s+) - \partial_x F(s,R_s-))d\ell_s^0(R-\lambda^2). \end{aligned}$$

Since $\int_0^t \mathbb{I}_{\{R_s=\lambda^2(s)\}} ds = 0$ P-a.s. by Lemma 3.4 ,we obtain P-a.s.

$$\begin{aligned} F(t,R_t) &= F(t,R_0) + \int_0^t \partial_t F(s,R_s)ds \\ &+ \int_0^t \partial_x F(s,R_s)\sigma\sqrt{|R_s|}dW_s \\ &+ \int_0^t \frac{\sigma^2}{4}(\delta - bR_s)\partial_x F(s,R_s)ds \\ &+ \int_0^t \frac{1}{2}(\partial_x F(s,\lambda^2(s)+) + \partial_x F(s,\lambda^2(s)-))(2p-1)d\ell_s^0(R-\lambda^2) \\ &+ \int_0^t \frac{\sigma^2}{2}|R_s|\partial_{xx}F(s,R_s)ds \\ &+ \frac{1}{2}\int_0^t (\partial_x F(s,R_s+) - \partial_x F(s,R_s-))d\ell_s^0(R-\lambda^2). \end{aligned} \tag{31}$$

Since $F \in D(\mathcal{L})$ we have

$$\partial_x F(s,\lambda^2(s)+) - \partial_x F(s,\lambda^2(s)-) = \frac{2p-1}{2(1-p)}\partial_x F(s,\lambda^2(s)-),$$

and

$$\partial_x F(s,\lambda^2(s)+) + \partial_x F(s,\lambda^2(s)-) = \frac{1}{2(1-p)}\partial_x F(s,\lambda^2(s)-),$$

so that the expressions with $\ell_s^0(R-\lambda^2)$ in (31) cancel each other. Therefore

$$F(t,R_t) - F(0,R_0) - \int_0^t \mathcal{L}F(s,R_s)ds = \int_0^t \partial_x F(s,R_s)\sigma\sqrt{|R_s|}dW_s.$$

By our further assumptions on F the left hand side is square integrable and the result follows.

□

Remark 3.6 *Observe, $D(\mathcal{L})$ is an algebra of functions that separates the points of $\mathbb{R}^+ \times \mathbb{R}$, thus well suited as starting point to study uniqueness in law for R.*

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