

# A VANISHING RESULT FOR TELEMAN'S CASSON-TYPE INSTANTON INVARIANT

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ABSTRACT. Recently Andrei Teleman considered instanton moduli spaces over negative definite four-manifolds  $X$  with  $b_2(X) \geq 1$ . If  $b_2(X)$  is divisible by four and  $b_1(X) = 1$  a gauge-theoretic invariant can be defined; it is a count of flat connections modulo the gauge group. Our main result shows that if such a moduli space is non-empty and the manifold admits a connected sum decomposition  $X \cong X_1 \# X_2$  then both  $b_2(X_1)$  and  $b_2(X_2)$  are divisible by four.

## INTRODUCTION

In a recent article Andrei Teleman considers moduli spaces of (projectively) anti-selfdual instantons in Hermitian rank-2 bundles  $E \rightarrow X$  over negative definite four-manifolds with  $b_2(X) \geq 1$ . If we choose  $c_1(E)$  to be (a lift of) the sum of basis elements of  $H^2(X; \mathbb{Z})/Torsion$  that diagonalise the intersection form we are in the favourable situation that the moduli space contains no reducibles provided  $c_2(E)$  is chosen strictly negative. Furthermore, the moduli space is compact independently of the Riemannian metric provided the instantons are of ‘low energy’, i.e. for particular values of  $c_2(E)$ . In particular, if the manifold has second Betti-number  $b_2(X)$  divisible by four and first Betti-number  $b_1(X) = 1$  we can choose  $c_2(E)$  in such a way that the moduli space  $\mathcal{M}(E)$  consists of flat connections only and has expected dimension zero. The holonomy thus yields a correspondance to the space of representations of  $\pi_1(X)$  in  $SO(3)$  with fixed Stiefel-Whitney class  $w_2 = c_1(E) \pmod{2}$ . Therefore the invariant obtained by a count of the moduli space is some kind of ‘Casson - invariant’.

It turns out [T] that non-emptiness of the moduli space  $\mathcal{M}(E)$  requires that no (lift of an) element of a basis of  $H^2(X; \mathbb{Z})/Torsion$  can be Poincaré dual to an element representable by a sphere, i.e. to an element in the image of the Hurewicz homomorphism  $\pi_2(X) \rightarrow H_2(X; \mathbb{Z})$ . Prasad and Yeung [PY] constructed manifolds  $W$  which are rational-cohomology complex projective planes for which  $\pi_2(W) = 0$ , so clearly no element in  $H_2(W)$  can be represented by a sphere. Thus a candidate of a manifold for which the moduli space of flat connections  $\mathcal{M}(E)$  might be non-empty is given by the connected sum  $4\overline{W}$  of 4 copies of  $\overline{W}$ , and a candidate of a manifold for which the ‘Casson - invariant’ can be defined is given by  $4\overline{W} \# S^1 \times S^3$ .

We will first recall the gauge-theoretical situation considered in [T] and indicate how to define an invariant  $n_X(E)$  with value in  $\mathbb{Z}/2$  or  $\mathbb{Z}$ , according to whether one takes orientations into account. Second, we shall explain the holonomy correspondance. We then show that if the moduli space of flat connections  $\mathcal{M}(E)$  is non-empty, then we cannot have a connected sum decomposition  $X \cong X_1 \# X_2$  unless both  $b_2(X_1)$  and  $b_2(X_2)$  are divisible by four. In particular the moduli spaces for the above mentioned candidates are empty. On the other hand there cannot be

a general connected sum vanishing theorem for the moduli spaces of zero-energy instantons, for, if we have a non-empty moduli space of zero-energy instantons over  $X_1$  and as well on  $X_2$ , then there is a non-empty moduli space over  $X_1 \# X_2$ .

A similar invariant has been defined by Ruberman and Saveliev in the case where  $b_2(X) = 0$  [RS], going back to work of Furuta and Ohta [FO]. Our situation is simpler because of the absence of reducibles in the moduli space due to the condition on  $b_2(X)$ .

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#### 1. DONALDSON THEORY ON NEGATIVE DEFINITE FOUR-MANIFOLDS, LOW-ENERGY INSTANTONS

We will briefly recall the gauge theory of anti-selfdual instantons here, and will mainly focus on the case of underlying negative definite four-manifolds. As a general reference for the more technical aspects of the theory we refer to the book of Donaldson and Kronheimer [DK]. As we shall see, for particular values of the characteristic classes involved, the moduli space will not contain any reducible connections, will be compact, and will, after possible perturbations, consist of regular points only. Furthermore, for manifolds with the second Betti number divisible by four, the characteristic classes can be chosen in such a way that the moduli space will consist of flat connections. If in addition the expected dimension is zero, a condition equivalent to  $b_1(X) = 1$ , the count of these instantons (modulo 2 if we do not consider orientation issues), after possible perturbations, defines an invariant that is reasonably called ‘‘Casson-type invariant’’ because it is a count of representations of the fundamental group in  $SO(3)$  modulo conjugation if the unperturbed moduli space is already non-degenerate, see [T], [RS].

**1.1. Connections.** Let  $X$  be a smooth Riemannian four-manifold and  $E \rightarrow X$  a Hermitian rank-2 bundle on  $X$ . Let further  $a$  be a fixed unitary connection in the associated determinant line bundle  $\det(E) \rightarrow X$ . We define  $\mathcal{A}_a(E)$  to be the affine space of unitary connections on  $E$  which induce the fixed connection  $a$  in  $\det(E)$ . This is an affine space over  $\Omega^1(X; \mathfrak{su}(E))$ , the vector space of  $\mathfrak{su}(E)$ -valued one-forms on  $X$ . Let us denote by  $P$  the principal  $U(2)$  bundle of frames in  $E$ , and let  $\overline{P}$  be the bundle that is associated to  $P$  via the projection  $\pi : U(2) \rightarrow PU(2)$ ,  $\overline{P} = P \times_{\pi} PU(2)$ . The space  $\mathcal{A}(\overline{P})$  of connections in the  $PU(2)$  principal bundle  $\overline{P}$  and the space  $\mathcal{A}_a(E)$  are naturally isomorphic. If we interpret a connection  $A \in \mathcal{A}_a(E)$  as a  $PU(2)$  connection via this isomorphism it is commonly called a projective connection. The adjoint representation  $\text{ad} : SU(2) \rightarrow SO(\mathfrak{su}(2))$  descends to a group isomorphism  $PU(2) \rightarrow SO(\mathfrak{su}(2))$ . The associated real rank-3 bundle  $\overline{P} \times_{\text{ad}} \mathfrak{su}(2)$  is just the bundle  $\mathfrak{su}(E)$  of traceless skew-symmetric endomorphisms of  $E$ . Thus the space  $\mathcal{A}_a(E)$  is also isomorphic to the space  $\mathcal{A}(\mathfrak{su}(E))$  of linear connections in  $\mathfrak{su}(E)$  compatible with the metric. We shall write  $A \in \mathcal{A}(\overline{P})$  for

connections in the  $PU(2)$  principal bundle and denote the associated connection in  $\mathfrak{su}(E)$  by the same symbol. Should we mean the unitary connection which induces the connection  $a$  in  $\det(E)$  we will write  $A_a$  instead.

Let  $\mathcal{G}^0$  denote the group of automorphisms of  $E$  of determinant 1. It is called the ‘gauge group’. This group equals the group of sections  $\Gamma(X; P \times_{\text{Ad}} SU(2))$ , where  $\text{Ad} : U(2) \rightarrow \text{Aut}(SU(2))$  is given by conjugation. We shall write  $\mathcal{B}(E)$  for the quotient space  $\mathcal{A}(\overline{P}_E)/\mathcal{G}^0$ . A connection is called *reducible* if its stabiliser under the gauge group action equals the subgroup given by the centre  $\mathbb{Z}/2 = Z(SU(2))$  which always operates trivially, otherwise *irreducible*. Equivalently, a connection  $A_a$  is reducible if and only if there is a  $A_a$  - parallel splitting of  $E$  into two proper subbundles.

Let us point out that the characteristic classes of the bundle  $\mathfrak{su}(E)$  are given by

$$\begin{aligned} w_2(\mathfrak{su}(E)) &= c_1(E) \pmod{2} \\ p_1(\mathfrak{su}(E)) &= -4c_2(E) + c_1(E)^2 . \end{aligned} \tag{1}$$

**1.2. Moduli space of anti-selfdual connections.** For a connection  $A \in \mathcal{A}(\overline{P})$  we consider the anti-selfduality equation

$$F_A^+ = 0 , \tag{2}$$

where  $F_A$  denotes the curvature form of the connection  $A$ , and  $F_A^+$  its self-dual part with respect to the Hodge-star operator defined by the Riemannian metric on  $X$ . The moduli space  $\mathcal{M}(E) \subseteq \mathcal{B}(E)$  of anti-self-dual connections,

$$\mathcal{M}(E) = \{A \in \mathcal{A}(\overline{P}_E) \mid F_A^+ = 0\} / \mathcal{G}^0$$

is the central object of study in instanton gauge theory. This space is in general non-compact and there is a canonical ‘Uhlenbeck-compactification’ of it. The anti-selfduality equations are elliptic, so Fredholm theory provides finite dimensional local models for the moduli space. The often problematic aspect of Donaldson theory is the need to deal with reducible connections and with a non-trivial compactification. We will consider special situations where these problems do not occur.

**1.3. Low-energy instantons over negative definite four-manifolds.** We restrict now our attention to smooth Riemannian four-manifolds  $X$  with  $b_2^+(X) = 0$  and  $b_2(X) \geq 1$ . According to the theorem of Donaldson’s [D] the intersection form of such a four-manifold is equivalent to the diagonal one. Let  $\{e_i\}$  be a basis of  $H^2(X; \mathbb{Z})/\text{Torsion}$  which diagonalises the intersection form. Let  $E \rightarrow X$  now have first Chern-class  $c_1(E)$  which is a lift of  $\sum e_i$  to  $H^2(X; \mathbb{Z})$ . As an abuse of notation we shall write  $c_1(E) = \sum e_i$  if there is no reference to a particular lift.

**Lemma 1.1.** [T] *Suppose the Hermitian rank-2 bundle  $E \rightarrow X$  has first Chern class  $c_1(E) = \sum e_i$  and its second Chern class is strictly negative,  $c_2(E) < 0$ . Then  $E \rightarrow X$  does not admit any topological decomposition  $E = L \oplus K$  into the sum of two complex line bundles.*

*Proof:* Suppose  $E = L \oplus K$  and  $c_1(L) = \sum l_i e_i$ . Then

$$c_2(E) = c_1(L)(c_1(E) - c_1(L)) = \sum (l_i^2 - l_i) \geq 0 .$$

□

**Corollary 1.2.** *Let  $E \rightarrow X$  be as in the previous lemma. Then the moduli space  $\mathcal{M}(E)$  does not admit reducibles.*

For a connection  $A \in \mathcal{A}(\mathfrak{su}(E))$  Chern-Weil theory gives the following formula:

$$\frac{1}{8\pi^2}(\|F_A^-\|_{L^2(X)}^2 - \|F_A^+\|_{L^2(X)}^2) = -\frac{1}{4}p_1(\mathfrak{su}(E)) = c_2(E) - \frac{1}{4}c_1(E)^2 \quad (3)$$

In particular, for anti-selfdual connections the left hand side of this equation is always positive, and we can draw the following observation from the formula:

**Observation.** 1. For  $c_2(E) - 1/4 c_1(E)^2 \in \{0, 1/4, 2/4, 3/4\}$  the moduli space  $\mathcal{M}(E)$  is always compact, independently of the chosen metric or any genericity argument. In fact, the lower strata in the Uhlenbeck-compactification consist of anti-selfdual connections in bundles  $E_k$  with  $c_1(E_k) = c_1(E)$  and  $c_2(E_k) = c_2(E) - k$  for  $k \geq 1$ .

2. For  $c_1(E) = \sum e_i$  we have  $c_1(E)^2 = -b_2(X)$ . Thus, if  $b_2(X) \equiv 0 \pmod{4}$  and  $c_2(E) = -\frac{1}{4}b_2(X)$  the moduli space  $\mathcal{M}(E)$  will consist of flat connections only. We will call such connections to be of “zero energy”.

We recall the expected dimension of the moduli space  $\mathcal{M}(E)$ . It is given by the formula

$$d(E) = -2p_1(\mathfrak{su}(E)) + 3(b_1(X) - b_2^+(X) - 1)$$

In particular it can happen that  $d(E) \geq 0$  in the situation we consider, namely,  $b_2^+(X) = 0$ ,  $c_1(E) = \sum e_i$ , and  $c_2(E) < 0$ , the latter condition assuring that we are in the favorable situation of Lemma 1.1.

Interesting is the following special case that we will consider from now on:

**Proposition 1.3.** *Let  $X$  be a negative definite Riemannian four-manifold with strictly positive second Betti-number  $b_2(X)$  divisible by four, and  $b_1(X) = 1$ . Let  $E \rightarrow X$  be a Hermitian rank-2 bundle with  $c_1(E)$  being a lift of  $\sum e_i$ , where  $\{e_i\}$  is a basis of  $H^2(X; \mathbb{Z})/\text{Torsion}$  that diagonalises the intersection form of  $X$ , and with  $c_2(E) = -1/4 b_2(X)$ . Then the moduli space  $\mathcal{M}(E)$  of projectively anti-selfdual connections in  $E$  is compact and consists of irreducible flat connections only, and is of expected dimension zero.*

In order to define an invariant one has to proceed as follows: First, one has to perturb the moduli space so that it consists of a finite number of points only. This should be possible by using holonomy perturbations as in [RS] or [K]. Second, one has to orient the moduli space in order to get a signed count of the moduli space. This is technical but standard [DK]. Because of the absence of reducibles in the ‘parametrised moduli space’ the count of these perturbed moduli spaces should not depend on the chosen perturbation. Furthermore, it seems likely that the count we get is even or even divisible by four, see [RS], [KLS]. As soon as we have found a non-empty moduli space in the situation of the above Proposition 1.3 a more accurate investigation of the convenient definition of this invariant will follow.

**Remark.** *Note that if  $H^2(X; \mathbb{Z})$  has no torsion then there is a unique lift of  $\sum e_i$  in the previous proposition. In this case there is, up to isomorphism, a unique Hermitian bundle  $E \rightarrow X$  with  $c_2(E) = -1/4 b_2(X)$  and  $c_1(E) = \sum e_i$ . In this case one therefore gets an invariant of the underlying smooth four-manifold only.*

**1.4. Flat connections, holonomy and representations of the fundamental group.** Suppose we are in the situation that our moduli space  $\mathcal{M}(E)$  consists of

flat connections in  $\mathfrak{su}(E) \rightarrow X$ , as for instance in the last proposition. Then we must have  $p_1(\mathfrak{su}(E)) = 0$  by Chern-Weil theory.

The holonomy establishes a correspondance between flat connections in the oriented real rank-3 bundle  $V \rightarrow X$  and representations of the fundamental group  $\pi_1(X)$  in  $SO(3)$  with a prescribed Stiefel-Whitney class. More precisely, let  $\rho : \pi_1(X) \rightarrow SO(3)$  be a representation of the fundamental group. Let  $\tilde{X}$  be the universal covering of  $X$ ; it is a  $\pi_1(X)$  principal bundle over  $X$ . We can form the associated oriented rank-3-bundle

$$V_\rho := \tilde{X} \times_\rho \mathbb{R}^3 .$$

It admits a flat connection as it is a bundle associated to a principal bundle with discrete structure group. Therefore it has vanishing first Pontryagin class,  $p_1(V_\rho) = 0$ , by Chern-Weil theory. Its only other characteristic class [DW] is its second Stiefel-Whitney class  $w_2(V_\rho)$ . Therefore we will say that the representation  $\rho$  has Stiefel-Whitney class  $w \in H^2(X; \mathbb{Z}/2)$  if  $w = w_2(V_\rho)$ . On the other hand, let  $V \rightarrow X$  be an oriented real rank-3 bundle with a flat connection  $A$ . Then the holonomy of  $A$  along a path only depends up to homotopy on the path, and therefore induces a representation  $Hol(A) : \pi_1(X) \rightarrow SO(3) = SO(V|_{x_0})$ . In particular, the holonomy defines a reduction of the structure group to  $\pi_1(X)$ , and the bundle can therefore be reconstructed as  $V \cong V_{Hol(A)}$ . In particular the representation  $Hol(A)$  has Stiefel-Whitney class  $w_2(V_{Hol(A)}) = w_2(V)$ .

The moduli space  $\mathcal{M}(E)$  has been obtained by quotienting the space of anti-self-dual connection in  $\mathcal{A}(P_E) \cong \mathcal{A}(\mathfrak{su}(E))$  by the gauge group  $\mathcal{G}^0$ . From the perspective of the  $PU(2)$  connections in  $\mathfrak{su}(E)$  this gauge group is not the most natural one. Instead, the group

$$\mathcal{G} := \Gamma(X; P \times_{\text{Ad}} PU(2))$$

is the natural group of automorphisms of connections in  $\mathfrak{su}(E)$ . Not any element  $g \in \mathcal{G}$  admits a lift to  $\mathcal{G}^0$ ; instead, there is a natural exact sequence

$$1 \rightarrow \mathcal{G}^0 \rightarrow \mathcal{G} \rightarrow H^1(X; \mathbb{Z}/2) \rightarrow 0 .$$

Quotienting by  $\mathcal{G}^0$  has the advantage of a simpler discussion of reducibles, as discussed above. Let us denote by

$$\mathcal{M}(\mathfrak{su}(E)) := \{A \in \mathcal{A}(\mathfrak{su}(E)) \mid F_A^+ = 0\} / \mathcal{G}$$

the moduli space of anti-self dual connections in  $\mathfrak{su}(E)$  modulo the full gauge group  $\mathcal{G}$ . Then there is a branched covering  $\mathcal{M}(E) \rightarrow \mathcal{M}(\mathfrak{su}(E))$  with ‘covering group’  $H^1(X; \mathbb{Z}/2)$ .

Let us denote by  $\mathcal{R}_w(\pi_1(X); SO(3))$  the space of representations of  $\pi_1(X)$  in  $SO(3)$  up to conjugation and of Stiefel-Whitney class  $w \in H^2(X; \mathbb{Z}/2)$ . The above discussion implies that there is a homeomorphism

$$Hol : \mathcal{M}(\mathfrak{su}(E)) \xrightarrow{\cong} \mathcal{R}_w(\pi_1(X); SO(3)) ,$$

where  $w = w_2(\mathfrak{su}(E))$ . In particular,  $\mathcal{M}(E)$  surjects onto  $\mathcal{R}_w(\pi_1(X); SO(3))$ .

## 2. REPRESENTATIONS OF THE FUNDAMENTAL GROUP IN $SO(3)$ AND THE MAIN RESULT

As we have seen moduli spaces of anti-self-dual instantons of zero energy correspond to representations of the fundamental group in  $SO(3)$ . We will use this

interpretation to derive our main result, namely the vanishing results for the invariant  $n_X(E)$  defined above by showing that the associated moduli space becomes empty in the connected sum situations mentioned in the introduction. More precisely, we will show that the representation space  $\mathcal{R}_w(\pi_1(X); SO(3))$  is empty for the given Stiefel-Whitney class  $w = w_2(E)$ .

Of course, a priori, there is no reason not to attempt a ‘count’ of representation spaces  $\mathcal{R}_w(X; SO(3))$  for a general four-manifold  $X$  and class  $w$ . However, Teleman’s approach to the particular situation we consider permits to borrow the above mentioned compactness and transversality results from gauge-theory to the representation theory.

The above construction of the bundle  $V_\rho$  associated to a representation  $\rho : \pi_1(X) \rightarrow SO(3)$  is functorial in the following sense:

**Lemma 2.1.** *Suppose we have a map  $f : W \rightarrow X$  between topological spaces, and  $\rho : \pi_1(X) \rightarrow SO(3)$  a representation of the fundamental group of  $X$ . Then there is a natural isomorphism*

$$f^*V_\rho \cong V_{\rho \circ f_*} \quad (4)$$

between the pull-back of the bundle  $V_\rho$  via  $f$  and the bundle  $V_{\rho \circ f_*} \rightarrow W$ , where  $f_* : \pi_1(W) \rightarrow \pi_1(X)$  is the map induced by  $f$  on the fundamental groups.

*Proof:* We have a commutative diagram

$$\begin{array}{ccc} \widetilde{W} & \xrightarrow{\widetilde{f}} & \widetilde{X} \\ \downarrow & & \downarrow \\ W & \xrightarrow{f} & X, \end{array}$$

where the vertical maps are the universal coverings, and where  $\widetilde{f}$  is the unique map turning the diagram commutative (we work in the category of pointed topological spaces here). It is elementary to check that the map  $\widetilde{f}$  is equivariant with respect to the action of  $\pi_1(W)$ , where this group acts on  $\widetilde{X}$  via  $f_* : \pi_1(W) \rightarrow \pi_1(X)$  and the Deck-transformation group of  $\widetilde{X}$ . The claimed isomorphism follows then from naturality of the associated bundle construction.  $\square$

**Proposition 2.2.** *Suppose the four-manifold  $X$  splits along the connected 3-manifold  $Y$  as  $X = X_1 \cup_Y X_2$  into two four-manifolds  $X_1$  and  $X_2$ . Then any representation  $\rho : \pi_1(X) \rightarrow SO(3)$  induces representations  $\rho_i : \pi_1(X_i) \rightarrow SO(3)$  via  $\rho \circ (j_i)_*$  where the map  $j_i : X_i \hookrightarrow X$  is the inclusion. For these representations we have*

$$V_\rho|_{X_i} = V_{\rho_i} . \quad (5)$$

*Conversely, given representations  $\rho_i : \pi_1(X_i) \rightarrow SO(3)$  such that  $\rho_1 \circ (k_1)_* = \rho_2 \circ (k_2)_* : \pi_1(Y) \rightarrow SO(3)$ , where  $k_i : Y \hookrightarrow X_i$  denotes the inclusion, there is a representation  $\rho : \pi_1(X) \rightarrow SO(3)$  inducing  $\rho_1$  and  $\rho_2$  via the respective restrictions.*

*Proof:* This follows from the Theorem of Seifert and van Kampen and the lemma above, or equivalently, by glueing connections.  $\square$

**Proposition 2.3.** *Let  $X$  be a four-manifold with  $b_2^+(X) = 0$ , and let  $w \in H^2(X; \mathbb{Z}/2)$  be the (mod 2) - reduction of (a lift of)  $\sum e_i \pmod{2}$ , where  $\{e_i\}$  is a basis of  $H^2(X; \mathbb{Z})/\text{Torsion}$  diagonalising the intersection form of  $X$ . Suppose there is a representation  $\rho : \pi_1(X) \rightarrow SO(3)$  with fixed second Stiefel-Whitney class  $w$ . Then none of the Poincaré dual of the basis elements  $e_i$  is in the image of the Hurewicz-homomorphism  $h : \pi_2(X) \rightarrow H_2(X; \mathbb{Z})$ .*

*Proof [T]:* Suppose we have a map  $f : S^2 \rightarrow X$  such that  $PD(e_i) = f_*[S^2]$ , where  $[S^2] \in H_2(S^2; \mathbb{Z})$  denotes the fundamental cycle of  $S^2$ , and  $PD(e_i)$  denotes the Poincaré dual of  $e_i$ . Then we have

$$\langle w, f_*[S^2] \rangle \equiv \langle \sum e_j, PD(e_i) \rangle = e_i^2 = -1 \pmod{2}. \quad (6)$$

On the other, by naturality of the cohomology-homology pairing, we get

$$\langle w, f_*[S^2] \rangle = \langle w_2(V_\rho), f_*[S^2] \rangle = \langle f^*w_2(V_\rho), [S^2] \rangle. \quad (7)$$

But the above Lemma 2.1 implies that  $f^*w_2(V_\rho) = w_2(f^*V_\rho) = w_2(V_{\rho \circ f_*})$ . As  $S^2$  has trivial fundamental group the bundle  $V_{\rho \circ f_*}$  is clearly the trivial bundle, so the number (7) must be zero modulo 2, a contradiction to equation (6).  $\square$

This proposition gives a topological significance of the zero-energy instantons: If the moduli space is non-empty then the elements  $PD(e_i)$  are not representable by spheres! One might wonder whether there exists any four-manifold where the elements  $PD(e_i)$  are not representable by spheres. Certainly this cannot be a simply connected four-manifold because of the Hurewicz-isomorphism theorem. Interestingly, the answer is affirmative. In [PY] Prasad and Yeung construct manifolds with the rational cohomology of the complex projective space  $\mathbb{C}P^2$  whose universal cover is the unit disc in  $\mathbb{C}^2$ . Such a manifold  $W$  is therefore an Eilenberg-MacLane space  $K(\pi_1(W), 1)$ .

Now let  $Z$  be the four-manifold that we obtain from four times the connected sum of  $\overline{W}$ , where we do again a connected sum of the last summand with the first. The so obtained “4- $\overline{W}$ -ring” is diffeomorphic to

$$Z := \overline{W} \# \overline{W} \# \overline{W} \# \overline{W} \# S^1 \times S^3 =: 4\overline{W} \# S^1 \times S^3.$$

This manifold has negative definite intersection form and has Betti-numbers  $b_1(Z) = 1$  and  $b_2(Z) = 4$ . In addition, no element of  $H_2(Z, \mathbb{Z})$  is representable by a 2-sphere as  $\pi_2(Z) = 0$ , so we get no obstruction to non-emptiness from Proposition 2.3. Thus the four-manifold  $Z$  is a prototype of a four-manifold on which to consider the moduli space of  $PU(2)$  instantons associated to the bundle  $E \rightarrow Z$  with  $c_1(E) = \sum e_i$  and  $c_2(E) = -\frac{1}{4}b_2(X)$  (and therefore of representations of  $\pi_1(X) \rightarrow SO(3)$  with fixed Stiefel-Whitney class  $w = \sum e_i \pmod{2}$ ). However, as we will see, there are no such instantons.

**Theorem 2.4.** *Let  $X$  be a negative definite four-manifold, and let  $\{e_i\}$  be a basis of  $H^2(X; \mathbb{Z})/\text{Tors}$  diagonalising the intersection form. If there is a representation  $\rho : \pi_1(X) \rightarrow SO(3)$  with Stiefel-Whitney class  $w := \sum e_i \pmod{2}$ , then the second Betti-number  $b_2(X)$  must be divisible by four.*

*Proof:* The bundle  $V_\rho$  has  $w_2(V_\rho) = w$  and vanishing first Pontryagin-class  $p_1(V_\rho) = 0$  because this bundle admits a flat connection. Now the Dold-Whitney

theorem states that the second Stiefel-Whitney class  $w_2$  and the first Pontryagin class  $p_1$  of any oriented real rank-3 bundle satisfy the equation

$$\text{P-Sq}(w_2) = p_1 \pmod{4} .$$

Here  $\text{P-Sq} : H^2(X; \mathbb{Z}/2) \rightarrow H^4(X; \mathbb{Z}/4)$  denotes the Pontryagin square, a lift of the cup-product squaring  $H^2(X; \mathbb{Z}/2) \rightarrow H^4(X; \mathbb{Z}/2)$  to the coefficient group  $\mathbb{Z}/4$ . If the class  $v \in H^2(X; \mathbb{Z}/2)$  is the mod-2 reduction of an integral class  $c \in H^2(X; \mathbb{Z})$  then the Pontryagin square is simply the mod-4 reduction of the square of  $c$ , i.e.

$$\text{P-Sq}(v) = c^2 \pmod{4} .$$

In our case the Dold-Whitney theorem thus implies that

$$0 = \text{P-Sq}(w) = \sum e_i^2 = -b_2(X) \pmod{4} .$$

□

As a corollary we obtain the following

**Theorem 2.5.** *Let  $X$  be a four-manifold with negative definite intersection form and suppose it admits a connected sum decomposition  $X_1 \# X_2$ , let further  $\{e_i\}$  be a basis of  $H^2(X; \mathbb{Z})/\text{Torsion}$  diagonalising the intersection form of  $X$ . Suppose  $\rho : \pi_1(X) \rightarrow SO(3)$  is a representation of the fundamental group of  $X$  with fixed Stiefel-Whitney class  $w$ , where  $w$  is the (mod 2) reduction of a lift of  $\sum e_i$ . Then both  $b_2(X_1)$  and  $b_2(X_2)$  must be divisible by four.*

*Proof:* Note first that the intersection form of both  $X_1$  and  $X_2$  must be diagonal. This follows from Eichler's theorem on unique decomposition of symmetric definite forms over  $\mathbb{Z}$ , see [HM]. Therefore the basis vectors  $\{e_i\}$  of  $H^2(X; \mathbb{Z})/\text{Torsion}$  are simply given by the union of basis vectors  $\{f_i\}$  of  $H^2(X_1; \mathbb{Z})/\text{Torsion}$  diagonalising the intersection form of  $X_1$ , and basis vectors  $\{g_i\}$  of  $H^2(X_2; \mathbb{Z})/\text{Torsion}$  diagonalising the intersection form of  $X_2$ .

Note that  $\pi_1(X_i \setminus B^4) \cong \pi_1(X_i)$ . The above proposition 2.2 now applies yielding representations  $\rho_i : \pi_1(X_i) \rightarrow SO(3)$ . Its second Stiefel-Whitney class computes, using the above equation (5),

$$w_2(V_{\rho_1}) = w_2(V_{\rho})|_{X_1 \setminus B^4} = \sum f_i \pmod{2} ,$$

and likewise for  $w_2(V_{\rho_2})$ . The above theorem therefore concludes the proof. □

**Remark.** *This corollary implies that the above considered manifold  $Z = 4\overline{W} \# S^1 \times S^3$  does not admit a representation  $\rho : \pi_1(X) \rightarrow SO(3)$  with Stiefel-Whitney class being the mod-2 reduction of the sum of basis elements diagonalising the intersection form.*

**Remark.** *Conversely, we would like to point out that there cannot be a general vanishing theorem for connected sums of Teleman's Casson-type invariant which relies on emptiness of the corresponding moduli space. In fact, suppose we are given a connected sum  $X = X_1 \# X_2$  and representations  $\rho_i : \pi_1(X_i) \rightarrow SO(3)$  with the desired Stiefel-Whitney classes. According to Proposition 2.2, we obtain the representation  $\rho = \rho_1 * \rho_2 : \pi_1(X) \rightarrow SO(3)$  which has the desired Stiefel-Whitney class.*

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