WHAT TO EXPECT FROM $U(n)$ SEIBERG-WITTEN MONOPOLES FOR $n > 1$

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ABSTRACT. We study generalisations to the structure groups U(n) of the familiar (abelian) Seiberg-Witten monopole equations on a four-manifold X and their moduli spaces. For $n = 1$ one obtains the classical monopole equations. For $n > 1$ our results indicate that there should not be any non-trivial gauge-theoretical invariants which are obtained by the scheme 'evaluation of cohomology classes on the fundamental cycle of the moduli space'. For, if b_2^+ is positive the moduli space should be 'cobordant' to the empty space because we can deform the equations so as the moduli space of the deformed equations is generically empty. Furthermore, on Kähler surfaces with $b_2^+ > 1$, the moduli spaces become empty as soon as we perturb with a non-vanishing holomorphic 2-form.

INTRODUCTION

In this paper we study generalisations of the familiar (abelian) Seiberg-Witten monopole equations to the structure groups $U(n)$ for $n > 1$. This is done by twisting a given $Spin^c$ structure $\mathfrak{s}$ on the four-manifold X with a Hermitian bundle E of rank n. The first variable in the theory will consist of sections Ψ of the twisted spinor bundle $S_{\mathfrak{s}}^+ \otimes E$. As for the second, instead of taking the $Spin^c$ connections in $\mathfrak{s}$ as variables in the theory, we keep one fixed as a parameter and take the $U(n)$ connections $\hat{A}$ in the bundle E as a variable. There are then straightforward generalisations of the classical Seiberg-Witten equations to this situation:

$$D_{\hat{A}}\Psi = 0$$
$$\gamma(F_{\hat{A}}^+) - \mu_{0,\tau}(\Psi) = \gamma(\eta) \, \mathrm{id} \ .$$

Here $D_{\hat{A}}$ is the associated Dirac-operator to $\hat{A}$, the map γ is derived from Clifford-multiplication, $F_{\hat{A}}^+$ is the self-dual part of the curvature of the connection $\hat{A}$, η is a self-dual 2-form which serves as a perturbation of the equations, and $\mu_{0,\tau}$ is a quadratic map in the spinor depending on a parameter $\tau \in [0,1]$, explicitly described below.

The involved analysis is much more difficult than in classical Seiberg-Witten theory. First, the associated moduli spaces are in general not compact anymore but have a natural compactification similar to the Uhlenbeck-compactification of the moduli spaces of instantons. Second, generic regularity is a much harder problem than in classical Seiberg-Witten theory and cannot be achieved by the perturbation with the self-dual 2-form η above (or the metric in addition). Third, we cannot avoid 'reducibles' in general, i.e. solutions $(\Psi, \hat{A})$ to the above equations which have positive-dimensional stabiliser under the action of the gauge group.

The aim of the present paper is easily stated: Without even solving all of the mentioned technical problems we shall show that it is not really worth to do so,

because we get quite strong evidence that there should not be interesting gauge-theoretical invariants envolved, at least none which are derived with the classical scheme 'evaluation of cohomology classes on the fundamental cycle of the moduli space'. This evidence is given by two main results. The first, Proposition 2.1 states that if we put $\tau = 0$ in the above equation then for a generic perturbation η the associated moduli space is empty if $b_2^+(X) \geq 1$. It should be pointed out that putting $\tau = 0$ is only sensible for $n > 1$ because otherwise we lose control over the compactness or compactification. But moduli spaces for different τ should be 'cobordant' if an invariant is defined at all. The second, Corollary 3.5, shows that on a Kähler surface the moduli space is empty as soon as one perturbs with a non-vanishing holomorphic 2-form, which is always possible if $b_2^+(X) > 1$.

Our reason for studying $U(n)$ monopoles consists in the fact that these appeared naturally when studying certain $PU(N)$ monopoles, for integers $1 \leq n < N$. The $PU(2)$ monopoles have been used extensively with the aim of proving Witten's conjecture [W] on the relation between the Seiberg-Witten and the Donaldson invariants, first by Pidstrigach-Tyurin [PT], Okonek-Teleman [OT], [T2], and then by Feehan and Leness [FL1], [FL2], [FL3], [FL4]. Feehan and Leness now seem to have proved the full conjecture [FL5].

Kronheimer has introduced instanton-type invariants associated to Hermitian bundles of rank N [K] which are a generalisation of the polynomial invariants of Donaldson appearing when $N = 2$. Before these invariants were even properly defined the physicists Mariño and Moore conjectured that such invariants should not contain new differential topological information and suggested a generalisation of Witten's conjecture to a relationship between these invariants and the Seiberg-Witten invariants. Kronheimer verified this conjecture for a large class of four-manifolds. We, instead, have investigated a generalisation of the above mentioned approach by means of $PU(N)$ monopoles [Z2]. The main results in this paper, Proposition 2.1 and Corollary 3.5 will be used in [Z2] in order to give first steps towards a proof of the mentioned conjecture. Aside this motivation, studying $U(n)$ monopoles is also interesting in itself.

In the first section we shall introduce our setting, define the above mentioned map $\mu_{0,\tau}$ and derive some important properness property which will imply the existence of an a-priori C^0 bound on the spinor component of a monopole. Given this bound one can show that there is a natural Uhlenbeck-type compactification of the moduli space. In the second section we discuss the implications of deforming the equations by $\tau \in [0, 1]$, yielding the above mentioned first main result. In the third section we discuss $U(n)$ moduli spaces on the Kähler surfaces yielding the mentioned vanishing result.

1. THE $U(n)$-MONOPOLE EQUATIONS

In this section we shall define the $U(n)$ monopole equations, study some of their basic properties and define the moduli space. The standard material in Seiberg-Witten theory ($Spin^c$ structures, $Spin^c$ connections etc.) can be found in one of the textbooks on the topics like [N], [M] or diverse lecture notes like [T3].

1.1. The configuration space.
Let X be a closed oriented Riemannian four-manifold with a $Spin^c$ structure $\mathfrak{s}$ on it. The $Spin^c$ structure consists of two Hermitian rank 2 vector bundles $S_{\mathfrak{s}}^{\pm}$ with identified determinant line bundles and a Clifford multiplication

$$\gamma : \Lambda^1(T^*X) \to \mathrm{Hom}_{\mathbb{C}}(S_{\mathfrak{s}}^+, S_{\mathfrak{s}}^-) \ .$$

Furthermore suppose we are given a Hermitian vector bundle E with determinant line bundle $w = \det(E)$ on X. We can then form spinor bundles

$$W_{\mathfrak{s},E}^{\pm} := S_{\mathfrak{s}}^{\pm} \otimes E.$$

Clifford multiplication extends by tensoring with the identity on E. This way we obtain a $Spin^c$ - structure 'twisted' by the hermitian bundle E.

We shall denote by $\mathscr{A}(E)$ the space of smooth unitary connections on E which is an affine space modelled on $\Omega^1(X; \mathfrak{u}(E))$. Here $\mathfrak{u}(E)$ denotes the bundle of skew-adjoint endomorphisms of E. Furthermore $\Gamma(X; W_{\mathfrak{s},E}^+)$ denotes the space of smooth sections of the spinor bundle $W_{\mathfrak{s},E}^+$. We define our configuration space to be the space

$$\mathscr{C}_{\mathfrak{s},E} := \Gamma(X; W_{\mathfrak{s},E}^+) \times \mathscr{A}(E) \ .$$

We denote by $\mathscr{G}$ the group of unitary automorphisms of E; it is the 'gauge group' of our problem. It acts in a canoncial way on sections of the spinor bundles, and as $(u, \nabla_A) \mapsto u\nabla_A u^{-1}$ on the connections, where u is a gauge transformation and ∇_A a unitary connection. The set $\mathscr{B}_{\mathfrak{s},E}$ is defined to be the configuration space up to gauge, i.e. the quotient space $\mathscr{C}_{\mathfrak{s},E}/\mathscr{G}$.

The reason we consider only smooth objects is purely a matter of simplicity here. Obviously, as soon as we wish to consider more analytical properties like transverality, we study suitable Sobolev-completions of these spaces.

A configuration $(\Psi, \hat{A})$ shall be called irreducible if its stabiliser $\Gamma(\Psi, \hat{A}) \subseteq \mathscr{G}$ inside the gauge group is trivial. We shall denote by $\mathscr{C}_{\mathfrak{s},E}^*$ the open subspace of configurations with trivial stabiliser and by $\mathscr{B}_{\mathfrak{s},E}^*$ its quotient space. If we consider Sobolev completions of our spaces it is standard to show that there are local slices for the $\mathscr{G}$ action on $\mathscr{C}_{\mathfrak{s},E}^*$, so that $\mathscr{B}_{\mathfrak{s},E}^*$ becomes a Banach manifold.

1.2. Algebraic preliminaries.
We shall now define a quadratic map in the spinor that will appear in the $U(n)$ monopole equations. We prove a properness property for this map that will be essential in proving a uniform bound for solutions to the monopole equations.

The twisted spinor bundles $W_{\mathfrak{s},E}^{\pm}$ are associated bundles of the fibre product of a $Spin^c$ principal bundle and a $U(n)$-principal bundle on X, with the standard fibre $\mathbb{C}^2 \otimes \mathbb{C}^n$. Let us consider the isomorphism

$$(p,q): \quad \mathfrak{gl}(\mathbb{C}^n) \rightarrow \mathfrak{sl}(\mathbb{C}^n) \oplus \mathbb{C} \, \mathrm{id}$$

$$a \mapsto \left(a - \frac{1}{n} \, \mathrm{tr}(a) \cdot \mathrm{id}, \frac{1}{n} \, \mathrm{tr}(a) \cdot \mathrm{id} \right) \ .$$

Both components p and q are orthogonal projections onto their images. Note that $\mathfrak{gl}(\mathbb{C}^2) \otimes \mathfrak{gl}(\mathbb{C}^n)$ and $\mathfrak{gl}(\mathbb{C}^2 \otimes \mathbb{C}^n)$ are canonically isomorphic. We define the orthogonal projections

$$P: \quad \mathfrak{gl}(\mathbb{C}^2 \otimes \mathbb{C}^n) \rightarrow \mathfrak{sl}(\mathbb{C}^2) \otimes \mathfrak{sl}(\mathbb{C}^n) \ ,$$

$$Q: \quad \mathfrak{gl}(\mathbb{C}^2 \otimes \mathbb{C}^n) \rightarrow \mathfrak{sl}(\mathbb{C}^2) \otimes \mathbb{C} \, \mathrm{id}$$

to be the tensor product $(\ )_0 \otimes p$ respectively $(\ )_0 \otimes q$, with $(\ )_0$ denoting the trace-free part of the endomorphism of the first factor $\mathbb{C}^2$.

For elements $\Psi, \Phi \in \mathbb{C}^2 \otimes \mathbb{C}^n$ we define the map

$$\mu_{0,\tau}(\Psi,\Phi) := P(\Psi\Phi^*) \ + \tau \, Q(\Psi\Phi^*),$$

where $(\Psi\Phi^*) \in \mathfrak{gl}(\mathbb{C}^2 \otimes \mathbb{C}^n)$ is defined to be the endomorphism $\Xi \mapsto (\Phi,\Xi) \cdot \Psi$. Furthermore $\tau \in [0,1]$.

With this notation $\mu_{0,1}(\Psi,\Phi)$ is simply the orthogonal projection of the endomorphism $\Psi\Phi^* \in \mathfrak{gl}(\mathbb{C}^2 \otimes \mathbb{C}^n)$ onto $\mathfrak{sl}(\mathbb{C}^2) \otimes \mathfrak{gl}(\mathbb{C}^n)$. We shall also write $\mu_{0,\tau}(\Psi) := \mu_{0,\tau}(\Psi,\Psi)$ for the associated quadratic map. In the case $n = 1$ the map $\mu_{0,1}(\Psi)$ is the quadratic map in the spinor usually occuring in the Seiberg-Witten equations [K].

Proposition 1.1. *Suppose $n > 1$. Then the quadratic map $\mu_{0,\tau}$ is uniformly proper. In other words, there is a positive constant $c > 0$ such that*

$$|\mu_{0,\tau}(\Psi)| \geq c|\Psi|^2 \ , \tag{1}$$

independently of $\tau \in [0,1]$. As a consequence we have the formula

$$(\mu_{0,\tau}(\Psi)\Psi, \Psi) \geq c^2|\Psi|^4 \tag{2}$$

whenever $\tau \geq 0$.

We defer the proof of this proposition to the appendix. $\qquad\qquad\square$

Because of the equivariance property of the map $\mu_{0,\tau}$ we get in a straightforward way corresponding maps between bundles, giving rise to

$$\mu_{0,\tau}: W_{\mathfrak{s},E}^{\pm} \times W_{\mathfrak{s},E}^{\pm} \ \rightarrow \mathfrak{sl}(S_{\mathfrak{s}}^{\pm}) \otimes_{\mathbb{C}} \mathfrak{gl}(E) \ ,$$

respectively, for the quadratic map,

$$\mu_{0,\tau}: W_{\mathfrak{s},E}^{\pm} \ \rightarrow \mathfrak{su}(S_{\mathfrak{s}}^{\pm}) \otimes_{\mathbb{R}} \mathfrak{u}(E) \ .$$

These maps on the bundle level satisfy the corresponding statement in the above proposition with the same constant c.

1.3. **The $U(n)$-monopole equations.** The Clifford map γ is, up to a universal constant, an isometry of the cotangent bundle onto a real form inside $\mathrm{Hom}_{\mathbb{C}}(S_{\mathfrak{s}}^+, S_{\mathfrak{s}}^-)$ which can be specified by the Pauli matrices. We extend γ to $\mathrm{End}(S_{\mathfrak{s}}^+ \oplus S_{\mathfrak{s}}^-)$ by $-\gamma^*$ on the negative Spinor bundle. It then naturally extends to exteriour powers of T^*X, and in particular its restriction to self-dual two-forms is zero on the negative Spinor bundle, and induces an isomorphism

$$\gamma : \Lambda_+^2(T^*X) \xrightarrow{\cong} \mathfrak{su}(S_{\mathfrak{s}}^+) \ .$$

Let's fix a background $Spin^c$ connection B on $\mathfrak{s}$. By composing the connection $\nabla_B \otimes \nabla_{\hat{A}}$ with the Clifford multiplication we get a Dirac operator

$$D_{\hat{A}} := \gamma \circ \left(\nabla_B \otimes \nabla_{\hat{A}}\right) : \Gamma(X; W_{\mathfrak{s},E}^{\pm}) \to \Gamma(X; W_{\mathfrak{s},E}^{\mp}) \ .$$

Its extension to sections of $W_{\mathfrak{s},E}^+ \oplus W_{\mathfrak{s},E}^-$ is a self-adjoint first order elliptic operator. We have oppressed the $Spin^c$ connection B from the notation because it will not be a variable in our theory.

For a configuration $(\Psi, \hat{A}) \in \mathscr{C}_{\mathfrak{s},E}$ the $U(n)$-monopole equations with parameter $\tau \in [0,1]$ and perturbation $\eta \in \Omega_+^2(X; i\mathbb{R})$ read

$$\begin{aligned} D_{\hat{A}}\Psi &= 0 \\ \gamma(F_{\hat{A}}^+) - \mu_{0,\tau}(\Psi) &= \gamma(\eta) \ \mathrm{id} \ . \end{aligned} \tag{3}$$

Here $F_{\hat{A}}$ designs the curvature of the connection $\hat{A}$ and $F_{\hat{A}}^+$ its selfdual part.

1.4. **The moduli space.** The left hand side of the above equations can be seen as a map $\mathscr{F}_\tau$ from the configuration space $\mathscr{C}_{\mathfrak{s},E}$ to the space $\Gamma(X; W_{\mathfrak{s},E}^-) \times \Gamma(X; \mathfrak{su}(S_{\mathfrak{s}}^+) \otimes \mathfrak{u}(E))$. This map satisfies the equivariance property

$$\mathscr{F}_\tau(u.(\Psi, \hat{A})) = (u \times \mathrm{ad}_u)(\mathscr{F}_\tau(\Psi, \hat{A}))$$

for $u \in \mathscr{G}$ and $(\Psi, \hat{A}) \in \mathscr{C}_{\mathfrak{s},E}$. In particular, the set of solutions to the above equations is gauge-invariant. The moduli space is then defined to be the space of solutions to the monopole equations modulo gauge:

$$M_{\mathfrak{s},E}(\tau,\eta) := \{[\Psi, \hat{A}] \in \mathscr{B}_{\mathfrak{s},E} | \mathscr{F}_\tau(\Psi, \hat{A}) = (0, \gamma(\eta))\} \ .$$

There is an elliptic deformation complex associated to a solution $x = (\Psi, \hat{A})$ of the monopole equations. Let us denote by $\mathscr{C}_x^0 = \Gamma(X; \mathfrak{u}(E))$ the Lie algebra of the gauge group, by $\mathscr{C}_x^1 = \Gamma(X; W_{\mathfrak{s},E}^+ \oplus \Lambda^1(X) \otimes \mathfrak{u}(E))$ the tangent space to the configuration space at x, and by $\mathscr{C}_x^2 = \Gamma(X; W_{\mathfrak{s},E}^- \oplus \Lambda_+^2(X) \otimes \mathfrak{u}(E))$ the target vector space of the monopole map $\mathscr{F}$. Deriving the map $u \mapsto \mathscr{F}_\tau(u(\Psi, \hat{A}))$ yields then an elliptic deformation complex

$$0 \longrightarrow \mathscr{C}_x^0 \xrightarrow{\lambda_x} \mathscr{C}_x^1 \xrightarrow{d_x\mathscr{F}} \mathscr{C}_x^2 \longrightarrow 0 \ .$$

Here λ_x is the derivative of the map $u \mapsto u(\Psi, \hat{A})$, and $d_x\mathscr{F}$ is the derivative of the monopole map $\mathscr{F}$ at the solution $x = (\Psi, \hat{A})$. Let us denote by H_x^i the associated cohomology groups. If the configuration $(\Psi, \hat{A})$ is irreducible, that is, has trivial stabiliser, than the cohomology group H_x^0 vanishes. A solution is called *regular* if the second cohomology group H_x^2 vanishes. If a solution $(\Psi, \hat{A})$ is regular and irreducible then the local Kuranishi models for the moduli space show that the moduli space is a smooth manifold in a neighbourhood of $[\Psi, \hat{A}]$, of dimension equal to minus the index of the above elliptic complex. This 'expected dimension'

of the moduli space is computed with the Atiyah-Singer index theorem and is given by the following formula:

$$d(\mathfrak{s}, E) := -2 \langle p_1(\mathfrak{su}(E)), [X] \rangle - n^2(b_2^+(X) - b_1(X) + 1)$$
$$- \frac{n}{4} \operatorname{sign}(X) + \langle c_1(E)^2 - 2 c_2(E) + c_1(L)c_1(E) + \frac{n}{4} c_1(L)^2, [X] \rangle ,$$

where $p_1(\mathfrak{su}(E))$ denotes the first Pontryagin class of the bundle $\mathfrak{su}(E)$, and $\operatorname{sign}(X)$ the signature of the intersection form on X. In this formula the expression in the first line of the right hand side is the expected dimension of the moduli space of $U(N)$ ASD-connections in E, and the second line is the index of the Dirac operator $D_{\hat{A}}$.

In general the moduli space $M_{\mathfrak{s},E}(\tau, \eta)$ will not consist only of irreducible and regular solutions. One usually uses perturbations to the equations (or the map $\mathscr{F}$) in order to get a moduli space consisting of regular elements only. For getting a well-defined moduli problem the perturbations need to be equivariant as well. This makes 'generic regularity' a harder problem in the case $n > 1$ than in the abelian situation $n = 1$. The holonomy perturbations as appearing in [K] can be slightly modified to fit to our situation. It can then be shown that for a generic perturbation the moduli space is a smooth manifold in neighbourhoods of points $[\Psi, \hat{A}]$ for which we have $\Psi \neq 0$ and $\hat{A}$ is an irreducible connection. In the instaton situation [K] reducible connections can be generically avoided under suitable topological assumptions on E. This, however, does not seem to hold in the monopole situation in general.

1.5. **Uniform bound on the spinor.** For solutions to the monopole equations with $n \geq 2$ we will now deduce a uniform bound on the spinor, which can be taken independently of the parameter $\tau \geq 0$. For this, notice that the Weitzenböck formula for the Dirac-operator $D_{\hat{A}}$ reads

$$D_{\hat{A}} D_{\hat{A}} = \nabla_{B,\hat{A}}^* \nabla_{B,\hat{A}} + \frac{1}{2} \gamma(F_{B,\hat{A}})$$
$$= \nabla_{B,\hat{A}}^* \nabla_{B,\hat{A}} + \frac{s}{4} + \frac{1}{2} \gamma(\operatorname{tr} F_B) + \frac{1}{2} \gamma(F_{\hat{A}}) ,$$

where s denotes the scalar curvature of the Riemannian four-manifold X, $\nabla_{B,\hat{A}}$ the tensor product connection of the fixed $Spin^c$ connection B and the $U(N)$ connection $\hat{A}$, and $F_{B,\hat{A}}$ its curvature. Now suppose that we have a monopole $[\Psi, \hat{A}] \in M_{\mathfrak{s},E}(\tau, \eta)$. Using the Weitzenböck formula, the monopole equation and the inequalities in the above proposition 1.1 now yields the following inequality

$$\frac{1}{2}\Delta|\Psi|^2 = \left(\nabla_{B,\hat{A}}^* \nabla_{B,\hat{A}} \Psi, \Psi \right) - |\nabla_{B,\hat{A}}\Psi|^2$$
$$\leq -\frac{s}{4}|\Psi|^2 - \frac{1}{2}\left(\gamma(\operatorname{tr} F_B^+)\Psi, \Psi \right) - \frac{1}{2}\left(\mu_{0,\tau}(\Psi)\Psi, \Psi \right) - \frac{1}{2}\left(\gamma(\eta)\Psi, \Psi \right)$$
$$\leq \left(-\frac{s}{4} + \frac{1}{2}|\operatorname{tr} F_B^+| + |\eta| \right)|\Psi|^2 - \frac{c^2}{2}|\Psi|^4 .$$

Let K be the maximum over X of the coefficient of $|\Psi|^2$ in the last line. This quantity can be a priori negative. At a point x on the four-manifold where $|\Psi|$ admits its maximum the Laplacian $\Delta|\Psi|^2$ must be positive. If $|\Psi|^2(x) \neq 0$ we may devide the above inequality by $|\Psi|^2(x)$, yielding the desired uniform bound:

Proposition 1.2. *There are constants $c, K \in \mathbb{R}$, with $c > 0$, such that for any monopole $[\Psi, \hat{A}] \in M_{\mathfrak{s},E}(\tau, \eta)$, with $\tau \geq 0$, we have a C^0 bound:*

$$\max |\Psi|^2 \leq \max\{0, K/c^2\} \ .$$

Here the constant K depends on the Riemannian metric, the fixed background $Spin^c$ connection, and the perturbation form η only, whereas the constant c is universal. In particular the bound is uniform in $\tau \in [0,1]$. If the constant K is negative then there are only solutions with vanishing spinor component to the $U(N)$ monopole equations.

1.6. **Compactness.** Contrary to the Abelian case $n = 1$ the $U(n)$ moduli spaces $M_{\mathfrak{s},E}(\tau, \eta)$ are in general not compact. However, there is a natural compactification of these moduli spaces similar to the Uhlenbeck-compactification of instanton moduli spaces [DK]. This subject has been treated with in detail in [T2], [FL4] in the case of $PU(2)-$ monopoles. The main reason why the Uhlenbeck-compactification carries over to the monopole situation is the uniform bound on the spinor which we have dealt with in Proposition 1.2 above. We will only describe this compactification here and refer to the above mentioned references for the highly technical proofs. An outline of the proof in the $PU(n)$ situation can also be found in [Z1].

Let $\mathfrak{s}$ be a $Spin^c$-structure on X and let $E \to X$ be a unitary bundle on X. We denote by E_{-k} a bundle which has first Chern class $c_1(E_{-k}) = c_1(E)$ and whose second Chern class satisfies

$$\langle c_2(E_{-k}), [X] \rangle = \langle c_2(E), [X] \rangle - k \ .$$

Such a bundle is unique up to isomorphism on a four-manifold.

Definition 1.3. *An ideal monopole associated to the data $(\mathfrak{s}, E)$ is given by a pair $([\Psi, \hat{A}], \mathbf{x})$, where $[\Psi, \hat{A}] \in M_{\mathfrak{s}, E_{-k}}(\tau, \eta)$ is a $(\mathfrak{s}, E_{-k})-$ monopole, and $\mathbf{x}$ is an element in the k-th symmetric power $Sym^k(X)$ of X (that is, an unordered set of k points in X, $\mathbf{x} = \{x_1, \ldots, x_k\}$). The curvature density of $([\Psi, \hat{A}], \mathbf{x})$ is defined to be the measure*

$$|F_{\hat{A}}|^2 + 8\pi^2 \sum_{x_i \in \mathbf{x}} \delta_{x_i} \ .$$

The set of ideal monopoles associated to the data $(\mathfrak{s}, E)$ and parameters (τ, η) is

$$IM_{\mathfrak{s},E}(\tau, \eta) := \coprod_{k \geq 0} M_{\mathfrak{s}, E_{-k}}(\tau, \eta) \times Sym^k(X) \tag{4}$$

The set of ideal monopoles is then endowed with a convenient topology. This is possible by specifying the underlying notion of convergence. In fact, it can even be shown (cf. for instance [T2], p. 433, [DK]) that this topology can be induced by a metric on the set of ideal monopoles (but this metric is not an extension of the natural metric induced by the L^2-metric on the slices of the gauge-group on the main-stratum). Either way, in this topology each stratum has its natural topology, and we have the following notion of convergence of a sequence in the main-stratum:

Definition 1.4. *The sequence of monopoles $[\Psi_n, \hat{A}_n] \in \mathscr{M}_{\mathfrak{s},E}$ converges to the ideal monopole $([\Psi, \hat{A}], \mathbf{x}) \in M_{\mathfrak{s}, E_{-k}}(\tau, \eta) \times Sym^k(X)$ if we have:*

(1) *The sequence of measures $|F_{\hat{A}_n}|^2 \, vol_g$ converges as measure to*

$$|F_{\hat{A}}|^2 \, vol_g + 8\pi^2 \sum_{x_i \in \mathbf{x}} \delta_{x_i}$$

(2) *On the complement $\Omega := X - \{x \,|\, x \in \mathbf{x}\}$ there are bundle isomorphisms $u_n : E_{-k}|_\Omega \to E|_\Omega$ such that the sequence*

$$u_n^*((\Psi_n, \hat{A}_n)|_\Omega)$$

converges in the C^∞-topology to $(\Psi, \hat{A})|_\Omega$.

In a similar way the convergence of sequences in the lower strata are defined.

Theorem 1.5. *Let $[\Psi_n, \hat{A}_n] \in M_{\mathfrak{s},E}(\tau, \eta)$ be a sequence of $U(N)$ monopoles. Then there is an integer $k \geq 0$, a multiset $\mathbf{x} \in Sym^k(X)$ and a $U(N)$ monopole $[\Psi', \hat{A}'] \in M_{\mathfrak{s},E_{-k}}(\tau, \eta)$ such that the following is true: There is a subsequence (n_k) such that the sequence $[\Psi_{n_k}, \hat{A}_{n_k}]$ converges to the ideal monopole $([\Psi', \hat{A}'], \mathbf{x}) \in M_{\mathfrak{s},E_{-k}}(\tau, \eta) \times Sym^k(X)$ in the above sense.*

Corollary 1.6. *(Compactness-Theorem) The closure of $M_{\mathfrak{s},E}$ inside the space of ideal monopoles $IM_{\mathfrak{s},E}(\tau, \eta)$ is compact. In fact, with the topology specified on $IM_{\mathfrak{s},E}(\tau, \eta)$ this space is itself compact.*

Suppose that we have a monopole $[\Psi, \hat{A}] \in M_{\mathfrak{s},E}(\tau, \eta)$. We will show that the L^2−norm of $F_{\hat{A}}^+$ is uniformly bounded independently of the topological data $(\mathfrak{s}, E)$. For $\tau \in [0, 1]$ we obviously have $|\mu_{0,\tau}(\Psi)| \leq C|\Psi|^2$ for a universal positive constant C (depending on n only). From the curvature equation of the $U(n)$−monopole equations (3) we therefore get, together with the uniform bound (1.2), a pointwise inequality which integrated yields

$$\|F_{\hat{A}}^+\|_{L^2(X)}^2 \leq \left(\frac{CK}{2c^2} + \|\eta\|_\infty\right)^2 vol_g(X) \ . \tag{5}$$

From this and the Chern-Weil formulae it follows that the L^2−norm of the total curvature $F_{\hat{A}}$ is also bounded. This fact is an essential input for the proof of the compactness theorem.

Remark. *We might also consider moduli spaces $\widetilde{M}_{\mathfrak{s},E}(\eta)$ of 'parametrised' $U(n)$− monopoles, where as parameter we take $\tau \in [0, 1]$. This parametrised moduli space then fibers over the interval $[0, 1]$. The fact that the uniform bound in (1.2) can be taken independently of τ indicates that we can also compactify the parametrised moduli space, and that fibrewise the compactification coincides with the one considered above. Thus, heuristically, the compactified moduli spaces $\overline{M}_{\mathfrak{s},E}(\tau, \eta)$ and $\overline{M}_{\mathfrak{s},E}(\tau', \eta)$ for $\tau, \tau' \in [0, 1]$ are 'cobordant'.*

2. A deformation of the equations for $n > 1$

A natural idea is to study the dependence of the moduli space on the fixed parameter $\tau \in [0, 1]$. Surprisingly, we have the following result for the case $\tau = 0$:

Proposition 2.1. *Suppose the 4-manifold X has $b_2^+(X)$ non-zero. Then for a generic imaginary-valued self-dual 2-form η the deformed moduli space $M_{\mathfrak{s},E}(0, \eta)$ is empty.*

Proof: Suppose $[\Psi, \hat{A}]$ belongs to the moduli space $M_{\mathfrak{s},E}(0, \eta)$. In particular, the configuration $(\Psi, \hat{A})$ solves the $U(n)$ - monopole equations (3) with parameter

$\tau = 0$. Let us take the trace of the curvature equation in (3). We get, after applying the isomorphism γ^{-1} the following formula:

$$\text{tr}(F_{\hat{A}}^+) = n \cdot \eta \ . \tag{6}$$

But $\text{tr}(F_{\hat{A}})$ is precisely the curvature $F_{\det(\hat{A})}$ of the connection $\det(\hat{A})$ that $\hat{A}$ induces in the determinant line bundle $\det(E)$. Therefore the equation (6) can be seen as a perturbed ASD - equation for connections in a line bundle. Now, if A_0 is a fixed connection in the line bundle $\det(E)$, then any other connection A is given by $A_0 + a$, where a is an imaginary valued one-form. Its curvature is given by $F_A = F_{A_0} + da$, hence the equation $F_A^+ = n \cdot \eta$ is equivalent to

$$d^+ a = -F_{A_0}^+ + n \cdot \eta \ .$$

But $d^+ : \Omega^1(X; i\mathbb{R}) \to \Omega^2_+(X; i\mathbb{R})$ has cokernel isomorphic to the space of self-dual harmonic imaginary-valued 2-forms, which is of dimension $b_2^+(X)$. Thus, for generic $\eta \in \Omega^2_+(X; i\mathbb{R})$ this equation has no solution. $\square$

Remark. *The Seiberg-Witten and Donaldson invariants are obtained by evaluating canonical cohomology classes on the 'fundamental cycle' given by the moduli space. Moduli spaces associated to different perturbations prove to be cobordant under the condition $b_2^+(X) > 1$ and the canonical cohomology classes extend to the cobordism. The above proposition and remark 1.6 suggests that no non-trivial invariants of that type should be expected from the $U(n)$ moduli spaces $M_{\mathfrak{s},E}(\tau,\eta)$.*

3. $U(n)$ MODULI SPACES ON KÄHLER SURFACES

In classical Seiberg-Witten theory Kähler surfaces are of a significant importance. Indeed, they provided the first examples of 4-manifolds with non-trivial Seiberg-Witten invariants [W]. This was generalised to symplectic manifolds [Ta]. All other non-vanishing results known to the author are derived from these manifolds by various kinds of glueing results for the Seiberg-Witten invariants [Ta], [Fr].

As the $U(n)$ monopole equations are a generalisation of the classical Seiberg-Witten equations it is therefore most natural to study the $U(n)$ monopole moduli spaces for Kähler surfaces. Whereas the analysis of the $U(n)$ monopole equations on Kähler surfaces is very analogous to the classical situation the final conclusion is in sharp contrast to the classical situation. Indeed, we will show in Corollary 3.5 that if we perturb the monopole moduli space on a Kähler surface with a non-vanishing holomorphic 2-form then the associated moduli space is empty.

Non-abelian monopoles on Kähler surfaces have also been studied by Teleman [T], Okonek and Teleman [OT3] and by Bradlow and Garcia-Prada [BG], but with a rather complex geometric motivation. Corollary 3.5 seems to appear here for the first time.

3.1. **The $U(n)$ - monopole equations on Kähler surfaces.** We will quickly recall now the canonical $Spin^c-$ structure on an almost complex surface. The additional condition of X being Kähler implies that there is a canonical $Spin^c$ connection induced by the Levi-Civita connection. This will be our fixed background $Spin^c$ connection and it is then simple to determine the Dirac-operator associated to this fixed connection and a $U(n)-$ connection in a Hermitian bundle

E. We will then write down the $U(n)$ monopole equations in this particular setting.

Suppose we have an almost complex structure $J : TX \to TX$ on the closed, oriented Riemannian 4-manifold X which is isometric. The associated Kähler form ω is defined by the formula

$$\omega_g(v, w) := g(Jv, w) \ .$$

This is an anti-symmetric form of type $(1, 1)$ when extended to the complexification $TX^{\mathbb{C}} := TX \otimes_{\mathbb{R}} \mathbb{C}$. It is a fundamental fact that the complexification of the bundle of self-dual two forms is given by

$$\Lambda_+^2 \otimes \mathbb{C} = \mathbb{C}\omega_g \oplus \Lambda^{2,0} \oplus \Lambda^{0,2} \ .$$

Let $e(u)$ denotes exterior multiplication with the form $u \in \Lambda(T^*X^{\mathbb{C}})$ and $e^*(u)$ its adjoint with respect to the inner product induced by the Riemannian metric.

There is a canonical $Spin^c$-structure associated to an almost-complex structure J on X [Hi]. We shall denote it by $\mathfrak{c}$. The spinor bundles are defined to be

$$S_{\mathfrak{c}}^+ := \Lambda^{0,0}(X) \oplus \Lambda^{0,2}(X) \ ,$$
$$S_{\mathfrak{c}}^- := \Lambda^{0,1}(X) \ ,$$

and the Clifford multiplication is given by

$$\gamma : \Lambda^1(T^*X) \to \mathrm{Hom}_{\mathbb{C}}(S_{\mathfrak{c}}^+, S_{\mathfrak{c}}^-)$$
$$u \mapsto \sqrt{2}(e(u^{0,1}) - e^*(u^{0,1})) \ .$$

The induced isomorphism

$$\gamma : \Lambda_+^2(X) \otimes \mathbb{C} \to \mathfrak{sl}(S_{\mathfrak{c}}^+)$$

is then seen to be given by the formula

$$\gamma(\eta^{1,1} + \eta^{2,0} + \eta^{0,2}) = 4 \begin{pmatrix} -i\Lambda_g(\eta^{1,1}) & - * (\eta^{2,0} \wedge \_) \\ \eta^{0,2} & i\Lambda_g(\eta^{1,1}) \end{pmatrix} \ . \tag{7}$$

Here we use the commonly used convention to denote contraction with ω_g, that is $e^*(\omega_g)$, by the symbol Λ_g.

Now suppose that X is a Kähler surface. This means that first the almost complex structure J is integrable to a complex structure, and second that the Kähler form ω_g is closed, $d\omega_g = 0$. The condition of closedness implies (cf. [KN], p. 148) that the the almost complex structure J is parallel with respect to the Levi-Civita-connection ∇_g. As a consequence, the splittings

$$\Lambda^k(X) \otimes \mathbb{C} = \oplus_{p+q=k}\Lambda^{p,q}(X)$$

are ∇_g-parallel, where we also denote by ∇_g the connection induced by the Levi-Civita connection on all exterior powers of T^*X. The canonical $Spin^c$-connection is now simply given by the the connection ∇_g in the bundles $\Lambda^{0,0}, \Lambda^{0,1}$ and $\Lambda^{0,2}$.

Let E be a Hermitian vector bundle on X, and further $\nabla_{\hat{A}}$ a unitary connection on E. We shall use the notation convention $\Lambda^{p,q}(E) := \Lambda^{p,q}(X) \otimes E$, and by $\Omega^{p,q}(E)$ we shall denote the space of sections of the latter bundle, $\Omega^{p,q}(E) = \Gamma(\Lambda^{p,q}(E)) \ .$

Definition 3.1. *The operator $\overline{\partial}_{\hat{A}} : \Omega^{p,q}(E) \to \Omega^{p,q+1}(E)$ is defined to be the composition of $d_{\hat{A}} : \Omega^{p+q}(E) \to \Omega^{p+q+1}(E)$, the extension of the exteriour derivative to forms with values in E by means of the connection $\nabla_{\hat{A}}$, with the bundle projection $\Lambda^{p+q+1}(E) \to \Lambda^{p,q+1}(E)$.*

The Dirac operator associated to the canonical $Spin^c$-connection ∇_g in the canonical $Spin^c$-structure $\mathfrak{s}_c$ and the unitary connection $\hat{A}$ in the Hermitian bundle E is expressible in terms of the above operator $\overline{\partial}_{\hat{A}}$ and its formal L^2-adjoint $\overline{\partial}_{\hat{A}}^*$ as follows:

$$D_{\hat{A}} = \sqrt{2}\left(\overline{\partial}_{\hat{A}} + \overline{\partial}_{\hat{A}}^*\right) \ . \tag{8}$$

This is a well-known fact in the case $n = 1$ [Hi]. The proof of the general case follows along the same lines. In particular, the proof given in the lecture notes [T3] is directly applicable to our situation. $\square$

We will now study the $U(n)$ monopoles associated to the data $(\mathfrak{c}, E)$ with spinor bundles $W_{\mathfrak{c},E}^{\pm} = S^{\pm} \otimes E$. Note that, up to tensoring E with a line bundle, we can always assume that general data $(\mathfrak{s}, E)$ is of the particular form $(\mathfrak{c}, E)$. Now according to the isomorphism $W_{\mathfrak{c},E}^{+} \cong \Lambda^{0,0}(E) \oplus \Lambda^{0,2}(E)$ a spinor $\Psi \in \Gamma(X; W_{\mathfrak{c},E}^{+})$ can be written as $\Psi = (\alpha, \beta)$ with $\alpha \in \Omega^{0,0}(X; E)$ a section of E and $\beta \in \Omega^{0,2}(X; E)$ a 2-form of type $(0,2)$ with values in E. We introduce the following notations. We denote by $^{-} : \Lambda^{p,q}(E) \rightarrow \Lambda^{q,p}(E^*)$ the conjugate linear isomorphism which is the tensor product of complex conjugation on the forms and the conjugate linear isomorphism specified by the hermitian structure on the bundle E. We denote by $^* : \Lambda^{p,q}(E) \rightarrow \mathrm{Hom}(\Lambda^{p,q}(E), \mathbb{C})$ the conjugate linear isomorphism specified by the Hermitian structure on $\Lambda^{p,q}(E)$. For an endomorphism $f \in \mathrm{End}(E)$ we denote $\{f\}_\tau := (f)_0 + \frac{\tau}{n}\mathrm{tr}(f)\mathrm{id}_E$, where $(f)_0$ denotes the trace-free part of f. Thus we simply have $\{f\}_1 = f$. With this said we can write $\mu_{0,\tau}(\Psi)$ according to the above isomorphism as

$$\mu_{0,\tau}(\Psi) = \begin{pmatrix} \frac{1}{2}\left(\{\alpha\alpha^*\}_\tau - \{*\beta \wedge \overline{\beta}\}_\tau\right) & \{\alpha\beta^*\}_\tau \\ \{\beta\alpha^*\}_\tau & \frac{1}{2}\left(\{\beta\beta^*\}_\tau - \{\alpha\alpha^*\}_\tau\right) \end{pmatrix} \ . \tag{9}$$

It is worth pointing out here that we have $\beta\beta^* = *\beta \wedge \overline{\beta}$ which is true because $\Lambda^{0,2}$ is 1-dimensional. In other words, the two diagonal entries only "look" differently.

With the above formulae (7) we can now write down the monopole equations (3) with parameter τ and as perturbation the imaginary-valued self-dual 2-form η for the pair consisting of the spinor $\Psi = (\alpha, \beta) \in \Gamma(X; \Lambda^{0,0}(E) \oplus \Lambda^{0,2}(E))$ and the connection $\hat{A}$ in E:

$$\overline{\partial}_{\hat{A}}\alpha + \overline{\partial}_{\hat{A}}^*\beta = 0$$
$$F_{\hat{A}}^{0,2} = \frac{1}{4}\{\beta\alpha^*\}_\tau + 4\eta^{0,2} \tag{10}$$
$$-i\Lambda_g(F_{\hat{A}}) = \frac{1}{8}\left\{\alpha\alpha^* - *(\beta \wedge \overline{\beta})\right\}_\tau - i\Lambda_g(\eta) \ .$$

Indeed, the curvature equation of (3) splits into four equations according to the above splitting, but the two equations resulting from the diagonal entries are equivalent, and, using that $\overline{F_{\hat{A}}^{0,2}} = -F_{\hat{A}}^{2,0}$ (here again, $^{-}$ denotes the complex-conjugation on the forms and the hermitian adjoint on $\mathrm{End}(E)$), the two off-diagonal equations also prove to be equivalent.

3.2. **Decoupling phenomena, moduli spaces for $b_2^+(X) > 1$ and holomorphic 2-forms.** As mentioned before a lot of the analysis of the classical monopole equations on Kähler surfaces carries over to our situation. Before we consider the perturbed monopole equations we shall first draw some intermediate conclusions from the unperturbed monopole equations. In particular there is a decoupling result completely analogous to the classical situation, interpreting monopoles as 'vortices', c.f. also [BG], [T].

Proposition 3.2. *Let X be a Kähler surface. Suppose that the configuration $(\Psi, \hat{A}) \in \Gamma(X; S_{\mathfrak{c}}^+ \otimes E) \times \mathscr{A}(E)$ solves the* unperturbed $U(n)$ *monopole equations with parameter $\tau \in [0, 1]$. If we write the spinor as $\Psi = (\alpha, \beta)$ according to the decomposition $S_{\mathfrak{c}}^+ \otimes E \cong \Lambda^{0,0}(E) \oplus \Lambda^{0,2}(E)$ then one of the following two statements holds:*

(1) *The second factor of the spinor vanishes identically, $\beta \equiv 0$. Furthermore the pair $(\alpha, \hat{A})$ satisfies the following 'Vortex-type' equations*

$$
\begin{aligned}
\overline{\partial}_{\hat{A}} \alpha &= 0 \\
F_{\hat{A}}^{0,2} &= 0 \\
i\Lambda_g(F_{\hat{A}}) &= -\frac{1}{8}\{\alpha\alpha^*\}_\tau \ .
\end{aligned}
\tag{11}
$$

(2) *The first factor of the spinor vanishes identically, $\alpha \equiv 0$. Furthermore the pair $(\beta, \hat{A})$ satisfies the following equations*

$$
\begin{aligned}
\overline{\partial}_{\hat{A}}^* \beta &= 0 \\
F_{\hat{A}}^{0,2} &= 0 \\
i\Lambda_g(F_{\hat{A}}) &= +\frac{1}{8}\{\beta\beta^*\}_\tau \ .
\end{aligned}
\tag{12}
$$

Proof: Using the first two of the monopole equations (10) we get:

$$
\overline{\partial}_{\hat{A}} \overline{\partial}_{\hat{A}}^* \beta = -\overline{\partial}_{\hat{A}} \overline{\partial}_{\hat{A}} \alpha = -F_{\hat{A}}^{0,2} \alpha = -\frac{1}{4}\{\beta\alpha^*\}_\tau \alpha \ .
$$

We take the inner product with β to get now:

$$
\begin{aligned}
\left(\beta, \overline{\partial}_{\hat{A}} \overline{\partial}_{\hat{A}}^* \beta\right) &= -\frac{1}{4} \left(\beta, \{\beta\alpha^*\}_\tau \alpha\right) \\
&= -\frac{1}{4} \left(|\beta|^2|\alpha|^2 - \frac{1-\tau}{n}\left(\beta, \operatorname{tr}(\beta\alpha^*)\alpha\right)\right) \\
&\leq \left(-\frac{1}{4} + \frac{1-\tau}{4n}\right)|\alpha|^2|\beta|^2 \\
&\leq 0 \ .
\end{aligned}
$$

Here we have used the Cauchy-Schwarz inequality, noting also that $|\operatorname{tr}(\beta\alpha^*)| \leq |\beta\alpha^*| = |\beta||\alpha|$. Integrating now the latter inequality over the whole manifold X yields the following:

$$
0 \ \leq \|\overline{\partial}_{\hat{A}}^* \beta\|^2 \ \leq \left(-\frac{1}{4} + \frac{1-\tau}{4n}\right) \int_X |\alpha|^2|\beta|^2 vol_g \leq 0
\tag{13}
$$

Thus we get $\overline{\partial}_{\hat{A}}^* \beta = 0$ and from the Dirac equation also $\overline{\partial}_{\hat{A}} \alpha = 0$. If further we have $\tau > 1 - n$ then we see from the last inequality that at any point of the manifold X

we have $\alpha = 0$ or $\beta = 0$. But we have $0 = \overline{\partial}_{\hat{A}}^{*}\overline{\partial}_{\hat{A}}\alpha = \Delta_{\overline{\partial}_{\hat{A}}}\alpha$, and because $\Delta_{\overline{\partial}_{\hat{A}}}$ is an elliptic second order operator with scalar symbol it follows from Aronaszajin's theorem [A] that solutions to $\Delta_{\overline{\partial}_{\hat{A}}}\alpha = 0$ satisfy a unique continuation theorem. Similarly we have $0 = \overline{\partial}_{\hat{A}}\overline{\partial}_{\hat{A}}^{*}\beta = \Delta_{\overline{\partial}_{\hat{A}}}\beta$, so the same holds for β. Therefore, if one of α or β vanishes on an open subset of X, then it vanishes on the whole of X. The conclusions now follow from (10). $\qquad\square$

Remark. *If $\tau \neq 0$ the moduli space $M_{\mathfrak{c},E}(\tau, 0)$ can only contain either solution with $\alpha \neq 0$ or with $\beta \neq 0$. This follows from taking the trace of the third equation of (10) and then integrating it over the whole manifold. The left hand term yields then the topological quantity $-2\pi \langle c_1(E) \smile [\omega_g], [X]\rangle$.*

On a Kähler surface we have $\Delta = 2\Delta_{\overline{\partial}}$, just reflecting again the compatibility between the complex structure and the Riemannian metric. Therefore the harmonic differential forms are also $\overline{\partial}$-harmonic and vice versa. In particular, we get the following decomposition from the Hodge-theorem:

$$H^2_{dR}(X; \mathbb{C}) = H^{2,0}_{\overline{\partial}}(X) \oplus H^{1,1}_{\overline{\partial}}(X) \oplus H^{0,2}_{\overline{\partial}}(X) . \tag{14}$$

Corollary 3.3. *If there are solutions to the unperturbed $U(n)$-monopole equations associated to the data $(\mathfrak{c}, E)$ and to the parameter $\tau \in [0, 1]$, then the image $c_1^{\mathbb{R}}(E)$ in real (complex) cohomology of the first Chern-class $c_1(E) \in H^2(X; \mathbb{Z})$ is of type $(1, 1)$ according to the above decomposition (14).*

Proof: Under these conditions there is a connection $\hat{A}$ on E with $F_{\hat{A}}^{0,2} = 0 = F_{\hat{A}}^{2,0}$. From the Chern-Weil formula we have that $\frac{-1}{2\pi i}\left[\text{tr}(F_{\hat{A}})\right] = c_1^{\mathbb{R}}(E)$. There is a 1-form λ such that $\omega := \text{tr}(F_{\hat{A}}) - \overline{\partial}\lambda$ is $\overline{\partial}$-harmonic and this class also represents $c_1^{\mathbb{R}}(E)$. We have $\omega^{2,0} = 0$ and $\omega^{0,2} = \overline{\partial}\lambda$. But a class is $\overline{\partial}$ - harmonic if and only if each component according to $\Omega^{p+q}(X) = \oplus\Omega^{p,q}(X)$ is $\overline{\partial}$ - harmonic. But then the harmonic form $\omega^{2,0} = \overline{\partial}\lambda$ must be zero, as it is a $\overline{\partial}$ - exact form also. $\qquad\square$

In the classical theory a common perturbation of the monopole equations was to perturb with imaginary-valued self-dual 2-forms η such that $\eta^{2,0}$ is a holomorphic form [W] [Bq]. There are such forms with $\eta^{2,0} \neq 0$ precisely if $b_2^+(X) > 1$. We will now consider this type of perturbation in the general case of $U(n)$ monopoles even though these perturbations are not enough to get generic regularity of the moduli space in the case $n > 1$. However, it will turn out that the moduli spaces perturbed in this way are empty in the case $n > 1$ as soon as the perturbing form η is non-zero.

If the unperturbed $U(n)$ monopole moduli space is empty then any invariant derived by the scheme 'evaluation of cohomology classes on the fundamental cycle of the moduli space' should be zero. Indeed, that kind of invariant would be defined with a 'generic' moduli space, i.e. one which is cut out transversally by the suitably perturbed monopole equations. An empty moduli space is always generic. Thus if there is a non-trivial invariant derived from some generic moduli space then the associated unperturbed moduli space may be not generic, but it could not be empty. Therefore it is natural to consider topological data $(\mathfrak{s}, E)$ only for situations where the unperturbed $U(n)$ monopole moduli spaces are a priori non-empty. As

we have seen, this can only be the case if the first Chern-class $c_1^{\mathbb{R}}(E)$ is of type $(1,1)$ according to the decomposition (14). Therefore we shall include this hypothesis to the next two results, the following theorem and its corollary:

Theorem 3.4. *Let X be a Kähler surface and let E be a bundle such that its first Chern-class $c_1^{\mathbb{R}}(E)$ is of type $(1,1)$. Further let η be an imaginary-valued 2-form with $\eta^{2,0}$ holomorphic. Then the $U(n)-$ monopole equations (10) associated to the data $(\mathfrak{c}, E)$, to the perturbation form η, and to the parameter $\tau \in (0,1]$ are equivalent to the following system of equations:*

$$
\begin{aligned}
\overline{\partial}_{\hat{A}}\alpha &= 0 \\
\overline{\partial}_{\hat{A}}^{*}\beta &= 0 \\
F_{\hat{A}}^{0,2} &= 0 \\
\frac{1}{4}\{\beta\alpha^*\}_\tau &= \eta^{0,2} \\
-i\Lambda_g(F_{\hat{A}}) &= \frac{1}{8}\{\alpha\alpha^* - \beta\beta^*\}_\tau - i\Lambda_g(\eta)
\end{aligned}
\tag{15}
$$

Proof: We will derive the following formula for a solution $((\alpha, \beta), \hat{A}))$ to the $U(n)-$monopole equations (10) with parameter τ and perturbation η:

$$
\begin{aligned}
0 =&\, 4\,\|F_{\hat{A}}^{0,2}\|_{L^2(X)}^2 \ + 4\,\frac{1-\tau}{\tau n}\,\|\mathrm{tr}F_{\hat{A}}^{0,2}\|_{L^2(X)}^2 + \|\overline{\partial}_{\hat{A}}^{*}\beta\|_{L^2(X)}^2 \\
&- \frac{4}{\tau}\,\langle 2\pi i\,[\eta^{2,0}] \smile c_1(E), [X]\rangle \ .
\end{aligned}
\tag{16}
$$

The conclusion then clearly follows as the topological term vanishes by assumption.

Provided that we have $\tau \neq 0$ the endomorphism $\beta\alpha^*$ can be expressed as

$$
\begin{aligned}
\beta\alpha^* &= \{\beta\alpha^*\}_\tau \ + \frac{1-\tau}{n}\mathrm{tr}(\beta\alpha^*) \\
&= \{\beta\alpha^*\}_\tau \ + \frac{1-\tau}{\tau n}\mathrm{tr}(\{\beta\alpha^*\}_\tau) \\
&= 4\,F_{\hat{A}}^{0,2} - 4\,\eta^{0,2} + 4\,\frac{1-\tau}{n\tau}\,\mathrm{tr}(F_{\hat{A}}^{0,2}) - 4\,\frac{1-\tau}{\tau}\,\eta^{0,2} \ ,
\end{aligned}
\tag{17}
$$

where the last equation used the second of the monopole equations (10) and the trace of it.

Again we get from the Dirac-equation that $\overline{\partial}_{\hat{A}}\overline{\partial}_{\hat{A}}^{*}\beta + F_{\hat{A}}^{0,2}\alpha = 0$, so that after taking the pointwise inner-product with β and using the above equation (17) we get:

$$
\begin{aligned}
0 &= \left(\beta, F_{\hat{A}}^{0,2}\alpha\right) \ + \left(\beta, \overline{\partial}_{\hat{A}}\overline{\partial}_{\hat{A}}^{*}\beta\right) \\
&= \left(\beta\alpha^*, F_{\hat{A}}^{0,2}\right) \ + \left(\beta, \overline{\partial}_{\hat{A}}\overline{\partial}_{\hat{A}}^{*}\beta\right) \\
&= 4\,|F_{\hat{A}}^{0,2}|^2 \ - 4\left(\eta^{0,2}, F_{\hat{A}}^{0,2}\right) \\
&\quad + 4\frac{1-\tau}{n\tau}|\mathrm{tr}(F_{\hat{A}}^{0,2})|^2 \ - 4\frac{1-\tau}{\tau}\left(\eta^{0,2}, F_{\hat{A}}^{0,2}\right) + \left(\beta, \overline{\partial}_{\hat{A}}\overline{\partial}_{\hat{A}}^{*}\beta\right) \\
&= 4\,|F_{\hat{A}}^{0,2}|^2 \ + 4\frac{1-\tau}{n\tau}|\mathrm{tr}(F_{\hat{A}}^{0,2})|^2 - \frac{4}{\tau}\left(\eta^{0,2}, F_{\hat{A}}^{0,2}\right) + \left(\beta, \overline{\partial}_{\hat{A}}\overline{\partial}_{\hat{A}}^{*}\beta\right)
\end{aligned}
\tag{18}
$$

As the next step we will integrate this whole equation over X. Beforehand we shall remark that $\eta^{2,0}$ is closed, and therefore the following integral is of topological nature:

$$
\begin{aligned}
\int_X \left(\eta^{0,2}, F_{\hat{A}}^{0,2} \right) vol_g &= \int_X \overline{\eta^{0,2}} \wedge *\mathrm{tr}(F_{\hat{A}}^{0,2}) \\
&= - \int_X \eta^{2,0} \wedge \mathrm{tr}(F_{\hat{A}}^{0,2}) \\
&= 2\pi i \, \langle [\eta^{2,0}] \smile c_1(E), [X] \rangle
\end{aligned}
\tag{19}
$$

With this said the integral of the formula (18) clearly yields the above formula (16).
$\square$

Corollary 3.5. *Let X be a Kähler surface with $b_2^+(X) > 1$ and let E be a bundle such that its first Chern-class $c_1^{\mathbb{R}}(E)$ is of type $(1,1)$. Then for any self-dual imaginary valued 2-form η with $\eta^{2,0}$ holomorphic and non-zero and constant $\tau \in (0,1]$ the moduli space $M_{\mathfrak{c},E}(\eta, \tau)$ is empty.*

Proof: Under the given hypothesis the preceeding theorem implies that

$$
\{\beta\alpha^*\}_\tau = 4\eta^{0,2} \, \mathrm{id}_E \ .
\tag{20}
$$

But using the definition of $\{\beta\alpha^*\}_\tau$ it is a pure matter of linear algebra to check that for $\eta^{0,2} \neq 0$ this is impossible if $n \geq 2$, because the left hand side of the equation (20) can never be a mutliple of the identity, unless $\alpha = 0$ or $\beta = 0$.

4. Appendix

Proof of Proposition 1.1: Obviously we have $|\mu_{0,\tau}(\Psi)| \geq |\mu_{0,0}(\Psi)|$, so for the first assertion it will be enough to consider $\mu_{0,0}$ alone. We will show that $\mu_{0,0}(\Psi) = 0$ implies $\Psi = 0$. Because $\mu_{0,0}$ is quadratic and the unit sphere inside $\mathbb{C}^2 \otimes \mathbb{C}^n$ is compact we then get the claimed uniform properness-inequality (1).

We shall use the canonical isomorphism $\mathbb{C}^2 \otimes \mathbb{C}^n \cong \mathbb{C}^n \oplus \mathbb{C}^n$, which permits to write a general element

$$
\Psi = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \alpha + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \beta
$$

as

$$
\Psi = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \ .
$$

We then have

$$
\begin{aligned}
\mu_{0,0}(\Psi) &= P(\Psi\Psi^*) \\
&= P\left(\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \begin{pmatrix} \alpha^* & \beta^* \end{pmatrix} \right) \\
&= P\begin{pmatrix} \alpha\alpha^* & \alpha\beta^* \\ \beta\alpha^* & \beta\beta^* \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{2}(\alpha\alpha^* - \beta\beta^*)_0 & (\alpha\beta^*)_0 \\ (\beta\alpha^*)_0 & \frac{1}{2}(\beta\beta^* - \alpha\alpha^*)_0 \end{pmatrix} \ .
\end{aligned}
$$

In particular, if $\mu_{0,0}(\Psi) = 0$, then we have $(\alpha\beta^*)_0 = 0$.

Lemma 4.1. *The equation $(\alpha\beta^*)_0 = 0$ implies that $\alpha = 0$ or $\beta = 0$. In other words, the bilinear map $(\alpha, \beta) \rightarrow (\alpha\beta^*)_0$ is without zero-divisors (here $n \geq 2$ is implicitly understood).*

Proof of Lemma 4.1: Write the elements α and β as

$$\alpha = (\alpha_i)_{i=1}^n \ , \quad \beta = (\beta_i)_{i=1}^n \ .$$

Then the equation $(\alpha\beta^*)_0 = 0$ reads in matrix-form

$$\begin{pmatrix} \alpha_1\overline{\beta_1} - \frac{1}{n}\sum \alpha_i\overline{\beta_i} & \dots & & \alpha_1\overline{\beta_n} \\ \vdots & \ddots & & \vdots \\ \alpha_n\overline{\beta_1} & \dots & \alpha_n\overline{\beta_n} - \frac{1}{n}\sum \alpha_i\overline{\beta_i} \end{pmatrix} = 0 \ .$$

Suppose $\beta \neq 0$, for instance $\beta_j \neq 0$. Then the jth column implies that $\alpha_i = 0$ for all $i \neq j$. Thus the jth element in the jth column simplifies,

$$\alpha_j\overline{\beta_j} - \frac{1}{n}\sum_{i=1}^n \alpha_i\overline{\beta_i} \ = \ (1 - \frac{1}{n})\alpha_j\overline{\beta_j} \ .$$

Therefore we have $\alpha_j = 0$ as well, so that we have $\alpha = 0$. The case $\alpha \neq 0$ is analogous. $\square$

Returning to the problem

$$\begin{pmatrix} \frac{1}{2}(\alpha\alpha^* - \beta\beta^*)_0 & (\alpha\beta^*)_0 \\ (\beta\alpha^*)_0 & \frac{1}{2}(\beta\beta^* - \alpha\alpha^*)_0 \end{pmatrix} = 0 \ ,$$

we see that the lemma gives $\alpha = 0$ or $\beta = 0$. Suppose, without loss of generality, that the first is the case. Then we are left with $(\beta\beta^*)_0 = 0$. Now again with lemma 4.1 we see that this also implies $\beta = 0$. Therefore

$$\Psi = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = 0 \ .$$

The second assertion now follows from the first, remembering that P and Q are both orthogonal projections. For non-negative τ we have the inequality

$$\begin{aligned} (\mu_{0,\tau}(\Psi)\Psi, \Psi) &= (P(\Psi\Psi^*)\Psi, \Psi) + \tau\,(Q(\Psi\Psi^*)\Psi, \Psi) \\ &= (P(\Psi\Psi^*), \Psi\Psi^*) + \tau\,(Q(\Psi\Psi^*), \Psi\Psi^*) \\ &= (P(\Psi\Psi^*), P(\Psi\Psi^*)) + \tau\,(Q(\Psi\Psi^*), Q(\Psi\Psi^*)) \\ &\geq |\mu_{0,0}(\Psi)|^2 \\ &\geq c^2|\Psi|^4 \ . \end{aligned}$$

$\square$

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