ON THE SECOND-ORDER CORRELATION FUNCTION OF THE CHARACTERISTIC POLYNOMIAL OF A HERMITIAN WIGNER MATRIX

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Abstract. We consider the asymptotics of the second-order correlation function of the characteristic polynomial of a random matrix. We show that the known result for a random matrix from the Gaussian Unitary Ensemble essentially continues to hold for a general Hermitian Wigner matrix. Our proofs rely on an explicit formula for the exponential generating function of the second-order correlation function of the characteristic polynomial. Furthermore, we show that the second-order correlation function of the characteristic polynomial is closely related to that of the permanental polynomial.

1. Introduction

The characteristic polynomials of random matrices have attracted considerable interest in the last years, a major reason being the striking similarities between the (asymptotic) moments of the characteristic polynomial of a random matrix from the Circular Unitary Ensemble (CUE) and the (asymptotic) moments of the value distribution of the Riemann zeta function along its critical line (see Keating and Snaith [KS]). These findings have inspired several authors to investigate the moments and correlation functions of the characteristic polynomial also for other random matrix ensembles (see e.g. Brézin and Hikami [BH1, BH2], Mehta and Normand [MN], Fyodorov and Strahov [FS], Strahov and Fyodorov [SF], Baik, Deift and Strahov [BDS], Borodin and Strahov [BS]).

In this paper, we consider the second-order moment and correlation function of the characteristic polynomial of a general (Hermitian) Wigner matrix: Let Q be a probability distribution on the real line such that

$$\int x \; Q(dx) = 0 \,, \quad a := \int x^2 \; Q(dx) = 1/2 \,, \quad b := \int x^4 \; Q(dx) < \infty \,, \tag{1.1}$$

and let $(X_{ii}/\sqrt{2})_{i \in \mathbb{N}}$, $(X_{ij}^{\mathrm{Re}})_{i<j,\, i,j \in \mathbb{N}}$ and $(X_{ij}^{\mathrm{Im}})_{i<j,\, i,j \in \mathbb{N}}$ be independent families of independent random variables with distribution Q on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Also, let $X_{ij} := X_{ij}^{\mathrm{Re}} + \mathbf{i}\, X_{ij}^{\mathrm{Im}}$ and $X_{ji} := X_{ij}^{\mathrm{Re}} - \mathbf{i}\, X_{ij}^{\mathrm{Im}}$ for $i < j$, $i,j \in \mathbb{N}$. Then, for any $N \in \mathbb{N}$, the (Hermitian) Wigner matrix of size $N \times N$ is given by $X_N = (X_{ij})_{1 \le i,j \le N}$, and the second-order correlation function of the characteristic polynomial is given by

$$f(N; \mu, \nu) := \mathbb{E}(\det(X_N - \mu I_N) \cdot \det(X_N - \nu I_N)) \,, \tag{1.2}$$

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where μ, ν are real numbers and I_N denotes the identity matrix of size $N \times N$. We are interested in the asymptotics of the values $f(N; \mu_N, \nu_N)$ as $N \to \infty$, where μ_N, ν_N depend on N in some suitable fashion.

In the special case where Q is the Gaussian distribution with mean 0 and variance 1/2, the distribution of the random matrix X_N is the so-called Gaussian Unitary Ensemble (GUE). (See e.g. FORRESTER [Fo] or MEHTA [Me] and the references cited therein. However, let it be noted that these authors use the variance 1/4 instead of 1/2, so that we have to do some rescalings when using their results.) A remarkable feature of the GUE is that the joint distribution of the eigenvalues of the random matrix X_N is known explicitly: It is given by

$$P_N(d\lambda_1, \ldots, d\lambda_N) = Z_N^{-1} \cdot \prod_{1 \le i < j \le N} (\lambda_j - \lambda_i)^2 \cdot \prod_{i=1}^{N} e^{-\lambda_i^2/2} \, \lambda\!\!\lambda^N(d\lambda_1, \ldots, d\lambda_N)\,,$$

where $\lambda\!\!\lambda^N$ denotes the N-dimensional Lebesgue measure on $\mathbb{R}^N$ and Z_N denotes the normalizing factor $Z_N = (2\pi)^{N/2} \cdot \prod_{k=1}^N k!$. Thus, the correlation function of the characteristic polynomial can be written as

$$f_{\mathrm{GUE}}(N; \mu, \nu) := \int_{\mathbb{R}^N} \prod_{i=1}^{N} (\lambda_i - \mu) \prod_{i=1}^{N} (\lambda_i - \nu) \; P_N(d\lambda_1, \ldots, d\lambda_N)\,,$$

from which it follows (see e.g. the proof of Proposition 4.3 in FORRESTER [Fo]) that

$$f_{\mathrm{GUE}}(N; \mu, \nu) = \frac{\langle p_N, p_N \rangle}{e^{-(\mu^2+\nu^2)/4}} \cdot K_{N+1}(\mu, \nu)\,.$$

Here, the scalar product $\langle \,\cdot\,, \cdot\, \rangle$ is given by $\langle \varphi, \psi \rangle := \int_{-\infty}^{+\infty} \varphi(x)\, \psi(x)\, e^{-x^2/2}\, dx$, the p_k are the monic orthogonal polynomials associated with this scalar product (i.e., up to scaling, the Hermite polynomials), and the kernel K_N is given by

$$K_N(x, y) := e^{-(x^2+y^2)/4} \sum_{k=1}^{N} \frac{p_{k-1}(x) p_{k-1}(y)}{\langle p_{k-1}, p_{k-1} \rangle}\,.$$

Using this representation, it is possible to obtain asymptotic approximations of the values $f_{\mathrm{GUE}}(N; \mu_N, \nu_N)$ from the corresponding asymptotics of the Hermite polynomials (see e.g. Section 8.22 in SZEGÖ [Sz]). It turns out that

$$\lim_{N\to\infty} \sqrt{\frac{\pi}{2N}} \cdot \frac{1}{N!} \cdot f_{\mathrm{GUE}}\left(N; \frac{\pi\mu}{\sqrt{N}}, \frac{\pi\nu}{\sqrt{N}}\right) = \frac{\sin \pi(\mu - \nu)}{\pi(\mu - \nu)}$$

(see e.g. the proof of Proposition 4.14 in FORRESTER [Fo]) and, for $\xi \in (-2, +2)$,

$$\lim_{N\to\infty} \sqrt{\frac{1}{2\pi N}} \cdot \frac{1}{N!} \cdot e^{-N\xi^2/2} \cdot f_{\mathrm{GUE}}(N; \sqrt{N}\xi, \sqrt{N}\xi) = \frac{1}{2\pi} \sqrt{4 - \xi^2}$$

(see e.g. the derivation of the semi-circle law in Chapter 4.3 in FORRESTER [Fo]). More generally, it is known that, for $\xi \in (-2, +2)$,

$$\lim_{N\to\infty} \sqrt{\frac{1}{2\pi N}} \cdot \frac{1}{N!} \cdot e^{-N\xi^2/2} \cdot f_{\mathrm{GUE}}\left(N; \sqrt{N}\xi + \frac{\mu}{\sqrt{N}\varrho(\xi)}, \sqrt{N}\xi + \frac{\nu}{\sqrt{N}\varrho(\xi)}\right)$$
$$= e^{\xi(\mu+\nu)/2\varrho(\xi)} \cdot \varrho(\xi) \cdot \frac{\sin \pi(\mu - \nu)}{\pi(\mu - \nu)}$$

(see e.g. Section 2.1 in Strahov and Fyodorov [SF]), where $\varrho(\xi) := \frac{1}{2\pi}\sqrt{4-\xi^2}$ denotes the density of the semi-circle law. Note that this formula includes the preceding two formulas as special cases.

Even more, it turns out that a similar result holds for the correlation function (of any even order $2, 4, 6, \ldots$) of the characteristic polynomial of a random matrix from the larger class of unitary-invariant ensembles (see e.g. Section 2.1 in Strahov and Fyodorov [SF]). In this respect, it is interesting to note that the emergence of the sine kernel is "universal" in that it is independent of the particular choice of the potential function of the unitary-invariant ensemble. In contrast to that, most of the other factors in the above result for the GUE have to be replaced by potential-specific factors.

It is well-known that the GUE is a special case not only of a unitary-invariant ensemble but also of a (Hermitian) Wigner ensemble as described at the beginning of this section. The purpose of this paper is to show that the above result for the GUE can also be generalized in this direction. More precisely, our main result is as follows:

Theorem 1.1. *Let Q be a probability distribution on the real line satisfying (1.1), let f be defined as in (1.2), let $\xi \in (-2, +2)$, and let $\mu, \nu \in \mathbb{R}$. Then we have*

$$\lim_{N\to\infty} \sqrt{\frac{1}{2\pi N}} \cdot \frac{1}{N!} \cdot e^{-N\xi^2/2} \cdot f\left(N; \sqrt{N}\xi + \frac{\mu}{\sqrt{N}\varrho(\xi)}, \sqrt{N}\xi + \frac{\nu}{\sqrt{N}\varrho(\xi)}\right) \\ = \exp\left(b - \tfrac{3}{4}\right) \cdot e^{\xi(\mu+\nu)/2\varrho(\xi)} \cdot \varrho(\xi) \cdot \frac{\sin \pi(\mu-\nu)}{\pi(\mu-\nu)}\,,$$

where $\varrho(\xi) := \frac{1}{2\pi}\sqrt{4-\xi^2}$ *and* $\sin 0/0 := 1$.

Specifically for the Gaussian distribution with mean 0 and variance 1/2, we have $b = \frac{3}{4}$, so that we re-obtain the above result for the GUE.

Furthermore, we see that for general Wigner matrices, the appropriately rescaled correlation function of the characteristic polynomial asymptotically factorizes into the universal sine kernel, a universal factor involving the density of the semi-circle law, and a non-universal factor depending only on the fourth moment b, or the fourth cumulant $b - \frac{3}{4}$, of the underlying distribution Q.

In particular, it follows immediately that if we normalize the correlation function of the characteristic polynomial by means of its second moment, we obtain the following universality result:

Corollary 1.2. *Under the assumptions of Theorem 1.1, we have*

$$\lim_{N\to\infty} \frac{\mathbb{E}\big(D_N(\xi,\mu)\, D_N(\xi,\nu)\big)}{\sqrt{\mathbb{E}D_N(\xi,\mu)^2}\,\sqrt{\mathbb{E}D_N(\xi,\nu)^2}} = \frac{\sin \pi(\mu-\nu)}{\pi(\mu-\nu)}\,,$$

where $D_N(\xi,\eta) := \det\left(X_N - \left(\sqrt{N}\xi + \frac{\eta}{\sqrt{N}\varrho(\xi)}\right) I_N\right)$.

Moreover, it can be shown that the asymptotics remain unchanged if we replace the correlation function $f(N;\mu,\nu)$ by the "true" correlation (in the sense of probability) of the characteristic polynomial,

$$\begin{aligned}\widetilde{f}(N;\mu,\nu) := \mathbb{E}((\det(X_N-\mu I_N) - \mathbb{E}\det(X_N-\mu I_N))\\ \cdot (\det(X_N-\nu I_N) - \mathbb{E}\det(X_N-\nu I_N)))\,. \qquad (1.3)\end{aligned}$$

We then have the following result:

Proposition 1.3. *Let Q be a probability distribution on the real line satisfying (1.1), let $\widetilde{f}$ be defined as in (1.3), let $\xi \in (-2,+2)$, and let $\mu,\nu \in \mathbb{R}$. Then we have*

$$\begin{aligned}\lim_{N\to\infty} \sqrt{\frac{1}{2\pi N}} \cdot \frac{1}{N!} \cdot e^{-N\xi^2/2} \cdot \widetilde{f}\left(N; \sqrt{N}\xi + \frac{\mu}{\sqrt{N}\varrho(\xi)}, \sqrt{N}\xi + \frac{\nu}{\sqrt{N}\varrho(\xi)}\right)\\ = \exp\left(b - \tfrac{3}{4}\right) \cdot e^{\xi(\mu+\nu)/2\varrho(\xi)} \cdot \varrho(\xi) \cdot \frac{\sin\pi(\mu-\nu)}{\pi(\mu-\nu)}\,,\end{aligned}$$

where $\varrho(\xi) := \frac{1}{2\pi}\sqrt{4-\xi^2}$ and $\sin 0/0 := 1$.

Similarly as before, normalizing the correlation of the characteristic polynomial by means of its variance leads to the following universality result for the correlation coefficient:

Corollary 1.4. *Under the assumptions of Proposition 1.3, we have*

$$\lim_{N\to\infty} \frac{\mathbb{E}\big(\widetilde{D}_N(\xi,\mu)\,\widetilde{D}_N(\xi,\nu)\big)}{\sqrt{\mathbb{E}\widetilde{D}_N(\xi,\mu)^2}\,\sqrt{\mathbb{E}\widetilde{D}_N(\xi,\nu)^2}} = \frac{\sin\pi(\mu-\nu)}{\pi(\mu-\nu)}\,,$$

where $\widetilde{D}_N(\xi,\eta) := \det\left(X_N - \left(\sqrt{N}\xi + \frac{\eta}{\sqrt{N}\varrho(\xi)}\right) I_N\right) - \mathbb{E}\det\left(X_N - \left(\sqrt{N}\xi + \frac{\eta}{\sqrt{N}\varrho(\xi)}\right) I_N\right)$.

The proofs of the above-mentioned results on the correlation function of the characteristic polynomial of a random matrix from the GUE (or another unitary-invariant ensemble) heavily depend on the special structure of the joint distribution of the eigenvalues. However, such a structure seems not to be available for general Wigner matrices.

Instead, we start from recursive equations for the correlation function of the characteristic polynomial (as well as some closely related correlation functions), derive an explicit expression for the associated exponential generating function and deduce all our asymptotic results from this expression. For the determinant of a real symmetric Wigner matrix, a similar analysis was carried out by Zhurbenko [Zh].

The crucial step in our analysis is to obtain an expression in closed form for the exponential generating function of the correlation function. Unfortunately, this approach has proven successful so far only for the second-order correlation function of the characteristic polynomial, which explains why we do not have any results for the higher-order correlation functions of the characteristic polynomial. Note however that for any distribution Q with the first $2k$ moments identical to the Gaussian moments, the correlation function of order k of the characteristic polynomial must be the same as that for the GUE.

Our approach also has some interesting implications for permanental polynomials of (Hermitian) Wigner matrices (see Section 5 for the definition). The analysis of permanental polynomials of random matrices from various well-known ensembles was recently initiated by FYODOROV [Fy], who established some striking similarities between permanental polynomials and characteristic polynomials. In particular, he proved that in the special case of the GUE, the second-order correlation functions of the permanental polynomial and the characteristic polynomial almost coincide, up to a simple transformation of the shift parameters. It follows from our approach that the same is true for arbitrary (Hermitian) Wigner matrices (see Proposition 5.4).

This paper is organized as follows. In Section 2, we start with the analysis of the recursive equations for the correlation function of the characteristic polynomial and derive the explicit expression for its exponential generating function. Sections 3 and 4 are devoted to the proofs of Theorem 1.1 and Proposition 1.3, respectively. In Section 5, we discuss the relationship between the second-order correlation function of the characteristic polynomial and that of the permanental polynomial.

Throughout this paper, K denotes an absolute constant which may change from one occurrence to the next.

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2. GENERATING FUNCTIONS

To simplify the notation, we adopt the following conventions: The determinant of the "empty" (i.e., 0×0) matrix is taken to be 1. If A is an $n \times n$ matrix and z is a real or complex number, we set $A - z := A - zI_n$, where I_n denotes the $n \times n$ identity matrix. Furthermore, if A is an $n \times n$ matrix and $i_1, \ldots, i_m$ and $j_1, \ldots, j_m$ are families of pairwise different indices from the set $\{1, \ldots, n\}$, we write $A^{[i_1,\ldots,i_m:j_1,\ldots,j_m]}$ for the $(n-m) \times (n-m)$-matrix obtained from A by removing the rows indexed by $i_1, \ldots, i_m$ and the columns indexed by $j_1, \ldots, j_m$. Thus, for any $n \times n$ matrix $A = (a_{ij})_{1 \le i,j \le n}$ $(n \ge 1)$, we have the identity

$$\det(A) = \sum_{i,j=1}^{n-1} (-1)^{i+j-1}\, a_{i,n}\, a_{n,j}\, \det(A^{[n,i:n,j]}) + a_{n,n} \det(A^{[n:n]})\,, \tag{2.1}$$

as follows by expanding the determinant about the last row and the last column. (For $n = 1$, note that the big sum vanishes.)

Recall that we write X_N for the random matrix $(X_{ij})_{1 \le i,j \le N}$, where the X_{ij} are the random variables introduced below (1.1). We will analyze the function

$$f(N;\mu,\nu) := \mathbb{E}\left(\det(X_N - \mu) \cdot \det(X_N - \nu)\right) \qquad (N \ge 0)\,.$$

To this purpose, we will also need the auxiliary functions

$$
\begin{aligned}
f_{11}(N;\mu,\nu) &:= \mathbb{E}(\det((X_N-\mu)^{[1:1]})\cdot\det((X_N-\nu)^{[2:2]})) && (N\geq 2)\,,\\
f_{11}^{\chi}(N;\mu,\nu) &:= \mathbb{E}(\det((X_N-\mu)^{[1:2]})\cdot\det((X_N-\nu)^{[2:1]})) && (N\geq 2)\,,\\
f_{10}(N;\mu,\nu) &:= \mathbb{E}(\det(X_{N-1}-\mu)\cdot\det(X_N-\nu)) && (N\geq 1)\,,\\
f_{01}(N;\mu,\nu) &:= \mathbb{E}(\det(X_N-\mu)\cdot\det(X_{N-1}-\nu)) && (N\geq 1)\,.
\end{aligned}
$$

Since μ and ν can be regarded as constants for the purposes of this section, we will only write $f(N)$ instead of $f(N;\mu,\nu)$ in the sequel, etc.

We have the following recursive equations:

Lemma 2.1.

$$
\begin{aligned}
f(0) &= 1\,,\\
f(N) &= (1+\mu\nu)\,f(N-1)+(2b+\tfrac{1}{2})(N-1)\,f(N-2)\\
&\quad +\,(N-1)(N-2)\,f_{11}(N-1)\\
&\quad +\,(N-1)(N-2)\,f_{11}^{\chi}(N-1)\\
&\quad +\,\nu(N-1)\,f_{10}(N-1)\\
&\quad +\,\mu(N-1)\,f_{01}(N-1) && (N\geq 1)\,, && (2.2)\\
f_{11}(N) &= \mu\nu\,f(N-2)+(N-2)\,f(N-3)\\
&\quad +\,(N-2)(N-3)\,f_{11}(N-2)\\
&\quad +\,\nu(N-2)\,f_{10}(N-2)\\
&\quad +\,\mu(N-2)\,f_{01}(N-2) && (N\geq 2)\,, && (2.3)\\
f_{11}^{\chi}(N) &= f(N-2)+(N-2)\,f(N-3)\\
&\quad +\,(N-2)(N-3)\,f_{11}^{\chi}(N-2) && (N\geq 2)\,, && (2.4)\\
f_{10}(N) &= -(N-1)\,f_{01}(N-1)-\nu\,f(N-1) && (N\geq 1)\,, && (2.5)\\
f_{01}(N) &= -(N-1)\,f_{10}(N-1)-\mu\,f(N-1) && (N\geq 1)\,. && (2.6)
\end{aligned}
$$

For the sake of clarity, note that these recursive equations may contain some terms which have not been defined (such as $f_{11}(N-1)$ for $N=1$), but this is not a problem since these terms occur in combination with the factor zero only.

Proof. We begin with the proof of (2.2). For $N=0$, the result is clear. For $N\geq 1$, we expand the determinants of the matrices $(X_N-\mu)$ and $(X_N-\nu)$ as in (2.1) and use the independence of the random variables $X_{ij}=\overline{X}_{ji}$ $(i\leq j)$, to the effect that

$$\begin{aligned}
&f(N) \\
&= \sum_{i,j=1}^{N-1} \sum_{k,l=1}^{N-1} (-1)^{i+j+k+l}\, \mathbb{E}\left(X_{i,N} X_{N,j} X_{k,N} X_{N,l}\right) \cdot \mathbb{E}\left(\det(X_{N-1}-\mu)^{[i:j]} \cdot \det(X_{N-1}-\nu)^{[k:l]}\right) \\
&\quad + \sum_{i,j=1}^{N-1} (-1)^{i+j+1}\, \mathbb{E}\left(X_{i,N} X_{N,j}\right) \cdot \mathbb{E}\left(X_{N,N}-\nu\right) \cdot \mathbb{E}\left(\det(X_{N-1}-\mu)^{[i:j]} \cdot \det(X_{N-1}-\nu)\right) \\
&\quad + \sum_{k,l=1}^{N-1} (-1)^{k+l+1}\, \mathbb{E}\left(X_{k,N} X_{N,l}\right) \cdot \mathbb{E}\left(X_{N,N}-\mu\right) \cdot \mathbb{E}\left(\det(X_{N-1}-\mu) \cdot \det(X_{N-1}-\nu)^{[k:l]}\right) \\
&\quad + \mathbb{E}\left((X_{N,N}-\mu)(X_{N,N}-\nu)\right) \cdot \mathbb{E}\left(\det(X_{N-1}-\mu) \cdot \det(X_{N-1}-\nu)\right)\,.
\end{aligned}$$

Since the (complex-valued) random variables $X_{ij} = \overline{X_{ji}}$ $(i \leq j)$ are independent with $\mathbb{E}(X_{ij}) = 0$ $(i \leq j)$ and $\mathbb{E}(X_{ij}^2) = 0$ $(i < j)$, several of the expectations vanish, and the sum reduces to

$$\begin{aligned}
&f(N) \\
&= (\mathbb{E}X_{N,N}^2 + \mu\nu) \cdot \mathbb{E}\left(\det(X_{N-1}-\mu) \cdot \det(X_{N-1}-\nu)\right) \\
&\quad + \sum_{i=j=k=l} \mathbb{E}|X_{i,N}|^4 \cdot \mathbb{E}\left(\det(X_{N-1}-\mu)^{[i:j]} \cdot \det(X_{N-1}-\nu)^{[k:l]}\right) \\
&\quad + \sum_{i=j\neq k=l} \mathbb{E}|X_{i,N}|^2 \cdot \mathbb{E}|X_{k,N}|^2 \cdot \mathbb{E}\left(\det(X_{N-1}-\mu)^{[i:j]} \cdot \det(X_{N-1}-\nu)^{[k:l]}\right) \\
&\quad + \sum_{i=l\neq j=k} \mathbb{E}|X_{i,N}|^2 \cdot \mathbb{E}|X_{k,N}|^2 \cdot \mathbb{E}\left(\det(X_{N-1}-\mu)^{[i:j]} \cdot \det(X_{N-1}-\nu)^{[k:l]}\right) \\
&\quad + \nu \sum_{i=j} \mathbb{E}|X_{i,N}|^2 \cdot \mathbb{E}\left(\det(X_{N-1}-\mu)^{[i:j]} \cdot \det(X_{N-1}-\nu)\right) \\
&\quad + \mu \sum_{k=l} \mathbb{E}|X_{k,N}|^2 \cdot \mathbb{E}\left(\det(X_{N-1}-\mu) \cdot \det(X_{N-1}-\nu)^{[k:l]}\right)\,.
\end{aligned}$$

(2.2) now follows by noting that $\mathbb{E}X_{N,N}^2 = 1$, $\mathbb{E}|X_{i,N}|^2 = 1$, $\mathbb{E}|X_{i,N}|^4 = 2b + \frac{1}{2}$, and by using symmetry.

To prove (2.3), we apply the analogue of (2.1) for the first row and the first column to the matrices $(X_N - \mu)^{[1:1]}$ and $(X_N - \nu)^{[2:2]}$. Using similar arguments as in the proof of (2.2) afterwards, we obtain

$$\begin{aligned}
&f_{11}(N) \\
&= \sum_{i,j=3}^{N} \sum_{k,l=3}^{N} (-1)^{i+j+k+l}\, \mathbb{E}\left(X_{i,2} X_{2,j}\right) \cdot \mathbb{E}\left(X_{k,1} X_{1,l}\right) \cdot \mathbb{E}\left(\det(X_N-\mu)^{[1,2,i:1,2,j]} \cdot \det(X_N-\nu)^{[1,2,k:1,2,l]}\right) \\
&\quad + \sum_{i,j=3}^{N} (-1)^{i+j+1}\, \mathbb{E}\left(X_{i,2} X_{2,j}\right) \cdot \mathbb{E}\left(X_{1,1}-\nu\right) \cdot \mathbb{E}\left(\det(X_N-\mu)^{[1,2,i:1,2,j]} \cdot \det(X_N-\nu)^{[1,2:1,2]}\right) \\
&\quad + \sum_{k,l=3}^{N} (-1)^{k+l+1}\, \mathbb{E}\left(X_{k,1} X_{1,l}\right) \cdot \mathbb{E}\left(X_{2,2}-\mu\right) \cdot \mathbb{E}\left(\det(X_N-\mu)^{[1,2:1,2]} \cdot \det(X_N-\nu)^{[1,2,k:1,2,l]}\right) \\
&\quad + \mathbb{E}\left(X_{2,2}-\mu\right) \cdot \mathbb{E}\left(X_{1,1}-\nu\right) \cdot \mathbb{E}\left(\det(X_N-\mu)^{[1,2:1,2]} \cdot \det(X_N-\nu)^{[1,2:1,2]}\right) \\
&= \mu\nu \cdot \mathbb{E}\left(\det(X_N-\mu)^{[1,2:1,2]} \cdot \det(X_N-\nu)^{[1,2:1,2]}\right) \\
&\quad + \sum_{i=j=k=l} \mathbb{E}|X_{i,2}|^2 \cdot \mathbb{E}|X_{k,1}|^2 \cdot \mathbb{E}\left(\det(X_N-\mu)^{[1,2,i:1,2,j]} \cdot \det(X_N-\nu)^{[1,2,k:1,2,l]}\right) \\
&\quad + \sum_{i=j\neq k=l} \mathbb{E}|X_{i,2}|^2 \cdot \mathbb{E}|X_{k,1}|^2 \cdot \mathbb{E}\left(\det(X_N-\mu)^{[1,2,i:1,2,j]} \cdot \det(X_N-\nu)^{[1,2,k:1,2,l]}\right) \\
&\quad + \nu \sum_{i=j} \mathbb{E}|X_{i,2}|^2 \cdot \mathbb{E}\left(\det(X_N-\mu)^{[1,2,i:1,2,j]} \cdot \det(X_N-\nu)^{[1,2:1,2]}\right) \\
&\quad + \mu \sum_{k=l} \mathbb{E}|X_{k,1}|^2 \cdot \mathbb{E}\left(\det(X_N-\mu)^{[1,2:1,2]} \cdot \det(X_N-\nu)^{[1,2,k:1,2,l]}\right)
\end{aligned}$$

and hence (2.3).

The proof of (2.4) is similar to that of (2.3):

$$\begin{aligned}
&f_{11}^{\chi}(N) \\
&= \sum_{i,j=3}^{N} \sum_{k,l=3}^{N} (-1)^{i+j+k+l}\, \mathbb{E}\left(X_{i,1} X_{1,l}\right) \cdot \mathbb{E}\left(X_{2,j} X_{k,2}\right) \cdot \mathbb{E}\left(\det(X_N-\mu)^{[1,2,i:1,2,j]} \cdot \det(X_N-\nu)^{[1,2,k:1,2,l]}\right) \\
&\quad + \sum_{i,j=3}^{N} (-1)^{i+j+1}\, \mathbb{E}\left(X_{i,1} X_{2,j}\right) \cdot \mathbb{E}\left(X_{1,2}\right) \cdot \mathbb{E}\left(\det(X_N-\mu)^{[1,2,i:1,2,j]} \cdot \det(X_N-\nu)^{[1,2:1,2]}\right) \\
&\quad + \sum_{k,l=3}^{N} (-1)^{k+l+1}\, \mathbb{E}\left(X_{k,2} X_{1,l}\right) \cdot \mathbb{E}\left(X_{2,1}\right) \cdot \mathbb{E}\left(\det(X_N-\mu)^{[1,2:1,2]} \cdot \det(X_N-\nu)^{[1,2,k:1,2,l]}\right) \\
&\quad + \mathbb{E}\left(X_{2,1} X_{1,2}\right) \cdot \mathbb{E}\left(\det(X_N-\mu)^{[1,2:1,2]} \cdot \det(X_N-\nu)^{[1,2:1,2]}\right) \\
&= \mathbb{E}\left(\det(X_N-\mu)^{[1,2:1,2]} \cdot \det(X_N-\nu)^{[1,2:1,2]}\right) \\
&\quad + \sum_{i=l=k=j} \mathbb{E}|X_{i,1}|^2 \cdot \mathbb{E}|X_{k,2}|^2 \cdot \mathbb{E}\left(\det(X_{N-1}-\mu)^{[1,2,i:1,2,j]} \cdot \det(X_{N-1}-\nu)^{[1,2,k:1,2,l]}\right) \\
&\quad + \sum_{i=l\neq k=j} \mathbb{E}|X_{i,1}|^2 \cdot \mathbb{E}|X_{k,2}|^2 \cdot \mathbb{E}\left(\det(X_{N-1}-\mu)^{[1,2,i:1,2,j]} \cdot \det(X_{N-1}-\nu)^{[1,2,k:1,2,l]}\right) .
\end{aligned}$$

For the proof of (2.5), we expand the determinant of the matrix $(X_N - \nu)$ as in (2.1) and use similar arguments as above to obtain

$$\begin{aligned} f_{10}(N) &= \sum_{k,l=1}^{N-1} (-1)^{k+l+1} \, \mathbb{E}\left(X_{k,N} X_{N,l}\right) \cdot \mathbb{E}\left(\det(X_{N-1} - \mu) \cdot \det(X_{N-1} - \nu)^{[k:l]}\right) \\ &\quad + \mathbb{E}\left(X_{N,N} - \nu\right) \cdot \mathbb{E}\left(\det(X_{N-1} - \mu) \cdot \det(X_{N-1} - \nu)\right) \\ &= -\sum_{k=l} \mathbb{E}|X_{k,N}|^2 \cdot \mathbb{E}\left(\det(X_{N-1} - \mu) \cdot \det(X_{N-1} - \nu)^{[k:l]}\right) \\ &\quad - \nu \cdot \mathbb{E}\left(\det(X_{N-1} - \mu) \cdot \det(X_{N-1} - \nu)\right) \end{aligned}$$

and hence (2.5).

The proof of (2.6) is completely analogous to that of (2.5). □

The interesting (although apparently rather special) phenomenon is that the preceding recursions can be combined into a single recursion involving only the values $f(N)$. To shorten the notation, we put

$$c(N) := \frac{f(N)}{N!} \qquad (N \geq 0)$$

and

$$s(N) := \sum_{\substack{k=0,\ldots,N \\ k \text{ even}}} c(N-k) \qquad (N \geq 0)\,.$$

We then have the following result:

Lemma 2.2. *The values $c(N)$ satisfy the recursive equation*

$$\begin{aligned} c(0) &= 1\,, && (2.7) \\ N c(N) &= c(N-1) + N \cdot c(N-2) \\ &\quad + \mu\nu \cdot \big(s(N-1) + s(N-3)\big) \\ &\quad - (\mu^2 + \nu^2) \cdot s(N-2) \\ &\quad + (2b - \tfrac{3}{2}) \cdot \big(c(N-2) - c(N-4)\big) \qquad (N \geq 1)\,, && (2.8) \end{aligned}$$

where all terms $c(\,\cdot\,)$ and $s(\,\cdot\,)$ with a negative argument are taken to be zero.

Proof. It is immediate from Lemma 2.1 that

$$f(N) = f_{11}(N+1) + f_{11}^{\chi}(N+1) + (2b - \tfrac{3}{2})(N-1)\, f(N-2)$$

for all $N \geq 1$. Using this relation for $N-2$ instead of N (where now $N \geq 3$), we can substitute $f_{11}(N-1) + f_{11}^{\chi}(N-1)$ on the right-hand side of (2.2) to obtain

$$\begin{aligned}
f(N) &= (1+\mu\nu)\, f(N-1) + (2b+\tfrac{1}{2})(N-1)\, f(N-2) \\
&\qquad + (N-1)(N-2)\Big(f(N-2) - (2b-\tfrac{3}{2})(N-3) f(N-4)\Big) \\
&\qquad + \nu(N-1)\, f_{10}(N-1) \\
&\qquad + \mu(N-1)\, f_{01}(N-1) \\
&= (1+\mu\nu)\, f(N-1) + N(N-1)\, f(N-2) \\
&\qquad + \nu(N-1)\, f_{10}(N-1) \\
&\qquad + \mu(N-1)\, f_{01}(N-1) \\
&\qquad + (2b-\tfrac{3}{2}) \cdot \Big((N-1)\, f(N-2) - (N-1)(N-2)(N-3)\, f(N-4)\Big)
\end{aligned}$$

for all $N \geq 3$. Dividing by $(N-1)!$, it follows that

$$\begin{aligned}
Nc(N) &= (1+\mu\nu) \cdot c(N-1) + Nc(N-2) \\
&\qquad + \nu \cdot f_{10}(N-1) \,/\, (N-2)! \\
&\qquad + \mu \cdot f_{01}(N-1) \,/\, (N-2)! \\
&\qquad + (2b-\tfrac{3}{2}) \cdot \Big(c(N-2) - c(N-4)\Big)
\end{aligned}$$

for all $N \geq 3$. (For $N = 3$, note that the second term in the large bracket vanishes.) A straightforward induction using (2.5) and (2.6) shows that

$$f_{10}(N-1) \,/\, (N-2)! = -\nu s(N-2) + \mu s(N-3)\,,$$

$$f_{01}(N-1) \,/\, (N-2)! = -\mu s(N-2) + \nu s(N-3)\,,$$

for all $N \geq 3$, which yields the assertion for $N \geq 3$.

For $N < 3$, the assertion is verified by direct calculation, also making use of Lemma 2.1:

$$c(0) = f(0) = 1\,.$$

$$\begin{aligned}
1c(1) = f(1) &= (1+\mu\nu) f(0) \\
&= (1+\mu\nu) c(0) \\
&= c(0) + \mu\nu s(0)\,.
\end{aligned}$$

$$\begin{aligned}
2c(2) = f(2) &= (1+\mu\nu) f(1) + (2b+\tfrac{1}{2}) f(0) + \nu(-\nu f(0)) + \mu(-\mu f(0)) \\
&= (1+\mu\nu) f(1) + (2b+\tfrac{1}{2}) f(0) - (\mu^2+\nu^2) f(0) \\
&= (1+\mu\nu) c(1) + (2b+\tfrac{1}{2}) c(0) - (\mu^2+\nu^2) c(0) \\
&= c(1) + 2c(0) + \mu\nu c(1) - (\mu^2+\nu^2) c(0) + (2b-\tfrac{3}{2}) c(0) \\
&= c(1) + 2c(0) + \mu\nu s(1) - (\mu^2+\nu^2) s(0) + (2b-\tfrac{3}{2}) c(0)\,.
\end{aligned}$$

□

We can now determine the exponential generating function of the sequence $(f(N))_{N \geq 0}$:

Lemma 2.3. *The exponential generating function* $F(x) := \sum_{N=0}^{\infty} f(N)\, x^N \,/\, N!$ *of the sequence* $(f(N))_{N \geq 0}$ *is given by*

$$F(x) = \frac{\exp\left(\mu\nu \cdot \frac{x}{1-x^2} - \frac{1}{2}(\mu^2+\nu^2) \cdot \frac{x^2}{1-x^2} + b^* x^2\right)}{(1-x)^{3/2} \cdot (1+x)^{1/2}}\,,$$

where $b^* := b - \frac{3}{4}$.

Proof. It is straightforward to obtain $F(x)$ starting from (2.7) and (2.8) and using the basic properties of generating functions. For the sake of completeness, we provide a detailed proof.

To begin with, recall that $f(N)/N! = c(N)$. Multiplying (2.8) by x^{N-1}, summing over N and recalling our convention concerning negative arguments, we have

$$\begin{aligned}\sum_{N=1}^{\infty} Nc(N)x^{N-1} = {} & \sum_{N=1}^{\infty} c(N-1)x^{N-1} + \sum_{N=2}^{\infty} Nc(N-2)x^{N-1} \\ & + \mu\nu \left(\sum_{N=1}^{\infty} s(N-1)x^{N-1} + \sum_{N=3}^{\infty} s(N-3)x^{N-1} \right) \\ & - (\mu^2+\nu^2) \sum_{N=2}^{\infty} s(N-2)x^{N-1} \\ & + 2b^* \left(\sum_{N=2}^{\infty} c(N-2)x^{N-1} - \sum_{N=4}^{\infty} c(N-4)x^{N-1} \right),\end{aligned}$$

whence

$$\begin{aligned}F'(x) = {} & F(x) + (2xF(x) + x^2 F'(x)) \\ & + \mu\nu \frac{1+x^2}{1-x^2} F(x) - (\mu^2+\nu^2) \frac{x}{1-x^2} F(x) + 2b^* \left(xF(x) - x^3 F(x) \right).\end{aligned}$$

We therefore obtain the differential equation

$$F'(x) = \left(\frac{1+2x}{1-x^2} + \mu\nu \frac{1+x^2}{(1-x^2)^2} - (\mu^2+\nu^2) \frac{x}{(1-x^2)^2} + 2b^* x \right) F(x)\,,$$

which has the solution

$$F(x) = \frac{F_0}{(1-x)^{3/2} \cdot (1+x)^{1/2}} \exp\left(\mu\nu \frac{x}{1-x^2} - \tfrac{1}{2}(\mu^2+\nu^2) \frac{1}{1-x^2} + b^* x^2 \right).$$

Here, F_0 denotes a multiplicative constant which is determined by (2.7):

$$F_0 = \exp(\tfrac{1}{2}(\mu^2+\nu^2))\,.$$

We therefore obtain

$$F(x) = \frac{1}{(1-x)^{3/2} \cdot (1+x)^{1/2}} \exp\left(\mu\nu \frac{x}{1-x^2} - \tfrac{1}{2}(\mu^2+\nu^2) \frac{x^2}{1-x^2} + b^* x^2 \right),$$

which completes the proof. □

3. The Proof of Theorem 1.1

To prove Theorem 1.1, we will establish the following slightly more general result:

Proposition 3.1. *Let* Q *be a probability distribution on the real line satisfying (1.1), let* f *be defined as in (1.2), let* $(\xi_N)_{N\in\mathbb{N}}$ *be a sequence of real numbers such that* $\lim_{N\to\infty}\xi_N/\sqrt{N}=\xi$ *for some* $\xi\in(-2,+2)$*, and let* $\eta\in\mathbb{C}$*. Then we have*

$$\lim_{N\to\infty}\sqrt{\frac{2\pi}{N}}\cdot\frac{1}{N!}\cdot\exp(-\xi_N^2/2)\cdot f\left(N;\xi_N+\frac{\eta}{\sqrt{N}},\xi_N-\frac{\eta}{\sqrt{N}}\right)$$
$$=\exp\left(b-\tfrac{3}{4}\right)\cdot\sqrt{4-\xi^2}\cdot\frac{\sin(\sqrt{4-\xi^2}\cdot\eta)}{(\sqrt{4-\xi^2}\cdot\eta)}\,,$$

where $\sin 0/0:=1$.

It is easy to deduce Theorem 1.1 from Proposition 3.1:

Proof of Theorem 1.1. Taking

$$\xi_N:=\sqrt{N}\xi+\frac{\pi(\mu+\nu)}{\sqrt{N}\cdot\sqrt{4-\xi^2}}\qquad\text{and}\qquad\eta:=\frac{\pi(\mu-\nu)}{\sqrt{4-\xi^2}}$$

in Proposition 3.1, we have

$$\lim_{N\to\infty}\sqrt{\frac{2\pi}{N}}\cdot\frac{1}{N!}\cdot\exp\left(-N\xi^2/2-\pi\xi(\mu+\nu)/\sqrt{4-\xi^2}\right)$$
$$\cdot\, f\left(N;\sqrt{N}\xi+\frac{2\pi\mu}{\sqrt{N}\sqrt{4-\xi^2}},\sqrt{N}\xi+\frac{2\pi\nu}{\sqrt{N}\sqrt{4-\xi^2}}\right)$$
$$=\exp\left(b-\tfrac{3}{4}\right)\cdot\sqrt{4-\xi^2}\cdot\frac{\sin\pi(\mu-\nu)}{\pi(\mu-\nu)}\,.$$

Multiplying by $\frac{1}{2\pi}\exp\left(\pi\xi(\mu+\nu)/\sqrt{4-\xi^2}\right)$ yields Theorem 1.1. □

It therefore remains to prove Proposition 3.1:

Proof of Proposition 3.1. By Lemma 2.3, we have

$$\sum_{N=0}^{\infty}\frac{f(N;\mu,\nu)}{N!}z^N=\frac{\exp\left(\mu\nu\cdot\frac{z}{1-z^2}-\frac{1}{2}(\mu^2+\nu^2)\cdot\frac{z^2}{1-z^2}+b^*z^2\right)}{(1-z)^{3/2}\cdot(1+z)^{1/2}}\,.$$

Thus, by Cauchy's formula, we have the integral representation

$$\frac{f(N;\mu,\nu)}{N!}=\frac{1}{2\pi i}\int_\gamma\frac{\exp\left(\mu\nu\cdot\frac{z}{1-z^2}-\frac{1}{2}(\mu^2+\nu^2)\cdot\frac{z^2}{1-z^2}+b^*z^2\right)}{(1-z)^{3/2}\cdot(1+z)^{1/2}}\,\frac{dz}{z^{N+1}}\,,\qquad(3.1)$$

where $\gamma\equiv\gamma_N$ denotes the counterclockwise circle of radius $R\equiv R_N=1-1/N$ around the origin. (We may and do assume that $N\geq 2$ for the rest of the proof.)

Setting $\mu = \xi_N + \eta/\sqrt{N}$ and $\nu = \xi_N - \eta/\sqrt{N}$, we have

$$\begin{aligned}
&\exp\left(\mu\nu \cdot \frac{z}{1-z^2} - \tfrac{1}{2}(\mu^2+\nu^2) \cdot \frac{z^2}{1-z^2} + b^* z^2\right) \\
&= \exp\left((\xi_N^2 - \eta^2/N) \cdot \frac{z}{1-z^2} - (\xi_N^2 + \eta^2/N) \cdot \frac{z^2}{1-z^2} + b^* z^2\right) \\
&= \exp\left(\xi_N^2 \cdot \frac{z}{1+z} - (\eta^2/N) \cdot \frac{z}{1-z} + b^* z^2\right) \\
&= \exp\left(\tfrac{1}{2}\xi_N^2 + \eta^2/N\right) \cdot \exp\left(-\tfrac{1}{2}\xi_N^2 \cdot \frac{1-z}{1+z} - (\eta^2/N) \cdot \frac{1}{1-z} + b^* z^2\right) .
\end{aligned}$$

We therefore obtain

$$\begin{aligned}
\frac{1}{N!} \cdot f\left(N; \xi_N + \frac{\eta}{\sqrt{N}}, \xi_N - \frac{\eta}{\sqrt{N}}\right) &= \exp\left(\tfrac{1}{2}\xi_N^2 + \eta^2/N\right) \\
&\cdot \frac{1}{2\pi i} \int_\gamma \frac{\exp\left(-\tfrac{1}{2}\xi_N^2 \cdot \tfrac{1-z}{1+z} - (\eta^2/N) \cdot \tfrac{1}{1-z} + b^* z^2\right)}{(1-z)^{3/2} \cdot (1+z)^{1/2}} \, \frac{dz}{z^{N+1}} . \qquad (3.2)
\end{aligned}$$

The idea is that the main contribution to the integral in (3.2) comes from a small neighborhood of $z = 1$, where the function

$$h(z) := \frac{\exp\left(-\tfrac{1}{2}\xi_N^2 \cdot \tfrac{1-z}{1+z} - (\eta^2/N) \cdot \tfrac{1}{1-z} + b^* z^2\right)}{(1-z)^{3/2} \cdot (1+z)^{1/2}}$$

can be well approximated by the simpler function

$$h_0(z) := \frac{\exp(b^*)}{\sqrt{2}} \cdot \frac{\exp\left(-\tfrac{1}{4}\xi_N^2 \cdot (1-z) - (\eta^2/N) \cdot \tfrac{1}{1-z}\right)}{(1-z)^{3/2}} .$$

We therefore rewrite the integral in (3.2) as

$$\frac{1}{2\pi i} \int_\gamma h(z) \, \frac{dz}{z^{N+1}} = I_1 + I_2 + I_3 - I_4 \qquad (3.3)$$

with

$$I_1 := \frac{1}{2\pi i} \int_\gamma h_0(z) \, \frac{dz}{z^{N+1}} , \qquad (3.4)$$

$$I_2 := \frac{1}{2\pi} \int_{1/\sqrt{N}}^{2\pi - 1/\sqrt{N}} h(Re^{it}) \, \frac{dt}{(Re^{it})^N} , \qquad (3.5)$$

$$I_3 := \frac{1}{2\pi} \int_{-1/\sqrt{N}}^{+1/\sqrt{N}} \left(h(Re^{it}) - h_0(Re^{it})\right) \, \frac{dt}{(Re^{it})^N} , \qquad (3.6)$$

$$I_4 := \frac{1}{2\pi} \int_{1/\sqrt{N}}^{2\pi - 1/\sqrt{N}} h_0(Re^{it}) \, \frac{dt}{(Re^{it})^N} . \qquad (3.7)$$

We will show that the integral I_1 is the asymptotically dominant term.

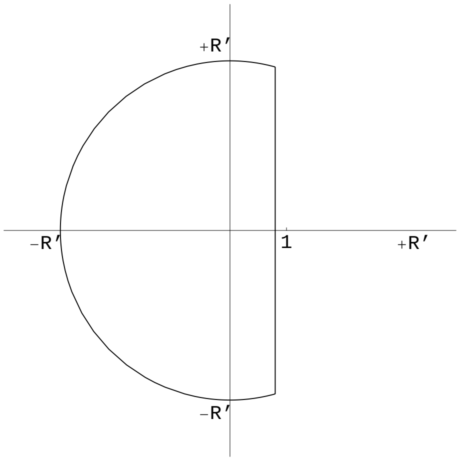

FIGURE 1. The contour δ.

First of all, note that since $\xi_N \in \mathbb{R}$, we have

$$\left|\exp\left(-\tfrac{1}{4}\xi_N^2 \cdot (1-z)\right)\right| = \exp\left(-\tfrac{1}{4}\xi_N^2 \cdot \mathrm{Re}\,(1-z)\right) \le 1 \tag{3.8}$$

for any $z \in \mathbb{C}$ with $\mathrm{Re}\,(z) \le 1$. Plugging in the series expansion

$$\exp\left(-(\eta^2/N) \cdot \frac{1}{1-z}\right) = \sum_{l=0}^{\infty} \frac{(-1)^l\,\eta^{2l}}{l!\,N^l}\,\frac{1}{(1-z)^l}$$

and using uniform convergence on the contour γ (for fixed $N \ge 2$ and $\eta \in \mathbb{C}$), we obtain

$$\frac{I_1}{\sqrt{N}} = \frac{\exp(b^*)}{\sqrt{2}} \cdot \sum_{l=0}^{\infty} \frac{(-1)^l\,\eta^{2l}}{l!\,N^{l+1/2}} \cdot \frac{1}{2\pi i}\int_\gamma \frac{\exp\left(-\frac{1}{4}\xi_N^2 \cdot (1-z)\right)}{(1-z)^{l+3/2}}\,\frac{dz}{z^{N+1}}\,. \tag{3.9}$$

We will show that for each $l = 0, 1, 2, 3, \ldots$,

$$\lim_{N\to\infty}\left(\frac{(-1)^l\,\eta^{2l}}{l!\,N^{l+1/2}} \cdot \frac{1}{2\pi i}\int_\gamma \frac{\exp\left(-\frac{1}{4}\xi_N^2 \cdot (1-z)\right)}{(1-z)^{l+3/2}}\,\frac{dz}{z^{N+1}}\right) = \frac{1}{\sqrt{\pi}} \cdot \frac{(-1)^l\,\eta^{2l}}{(2l+1)!} \cdot (4-\xi^2)^{l+1/2}\,. \tag{3.10}$$

To begin with,

$$\frac{1}{2\pi i}\int_\gamma \frac{\exp\left(-\frac{1}{4}\xi_N^2 \cdot (1-z)\right)}{(1-z)^{l+3/2}}\,\frac{dz}{z^{N+1}} = \frac{1}{2\pi i}\int_{(1-1/N)-\mathbf{i}\,\infty}^{(1-1/N)+\mathbf{i}\,\infty} \frac{\exp\left(-\frac{1}{4}\xi_N^2 \cdot (1-z)\right)}{(1-z)^{l+3/2}}\,\frac{dz}{z^{N+1}}\,. \tag{3.11}$$

In fact, for any $R' > 1$, we can replace the contour γ by the contour δ which consists of the line segment between the points $R - \mathbf{i}\,\sqrt{(R')^2 - R^2}$ and $R + \mathbf{i}\,\sqrt{(R')^2 - R^2}$, and the arc of radius R' around the origin to the left of this line segment (see Figure 1). Now, it is easy to see that the integral along this arc is bounded above by

$$\frac{1}{2\pi} \cdot 2\pi R' \cdot \frac{1}{(R'-1)^{l+3/2}} \cdot \frac{1}{(R')^{N+1}}$$

and therefore tends to zero as $R' \to \infty$, whence (3.11).

Next, performing a change of variables, we find that the right-hand side in (3.11) is equal to

$$N^{l+1/2} \cdot \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{\exp\left(-\frac{1}{4}(\xi_N^2/N) \cdot (1-iu)\right)}{(1-iu)^{l+3/2}} \, \frac{du}{(1-\frac{1-iu}{N})^{N+1}} \, .$$

Since $\lim_{N\to\infty} \xi_N/\sqrt{N} = \xi$, it follows by the dominated convergence theorem that

$$\lim_{N\to\infty} \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{\exp\left(-\frac{1}{4}(\xi_N^2/N) \cdot (1-iu)\right)}{(1-iu)^{l+3/2}} \, \frac{du}{(1-\frac{1-iu}{N})^{N+1}} \\ = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{\exp\left((1-\frac{1}{4}\xi^2) \cdot (1-iu)\right)}{(1-iu)^{l+3/2}} \, du \, ,$$

which is equal to

$$\frac{(1-\frac{1}{4}\xi^2)^{l+1/2}}{\Gamma(l+3/2)} = \frac{1}{\sqrt{\pi}} \cdot \frac{l!}{(2l+1)!} \cdot (4-\xi^2)^{l+1/2}$$

by Laplace inversion (see e.g. Chapter 24 in Doetsch [Do]) and the functional equation of the Gamma function. This proves (3.10).

Let $\varepsilon > 0$ denote a constant such that $\cos t \leq 1 - \varepsilon^2 t^2$ for $-\pi \leq t \leq +\pi$. Then, for any $\alpha > 1$, we have the estimate

$$\begin{aligned} \int_{-\pi}^{+\pi} \frac{1}{|1-Re^{it}|^\alpha} \, dt &= \int_{-\pi}^{+\pi} \frac{1}{(1+R^2-2R\cos t)^{\alpha/2}} \, dt \\ &\leq \int_{-\pi}^{+\pi} \frac{1}{((1-R)^2+\varepsilon^2 t^2)^{\alpha/2}} \, dt \\ &= N^\alpha \int_{-\pi}^{+\pi} \frac{1}{(1+N^2\varepsilon^2 t^2)^{\alpha/2}} \, dt \\ &= K N^{\alpha-1} \int_{-N\varepsilon\pi}^{+N\varepsilon\pi} \frac{1}{(1+u^2)^{\alpha/2}} \, du \\ &\leq K N^{\alpha-1} \left(1 + \int_1^\infty \frac{1}{u^\alpha} \, du\right) \\ &\leq K N^{\alpha-1} \left(1 + \frac{1}{\alpha-1}\right) \, , \end{aligned} \tag{3.12}$$

where K denotes some absolute constant which may change from line to line. We therefore obtain the bound

$$\begin{aligned} &\sum_{l=0}^\infty \left| \frac{(-1)^l \, \eta^{2l}}{l! \, N^{l+1/2}} \cdot \frac{1}{2\pi i} \int_\gamma \frac{\exp\left(-\frac{1}{4}\xi_N^2 \cdot (1-z)\right)}{(1-z)^{l+3/2}} \, \frac{dz}{z^{N+1}} \right| \\ &\quad \leq \sum_{l=0}^\infty \frac{|\eta|^{2l}}{l!} \cdot \frac{1}{N^{l+1/2}} \cdot \frac{1}{2\pi} \int_{-\pi}^{+\pi} \frac{1}{|1-Re^{it}|^{l+3/2}} \, \frac{dt}{|Re^{it}|^N} \\ &\quad \leq K \cdot \sum_{l=0}^\infty \frac{|\eta|^{2l}}{l!} \cdot \left(1 + \frac{1}{l+1/2}\right) < \infty \, , \end{aligned}$$

uniformly in $N \geq 2$. Thus, the term-by-term convergence established in (3.10) entails the convergence of the complete series in (3.9), and we obtain

$$\begin{aligned}\lim_{N\to\infty} \frac{I_1}{\sqrt{N}} &= \sqrt{\frac{1}{2\pi}} \cdot \exp\left(b - \tfrac{3}{4}\right) \cdot \sqrt{4-\xi^2} \cdot \sum_{l=0}^{\infty} \frac{(-1)^l \left(\sqrt{4-\xi^2}\cdot\eta\right)^{2l}}{(2l+1)!} \\ &= \sqrt{\frac{1}{2\pi}} \cdot \exp\left(b - \tfrac{3}{4}\right) \cdot \sqrt{4-\xi^2} \cdot \frac{\sin(\sqrt{4-\xi^2}\cdot\eta)}{(\sqrt{4-\xi^2}\cdot\eta)} \,.\end{aligned}$$

Hence, in view of (3.2) and (3.3), the proof of Proposition 3.1 will be complete once we have shown that the integrals I_2, I_3, I_4 are asymptotically negligible in the sense that they are of order $o(\sqrt{N})$.

For the integral I_2, we use the estimates ($R = 1 - 1/N$, $t \in \mathbb{R}$)

$$\begin{aligned}\left|\exp\left(-\tfrac{1}{2}\xi_N^2 \cdot \frac{1-Re^{it}}{1+Re^{it}}\right)\right| &= \exp\left(-\tfrac{1}{2}\xi_N^2 \cdot \mathrm{Re}\left(\frac{1-Re^{it}}{1+Re^{it}}\right)\right) \\ &= \exp\left(-\tfrac{1}{2}\xi_N^2 \cdot \frac{1-R^2}{1+R^2+2R\cos t}\right) \leq 1 \,, \end{aligned} \tag{3.13}$$

$$\begin{aligned}\left|\exp\left(-(\eta^2/N)\cdot\frac{1}{1-Re^{it}}\right)\right| &\leq \exp\left((|\eta|^2/N)\cdot\frac{1}{|1-Re^{it}|}\right) \\ &\leq \exp\left((|\eta|^2/N)\cdot\frac{1}{1-R}\right) = \exp(|\eta|^2) \,, \end{aligned} \tag{3.14}$$

$$\left|\exp\left(b^*(Re^{it})^2\right)\right| \leq \exp\left(|b^*||Re^{it}|^2\right) \leq \exp(|b^*|) \,, \tag{3.15}$$

$$|1+Re^{it}| \geq 1 + R\cos t \geq 1 \quad \text{for} \quad \cos t \geq 0 \,, \tag{3.16}$$

$$|1-Re^{it}| \geq 1 - R\cos t \geq 1 \quad \text{for} \quad \cos t \leq 0 \,. \tag{3.17}$$

Using these estimates, it follows that

$$\begin{aligned}|I_2| \leq{}& \frac{1}{2\pi}\int_{1/\sqrt{N}}^{\pi/2} \frac{\exp(|\eta|^2+|b^*|)}{|1-Re^{it}|^{3/2}}\ \frac{dt}{R^N} \\ &+ \frac{1}{2\pi}\int_{\pi/2}^{3\pi/2} \frac{\exp(|\eta|^2+|b^*|)}{|1+Re^{it}|^{1/2}}\ \frac{dt}{R^N} \\ &+ \frac{1}{2\pi}\int_{3\pi/2}^{2\pi-1/\sqrt{N}} \frac{\exp(|\eta|^2+|b^*|)}{|1-Re^{it}|^{3/2}}\ \frac{dt}{R^N} \,.\end{aligned}$$

Similarly as in (3.12), we have the estimates

$$\begin{aligned} \int_{1/\sqrt{N}}^{\pi/2} \frac{1}{|1-Re^{it}|^{3/2}}\, dt &\le KN^{1/2} \int_{\sqrt{N}\varepsilon}^{\infty} \frac{1}{u^{3/2}}\, du \le KN^{1/4}\,, \\ \int_{\pi/2}^{3\pi/2} \frac{1}{|1+Re^{it}|^{1/2}}\, dt &\le KN^{-1/2} \left(1 + \int_{1}^{N\varepsilon\pi/2} \frac{1}{u^{1/2}}\, du \right) \le K\,, \\ \int_{3\pi/2}^{2\pi-1/\sqrt{N}} \frac{1}{|1-Re^{it}|^{3/2}}\, dt &\le KN^{1/2} \int_{\sqrt{N}\varepsilon}^{\infty} \frac{1}{u^{3/2}}\, du \le KN^{1/4}\,. \end{aligned}$$

(Recall our convention that the constant K may change from one occurrence to the next.) It follows that

$$|I_2| \le K \exp(|\eta|^2 + |b^*|)\, N^{1/4} = o(\sqrt{N})\,.$$

For the integral I_3, we write

$$h(z) - h_0(z) = \frac{\exp\left(-\frac{1}{4}\xi_N^2 \cdot (1-z) - (\eta^2/N) \cdot \frac{1}{1-z}\right)}{(1-z)^{3/2}} \cdot \left(\tilde{h}(z) - \tilde{h}(1)\right)\,,$$

where

$$\tilde{h}(z) = \frac{\exp\left(-\frac{1}{4}\xi_N^2 \cdot \frac{(1-z)^2}{1+z} + b^* z^2\right)}{(1+z)^{1/2}}\,,$$

so that

$$\begin{aligned} \tilde{h}'(z) = \Bigg(\frac{\frac{1}{2}\xi_N^2 \cdot \frac{1-z}{1+z} + \frac{1}{4}\xi_N^2 \cdot \frac{(1-z)^2}{(1+z)^2} + 2b^* z}{(1+z)^{1/2}} &- \frac{1/2}{(1+z)^{3/2}} \Bigg) \\ &\cdot \exp\left(-\tfrac{1}{4}\xi_N^2 \cdot \frac{(1-z)^2}{1+z} + b^* z^2\right)\,. \end{aligned}$$

Let

$$Z := \left\{ z \in \mathbb{C} \mid z = re^{i\varphi}, 1 - 1/N \le r \le 1, |\varphi| \le 1/\sqrt{N} \right\}$$

and note that for $z \in Z$, we have the estimate

$$\operatorname{Re}\left(-\frac{(1-z)^2}{1+z}\right) \le 4/N\,. \tag{3.18}$$

Indeed, with $z = re^{i\varphi}$, a simple calculation yields

$$\operatorname{Re}\left(-\frac{(1-z)^2}{1+z}\right) = \frac{-1 + r\cos\varphi + 2r^2 - r^2\cos 2\varphi - r^3\cos\varphi}{1 + r^2 + 2r\cos\varphi}\,,$$

where the denominator is clearly larger than 1 and the numerator is bounded above by

$$r\cos\varphi - r^3\cos\varphi + r^2 - r^2\cos 2\varphi \le r(1-r^2)\cos\varphi + r^2 - r^2(1 - 2\varphi^2) \le 4/N\,.$$

Since for $z = Re^{it}$ with $|t| \le 1/\sqrt{N}$, the line segment between the points z and 1 is contained in the set Z, it follows that

$$\begin{aligned}
\left|\tilde{h}(z) - \tilde{h}(1)\right| &\le |z-1| \sup_{\alpha \in [0;1]} \left|\tilde{h}'((1-\alpha)z+\alpha)\right| \le |z-1| \sup_{\zeta \in Z} \left|\tilde{h}'(\zeta)\right| \\
&\le K|z-1| \sup_{\zeta \in Z} \left\{ \left(\xi_N^2 |1-\zeta| + |b^*| + 1 \right) \exp\left(\tfrac{1}{4} \xi_N^2 \cdot \operatorname{Re}\left(-\frac{(1-\zeta)^2}{1+\zeta} \right) + |b^*| \right) \right\} \\
&\le K|z-1| \left(\xi_N^2/\sqrt{N} + |b^*| + 1 \right) \exp\left(\xi_N^2/N + |b^*| \right) \\
&\le K(b^*,\xi^*)\, \sqrt{N}\, |z-1| \,,
\end{aligned}$$

where the last step uses that $\lim_{N\to\infty} \xi_N/\sqrt{N} = \xi$, and $K(b^*,\xi^*)$ denotes some constant depending only on b^* and $\xi^* := (\xi_N)_{N\in\mathbb{N}}$. Using (3.8), (3.14) as well as a similar estimate as in (3.12), we therefore obtain

$$\begin{aligned}
|I_3| &\le \frac{1}{2\pi} \int_{-1/\sqrt{N}}^{+1/\sqrt{N}} \frac{\exp(|\eta|^2)}{|1-Re^{it}|^{3/2}} \cdot \left|\tilde{h}(Re^{it}) - \tilde{h}(1)\right| \frac{dt}{R^N} \\
&\le K(b^*,\xi^*,\eta)\, \sqrt{N} \int_{-1/\sqrt{N}}^{+1/\sqrt{N}} \frac{1}{|1-Re^{it}|^{1/2}}\, dt \\
&\le K(b^*,\xi^*,\eta) \left(1 + \int_1^{\sqrt{N}\varepsilon} \frac{1}{u^{1/2}}\, du \right) \\
&\le K(b^*,\xi^*,\eta)\, N^{1/4} = o(\sqrt{N}) \,,
\end{aligned}$$

where $K(b^*,\xi^*,\eta)$ denotes some constant which depends only on b^*, ξ^*, and η (and which may change from line to line as usual).

For the integral I_4, we can use (3.8), (3.14) as well as a similar estimate as in (3.12) to obtain

$$\begin{aligned}
|I_4| &\le \frac{1}{2\pi} \int_{1/\sqrt{N}}^{2\pi-1/\sqrt{N}} \frac{\exp(|\eta|^2 + |b^*|)}{|1-Re^{it}|^{3/2}} \frac{dt}{R^N} \\
&\le K \exp(|\eta|^2 + |b^*|) \left(\int_{1/\sqrt{N}}^{\pi} \frac{1}{|1-Re^{it}|^{3/2}}\, dt + \int_{\pi}^{2\pi-1/\sqrt{N}} \frac{1}{|1-Re^{it}|^{3/2}}\, dt \right) \\
&\le K \exp(|\eta|^2 + |b^*|) \left(N^{1/2} \int_{\sqrt{N}\varepsilon}^{\infty} \frac{1}{u^{3/2}}\, du + N^{1/2} \int_{\sqrt{N}\varepsilon}^{\infty} \frac{1}{u^{3/2}}\, du \right) \\
&\le K \exp(|\eta|^2 + |b^*|)\, N^{1/4} = o(\sqrt{N}) \,.
\end{aligned}$$

This completes the proof of Proposition 3.1. □

4. THE PROOF OF PROPOSITION 1.3

Let $f(N;\mu,\nu)$ and $\widetilde{f}(N;\mu,\nu)$ be defined as in (1.2) and (1.3), respectively, and note that

$$\begin{aligned}
&\widetilde{f}(N;\mu,\nu)\\
&= \mathbb{E}\big((\det(X_N-\mu)-\mathbb{E}\det(X_N-\mu))\cdot(\det(X_N-\nu)-\mathbb{E}\det(X_N-\nu))\big)\\
&= \mathbb{E}\big(\det(X_N-\mu)\cdot\det(X_N-\nu)\big)-\mathbb{E}\det(X_N-\mu)\cdot\mathbb{E}\det(X_N-\nu)\\
&= f(N;\mu,\nu)-g(N;\mu)\,g(N;\nu)\,, \qquad (4.1)
\end{aligned}$$

where

$$g(N;\mu) := \mathbb{E}\det(X_N-\mu)$$

for any $\mu\in\mathbb{R}$. We will deduce Proposition 1.3 from Theorem 1.1 by showing that $f(N;\mu,\nu)$ is asymptotically much larger than $g(N;\mu)\,g(N;\nu)$.

To this end, we need some more information about the values $g(N;\mu)$. Similarly as in Section 2, we have the following recursive equation:

Lemma 4.1.

$$g(0;\mu)=1,\quad g(N;\mu)=-\mu g(N-1;\mu)-(N-1)g(N-2;\mu)\quad (N\geq 1)\,.$$

Proof. For $N=0$, the claim follows from our convention that the determinant of the empty matrix is 1. For $N\geq 1$, we expand the determinant of the matrix $(X_N-\mu)$ as in (2.1) and use independence and symmetry to get

$$\begin{aligned}
g(N;\mu) &= \sum_{i,j=1}^{N-1}(-1)^{i+j+1}\,\mathbb{E}\left(X_{i,N}X_{N,j}\right)\cdot\mathbb{E}\left(\det(X_{N-1}-\mu)^{[i:j]}\right)\\
&\qquad + \mathbb{E}\left(X_{NN}-\mu\right)\cdot\mathbb{E}\left(\det(X_{N-1}-\mu)\right)\\
&= \sum_{i=j}(-1)\,\mathbb{E}|X_{i,N}|^2\cdot\mathbb{E}\left(\det(X_{N-1}-\mu)^{[i:i]}\right)\\
&\qquad + \mathbb{E}\left(X_{NN}-\mu\right)\cdot\mathbb{E}\left(\det(X_{N-1}-\mu)\right)\\
&= -(N-1)g(N-2;\mu)-\mu g(N-1;\mu)\,.
\end{aligned}$$

This completes the proof of Lemma 4.1. □

It follows from Lemma 4.1 that the polynomials $g(N;\mu)$ coincide, up to scaling, with the Hermite polynomials $H_N(x)$ (see e.g. Section 5.5 in SZEGÖ [Sz]), which satisfy the recursive equation

$$H_0(x)=1,\quad H_N(x)=2xH_{N-1}(x)-2(N-1)H_{N-2}(x)\quad (N\geq 1)\,.$$

(Specifically for the GUE, this is well-known, see e.g. Chapter 4 in FORRESTER [Fo].) The precise relationship is as follows:

Lemma 4.2. *For any $N=0,1,2,3,\ldots$,*

$$g(N;\mu)=(-1)^N\,2^{-N/2}\,H_N(\mu/\sqrt{2})\,.$$

Proof. This follows from the recursive equations for $g(N;\mu)$ and $H_N(x)$ by a straightforward induction on N. □

Due to Lemma 4.2, it is easy to obtain the asymptotics of the values $g(N;\sqrt{N}\xi+\mu/\sqrt{N}\varrho(\xi))$ from the corresponding asymptotics of the Hermite polynomials (see e.g. Section 8.22 in Szegö [Sz]). For our purposes, the following estimate will be sufficient:

Lemma 4.3. *For $\xi \in (-2,+2)$, $\mu \in \mathbb{R}$ fixed,*

$$\left| e^{-N\xi^2/4}\, g\left(N;\sqrt{N}\xi+\frac{\mu}{\sqrt{N}\varrho(\xi)}\right)\right| \le K(\xi,\mu)\, N^{-1/4}\, N!^{1/2}\,,$$

where $K(\xi,\mu)$ is some constant depending only on ξ and μ.

Proof. By Theorem 8.22.9 (a) in Szegö [Sz], we have, for $x=\sqrt{2N+1}\,\cos\varphi$,

$$\begin{aligned} e^{-x^2/2}\, H_N(x) &= 2^{(N/2)+(1/4)}(N!)^{1/2}(N\pi)^{-1/4}(\sin\varphi)^{-1/2} \\ &\quad\cdot \big(\sin\left(((2N+1)/4)\cdot(\sin 2\varphi-2\varphi)+3\pi/4\right)+\mathcal{O}(N^{-1})\big)\,, \end{aligned}$$

where the $\mathcal{O}$-bound holds uniformly in $\varphi\in[\varepsilon,\pi-\varepsilon]$, for any $\varepsilon>0$.

Combining this result with Lemma 4.2, we obtain, for N sufficiently large,

$$\begin{aligned} &e^{-(\sqrt{N}\xi+\mu/\sqrt{N}\varrho(\xi))^2/4}\, g(N;\sqrt{N}\xi+\mu/\sqrt{N}\varrho(\xi)) \\ &\qquad = (-1)^N\, 2^{1/4}\,(N!)^{1/2}\,(N\pi)^{-1/4}\,(\sin\varphi)^{-1/2} \\ &\qquad\quad \cdot \big(\sin\left(((2N+1)/4)\cdot(\sin 2\varphi-2\varphi)+3\pi/4\right)+\mathcal{O}(N^{-1})\big)\,, \end{aligned}$$

where

$$\varphi \equiv \varphi_N := \arccos\left((\sqrt{N}\xi+\mu/\sqrt{N}\varrho(\xi))\,/\,\sqrt{4N+2}\right)$$

is contained in an interval of the form $[-\varepsilon,\pi-\varepsilon]$ with $\varepsilon>0$. From this, Lemma 4.3 easily follows. ☐

After these preparations we can turn to the proof of Proposition 1.3:

Proof of Proposition 1.3. By Equation (4.1) and Lemma 4.3, the difference between the left-hand sides in Theorem 1.1 and Proposition 1.3 is bounded by

$$\begin{aligned} &\left|\sqrt{\frac{1}{2\pi N}}\cdot\frac{1}{N!}\cdot e^{-N\xi^2/2}\cdot g\left(N;\sqrt{N}\xi+\frac{\mu}{\sqrt{N}\varrho(\xi)}\right) g\left(N;\sqrt{N}\xi+\frac{\nu}{\sqrt{N}\varrho(\xi)}\right)\right| \\ &\qquad\qquad \le K(\xi,\mu,\nu)\, N^{-1/2}\, N!^{-1}\left(N^{-1/4}\, N!^{1/2}\right)^2 = K(\xi,\mu,\nu)\, N^{-1}\,, \end{aligned}$$

where $K(\xi,\mu,\nu)$ is some constant depending only on ξ, μ and ν. Thus, Proposition 1.3 follows from Theorem 1.1. ☐

5. THE PERMANENTAL POLYNOMIAL

In this section, we discuss the implications of our approach for permanental polynomials of Hermitian Wigner matrices.

For any $n\times n$ matrix $A=(a_{ij})_{i,j=1,\ldots,n}$, the permanent is defined by

$$\operatorname{per}(A) := \sum_{\pi\in\mathcal{S}_n}\prod_{j=1}^{n} a_{j,\pi(j)}\,,$$

where the sum is taken over the set $\mathcal{S}_n$ of permutations of the set $\{1,\ldots,n\}$.

Similarly as in Section 2, we adopt the convention that the permanent of the "empty" (i.e., 0×0) matrix is taken to be 1. The definition of the permanent is obviously analogous to that of the determinant, except that the sign factors are absent. It is easy to see that this analogy extends to row and column expansions. Thus, in analogy to (2.1), the permanent satisfies the identity

$$\operatorname{per}(A) = \sum_{i,j=1}^{n-1} a_{i,n}\, a_{n,j}\ \operatorname{per}(A^{[n,i:n,j]}) + a_{n,n} \operatorname{per}(A^{[n:n]})\,. \tag{5.1}$$

The permanental polynomial of the matrix A is defined by $\operatorname{per}(A - z)$, where $A - z$ is defined as in Section 2, and the second-order correlation function of the permanental polynomial is defined by

$$\hat{f}(N;\mu,\nu) := \mathbb{E}\left(\operatorname{per}(X_N-\mu)\cdot\operatorname{per}(X_N-\nu)\right) \qquad (N \geq 0)\,,$$

where μ and ν are complex numbers. We will also need the auxiliary functions $\hat{f}_{11}(N;\mu,\nu)$, $\hat{f}_{11}^{\chi}(N;\mu,\nu)$, $\hat{f}_{10}(N;\mu,\nu)$, $\hat{f}_{01}(N;\mu,\nu)$, which are defined in the same way as the auxiliary functions from Section 2, but with the determinant replaced by the permanent.

Starting from the identity (5.1), it is easy to see that all our results from Section 2 carry over to the correlation function of the permanental polynomial, provided that one adjusts the signs appropriately. Since the proofs are virtually the same, we confine ourselves to stating the results.

Lemma 5.1.

$$\begin{aligned}
\hat{f}(0) &= 1\,,\\
\hat{f}(N) &= (1+\mu\nu)\,\hat{f}(N-1) + (2b+\tfrac{1}{2})(N-1)\,\hat{f}(N-2)\\
&\quad + (N-1)(N-2)\,\hat{f}_{11}(N-1)\\
&\quad + (N-1)(N-2)\,\hat{f}_{11}^{\chi}(N-1)\\
&\quad - \nu(N-1)\,\hat{f}_{10}(N-1)\\
&\quad - \mu(N-1)\,\hat{f}_{01}(N-1) && (N \geq 1)\,, && (5.2)\\
\hat{f}_{11}(N) &= \mu\nu\,\hat{f}(N-2) + (N-2)\,\hat{f}(N-3)\\
&\quad + (N-2)(N-3)\,\hat{f}_{11}(N-2)\\
&\quad - \nu(N-2)\,\hat{f}_{10}(N-2)\\
&\quad - \mu(N-2)\,\hat{f}_{01}(N-2) && (N \geq 2)\,, && (5.3)\\
\hat{f}_{11}^{\chi}(N) &= \hat{f}(N-2) + (N-2)\,\hat{f}(N-3)\\
&\quad + (N-2)(N-3)\,\hat{f}_{11}^{\chi}(N-2) && (N \geq 2)\,, && (5.4)\\
\hat{f}_{10}(N) &= (N-1)\,\hat{f}_{01}(N-1) - \nu\,\hat{f}(N-1) && (N \geq 1)\,, && (5.5)\\
\hat{f}_{01}(N) &= (N-1)\,\hat{f}_{10}(N-1) - \mu\,\hat{f}(N-1) && (N \geq 1)\,. && (5.6)
\end{aligned}$$

Let $\hat{c}(N) := \dfrac{\hat{f}(N)}{N!}$ $(N \geq 0)$ and $\hat{s}(N) := \displaystyle\sum_{\substack{k=0,\ldots,N \\ k \text{ even}}} \hat{c}(N-k)$ $(N \geq 0)$.

Lemma 5.2. *The values* $\hat{c}(N)$ *satisfy the recursive equation*

$$\hat{c}(0) = 1\,, \tag{5.7}$$

$$\begin{aligned} N\hat{c}(N) &= \hat{c}(N-1) + N \cdot \hat{c}(N-2) \\ &\quad + \mu\nu \cdot \big(\hat{s}(N-1) + \hat{s}(N-3)\big) \\ &\quad + (\mu^2 + \nu^2) \cdot \hat{s}(N-2) \\ &\quad + (2b - \tfrac{3}{2}) \cdot \big(\hat{c}(N-2) - \hat{c}(N-4)\big) \qquad (N \geq 1)\,, \end{aligned} \tag{5.8}$$

where all terms $\hat{c}(\,\cdot\,)$ *and* $\hat{s}(\,\cdot\,)$ *with a negative argument are taken to be zero.*

Lemma 5.3. *The exponential generating function* $\hat{F}(x) := \sum_{N=0}^{\infty} \hat{f}(N)\, x^N \,/\, N!$ *of the sequence* $(\hat{f}(N))_{N \geq 0}$ *is given by*

$$\hat{F}(x) = \frac{\exp\Big(\mu\nu \cdot \frac{x}{1-x^2} + \frac{1}{2}(\mu^2 + \nu^2) \cdot \frac{x^2}{1-x^2} + b^* x^2\Big)}{(1-x)^{3/2} \cdot (1+x)^{1/2}}\,,$$

where $b^* := b - \frac{3}{4}$.

Note that Lemmas 5.2 and 5.3 are completely analogous to Lemmas 2.2 and 2.3, except that the sign associated with the factor $(\mu^2 + \nu^2)$ is different. Clearly, this can also be effectuated by making the replacements $\mu \to -i\mu$ and $\nu \to +i\nu$ in Lemmas 2.2 and 2.3. This implies that the second-order correlation functions of the permanental polynomial and of the characteristic polynomial are related as follows:

Proposition 5.4. *Under the moment conditions (1.1), we have*

$$\mathbb{E}\left(\mathrm{per}(X_N - \mu)\,\mathrm{per}(X_N - \nu)\right) = \mathbb{E}\left(\det(X_N + i\mu)\det(X_N - i\nu)\right) \tag{5.9}$$

for all $N \geq 0$ *and all* $\mu, \nu \in \mathbb{C}$.

Proposition 5.4 generalizes a result by FYODOROV [Fy], who obtained Equation (5.9) by a completely different approach for GUE random matrices (see Equation (1.15) in FYODOROV [Fy]). For the latter FYODOROV [Fy] conjectured that, as the matrix size tends to infinity, the suitably rescaled roots of the permanental polynomial concentrate around the imaginary axis in the complex plane, with a semi-circular density profile.

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