

# DIFFRACTION OF STOCHASTIC POINT SETS: EXPLICITLY COMPUTABLE EXAMPLES

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ABSTRACT. Stochastic point processes relevant to the theory of long-range aperiodic order are considered that display diffraction spectra of mixed type, with special emphasis on explicitly computable cases together with a unified approach of reasonable generality. The latter is based on the classical theory of point processes and the Palm distribution. Several pairs of autocorrelation and diffraction measures are discussed which show a duality structure analogous to that of the Poisson summation formula for lattice Dirac combs.

## 1. INTRODUCTION

The discoveries of quasicrystals [53], aperiodic tilings [47, 39], and complex metallic alloys [60] have greatly increased our awareness that there is a substantial difference between the notions of periodicity and long-range order. Although pinning an exact definition to the concept of long-range order is not yet possible (nor perhaps desirable at this intermediate stage, compare the discussion in [56]), there is still some general agreement that the appearance of a substantial point-like component in the diffraction of a structure is a strong, though not a necessary, indicator of the phenomenon.

Mathematically, the diffraction – say of a point set  $\Lambda$  in  $\mathbb{R}^3$  – is the measure  $\widehat{\gamma}$  on  $\mathbb{R}^3$  which is the Fourier transform of the volume averaged autocorrelation  $\gamma$  of  $\Lambda$  (or, more precisely, of its Dirac comb  $\delta_\Lambda = \sum_{x \in \Lambda} \delta_x$ ). Over the past 20 years or so, considerable effort has been put into understanding the mathematics of diffraction, especially conditions under which  $\Lambda$  is pure point diffractive, in the sense that  $\widehat{\gamma}$  is a pure point measure, compare [32, 16, 55, 2, 8, 29, 42, 43]. At this point in time, we have a good collection of models for producing pure point diffraction, particularly the cut and project sets (or model sets). Under certain types of discreteness conditions, one can even go as far as to say that these types of sets essentially characterise the pure point diffractive point sets [6].

But real life structures are not perfectly pure point diffractive, and in order to gain further insight into the possible structures of materials, and more generally into the whole concept of long-range order, it is necessary to widen the scope of this study to include mixed diffraction spectra. In particular, this means that one has to consider structures whose diffraction measures contain at least some continuous part. Note that distinct structures may have the same diffraction, which makes the corresponding inverse problem difficult (and generally unsolvable without further information). In fact, this gets worse outside the realm of pure point spectra, and any improvement requires a better understanding of the diffraction of structures with some form of randomness.

However, when it comes to mixed spectra, relatively little is known, although there are many particular examples [23, 4, 33, 40, 30, 63, 21]. Even deterministic sets can have mixed diffraction spectra, and once any randomness is introduced, this is the norm. Determining the exact nature of the diffraction is usually difficult and often simply not known. No doubt, the possibilities, both in Nature and in mathematics, for structures with long-range order are well beyond what we have presently imagined. This is also made apparent by systems such as the pinwheel tiling, compare [49] and references therein, which looks like an amorphous structure in diffraction, in spite of being completely regular. In particular, except for the trivial Bragg peak at 0, there is no pure point part in the diffraction.

The primary goal of this paper is to show how the techniques from the theory of stochastic point processes may be used when systems with long-range order are subjected to modifications in the form of stochastic perturbations. In as much as our primary goal is the study of long-range order, the types of point processes of most interest to us are quite different from those usually studied in stochastic geometry. For instance, as mentioned above, a phenomenon of fundamental importance in long-range order is the appearance of Bragg peaks in diffraction, which refers to a non-trivial pure point part of the diffraction measure. For ergodic point processes, this implies the existence of non-trivial (dynamical) eigenvalues and thus excludes weak mixing [62]. Hence, we are particularly interested in systems that lie between ergodic and weak mixing. Furthermore, a number of basic and influential mathematical models of long-range order are deterministic. Even so, the theory of point processes is relevant [28] and yields considerable insight.

To elaborate on this a little, consider the classic Penrose tilings [47]. Fix a set of two generating Penrose prototiles (with matching rule markers) and a set of overall orientations for them (so each prototile comes in 10 distinct orientations). The resulting set of admissible tilings of the plane, even if one vertex of the first tile laid down is fixed, is uncountable. Replacing each of these tilings by its corresponding vertex point set and allowing all (global) translations, collectively we obtain the set  $\mathbb{X}$  of all Penrose point sets in the given orientation. The set  $\mathbb{X}$ , which is also called the *hull*, has a natural topology in which it is compact, and it carries a unique (and hence ergodic) stationary Borel probability measure  $\mu$ . The pair  $(\mathbb{X}, \mu)$  can now be viewed both as a dynamical system and as a stationary ergodic point process, which permits the powerful tools from both subjects to be applied.

In particular, the diffraction and the dynamical spectrum are linked by the dynamics [6], while the diffraction is also the Fourier transform of the first moment measure of the Palm measure of the point process [28]. This type of scenario, which applies to many models of long-range order, deterministic or otherwise, has not yet been investigated in much detail. It would seem desirable then, as a first step, to establish methods, capable of being explicitly computable, that would cover typical and much-studied situations and also suggest ways in which to generalise what is known, and even move into yet unexplored territory.

As already implicitly mentioned, mathematical diffraction theory is based on the approach set out by Hof in [32, 33], namely via autocorrelations and their measures. Our paper is primarily guided by examples, set in as great a generality as we can manage without becoming too technical. The examples are selected under the consistent theme that they are computable,

while they unify and extend existing results in a systematic way. Briefly, the types of situations that we consider are these:

- (i) renewal processes on the real line (Sec. 3), as a versatile, elementary approach to one-dimensional phenomena; the main result here is Theorem 1;
- (ii) randomisation of a given point set  $\Lambda$  (with a certain discreteness restriction) whose diffraction is known, by complex, identically distributed, finite random measures that are independently centred at each point of  $\Lambda$  (Sec. 4); see Theorem 2;
- (iii) randomisation of a random point process  $\Phi$  (with known law) by identically distributed, finite, random measures (positive or signed) which are independently centred at each point of any realisation of  $\Phi$  (Sec. 5.1–5.4); the main results are formulated in Theorems 3 and 4;
- (iv) equilibria of critical branching Brownian motions (Sec. 5.5, with Theorem 6).

While (i) and (ii) have a bit of a review character, (iii) and (iv) are new in this context. The results for (i) and (ii) are included with details and several examples because they have immediate applications to practical diffraction analysis, while the results are scattered over the literature or only covered implicitly. Since the transfer of methods and results from stochastic geometry to mathematical diffraction theory requires a somewhat unusual view on measures and their Fourier transforms, we begin with a brief recapitulation of concepts needed here. For the sake of completeness and readability (as well as lack of reference), some details on the ergodic theorem that we need for (iii) are added as an appendix.

## 2. SOME RECOLLECTIONS FROM FOURIER ANALYSIS AND DIFFRACTION THEORY

Throughout the paper, we need various standard notions and results from Fourier analysis and measure theory, which we summarise here, introducing our notation at the same time.

First, let  $\mu$  be a finite, regular (and possibly complex) Borel measure on  $\mathbb{R}^d$ ; compare [51] for background. Its Fourier (or Fourier-Stieltjes) transform is a uniformly continuous function on  $\mathbb{R}^d$ , defined by

$$\widehat{\mu}(k) = \int_{\mathbb{R}^d} e^{-2\pi i k x} d\mu(x),$$

see [52] for details. This definition includes the Fourier transform of arbitrary Schwartz functions or integrable functions (the corresponding spaces being denoted by  $\mathcal{S}(\mathbb{R}^d)$  and  $L^1(\mathbb{R}^d)$ ) by viewing them as Radon-Nikodym densities for Lebesgue measure  $\lambda$ , hence as finite measures. In this version of the Fourier transform, with the factor  $2\pi$  included in the exponent, there is no need for prefactors (though the factor  $2\pi$  reappears up front under differentiation). In particular, one has the usual convolution theorem in the form  $\widehat{\mu * \nu} = \widehat{\mu} \widehat{\nu}$ , where

$$(\mu * \nu)(g) = \int_{\mathbb{R}^d \times \mathbb{R}^d} g(x + y) d\mu(x) d\nu(y)$$

for continuous functions  $g \in C_0(\mathbb{R}^d)$  (note that we identify finite, complex, regular Borel measures on  $\mathbb{R}^d$  with continuous linear functionals on  $C_0(\mathbb{R}^d)$ , in line with the Riesz-Markov representation theorem [51, Thm. 6.19]).

Below, we need to go beyond the situation of finite measures. For the introduction of unbounded measures, the linear functional point of view is advantageous. Here, an *unbounded measure* is thus understood as a linear functional on the space  $C_c(\mathbb{R}^d)$  of continuous functions with compact support such that, for any compact set  $K \subset \mathbb{R}^d$ , there is a constant  $a = a_K$  with  $|\mu(g)| \leq a_K \|g\|_\infty$  for all continuous  $g$  with support in  $K$ . The corresponding space  $\mathcal{M}(\mathbb{R}^d)$  is thus equipped with the vague topology, see [22, Ch. XIII] for details. As before, we can identify these measures with the locally finite, complex, regular Borel measures on  $\mathbb{R}^d$ , by an appropriate version of the Riesz-Markov representation theorem; see [12, Thm. 69.1] for a formulation for positive measures and use the polar representation [22, Thm. 13.16.3] for an extension to complex measures. The absolute value  $|\mu|$  of  $\mu$ , also known as the *total variation* measure of  $\mu$ , is the smallest positive measure such that  $|\mu(g)| \leq |\mu|(|g|)$  holds for all  $g \in C_c(\mathbb{R}^d)$ .

When an unbounded measure  $\mu$  also defines a tempered distribution, via  $\mu(\varphi) = \int_{\mathbb{R}^d} \varphi d\mu$  for  $\varphi \in \mathcal{S}(\mathbb{R}^d)$ , it is called a *tempered measure*. Its Fourier transform (as a distribution) is then defined via  $\widehat{\mu}(\varphi) = \mu(\widehat{\varphi})$  as usual [50], so that  $\widehat{\mu}$  is a tempered distribution. Below, we only consider situations where  $\widehat{\mu}$  is also a measure, hence a linear functional on  $C_c(\mathbb{R}^d)$ . Recall that a (complex) measure  $\mu$  is called *translation bounded* when

$$\sup_{t \in \mathbb{R}^d} |\mu|(t + K) < \infty$$

holds for arbitrary compact sets  $K \subset \mathbb{R}^d$ . Translation boundedness is a sufficient (though not a necessary) criterion for a measure to be tempered, see [50] for details. In this setting, we call a measure *transformable* when it is tempered and when its Fourier transform (as a distribution) is again a measure. Transformability of a measure is a difficult question in general; see [26] and references therein for details.

A measure  $\mu \in \mathcal{M}(\mathbb{R}^d)$  is called *positive definite*, when  $\mu(g * \widetilde{g}) \geq 0$  holds for every function  $g \in C_c(\mathbb{R}^d)$ ; here,  $\widetilde{g}$  is the function defined by  $\widetilde{g}(x) = \overline{g(-x)}$ . Positive definite measures have various nice properties, some of which can be summarised as follows; see [13, Sec. 4] for details and proofs.

**Lemma 1.** *For  $\mu \in \mathcal{M}(\mathbb{R}^d)$ , the following properties hold.*

- (i) *If  $\mu$  is positive and positive definite, it is translation bounded;*
- (ii) *If  $\mu$  is positive definite, it is Fourier transformable, and  $\widehat{\mu}$  is a positive, translation bounded measure;*
- (iii) *A transformable measure  $\mu$  is positive definite if and only if  $\widehat{\mu}$  is a positive measure;*

*Moreover, the mapping  $\mu \mapsto \widehat{\mu}$  defines a bijection between the positive definite and the transformable positive measures on  $\mathbb{R}^d$ .  $\square$*

Let us consider some examples that will reappear later. If  $\Gamma \subset \mathbb{R}^d$  is a *lattice* (meaning a discrete subgroup of  $\mathbb{R}^d$  with compact factor group  $\mathbb{R}^d/\Gamma$ ), we write  $\delta_\Gamma := \sum_{x \in \Gamma} \delta_x$  for the corresponding Dirac comb, with  $\delta_x$  the normalised point measure at  $x$ . It is well-known that  $\delta_\Gamma$  is a tempered measure, whose Fourier transform is again a tempered measure. The latter is explicitly given by the Poisson summation formula (PSF) in its version for lattice Dirac

combs [13, Ex. 6.22],

$$(1) \quad \widehat{\delta}_\Gamma = \text{dens}(\Gamma) \delta_{\Gamma^*},$$

where  $\Gamma^* := \{x \in \mathbb{R}^d \mid x \cdot y \in \mathbb{Z} \text{ for all } y \in \Gamma\}$  is the *dual lattice* of  $\Gamma$ ; see [16] for details. The density of  $\Gamma$  is well-defined and given by  $\text{dens}(\Gamma) = 1/|\det(\Gamma)|$ , where  $\det(\Gamma)$  is the oriented volume of a (measurable) fundamental domain of  $\Gamma$ . It can most easily be calculated as the determinant of a lattice basis. Observing  $|\det(\Gamma^*)| = 1/|\det(\Gamma)|$ , a more symmetric version of the PSF reads

$$(2) \quad (\sqrt{|\det(\Gamma)|} \delta_\Gamma)^\wedge = \sqrt{|\det(\Gamma^*)|} \delta_{\Gamma^*}.$$

In particular, one has  $\widehat{\delta}_{\mathbb{Z}^d} = \delta_{\mathbb{Z}^d}$ , so that the lattice Dirac comb of  $\mathbb{Z}^d$  is self-dual in this sense.

**Remark 1.** RADIALLY SYMMETRIC PSF. As an aside of independent interest, let us recall the following related formula for a radially symmetric situation in  $\mathbb{R}^d$ , which emerges from a simplified model of powder diffraction [3]. Let  $\Gamma$  and  $\Gamma^*$  be as before, and let  $\eta_\Gamma(r)$  and  $\eta_{\Gamma^*}(r)$  denote the numbers of points of  $\Gamma$  and  $\Gamma^*$  on centred spheres  $\partial B_r(0)$  of radius  $r$ . The (non-zero) numbers  $\eta_\Gamma(r)$  are also called the *shelling numbers* of the lattice  $\Gamma$ . If  $\mu_r$  denotes the uniform probability measure on  $\partial B_r(0)$ , with  $\mu_0 = \delta_0$ , one has the following radial analogue of the PSF in (1),

$$(3) \quad \left( \sum_{r \in \mathcal{D}_\Gamma} \eta_\Gamma(r) \mu_r \right)^\wedge = \text{dens}(\Gamma) \sum_{r \in \mathcal{D}_{\Gamma^*}} \eta_{\Gamma^*}(r) \mu_r,$$

where  $\mathcal{D}_\Gamma = \{r \geq 0 \mid \eta_\Gamma(r) > 0\}$  and analogously for  $\mathcal{D}_{\Gamma^*}$ , see [3] for a proof and further details. The formula can also be brought to a more symmetric form, as in Eq. (2).  $\diamond$

Another simple, but important, pair of mutual Fourier transforms follows from the relations  $\widehat{\delta}_0 = \lambda$  and  $\widehat{\lambda} = \delta_0$ , with  $\lambda$  being Lebesgue measure, so that we have

$$(4) \quad (\delta_0 + \lambda)^\wedge = \delta_0 + \lambda.$$

We shall meet this self-dual pair of measures below in Examples 1 and 9, in connection with the Poisson process.

A little less obvious is the following result.

**Lemma 2.** *Let  $\lambda$  denote Lebesgue measure on  $\mathbb{R}^d$  and  $0 < \alpha < d$ . The function  $x \mapsto 1/|x|^{d-\alpha}$  is locally integrable and, when seen as a Radon-Nikodym density for  $\lambda$ , defines an absolutely continuous and translation bounded measure on  $\mathbb{R}^d$ . This measure satisfies the identity*

$$\left( \frac{\Gamma(\frac{d-\alpha}{2})}{\pi^{\frac{d-\alpha}{2}}} \frac{\lambda}{|x|^{d-\alpha}} \right)^\wedge(k) = \frac{\Gamma(\frac{\alpha}{2})}{\pi^{\frac{\alpha}{2}}} \frac{\lambda}{|k|^\alpha},$$

where the transformed measure is again translation bounded and absolutely continuous. Moreover, both measures are positive and positive definite.

*Proof.* Local integrability of both measures on  $\mathbb{R}^d$  rests upon that of their densities around 0, which follows from rewriting the volume element in polar coordinates,  $d\lambda(x) = r^{d-1} dr d\Omega$ , with  $d\Omega$  the standard surface element of the unit sphere in  $\mathbb{R}^d$ . Absolute continuity and

translation boundedness are then clear, while the Fourier identity follows from a calculation with the heat kernel, see [48, Sec. 2.2.3]. As both measures are clearly positive, they are also positive definite by the Bochner-Schwartz theorem [50, Thm. IX.10], compare Lemma 1.  $\square$

Incidentally, dividing the identity in Lemma 2 by  $\Gamma(\alpha/2)/\pi^{\alpha/2}$  shows that

$$(5) \quad \frac{\Gamma\left(\frac{d-\alpha}{2}\right) \pi^{\frac{\alpha}{2}}}{\Gamma\left(\frac{\alpha}{2}\right) \pi^{\frac{d-\alpha}{2}}} \frac{\lambda}{|x|^{d-\alpha}} \xrightarrow{\alpha \rightarrow 0} \delta_0$$

in the vague topology, which follows from the corresponding Fourier transforms of the left hand side converging vaguely to  $\lambda$ .

Let us now briefly review the concept of the diffraction measure of a complex measure  $\omega$  as the Fourier transform of the autocorrelation  $\gamma$  of  $\omega$ . Its motivation comes from the physics of diffraction [17], while its precise mathematical formulation was pioneered by Hof [32].

In general, a complex measure  $\omega$  need not be transformable, and may thus not be a good object for harmonic analysis. In view of Lemma 1, it seems appealing to first attach a positive definite measure to  $\omega$ , which is possible as follows. If  $\omega_r$  denotes the restriction of  $\omega$  to the open ball  $B_r$  of radius  $r$  around 0, the natural *autocorrelation measure*  $\gamma = \gamma_\omega$  is defined as

$$(6) \quad \gamma := \lim_{r \rightarrow \infty} \frac{\omega_r * \widetilde{\omega}_r}{\text{vol}(B_r)},$$

provided the limit exists. Here,  $\widetilde{\mu}$  denotes the measure given by  $\widetilde{\mu}(g) = \overline{\mu(\widetilde{g})}$  for  $g \in C_c(\mathbb{R}^d)$ , with  $\widetilde{g}$  as before. If  $\omega$  is translation bounded, the one-parameter family of finite measures  $\left\{ \frac{\omega_r * \widetilde{\omega}_r}{\text{vol}(B_r)} \mid r > 0 \right\}$  is uniformly translation bounded and hence precompact in the vague topology by [32, Prop. 2.2]. One can thus always select converging subsequences to define an autocorrelation (which then depends on the sequence of averaging sets). As long as balls are used, one speaks of *natural* autocorrelations. More generally, one may work with any *averaging sequence*

$$\mathcal{A} = \{A_n \mid n \in \mathbb{N}\}$$

of relatively compact, open sets  $A_n \subset \mathbb{R}^d$  that satisfy  $\overline{A_n} \subset A_{n+1}$  for all  $n \in \mathbb{N}$  together with  $\bigcup_{n \in \mathbb{N}} A_n = \mathbb{R}^d$ . Again, for translation bounded measures  $\omega$ , the corresponding limit in (6) exists, at least along suitable subsequences.

An important further ingredient is the concept of a van Hove sequence, which is an averaging sequence with a restricted ‘surface to volume’ ratio. To formalise this, let  $K, C \subset \mathbb{R}^d$  be compact and define

$$(7) \quad \partial^K C := ((C + K) \setminus C^\circ) \cup ((\overline{\mathbb{R}^d \setminus C} - K) \cap C),$$

which may be viewed as a  $K$ -thickened boundary of  $C$ . Then,  $\mathcal{A}$  is called *van Hove* when, for every compact  $K \subset \mathbb{R}^d$ ,

$$(8) \quad \lim_{n \rightarrow \infty} \frac{\text{vol}(\partial^K \overline{A_n})}{\text{vol}(A_n)} = 0.$$

Now, the comparison of limits taken along different averaging sequences makes sense, and becomes independent of  $\mathcal{A}$  for ergodic systems; compare [55, Lemma 1.1]. Also, as follows

from [55, Lemma 1.2], translation bounded measures satisfy the relation

$$(9) \quad \lim_{n \rightarrow \infty} \frac{\omega_n * \widetilde{\omega}_n}{\text{vol}(A_n)} = \lim_{n \rightarrow \infty} \frac{\omega_n * \widetilde{\omega}}{\text{vol}(A_n)},$$

provided that  $\mathcal{A}$  is van Hove and one of the limits exists. Here,  $\omega_n = \omega|_{A_n}$ , and  $\omega_n * \widetilde{\omega}$  is well-defined by [13, Prop. 1.13]. This freedom will be used several times below.

The general situation for a translation bounded measure  $\omega$  is as follows. The van Hove property of  $\mathcal{A}$  implies that  $|\omega|(A_n) \leq c \text{vol}(A_n)$  with a constant  $c > 0$ . An obvious modification of [32, Prop. 2.2] in conjunction with [55, Lemma 1.2] then gives the following result.

**Lemma 3.** *Let  $\omega$  be a translation bounded measure, and  $\mathcal{A}$  a van Hove averaging sequence. With  $\gamma_n := \frac{\omega_n * \widetilde{\omega}_n}{\text{vol}(A_n)}$  and  $\gamma_{n;\text{mod}} := \frac{\omega_n * \widetilde{\omega}}{\text{vol}(A_n)}$ , the families  $\{\gamma_n \mid n \in \mathbb{N}\}$  and  $\{\gamma_{n;\text{mod}} \mid n \in \mathbb{N}\}$  are uniformly translation bounded and hence precompact in the vague topology. Any accumulation point of either family, of which there is at least one, is also an accumulation point of the other family, and a translation bounded, positive definite measure.  $\square$*

Lemma 1 applies to any autocorrelation measure, and the corresponding measure  $\widehat{\gamma}$  is then a positive, translation bounded measure. It is called the *diffraction measure* of  $\omega$ , relative to the averaging sequence  $\mathcal{A}$ . In ergodic situations, we have no dependence on  $\mathcal{A}$  and thus suppress it. Then, the diffraction measure is also related to the Bartlett spectrum known from stochastic geometry, though there are important differences to be discussed later; see Remark 15 below.

In general, an interesting initial question concerns the spectral type of  $\widehat{\gamma}$ , which follows from the spectral decomposition

$$(10) \quad \widehat{\gamma} = (\widehat{\gamma})_{\text{pp}} + (\widehat{\gamma})_{\text{sc}} + (\widehat{\gamma})_{\text{ac}}$$

of  $\widehat{\gamma}$  into its pure point, singular continuous and absolutely continuous parts relative to  $\lambda$ , the latter being the Haar measure on  $\mathbb{R}^d$  with  $\lambda([0, 1]^d) = 1$ . Lattices and regular model sets [55, 8] are examples with  $\widehat{\gamma} = (\widehat{\gamma})_{\text{pp}}$ , while the Thue-Morse and the Rudin-Shapiro sequence show singular continuous and absolutely continuous components, respectively; compare [35] and references given there. Absolutely continuous components appearing as a result of stochastic influence are the main theme below.

### 3. RENEWAL PROCESSES IN ONE DIMENSION

An illustrative class of examples is provided by the classical renewal process on the real line, defined by a probability measure  $\varrho$  on  $\mathbb{R}_+ = \{x > 0\}$  of finite mean as follows. Starting from some initial point, at an arbitrary position, a machine moves to the right with constant speed and drops a point on the line with a random waiting time that is distributed according to  $\varrho$ . When this happens, the clock is reset and the process resumes. In what follows, we assume that both the velocity of the machine and the expectation value of  $\varrho$  are 1, so that we end up (in the limit that we let the initial point move to  $-\infty$ ) with realisations that are almost surely point sets in  $\mathbb{R}$  of density 1.

Clearly, the process just described defines a stationary process. It can thus be analysed by considering all realisations which contain the point 0. Moreover, there is a clear (distributional) symmetry around this point, so that we can determine the autocorrelation (in the

sense of (6)) of almost all realisations from studying what happens to the right of 0 (we will make this approach rigorous in Proposition 1 below). Indeed, if we want to know the frequency per unit length of the occurrence of two points with distance  $x$  (or the corresponding density), we need to sum the contributions that  $x$  is the first point after 0, the second point, the third, and so on. In other words, we almost surely obtain the autocorrelation

$$(11) \quad \gamma = \delta_0 + \nu + \tilde{\nu}$$

with  $\nu = \varrho + \varrho * \varrho + \varrho * \varrho * \varrho + \dots = \sum_{n=1}^{\infty} \varrho^{*n}$  and  $\tilde{\nu}$  as defined above, provided that the sum in Eq. (11) converges properly. Note that the point measure at 0 simply reflects that the almost sure density of the resulting point set is 1. In the slightly more general case of a probability measure  $\varrho$  on  $\mathbb{R}_+ \cup \{0\}$ , one has the following convergence result. It is essentially a measure theoretic reformulation of the main lemma in [25, Sec. XI.1], but we prefer to give a complete proof that is adjusted to our setting.

**Lemma 4.** *Let  $\varrho$  be a probability measure on  $\mathbb{R}_+ \cup \{0\}$ , with  $\varrho(\mathbb{R}_+) > 0$ . Then,  $\nu_n := \varrho + \varrho * \varrho + \dots + \varrho^{*n}$  with  $n \in \mathbb{N}$  defines a sequence of positive measures that converges towards a translation bounded measure  $\nu$  in the vague topology.*

*Proof.* Note that the condition  $\varrho(\mathbb{R}_+) > 0$  implies  $0 \leq \varrho(\{0\}) < 1$ , hence excludes the case  $\varrho = \delta_0$ . When  $\varrho = \delta_a$  for some  $a > 0$ , one has  $\nu_n = \sum_{m=1}^n \delta_{ma}$  by a simple convolution calculation, and the claim is obvious. In all remaining cases, it is possible to choose some  $a \in \mathbb{R}_+$  with  $\varrho(\{a\}) = 0$  and  $0 < \varrho([0, a)) = p < 1$ , so that also  $\varrho([a, \infty)) = 1 - p < 1$ . Since the sequence  $\nu_n$  is monotonically increasing, the claimed vague convergence follows from showing that  $\limsup_{n \rightarrow \infty} \nu_n([0, x))$  is bounded by  $C_1 + C_2 x$  for some constants  $C_i$ . As there are at most countably many points  $y$  with  $\varrho(\{y\}) > 0$ , it is sufficient to show these estimates for all  $x \in \mathbb{R}_+$  with  $\varrho(\{x\}) = 0$ . In a second step, we then demonstrate that  $\sum_{n=1}^{\infty} \varrho^{*n}([b, b+x))$  is bounded by  $1 + C_1 + C_2 x$ , independently of  $b$ , which establishes translation boundedness.

If  $(X_i)_{i \in \mathbb{N}}$  denotes a family of i.i.d. random variables, with common distribution according to  $\varrho$  (and thus values in  $\mathbb{R}_+ \cup \{0\}$ ), one has

$$\mathbb{P}(X_1 + \dots + X_m < x) = \varrho^{*m}([0, x)).$$

On the other hand, for the  $a$  chosen above, one has the inequality

$$\mathbb{P}(X_1 + \dots + X_m < x) \leq \mathbb{P}(\text{card}\{1 \leq i \leq m \mid X_i \geq a\} \leq x/a) = \sum_{\ell=0}^{\lfloor x/a \rfloor} \binom{m}{\ell} (1-p)^\ell p^{m-\ell},$$

where  $\binom{m}{\ell} = 0$  whenever  $\ell > m$ . Observing  $\sum_{m=1}^{\infty} p^m = p/(1-p)$  and

$$\sum_{m=1}^{\infty} \binom{m}{\ell} (1-p)^\ell p^{m-\ell} = (1-p)^\ell \frac{1}{\ell!} \frac{d^\ell}{dp^\ell} \sum_{m=0}^{\infty} p^m = \frac{1}{1-p}$$

for all  $\ell \geq 1$ , the previous inequality implies, for arbitrary  $n \in \mathbb{N}$ ,

$$\nu_n([0, x)) \leq \sum_{m=1}^{\infty} \sum_{\ell=0}^{\lfloor x/a \rfloor} \binom{m}{\ell} (1-p)^\ell p^{m-\ell} = \frac{p + \lfloor x/a \rfloor}{1-p} \leq \frac{p}{1-p} + \frac{1}{a(1-p)} x,$$

which establishes the first claim.

For the second estimate, we choose  $b \geq 0$ ,  $x > 0$  and observe

$$\begin{aligned}
 \sum_{n=1}^{\infty} \varrho^{*n}([b, b+x]) &= \sum_{n=1}^{\infty} \mathbb{P}(b \leq X_1 + \cdots + X_n < b+x) \\
 &= \sum_{n=1}^{\infty} \sum_{k=1}^n \mathbb{P}(X_1 + \cdots + X_{k-1} \leq b \leq X_1 + \cdots + X_k \text{ and } b \leq X_1 + \cdots + X_n < b+x) \\
 &\leq \sum_{n=1}^{\infty} \sum_{k=1}^n \mathbb{P}(X_1 + \cdots + X_{k-1} \leq b \leq X_1 + \cdots + X_k) \mathbb{P}(X_{k+1} + \cdots + X_n < x) \\
 &= \sum_{k=1}^{\infty} \mathbb{P}(X_1 + \cdots + X_{k-1} \leq b \leq X_1 + \cdots + X_k) \sum_{n=k}^{\infty} \mathbb{P}(X_{k+1} + \cdots + X_n < x) \\
 &= 1 + \sum_{m=1}^{\infty} \mathbb{P}(X_1 + \cdots + X_m < x),
 \end{aligned}$$

with the convention to treat empty sums of random variables as 0. The last step used the i.i.d. property of the random variables together with  $\mathbb{P}(0 \leq X) = 1$  and

$$\sum_{k=1}^{\infty} \mathbb{P}(X_1 + \cdots + X_{k-1} \leq b \leq X_1 + \cdots + X_k) = 1.$$

In conjunction with our previous estimate, this completes the proof.  $\square$

When  $\varrho(\{0\}) > 0$ , we are outside the realm of (renewal) point processes, and formula (11) for the autocorrelation no longer applies. This case might nevertheless be analysed with the methods of Sections 4 and 5, see Example 7 and Corollary 1 in particular. For the remainder of this section, we assume  $\varrho(\{0\}) = 0$ , so that  $\varrho$  is a measure on  $\mathbb{R}_+$ ; see Remark 14 below for an alternative approach via random counting measures, or [25, Ch. XI.9].

**Proposition 1.** *Consider a renewal process on the real line, defined by a probability measure  $\varrho$  on  $\mathbb{R}_+$  with mean 1. This defines a stationary stochastic process, whose realisations are point sets that almost surely possess the autocorrelation measure  $\gamma = \delta_0 + \nu + \tilde{\nu}$  of (11).*

*Here,  $\nu = \sum_{n=1}^{\infty} \varrho^{*n}$  is a translation bounded positive measure. It satisfies the renewal equations*

$$\nu = \varrho + \varrho * \nu \quad \text{and} \quad (1 - \hat{\varrho}) \hat{\nu} = \hat{\varrho},$$

*where  $\hat{\varrho}$  is a uniformly continuous function on  $\mathbb{R}$ . In this setting, the measure  $\gamma$  is both positive and positive definite.*

*Proof.* The renewal process is a classic stochastic process on the real line which is known to be stationary and ergodic; compare [25, Ch. VI.6] for details. Consequently, the measure of occurrence of a pair of points at distance  $x + dx$  (or the corresponding density) can be calculated by fixing one point at 0 (due to stationarity) and then determining the ensemble average for another point at  $x + dx$  (due to ergodicity). This is the justification for the heuristic reasoning given above, prior to Eq. (11).

By Lemma 4,  $\nu$  is a translation bounded measure, so that the convolution  $\varrho * \nu$  is well defined by [13, Prop. 1.13]. The first renewal identity is then clear from the structure of  $\nu$

as a limit, while the second follows by Fourier transform and the convolution theorem. The autocorrelation is a positive definite measure by construction, though this is not immediate here on the basis of its form as a sum, see [1] for a related discussion.  $\square$

Let us now consider the spectral type of the resulting diffraction measure for the class of point sets generated by a renewal process. This requires a distinction on the basis of the support of  $\varrho$ . To this end, the second identity of Proposition 1 is helpful, because one has

$$(12) \quad \widehat{\nu}(k) = \frac{\widehat{\varrho}(k)}{1 - \widehat{\varrho}(k)}$$

at all positions  $k$  with  $\widehat{\varrho}(k) \neq 1$ . This is in line with summing  $\widehat{\nu}$  as a geometric series, which gives the same formula for  $\widehat{\nu}(k)$  for all  $k$  with  $|\widehat{\varrho}(k)| < 1$  and has (12) as the unique continuous extension to all  $k$  with  $|\widehat{\varrho}(k)| = 1 \neq \widehat{\varrho}(k)$ . In fact, one sees that  $\widehat{\nu}(k)$  is a continuous function on the complement of the set  $\{k \in \mathbb{R} \mid \widehat{\varrho}(k) = 1\}$ . For most  $\varrho$ , the latter set happens to be the singleton set  $\{0\}$ .

In general, a probability measure  $\mu$  on  $\mathbb{R}$  is called *lattice-like* when its support is a subset of a translate of a lattice, see [27] for details. We need a slightly stronger property here, and call  $\mu$  *strictly lattice-like* (called *arithmetic* in [25]) when its support is a subset of a lattice. So, the difference is that we do not allow any translates here; see [2] for related results.

**Lemma 5.** *If  $\mu$  is a probability measure on  $\mathbb{R}$ , its Fourier transform,  $\widehat{\mu}(k)$ , is a uniformly continuous and positive definite function on  $\mathbb{R}$ , with  $|\widehat{\mu}(k)| \leq \widehat{\mu}(0) = 1$ .*

*Moreover, the following three properties are equivalent.*

- (i)  $\text{card}\{k \in \mathbb{R} \mid \widehat{\mu}(k) = 1\} > 1$ ;
- (ii)  $\text{card}\{k \in \mathbb{R} \mid \widehat{\mu}(k) = 1\} = \infty$ ;
- (iii)  $\text{supp}(\mu)$  is strictly lattice like.

*Proof.* One has  $\widehat{\mu}(k) = \int_{\mathbb{R}} e^{-2\pi i k x} d\mu(x)$ , whence the first claims are standard consequences of Fourier analysis; compare [48, Prop. 5.2.1] and [52, Sec. 1.3.3].

If  $\mu = \sum_{x \in \Gamma} p(x) \delta_x$  for a lattice  $\Gamma \subset \mathbb{R}$ , with  $p(x) \geq 0$  and  $\sum_{x \in \Gamma} p(x) = 1$ , one has

$$\widehat{\mu}(k) = \sum_{x \in \Gamma} p(x) e^{-2\pi i k x},$$

so that  $\widehat{\mu}(k) = 1$  for any  $k \in \Gamma^*$ . In particular,  $\Gamma^* \subset \{k \in \mathbb{R} \mid \widehat{\mu}(k) = 1\}$ , so that we have the implications (iii)  $\Rightarrow$  (ii)  $\Rightarrow$  (i).

Conversely, if  $\widehat{\mu}(k) = 1$  for some  $k \neq 0$ , one has  $\int_{\mathbb{R}} e^{-2\pi i k x} d\mu(x) = 1$  and hence

$$(13) \quad \int_{\mathbb{R}} (1 - \cos(2\pi k x)) d\mu(x) = \int_{\text{supp}(\mu)} (1 - \cos(2\pi k x)) d\mu(x) = 0,$$

where  $\text{supp}(\mu)$ , the support of the probability measure  $\mu$ , is a closed subset of  $\mathbb{R}$  and measurable. The integrand is a continuous non-negative function that, due to  $k \neq 0$ , vanishes precisely on the set  $\frac{1}{k}\mathbb{Z}$ , which is a lattice.

Write  $\text{supp}(\mu) = A \dot{\cup} B$  as a disjoint union of measurable sets, with  $A = \text{supp}(\mu) \cap \frac{1}{k}\mathbb{Z}$  and  $B = \text{supp}(\mu) \cap (\mathbb{R} \setminus \frac{1}{k}\mathbb{Z})$ . We can now split the second integral in (13) into an integral over  $A$ , which vanishes because the integrand does, and one over the set  $B$ , which would give a positive

contribution by standard arguments, unless  $B = \emptyset$ . But this means  $\text{supp}(\mu) = A \subset \frac{1}{k}\mathbb{Z}$ , so that (i)  $\Rightarrow$  (iii), which establishes the result.  $\square$

At this point, we can state the main result of this section, the diffraction properties of renewal processes; compare [19, Ex. 8.2(b)] for a special case.

**Theorem 1.** *Let  $\varrho$  be a probability measure on  $\mathbb{R}_+$  with mean 1, and assume that  $\varrho$  is not strictly lattice-like. Assume further that a moment of  $\varrho$  of order  $1 + \varepsilon$  exists for some  $\varepsilon > 0$ . Then, the point sets obtained from the stationary renewal process based on  $\varrho$  almost surely have a diffraction measure of the form  $\hat{\gamma} = \delta_0 + (\hat{\gamma})_{\text{ac}}$  with*

$$(\hat{\gamma})_{\text{ac}} = \frac{1 - |\hat{\varrho}(k)|^2}{|1 - \hat{\varrho}(k)|^2} \lambda = (1 - h) \lambda,$$

where  $h$  is a continuous function on  $\mathbb{R} \setminus \{0\}$  that is locally integrable. It is given by

$$h(k) = \frac{2(|\hat{\varrho}(k)|^2 - \text{Re}(\hat{\varrho}(k)))}{|1 - \hat{\varrho}(k)|^2}$$

and measures the difference from a constant background as described by  $\lambda$ .

When  $\varrho$  is strictly lattice-like, the pure point part becomes a lattice Dirac comb, and the behaviour of  $h$  at 0 repeats at each point of the underlying lattice, see Remark 3 for details.

*Proof.* The process has a well-defined autocorrelation  $\gamma$ , by an application of Proposition 1, in the sense that almost every realisation of the process is a point set  $A$  with this autocorrelation. Since  $\gamma$  is a positive definite measure, it is Fourier transformable by Lemma 1(ii), with  $\hat{\gamma}$  being a positive measure on  $\mathbb{R}$ .

The point measure at 0 with intensity 1 reflects the fact that the resulting point set  $A$  almost surely has density 1. To see this, define  $g_n = \frac{1}{n} \mathbf{1}_{[-\frac{n}{2}, \frac{n}{2}]}$  and  $h_n = g_n * \tilde{g}_n$ . Here,  $h_n$  is a positive definite, tent-shaped function with support  $[-n, n]$  and maximal value  $\frac{1}{n}$  at 0. It has (inverse) Fourier transform  $\tilde{h}_n(k) = \left(\frac{\sin(\pi kn)}{\pi kn}\right)^2$ , which is a non-negative function (with maximum value 1 at  $k = 0$ ) that concentrates around 0 as  $n \rightarrow \infty$ . Let  $\omega_r := \delta_{A \cap [-r, r]}$ . Using (6) together with  $(\omega_r * \tilde{\omega}_r)(g_n * \tilde{g}_n) \geq 0$ , it is not difficult to see that  $\gamma(h_n) \xrightarrow{n \rightarrow \infty} (\text{dens}(A))^2$ , which is almost surely 1 (for this, assume first that  $r \gg n \gg 1$ , then take the limit  $r \rightarrow \infty$  followed by the limit  $n \rightarrow \infty$ ). On the other hand, one has

$$\gamma(h_n) = \tilde{\tilde{\gamma}}(h_n) = \hat{\gamma}(\tilde{h}_n) \xrightarrow{n \rightarrow \infty} \hat{\gamma}(\{0\}),$$

due to the concentration property of  $\tilde{h}_n$  (in particular, for all  $\varepsilon > 0$ , one verifies the relation  $\hat{\gamma}(B_\varepsilon(0)) \geq (\text{dens}(A))^2 > 0$ , which proves the existence of a point measure at 0).

Due to the assumption that  $\text{supp}(\varrho)$  is not contained in a lattice, we may invoke Lemma 5 to see that  $\hat{\varrho}(k) \neq 1$  whenever  $k \neq 0$ , so that we have pointwise convergence

$$\hat{\nu}_n(k) \xrightarrow{n \rightarrow \infty} \hat{\nu}(k) = \frac{\hat{\varrho}(k)}{1 - \hat{\varrho}(k)}$$

on  $\mathbb{R} \setminus \{0\}$ , and similarly for  $\hat{\tilde{\nu}}$ . Since  $\hat{\varrho}$  is uniformly continuous on  $\mathbb{R}$  and  $\hat{\varrho}(k) \neq 1$  on  $\mathbb{R} \setminus \{0\}$ , both  $\hat{\nu}$  and  $\hat{\tilde{\nu}}$  are represented, on  $\mathbb{R} \setminus \{0\}$ , by continuous Radon-Nikodym densities. Writing  $(\delta_0 + \nu + \tilde{\nu})^\wedge = (1 - h)\lambda$ , hence  $(\nu + \tilde{\nu})^\wedge = -h\lambda$ , the formula for  $h$  follows from  $\hat{\tilde{\nu}} = \overline{\hat{\nu}}$ .

It remains to show that  $1 - h$  is locally integrable near 0. Let  $X$  be a random variable with distribution  $\varrho$ . Since the latter has mean 1 and our assumption guarantees that  $\langle X^{1+\varepsilon} \rangle = \int_0^\infty x^{1+\varepsilon} d\varrho(x) < \infty$ , we have the Taylor series expansion

$$\widehat{\varrho}(k) = 1 - 2\pi ik + \mathcal{O}(|k|^{1+\varepsilon}), \quad \text{as } |k| \rightarrow 0,$$

by an application of [61, Thm. 1.5.4]. Inserting this into the expression for  $h$  results in

$$h(k) = 2 + \mathcal{O}(k^{-1+\varepsilon}), \quad \text{as } |k| \rightarrow 0,$$

which establishes integrability around 0, and thus absolute continuity of the measure  $(1 - h)\lambda$ .

As the contribution to the central peak is already completely accounted for by the term  $\delta_0$ , the claim follows.  $\square$

**Remark 2.** ASYMPTOTIC BEHAVIOUR OF  $h$ . When, in the setting of Theorem 1, the second moment of  $\varrho$  exists, one obtains from [61, Thm. 1.5.3] the slightly stronger expansion

$$\widehat{\varrho}(k) = 1 - 2\pi ik - 2\pi^2 \langle X^2 \rangle k^2 + \mathcal{O}(|k|^2), \quad \text{as } |k| \rightarrow 0.$$

This leads to the asymptotic behaviour

$$h(k) = 2 - \langle X^2 \rangle + \mathcal{O}(1), \quad \text{as } |k| \rightarrow 0,$$

which implies that  $h$  is bounded and can continuously be extended to  $h(0) = 2 - \langle X^2 \rangle = 1 - \sigma^2$ , where  $\sigma^2$  is the variance of  $\varrho$ . Clearly, the existence of higher moments implies stronger smoothness properties.  $\diamond$

**Remark 3.** COMPLEMENT OF THEOREM 1. When  $\varrho$  happens to be strictly lattice like, the  $\mathbb{Z}$ -span of the finite or uniformly discrete<sup>1</sup> set  $\text{supp}(\varrho)$  is a lattice of the form  $\Gamma = b\mathbb{Z}$ , where  $b > 0$  is unique (in other words,  $\Gamma$  is the coarsest lattice in  $\mathbb{R}$  that contains  $\text{supp}(\varrho)$ ). Then, one finds the diffraction

$$\widehat{\gamma} = \delta_{\mathbb{Z}/b} + (1 - h)\lambda,$$

with the function  $h$  from Theorem 1. Note that  $h$  is well-defined (and continuous) on  $\mathbb{R} \setminus \Gamma^*$ , with  $\Gamma^* = \mathbb{Z}/b$  being the dual lattice of  $\Gamma$ . Moreover, it is locally integrable around all points of  $\Gamma^*$ , so that  $(1 - h)\lambda$  is again an absolutely continuous measure. Note that, since the underlying point set is always a subset of  $\Gamma$ , the diffraction measure is periodic, with  $\Gamma^*$  as its lattice of periods; compare [2] for general results in this direction.

When  $\text{supp}(\varrho)$  is a finite set, one is in the situation of a random tiling with finitely many prototiles. A more detailed discussion, together with an explicit calculation of  $h$  for this case, is given in [4, Thm. 2]; see also Example 5 and Remark 7.  $\diamond$

Let us turn to some examples, for which we employ the Heavyside function,

$$(14) \quad \Theta(x) := \begin{cases} 1, & \text{if } x > 0, \\ \frac{1}{2}, & \text{if } x = 0, \\ 0, & \text{if } x < 0. \end{cases}$$

---

<sup>1</sup>Recall that a set  $S \subset \mathbb{R}^d$  is called *uniformly discrete* when there is a number  $s > 0$  such that the distance between any two distinct points of  $S$  is at least  $s$ .

This formulation of  $\Theta$  has some advantage for formal calculations around generalised functions and their Fourier transforms.

**Example 1.** POISSON PROCESS ON THE REAL LINE. The probably best-known stochastic process is the classical (homogeneous) Poisson process on the line, with intensity 1, where  $\varrho = f\lambda$  is given by the density

$$f(x) = e^{-x} \Theta(x).$$

It is easy to check that the convolution of  $n + 1$  copies of this function yields  $e^{-x} x^n \Theta(x)/n!$ , which gives  $\nu = \Theta\lambda$ . As the intensity is 1, this results in the autocorrelation

$$\gamma = \delta_0 + \nu + \tilde{\nu} = \delta_0 + \lambda$$

and thus in the diffraction  $\hat{\gamma} = \gamma$ , compare Eq. (4). ◇

**Remark 4.** CHARACTERISATION OF POISSON PROCESSES. Let  $N$  denote a homogeneous Poisson process on the real line, so that, for any measurable  $A \subset \mathbb{R}$ ,  $N(A)$  is the number of renewal points that fall into  $A$ . It is well-known that  $N(A)$  is then Poisson- $(\lambda(A))$ -distributed, which means that

$$\mathbb{P}(N(A) = k) = \frac{e^{-\lambda(A)} (\lambda(A))^k}{k!}$$

with  $k \in \mathbb{N}_0$ , and that, for any collection of pairwise disjoint sets  $A_1, A_2, \dots, A_m$ , the random numbers  $N(A_1), \dots, N(A_m)$  are independent. In fact, this property characterises the Poisson process (compare [19, Ch. 2.1]), and it can serve as a definition in higher dimensions or in more general measure spaces, to which the renewal process cannot be extended. ◇

**Example 2.** RENEWAL PROCESS WITH REPULSION. A perhaps more interesting example in this spirit is given by the density

$$f(x) = 4x e^{-2x} \Theta(x).$$

It is normalised and has mean 1, as in Example 1, but models a repulsion of points for small distances. Note that this distribution can be realised out of Example 1 by taking only every second point, followed by a rescaling of time.

By induction (or by using well-known properties of the gamma distributions, compare [25, Sec. II.2]), one checks that

$$f^{*n}(x) = \frac{4^n}{(2n-1)!} x^{2n-1} e^{-2x} \Theta(x),$$

which finally results in the autocorrelation

$$\gamma = \delta_0 + (1 - e^{-4|x|}) \lambda = \delta_0 + \lambda - e^{-4|x|} \lambda$$

and in the diffraction measure

$$\hat{\gamma} = \delta_0 + \frac{2 + (\pi k)^2}{4 + (\pi k)^2} \lambda = \delta_0 + \lambda - \frac{2\lambda}{4 + (\pi k)^2}.$$

This is illustrated in Figure 1. The ‘dip’ in the absolutely continuous part around 0, and thus the deviation from the previous example, reflects the effectively repulsive nature of the stochastic process when viewed from the perspective of neighbouring points. ◇

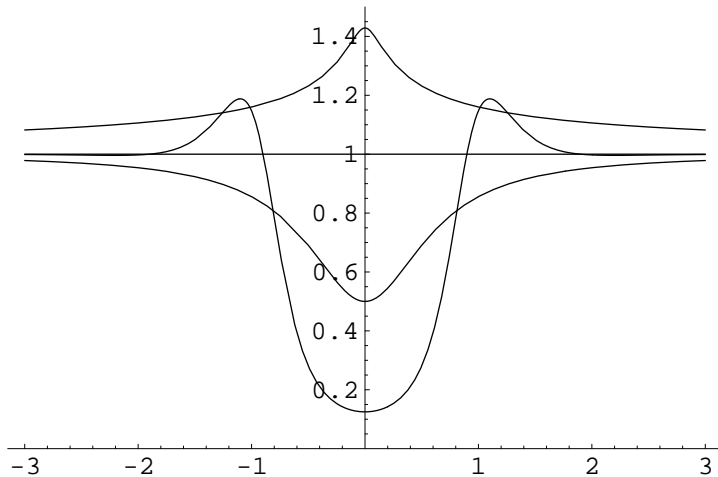


FIGURE 1. Absolutely continuous part of the diffraction measure from Example 3, for  $\alpha = 0.7$  (top curve),  $\alpha = 1$  (horizontal line, which also represents Example 1),  $\alpha = 2$  (see also Example 2) and  $\alpha = 8$  (overshooting curve).

**Example 3.** RENEWAL PROCESS WITH GAMMA LAW OF MEAN 1. The previous two examples are special cases of the gamma family of measures. For fixed mean 1, they are parametrised by a real number  $\alpha > 0$  via  $\varrho_\alpha = f_\alpha \lambda$  and the density

$$(15) \quad f_\alpha(x) := \frac{\alpha^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\alpha x} \Theta(x).$$

While  $\alpha = 1$  is the ‘interaction-free’ Poisson process of Example 1, the density implies an effectively attractive (repulsive) nature of the process for  $0 < \alpha < 1$  (for  $\alpha > 1$ ). When  $\alpha = k \in \mathbb{N}$ , the process can also be interpreted as a modified Poisson process where one keeps only every  $k$ th point (followed by an appropriate rescaling).

Observing  $f_\alpha^{*n}(x) = \frac{\alpha^{n\alpha}}{\Gamma(n\alpha)} x^{n\alpha-1} e^{-\alpha x} \Theta(x)$  for  $n \in \mathbb{N}$ , this leads to the measure

$$(16) \quad \nu_\alpha = g_\alpha \Theta \lambda \quad \text{with} \quad g_\alpha(x) = \alpha e^{-\alpha x} \sum_{n=1}^{\infty} \frac{(\alpha x)^{n\alpha-1}}{\Gamma(n\alpha)}.$$

Note that, for fixed  $\alpha$ , one has  $\lim_{x \rightarrow \infty} g_\alpha(x) = 1$ . The calculations result in the autocorrelation

$$\gamma_\alpha = \delta_0 + g_\alpha(|x|) \lambda$$

and in the diffraction  $\widehat{\gamma}_\alpha = \delta_0 + (1 - h_\alpha) \lambda$ , where  $h_\alpha$  is the symmetric function defined by

$$h_\alpha(k) = \frac{2(1 - \operatorname{Re}((1 + 2\pi ik/\alpha)^\alpha))}{|1 - (1 + 2\pi ik/\alpha)^\alpha|^2}.$$

The latter follows from the general form of  $h$  in Theorem 1, together with the observation that  $\widehat{f}_\alpha(k) = (1 + 2\pi ik/\alpha)^{-\alpha}$ .

It is easy to see that  $\lim_{k \rightarrow \pm\infty} h_\alpha(k) = 0$ , for any fixed  $\alpha > 0$ , which makes the role of  $h_\alpha$  as the deviation from the Poisson process diffraction more transparent, where  $\alpha = 1$  and  $h_1 \equiv 0$ . Note also that  $\lim_{\alpha \rightarrow \infty} \widehat{\gamma}_\alpha = \delta_{\mathbb{Z}}$  in the vague topology, in line with the limits mentioned

before. This can nicely be studied in a series of plots of the diffraction with growing value of the parameter  $\alpha$ . Figure 1 shows some initial cases.  $\diamond$

**Remark 5.** CONSTRUCTION OF DELONE SETS. Of particular interest in the applications are *Delone sets* (which are point sets that are both uniformly discrete and relatively dense), because points (representing atoms, say) should neither be too close nor too far apart. Such sets can also arise from a renewal process. In fact, if one considers a probability measure  $\varrho$  on  $\mathbb{R}_+$ , the resulting point sets are always Delone sets when  $\text{supp}(\varrho) \subset [a, b]$  with  $0 < a \leq b < \infty$ , and conversely. This equivalence does not depend on the nature of  $\varrho$  on  $[a, b]$ , while the local complexity of the resulting point sets does. In particular, if  $\varrho$  is absolutely continuous, the point sets will not have finite local complexity (see below for a definition).  $\diamond$

It is clear that no absolutely continuous  $\varrho$  is lattice-like, hence certainly not strictly lattice-like, so that all these examples match Theorem 1. Probability measures  $\varrho$  with  $\text{supp}(\varrho)$  contained in a lattice are covered by Remark 3. They are of interest because they form a link to point sets and tilings of finite local complexity, which have only finitely many patches of a given size (up to translations). Let us consider some examples.

**Example 4.** DETERMINISTIC LATTICE CASE. The simplest case is  $\varrho = \delta_1$ . From  $\delta_1 * \delta_1 = \delta_2$ , one sees that  $\nu = \delta_{\mathbb{N}}$  and hence

$$\gamma = \delta_0 + \delta_{\mathbb{N}} + \delta_{-\mathbb{N}} = \delta_{\mathbb{Z}},$$

which is a lattice Dirac comb, with Fourier transform

$$\widehat{\gamma} = \delta_{\mathbb{Z}}$$

according to the Poisson summation formula (1). This is the deterministic case of the integer lattice  $\mathbb{Z}$ , covered in this setting.  $\diamond$

**Remark 6.** DETERMINISTIC LIMIT OF EXAMPLE 3. The last example can also be seen as a limiting case of the measure  $\varrho_\alpha$  defined by Eq. (15). In particular, one has  $\lim_{\alpha \rightarrow \infty} \varrho_\alpha = \delta_1$  and  $\lim_{\alpha \rightarrow \infty} \nu_\alpha = \delta_{\mathbb{N}}$ , with  $\nu_\alpha$  as in (16) and both limits to be understood in the vague topology. This can also be seen by means of the strong law of large numbers. For each  $n \in \mathbb{N}$ , by well-known divisibility properties of the family of Gamma distributions,  $\varrho_n$  is the distribution of

$$\frac{1}{n} \sum_{i=1}^n X_i,$$

where the  $X_i$  are independent and exponentially distributed random variables with mean 1. This sum then concentrates around 1, with a standard deviation of order  $1/\sqrt{n}$ .  $\diamond$

**Example 5.** RANDOM TILINGS WITH FINITELY MANY PROTOTILES. Consider the measure

$$\varrho = \alpha\delta_a + (1-\alpha)\delta_b,$$

with  $\alpha \in (0, 1)$  and  $a, b > 0$ , subject to the restriction  $\alpha a + (1-\alpha)b = 1$  to ensure density 1. Each realisation of the corresponding renewal process results in a point set that can also be viewed as a random tiling on the line with two prototiles, of lengths  $a$  and  $b$ . As before, place a normalised point measure at each point of the realisation. Then, the diffraction (almost

surely) has a pure point and an absolutely continuous part, but no singular continuous one. The pure point part can be just  $\delta_0$  (when  $b/a$  is irrational) or a lattice comb (see Remark 3); details are given in [4], including an explicit formula for the AC part.

This has a straight-forward generalisation to any finite number of prototiles, with a similar result. Also in this case, there is an explicit formula for the diffraction measure, which was derived in [4] by a direct method, without using the renewal process.  $\diamond$

**Remark 7.** CONTINUOUS DIFFRACTION WITH ‘NEEDLES’. Looking back at Lemma 5, one realises that Example 5 revolves around the lattice condition in an interesting way. Namely, even if  $\varrho$  is *not* strictly lattice-like,  $\text{supp}(\varrho)$  for a random tiling example with finitely many prototiles is a finite set, and thus a subset of a *Meyer set* (which is a relatively dense set  $\Lambda$  whose difference set  $\Lambda - \Lambda$  is uniformly discrete). We then know from the harmonic analysis of Meyer sets, compare [44] and references therein, that  $\widehat{\varrho}(k)$  will come  $\varepsilon$ -close to 1 with bounded gaps in  $k$ . This means that the diffraction measure, though it is absolutely continuous apart from the central peak at  $k = 0$ , will develop sharp ‘needles’ that are close to point measures in the vague topology — a phenomenon that was also observed in [4] on the basis of the explicit solution.  $\diamond$

#### 4. ARBITRARY DIMENSIONS: ELEMENTARY APPROACH

Let us now develop some intuition for the influence of randomness on the diffraction of point sets and certain structures derived from them in Euclidean spaces of arbitrary dimension. In this section, our point of view is from a single point set  $\Lambda \subset \mathbb{R}^d$  that is being modified randomly, by replacing each point by a complex, finite, random cluster. This is still relatively easy as long as  $\Lambda$  is sufficiently ‘nice’. In Section 5, we revisit this situation from the point of view of a stationary ergodic point process, which treats almost all of its realisations at once and permits a larger generality for the sets  $\Lambda$ , though the clusters will then be restricted to positive or signed measures.

Let  $\Lambda \subset \mathbb{R}^d$  be a fixed point set, which we assume to be of *finite local complexity* (FLC). By definition, this means that there are only finitely many distinct patches of any given size (up to translations) in  $\Lambda$ . This property is equivalent to the difference set  $\Lambda - \Lambda$  being locally finite [55], the latter saying that  $K \cap (\Lambda - \Lambda)$  is a finite set for all compact  $K \subset \mathbb{R}^d$ . In particular, since 0 is then isolated in  $\Lambda - \Lambda$ , the set  $\Lambda$  itself is uniformly discrete; see Remark 11 for a possible extension. Attached to  $\Lambda$  is its *Dirac comb*  $\delta_\Lambda = \sum_{x \in \Lambda} \delta_x$ , which is a translation bounded measure, as a consequence of the FLC property. We associate to  $\delta_\Lambda$  the autocorrelation and the diffraction measure as explained in Section 2, for a suitably chosen averaging sequence  $\mathcal{A} = \{A_n \mid n \in \mathbb{N}\}$  of van Hove type. A natural choice is  $A_n = B_{r_n}(0)$ , with  $B_r(0)$  denoting the open ball of radius  $r$  around 0, for a non-decreasing series of radii with  $r_n \xrightarrow{n \rightarrow \infty} \infty$  (alternatively, nested cubes are also quite common).

Set  $\Lambda_n = \Lambda \cap A_n$  (so that  $\Lambda_n \nearrow \Lambda$  in the obvious local topology [55]) and consider

$$\gamma_{\Lambda,n} := \frac{\delta_{\Lambda_n} * \widetilde{\delta_{\Lambda_n}}}{\text{vol}(A_n)} = \frac{1}{\text{vol}(A_n)} \sum_{x,y \in \Lambda_n} \delta_{x-y}.$$

We now make the assumption that the limit

$$(17) \quad \lim_{n \rightarrow \infty} \gamma_{\Lambda, n} =: \gamma_{\Lambda}$$

exists in the vague topology, which is then the autocorrelation measure of the set  $\Lambda$  relative to the averaging sequence  $\mathcal{A}$ .

**Remark 8.** ACCUMULATION POINTS VERSUS LIMITS. Due to translation boundedness of  $\delta_{\Lambda}$ , the sequence of measures  $\gamma_{\Lambda, n}$  always has points of accumulation; see [32, Prop. 2.2] and Lemma 3. Consequently, one can always select a subsequence of  $\mathcal{A}$  for which the assumption (17) is satisfied. This remains true even if we relax the nesting condition for  $\mathcal{A}$ . In this sense, when the autocorrelation is not unique (as in the example of the visible lattice points without nesting [9]), we simply select *one* of the possible autocorrelations by a suitable choice of  $\mathcal{A}$ . Our results below apply to any autocorrelation of this kind separately. In this sense, the assumption made in (17) is not restrictive.  $\diamond$

As briefly explained in Section 2, see Lemma 3, the van Hove property of  $\mathcal{A}$  in the context of (17) implies that one also has

$$(18) \quad \lim_{n \rightarrow \infty} \gamma_{\Lambda, n; \text{mod}} := \lim_{n \rightarrow \infty} \frac{\delta_{\Lambda_n} * \widetilde{\delta_{\Lambda}}}{\text{vol}(A_n)} = \gamma_{\Lambda},$$

the difference between the two approximating measures in (17) and (18) being a ‘surface term’ that vanishes in the infinite volume limit  $n \rightarrow \infty$ . Eq. (17) explicitly shows that the measure  $\gamma_{\Lambda}$  is positive definite (hence transformable by Lemma 1), while (18) is easier to work with for (pointwise) calculations in the presence of random modifications as introduced below.

Since  $\Lambda - \Lambda$  is locally finite by assumption, Eq. (18) is equivalent to the existence of all the pointwise limits

$$(19) \quad \lim_{n \rightarrow \infty} \eta_n(z) =: \eta(z),$$

with the approximating coefficients

$$\eta_n(z) = \frac{\text{card}\{x \in A_n \mid x - z \in \Lambda\}}{\text{vol}(A_n)},$$

where  $\eta(z) = 0$  for any  $z \notin \Lambda - \Lambda$ . Clearly, the measure  $\gamma_{\Lambda}$  as well as the coefficients  $\eta(z)$  may (and generally will) depend on the averaging sequence; compare Remark 8.

The next step consists in modifying  $\Lambda$  by a random process in a local way. To come to a reasonably general formulation that includes several notions of randomness known from lattice theory, compare [30, 63], we employ a formulation with finite, random, complex measures. Let  $\Omega$  denote a measure-valued random variable, and  $Q$  the corresponding law, which is itself a probability measure on  $\mathcal{M}_{\text{bd}} = \mathcal{M}_{\text{bd}}(\mathbb{R}^d)$ , the space of finite complex measures on  $\mathbb{R}^d$ . To keep the notation compact, we use the symbol  $\mathbb{E}_Q$  for the various expectation values that arise in connection with  $(\Omega, Q)$ . In particular, we write  $\mathbb{E}_Q(\Omega) = \int_{\mathcal{M}_{\text{bd}}} \omega \, dQ(\omega)$ , where  $\omega$  refers to the realisations of  $\Omega$  as usual. Note that we also refer via the index  $Q$  to the underlying law for one random variable for more complicated expectation values, rather than using the underlying (though hidden) probability space. This will be explained in more detail in Section 5 below.

To proceed, we need a version of the strong law of large numbers (SLLN) for measures.

**Lemma 6.** *Let  $(\Omega_i)_{i \in \mathbb{N}}$  be a sequence of integrable, finite, i.i.d. random measures, with common law  $Q$ . Then, with probability 1, one has*

$$\frac{1}{n} \sum_{i=1}^n \Omega_i \xrightarrow{n \rightarrow \infty} \mathbb{E}_Q(\Omega_1)$$

in the vague topology.

*Proof.* By definition, integrability means that  $\mathbb{E}_Q(|\Omega_i|)$ , which is independent of  $i \in \mathbb{N}$ , is a finite measure. As the space of continuous functions  $C_c(\mathbb{R}^d)$  is separable, the almost sure convergence of the measures follows from the almost sure convergence of  $\frac{1}{n} \sum_{i=1}^n \Omega_i(\varphi)$  for an arbitrary (but fixed) bounded, continuous function  $\varphi$ . This, in turn, follows from the conventional SLLN [24], possibly after splitting the sums into their real and imaginary parts and applying the SLLN twice.  $\square$

Recall that  $\tilde{\omega}$  is the measure defined by  $\tilde{\omega}(\varphi) = \overline{\omega(\tilde{\varphi})}$ . Let  $\Omega$  and  $\Omega'$  be two independent random measures, with the same law  $Q$ , and such that  $\mathbb{E}_Q(|\Omega|)$  is a finite measure, and also assume the second moment condition  $\mathbb{E}_Q((|\Omega|(\mathbb{R}^d))^2) < \infty$ . Then, the convolution  $\Omega * \Omega'$  is well defined, and one obtains from elementary calculations the important relations

$$(20) \quad \mathbb{E}_Q(\tilde{\Omega}) = \widetilde{\mathbb{E}_Q(\Omega)} \quad \text{and} \quad \mathbb{E}_Q(\Omega * \tilde{\Omega}') = \mathbb{E}_Q(\Omega) * \widetilde{\mathbb{E}_Q(\Omega')},$$

the second due to the assumed independence.

Let us now fix an FLC set  $\Lambda$ , which is assumed to possess the autocorrelation measure  $\gamma_\Lambda$  relative to the van Hove averaging sequence  $\mathcal{A}$  chosen, and consider the family  $(\Omega_x)_{x \in \Lambda}$  of integrable, complex, i.i.d. random measures, with common law  $Q$  and subject to the moment conditions mentioned above. When  $\Omega$  is any representative of these random measures,  $\mathbb{E}_Q(|\Omega|)$  is a finite measure by assumption, and the measure-valued expectations  $\mathbb{E}_Q(\Omega)$  and  $\mathbb{E}_Q(\Omega * \tilde{\Omega})$  exist (note that also  $\mathbb{E}_Q(|\Omega * \tilde{\Omega}|)$  is a finite measure, due to the condition on the second moment). We are now interested in the random object

$$(21) \quad \delta_\Lambda^{(\Omega)} = \sum_{x \in \Lambda} \Omega_x * \delta_x,$$

which is almost surely a locally finite measure (though not necessarily translation bounded).

To see this, we observe that, for any bounded Borel set  $B \subset \mathbb{R}^d$ , the sum  $\sum_{x \in \Lambda} |(\Omega_x * \delta_x)(B)|$  converges almost surely, since

$$\mathbb{E}_Q(|(\Omega_x * \delta_x)(B)|) = \mathbb{E}_Q(|\Omega(B - x)|) \leq (\mathbb{E}_Q(|\Omega|))(B - x)$$

and the convolution  $\delta_\Lambda * \mathbb{E}_Q(|\Omega|)$  is a well-defined locally finite measure due to the translation boundedness of  $\delta_\Lambda$  (note that the summands in  $\sum_{x \in \Lambda} \Omega_x * \delta_x(B)$  are non-negative, hence convergence of the means implies almost sure convergence). As the Borel  $\sigma$ -algebra on  $\mathbb{R}^d$  is countably generated, we can find a set of  $Q$ -measure 1 on which the sum (21) converges (absolutely) for each Borel set  $B$ , and the limit is a measure.

Let  $\mathcal{A}$  be fixed and assume for simplicity that each  $A_n$  is invariant under  $x \mapsto -x$ . The ( $n$ -th) approximating autocorrelation of  $\delta_A^{(\Omega)}$  reads

$$(22) \quad \gamma_{\Lambda,n}^{(\Omega)} = \frac{1}{\text{vol}(A_n)} \delta_{\Lambda}^{(\Omega)}|_{A_n} * \widetilde{\delta_{\Lambda}^{(\Omega)}}|_{A_n} = \frac{1}{\text{vol}(A_n)} \sum_{x \in \Lambda} (\Omega_x * \delta_x)|_{A_n} * \sum_{y \in \Lambda} (\widetilde{\Omega}_y * \delta_{-y})|_{A_n}.$$

For certain pointwise calculations and arguments, it will be more convenient below to consider the modified approximating autocorrelation

$$(23) \quad \gamma_{\Lambda,n;\text{mod}}^{(\Omega)} := \frac{1}{\text{vol}(A_n)} \left( \sum_{x \in \Lambda_n} \Omega_x * \delta_x \right) * \left( \sum_{y \in \Lambda} \widetilde{\Omega}_y * \delta_{-y} \right).$$

To this end, we need a probabilistic analogue of Eq. (9).

**Proposition 2.** *Almost surely,  $\gamma_{\Lambda,n}^{(\Omega)}$  of (22) and  $\gamma_{\Lambda,n;\text{mod}}^{(\Omega)}$  of (23) define sequences of locally finite random measures. Moreover, we can choose a strictly increasing subsequence  $(n_k)_{k \in \mathbb{N}}$  such that, in the vague topology, we almost surely have*

$$\gamma_{\Lambda,n_k}^{(\Omega)} - \gamma_{\Lambda,n_k;\text{mod}}^{(\Omega)} \xrightarrow{k \rightarrow \infty} 0.$$

*In particular, if  $\gamma_{\Lambda,n}^{(\Omega)}$  or  $\gamma_{\Lambda,n;\text{mod}}^{(\Omega)}$  almost surely converges to  $\gamma$  along  $\mathcal{A}$  or along a subsequence of it, we can choose a subsequence  $\mathcal{A}'$  of  $\mathcal{A}$  so that both sequences almost surely converge to  $\gamma$  along  $\mathcal{A}'$ .*

*Proof.* We abbreviate  $\mu(\cdot) := \mathbb{E}(|\Omega|(\cdot))$ ,  $\nu(\cdot) := \mathbb{E}(|\Omega|(\mathbb{R}^d) |\Omega|(\cdot))$ . Due to the assumption  $\mathbb{E}((|\Omega|(\mathbb{R}^d))^2) < \infty$ , both  $\mu$  and  $\nu$  are finite, positive measures. Consequently, since  $\Lambda$  is uniformly discrete,  $\mu * \delta_{\Lambda}$  and  $\nu * \delta_{\Lambda}$  are translation bounded by [13, Prop. 1.13] (and thus certainly locally finite).

Let us first verify that the expression in (23) almost surely defines a locally finite measure (the estimate for (22) is completely analogous, with the same upper bound). If  $B \subset \mathbb{R}^d$  is a bounded Borel set, we have

$$\begin{aligned} \text{vol}(A_n) \mathbb{E}_Q(|\gamma_{\Lambda,n;\text{mod}}^{(\Omega)}|(B)) &\leq \sum_{\substack{x \in \Lambda_n \\ y \in \Lambda}} \mathbb{E}_Q \left[ \int_{\mathbb{R}^d \times \mathbb{R}^d} \mathbf{1}_B(u+v) d(|\Omega_x| * \delta_x)(u) d(|\widetilde{\Omega}_y| * \delta_{-y})(v) \right] \\ &\leq \sum_{\substack{x \in \Lambda_n \\ y \in \Lambda \setminus \{x\}}} \int_{\mathbb{R}^d \times \mathbb{R}^d} \mathbf{1}_B(u+v) d(\mu * \delta_x)(u) d(\widetilde{\mu} * \delta_{-y})(v) \\ &\quad + \sum_{x \in \Lambda_n} \mathbb{E}_Q \left[ \int_{\mathbb{R}^d \times \mathbb{R}^d} \mathbf{1}_B(u+v) d|\Omega_x|(u) d|\widetilde{\Omega}_x|(v) \right] \\ &\leq \left( (\mu * \delta_{\Lambda_n}) * (\widetilde{\mu} * \delta_{\Lambda}) \right)(B) + \nu(\mathbb{R}^d) \text{card}(\Lambda_n) < \infty. \end{aligned}$$

Thus, by arguments analogous to those used before, the sum on the right hand side of (23) almost surely converges absolutely when applied to any bounded Borel set  $B$ . Again, since the Borel  $\sigma$ -algebra on  $\mathbb{R}^d$  is countably generated, this suffices to ensure that  $\gamma_{\Lambda,n;\text{mod}}^{(\Omega)}$  is a locally finite, random measure.

Furthermore, again for a bounded Borel set  $B$ , one has

$$\begin{aligned}
(24) \quad & \text{vol}(A_n) \mathbb{E}_Q(|\gamma_{A,n}^{(\Omega)} - \gamma_{A,n;\text{mod}}^{(\Omega)}|(B)) \\
& \leq \sum_{x,y \in \Lambda} \mathbb{E}_Q \left[ \int_{\mathbb{R}^d \times \mathbb{R}^d} \mathbf{1}_B(u+v) |\mathbf{1}_{A_n}(u)\mathbf{1}_{A_n}(v) - \mathbf{1}_{A_n}(x)| \, \text{d}(|\Omega_x| * \delta_x)(u) \, \text{d}(|\widetilde{\Omega}_y| * \delta_{-y})(v) \right] \\
& = \sum_{\substack{x,y \in \Lambda \\ x \neq y}} \int_{\mathbb{R}^d \times \mathbb{R}^d} \mathbf{1}_B(u+v) |\mathbf{1}_{A_n}(u)\mathbf{1}_{A_n}(v) - \mathbf{1}_{A_n}(x)| \, \text{d}(\mu * \delta_x)(u) \, \text{d}(\widetilde{\mu} * \delta_{-y})(v) \\
& \quad + \sum_{x \in \Lambda} \mathbb{E}_Q \left[ \int_{\mathbb{R}^d \times \mathbb{R}^d} \mathbf{1}_B(u+v) |\mathbf{1}_{A_n}(u)\mathbf{1}_{A_n}(v) - \mathbf{1}_{A_n}(x)| \, \text{d}(|\Omega_x| * \delta_x)(u) \, \text{d}(|\widetilde{\Omega}_x| * \delta_{-x})(v) \right].
\end{aligned}$$

We estimate the term in the third line of (24) as follows:

$$\begin{aligned}
& \sum_{\substack{x,y \in \Lambda \\ x \neq y}} \int_{\mathbb{R}^d \times \mathbb{R}^d} \mathbf{1}_B(u+v) |\mathbf{1}_{A_n}(u)\mathbf{1}_{A_n}(v) - \mathbf{1}_{A_n}(x)| \, \text{d}(\mu * \delta_x)(u) \, \text{d}(\widetilde{\mu} * \delta_{-y})(v) \\
& = \sum_{x \in \Lambda_n} \sum_{\substack{y \in \Lambda \\ y \neq x}} \int_{\mathbb{R}^d \times \mathbb{R}^d} \mathbf{1}_B(u+v) |\mathbf{1}_{A_n}(u)\mathbf{1}_{A_n}(v) - 1| \, \text{d}(\mu * \delta_x)(u) \, \text{d}(\widetilde{\mu} * \delta_{-y})(v) \\
& \quad + \sum_{x \in \Lambda \setminus \Lambda_n} \sum_{\substack{y \in \Lambda \\ y \neq x}} \int_{\mathbb{R}^d \times \mathbb{R}^d} \mathbf{1}_B(u+v) \mathbf{1}_{A_n}(u)\mathbf{1}_{A_n}(v) \, \text{d}(\mu * \delta_x)(u) \, \text{d}(\widetilde{\mu} * \delta_{-y})(v) \\
& \leq \int_{\mathbb{R}^d \times \mathbb{R}^d} \mathbf{1}_B(u+v) |\mathbf{1}_{A_n}(u)\mathbf{1}_{A_n}(v) - 1| \, \text{d}(\mu * \delta_{\Lambda_n})(u) \, \text{d}(\widetilde{\mu * \delta_{\Lambda}})(v) \\
& \quad + \int_{\mathbb{R}^d \times \mathbb{R}^d} \mathbf{1}_B(u+v) \mathbf{1}_{A_n}(u)\mathbf{1}_{A_n}(v) \, \text{d}(\mu * \delta_{\Lambda \setminus \Lambda_n})(u) \, \text{d}(\widetilde{\mu * \delta_{\Lambda}})(v) \\
& \leq \left( (\mu * \delta_{\Lambda_n})|_{\mathbb{R}^d \setminus \Lambda_n} * (\widetilde{\mu * \delta_{\Lambda}}) \right)(B) + \left( (\mu * \delta_{\Lambda_n}) * (\widetilde{\mu * \delta_{\Lambda}})|_{\mathbb{R}^d \setminus \Lambda_n} \right)(B) \\
& \quad + \left( (\mu * \delta_{\Lambda \setminus \Lambda_n})|_{\Lambda_n} * (\widetilde{\mu * \delta_{\Lambda}})|_{\Lambda_n} \right)(B),
\end{aligned}$$

where, in the first inequality, we have used the fact that removing the restriction  $x \neq y$  in the summation only adds positive terms (note that  $\mu$  is a positive measure by definition), and employed the estimate  $|\mathbf{1}_{A_n}(u)\mathbf{1}_{A_n}(v) - 1| \leq \mathbf{1}_{\mathbb{R}^d \setminus \Lambda_n}(u) + \mathbf{1}_{\mathbb{R}^d \setminus \Lambda_n}(v)$  for the second inequality. There is a constant  $c_B$  that depends on  $B$  (as well as on  $\mu$  and the averaging sequence) and a sequence  $d_n \rightarrow 0$  that is independent of  $B$  such that the sum in the last two lines above is bounded by  $d_n c_B \text{vol}(A_n)$ . This comes from the fact that there are only contributions from ‘surface terms’; compare the arguments in [55].

By way of example, we verify that, for any  $R > 0$ ,

$$\begin{aligned} & \left( (\mu * \delta_{A_n})|_{\mathbb{R}^d \setminus A_n} * \widetilde{(\mu * \delta_A)} \right) (B) \\ & \leq \left( \mu(\mathbb{R}^d) |\{x \in A_n \mid d(x, \mathbb{R}^d \setminus A_n) \leq R\}| + \mu(\mathbb{R}^d \setminus B_R) |A_n| \right) \sup_{x \in \mathbb{R}^d} \widetilde{(\mu * \delta_A)}(B+x). \end{aligned}$$

Note that this (together with analogous statements for the two other summands above) yields the claim by the van Hove property of the averaging sequence  $\mathcal{A}$ .

Observe next that, for a (possibly) complex measure  $\xi$  with  $|\xi|(\mathbb{R}^d) < \infty$ , a translation bounded measure  $\nu$  and a bounded Borel set  $B$ , we have (by an application of [32, Prop. 2.2] and its proof) the estimate  $|(\xi * \nu)(B)| \leq |\xi|(\mathbb{R}^d) \sup_{x \in \mathbb{R}^d} |\nu|(B+x) < \infty$ . Finally, note that  $\widetilde{\mu * \delta_A}$  is translation bounded by [13, Prop. 1.13], and that

$$\begin{aligned} (\mu * \delta_{A_n})|_{\mathbb{R}^d \setminus A_n}(\mathbb{R}^d) &= \sum_{x \in A_n} (\mu|_{B_R} * \delta_x)(\mathbb{R}^d \setminus A_n) + \sum_{x \in A_n} (\mu|_{\mathbb{R}^d \setminus B_R} * \delta_x)(\mathbb{R}^d \setminus A_n) \\ &\leq \mu(\mathbb{R}^d) |\{x \in A_n \mid d(x, \mathbb{R}^d \setminus A_n) \leq R\}| + \mu(\mathbb{R}^d \setminus B_R) |A_n|. \end{aligned}$$

Similarly, the term in the fourth line of (24) is bounded from above by

$$\begin{aligned} & \sum_{x \in A_n} \mathbb{E}_Q \left[ \int_{\mathbb{R}^d \times \mathbb{R}^d} \mathbf{1}_B(u+v) (\mathbf{1}_{\mathbb{R}^d \setminus A_n}(u) + \mathbf{1}_{\mathbb{R}^d \setminus A_n}(v)) d(|\Omega_x| * \delta_x)(u) d(|\widetilde{\Omega}_x| * \delta_{-x})(v) \right] \\ & + \sum_{x \in A \setminus A_n} \mathbb{E}_Q \left[ \int_{\mathbb{R}^d \times \mathbb{R}^d} \mathbf{1}_B(u+v) \mathbf{1}_{A_n}(u) \mathbf{1}_{A_n}(v) d(|\Omega_x| * \delta_x)(u) d(|\widetilde{\Omega}_x| * \delta_{-x})(v) \right] \\ & \leq \sum_{x \in A_n} \mathbb{E}_Q \left( |\widetilde{\Omega}_x|(\mathbb{R}^d) \int_{\mathbb{R}^d} \mathbf{1}_{\mathbb{R}^d \setminus A_n}(u) d(|\Omega_x| * \delta_x)(u) \right) \\ & + \sum_{x \in A_n} \mathbb{E}_Q \left( |\Omega_x|(\mathbb{R}^d) \int_{\mathbb{R}^d} \mathbf{1}_{\mathbb{R}^d \setminus A_n}(v) d(|\widetilde{\Omega}_x| * \delta_{-x})(v) \right) \\ & + \sum_{x \in A \setminus A_n} \mathbb{E}_Q \left( |\widetilde{\Omega}_x|(\mathbb{R}^d) \int_{\mathbb{R}^d} \mathbf{1}_{A_n}(u) d(|\Omega_x| * \delta_x)(u) \right) \\ (25) \quad & = (\nu * \delta_{A_n})(\mathbb{R}^d \setminus A_n) + (\widetilde{\nu * \delta_{A_n}})(\mathbb{R}^d \setminus A_n) + (\nu * \delta_{A \setminus A_n})(A_n) \leq d'_n \text{vol}(A_n), \end{aligned}$$

with a sequence  $d'_n \rightarrow 0$ . Combining the above estimates, we obtain

$$(26) \quad \mathbb{E}_Q \left( \frac{|\gamma_{A,n}^{(\Omega)} - \gamma_{A,n;\text{mod}}^{(\Omega)}(B)|}{\text{vol}(A_n)} \right) \leq (c_B + 1) d''_n$$

for a sequence  $d''_n \rightarrow 0$ ; hence, for  $\varepsilon > 0$ ,

$$(27) \quad \mathbb{P} \left( \frac{|\gamma_{A,n}^{(\Omega)} - \gamma_{A,n;\text{mod}}^{(\Omega)}(B)|}{\text{vol}(A_n)} > \varepsilon \right) \leq \frac{c_B + 1}{\varepsilon} d''_n$$

by Markov's inequality. If we choose  $(n_k)_{k \in \mathbb{N}}$  such that  $\sum_k d''_{n_k} < \infty$ , we obtain from (27) and the (first) Borel-Cantelli lemma that

$$(28) \quad \frac{|\gamma_{A, n_k}^{(\Omega)} - \gamma_{A, n_k; \text{mod}}^{(\Omega)}|(B)}{\text{vol}(A_{n_k})} \longrightarrow 0, \quad \text{almost surely as } k \rightarrow \infty.$$

By (27), we may choose the subsequence  $(n_k)_{k \in \mathbb{N}}$  independently of  $B$  in such a way that, for each bounded Borel set  $B$ , (28) holds almost surely. Finally, since the Borel  $\sigma$ -algebra on  $\mathbb{R}^d$  is countably generated, this implies the main claim of the lemma. The last statement is then obvious.  $\square$

Let us now resume our study of  $\gamma_A^{(\Omega)}$ . Invoking Proposition 2 and replacing the averaging sequence  $\mathcal{A} = (A_n)_{n \in \mathbb{N}}$  by the subsequence chosen there, we may use the modified measures  $\gamma_{A, n; \text{mod}}^{(\Omega)}$  as our approximating measures. Observing

$$\widetilde{\delta}_A^{(\Omega)} = \sum_{y \in \Lambda} \widetilde{\Omega}_y * \delta_{-y},$$

the modified autocorrelation approximant reads

$$(29) \quad \begin{aligned} \gamma_{A, n; \text{mod}}^{(\Omega)} &= \frac{1}{\text{vol}(A_n)} \left( \sum_{x \in A_n} \Omega_x * \delta_x \right) * \left( \sum_{y \in \Lambda} \widetilde{\Omega}_y * \delta_{-y} \right) \\ &= \sum_{z \in \Lambda - \Lambda} \left( \frac{1}{\text{vol}(A_n)} \sum_{\substack{x \in A_n \\ x - z \in \Lambda}} \Omega_x * \widetilde{\Omega}_{x-z} \right) * \delta_z =: \sum_{z \in \Lambda - \Lambda} \zeta_{z, n}^{(\Omega)} * \delta_z, \end{aligned}$$

where we now need to analyse the behaviour of the random measures  $\zeta_{z, n}^{(\Omega)}$ .

Let us first look at  $z = 0$ , where we obtain

$$(30) \quad \zeta_{0, n}^{(\Omega)} = \frac{\text{card}(A_n)}{\text{vol}(A_n)} \frac{1}{\text{card}(A_n)} \sum_{x \in A_n} \Omega_x * \widetilde{\Omega}_x \xrightarrow{n \rightarrow \infty} \text{dens}(\Lambda) \cdot \mathbb{E}_Q(\Omega * \widetilde{\Omega}) \quad (\text{a.s.})$$

by an application of Lemma 6. Note that  $\text{dens}(\Lambda) = \eta(0)$  as introduced in Eq. (19). Next, assume  $z \in \Lambda - \Lambda$  with  $z \neq 0$ . Then, we split  $\zeta_{z, n}^{(\Omega)}$  into two sums,

$$(31) \quad \zeta_{z, n}^{(\Omega)} = \frac{1}{\text{vol}(A_n)} \left( \sum_{\substack{x \in A_n \\ x - z \in \Lambda}}^{(0)} \Omega_x * \widetilde{\Omega}_{x-z} + \sum_{\substack{y \in A_n \\ y - z \in \Lambda}}^{(1)} \Omega_y * \widetilde{\Omega}_{y-z} \right),$$

where the upper index stands for the following additional restriction: Given  $z$ , our point set  $\Lambda$  is the disjoint union of countably many maximal linear chains of the form

$$\{\dots, x + 2z, x + z, x, x - z, x - 2z, \dots\}$$

with *all* points lying in  $\Lambda$ , and  $x$  being chosen as its representative. Such a chain may be finite or infinite, but has no gaps by construction. For each of these chains, the random measures  $\Omega_{x+mz} * \widetilde{\Omega}_{x+(m-1)z}$ , are identically distributed, but not independent (due to the index overlap). However, those with  $m$  *even* (type (0)) are mutually independent, as are those with  $m$  *odd* (type (1)). This way, each element of  $\Lambda$  inherits the type as a label, and the terms in (31) are distributed to the two sums according to their type. This approach

guarantees that the terms for  $\zeta_{z,n}^{(\Omega)}$  which already showed up in  $\zeta_{z,n-1}^{(\Omega)}$  end up in the same sum as before, no matter what the detailed structure of the (nested) averaging sequence might be. We also split the number of terms

$$\text{card}\{x \in A_n \mid x - z \in A\} = N_n^{(0)} + N_n^{(1)}$$

accordingly. We can now rewrite our previous expression in the form

$$(32) \quad \zeta_{z,n}^{(\Omega)} = \frac{\text{card}\{x \in A_n \mid x - z \in A\}}{\text{vol}(A_n)} \left( \frac{N_n^{(0)}}{N_n^{(0)} + N_n^{(1)}} \frac{\sum^{(0)}}{N_n^{(0)}} + \frac{N_n^{(1)}}{N_n^{(0)} + N_n^{(1)}} \frac{\sum^{(1)}}{N_n^{(1)}} \right),$$

where the term in brackets is a convex combination of two random measures  $\sum^{(0)}/N_n^{(0)}$  and  $\sum^{(1)}/N_n^{(1)}$ . By (19), the factor in front of the bracket converges to  $\eta(z)$ . When this limit is non-zero, we know that  $N_n^{(i)} \xrightarrow{n \rightarrow \infty} \infty$  for  $i \in \{0, 1\}$ , so that Lemma 6 and Eq. (20) imply

$$(33) \quad \frac{1}{N_n^{(i)}} \sum^{(i)} \xrightarrow{n \rightarrow \infty} \mathbb{E}_Q(\Omega) * \widetilde{\mathbb{E}_Q(\Omega)} \quad (\text{a.s.}).$$

Although we do not know whether the rational prefactors in (32) converge, we have a convex combination of two sequences that each almost surely converge to the same limit, which must then also be the limit of their convex combination. Put together, this gives

$$(34) \quad \zeta_{z,n}^{(\Omega)} \xrightarrow{n \rightarrow \infty} \eta(z) \cdot \mathbb{E}_Q(\Omega) * \widetilde{\mathbb{E}_Q(\Omega)} \quad (\text{a.s.})$$

for all  $z \in \Lambda - \Lambda$  with  $z \neq 0$ .

These considerations will be sufficient when the random measures almost surely have a (deterministic) compact support. To formulate the main result of this section in greater generality, we need one further technical property. For brevity, we write  $B_r^c = \mathbb{R}^d \setminus B_r(0)$ .

**Lemma 7.** *If  $\mathcal{M}'$  is a set of uniformly translation bounded, positive measures and  $\nu$  a finite, positive measure on  $\mathbb{R}^d$ , there is a sequence  $R_k \nearrow \infty$  such that*

$$\sum_{k=1}^{\infty} \sup_{\omega \in \mathcal{M}'} (\omega|_{B_{R_k}^c} * \nu)(K) < \infty$$

holds for any compact set  $K \subset \mathbb{R}^d$ .

*Proof.* Since  $\nu$  is a finite, positive measure and  $\nu(B_r^c)$  a decreasing function that tends to 0 as  $r \rightarrow \infty$ , we can choose radii  $R_k$  with  $\nu(B_{R_k}^c) < 1/k^2$ , so that  $\sum_{k=1}^{\infty} \nu(B_{R_k}^c) < \pi^2/6 < \infty$ . Moreover, we may do this in such a way that the differences between consecutive radii do not decrease, meaning that  $R_2 \geq 2R_1$  and  $R_{k+2} - R_{k+1} \geq R_{k+1} - R_k$  for all  $k \in \mathbb{N}$ .

Uniform translation boundedness of  $\mathcal{M}'$  means that, for any compact  $K \subset \mathbb{R}^d$ , there is a positive constant  $\alpha_K$  with  $\omega(x + K) \leq \alpha_K$ , simultaneously for all  $x \in \mathbb{R}^d$  and all  $\omega \in \mathcal{M}'$ . If  $K \subset \mathbb{R}^d$  is compact, we have  $K \subset B_r(0)$  for some  $r > 0$ . If  $R > r$ , one has

$$\begin{aligned} (\omega|_{B_R^c} * \nu)(K) &= \int_{\mathbb{R}^d \times \mathbb{R}^d} \mathbf{1}_K(x + y) \mathbf{1}_{B_R^c}(x) d\omega(x) d\nu(y) \\ &= \int_{\mathbb{R}^d} \omega((K - y) \cap B_R^c) d\nu(y) \leq \alpha_K \nu(B_{R-r}^c), \end{aligned}$$

where the last step follows because  $\omega((K - y) \cap B_R^c)$  vanishes whenever  $|y| \leq R - r$ . For radii  $0 \leq R \leq r$ , the bound is simply given by  $\alpha_K \nu(\mathbb{R}^d)$ , which is finite.

Now, for some  $m \in \mathbb{N}$ , we have  $R_m - r \geq R_1$ , with the sequence of radii chosen before. The additional difference property of the radii makes sure that the radii  $R_{m+k}$  with  $k \in \mathbb{N}$  give a summable contribution, while the remaining terms are finite by construction. Since this argument is uniform in  $\omega \in \mathcal{M}'$  and holds for all compact  $K \subset \mathbb{R}^d$ , our claim follows.  $\square$

**Theorem 2.** *Let  $\Lambda \subset \mathbb{R}^d$  be an FLC point set such that its Dirac comb  $\delta_\Lambda$  possesses the autocorrelation measure  $\gamma_\Lambda$  of (17), relative to the fixed averaging sequence  $\mathcal{A}$ , and thus the diffraction measure  $\widehat{\gamma}_\Lambda$ . Let  $(\Omega_x)_{x \in \Lambda}$  be a family of integrable, complex, i.i.d. random measures with common law  $Q$  and finite second moment measure, with  $\Omega$  being any representative of this family, and consider the random measure  $\delta_\Lambda^{(\Omega)}$  of (21).*

*Then, possibly after replacing  $\mathcal{A}$  by a suitable subsequence  $\mathcal{A}'$ , the sequence of approximating measures  $\gamma_{\Lambda,n}^{(\Omega)}$  of (22) almost surely converges, as  $n \rightarrow \infty$ , to the positive definite, translation bounded autocorrelation measure*

$$\gamma_{\Lambda,Q} = (\mathbb{E}_Q(\Omega) * \widetilde{\mathbb{E}_Q(\Omega)}) * \gamma_\Lambda + \text{dens}(\Lambda) (\mathbb{E}_Q(\Omega * \widetilde{\Omega}) - \mathbb{E}_Q(\Omega) * \widetilde{\mathbb{E}_Q(\Omega)}) * \delta_0.$$

*This measure has the Fourier transform*

$$\widehat{\gamma}_{\Lambda,Q} = |\widehat{\mathbb{E}_Q(\Omega)}|^2 \cdot \widehat{\gamma}_\Lambda + \text{dens}(\Lambda) (\mathbb{E}_Q(\Omega * \widetilde{\Omega}) - \mathbb{E}_Q(\Omega) * \widetilde{\mathbb{E}_Q(\Omega)})^\wedge \cdot \lambda,$$

*which is the almost sure diffraction measure of the random measure  $\delta_\Lambda^{(\Omega)}$  relative to  $\mathcal{A}'$ .*

*Proof.* The previous calculations establish the individual almost sure convergence of the (countably many) measures  $\zeta_{z,n}^{(\Omega)}$ , with the limits as given in Eqs. (30) and (34). Our assumptions on  $\Omega$  ensure that  $\mathbb{E}_Q(\Omega) * \widetilde{\mathbb{E}_Q(\Omega)}$  in (34) is a finite, positive definite measure, which is concentrated at 0 in the sense that  $|\mathbb{E}_Q(\Omega) * \widetilde{\mathbb{E}_Q(\Omega)}|(\mathbb{R}^d \setminus B_r(0)) \xrightarrow{r \rightarrow \infty} 0$ , while  $(\mathbb{E}_Q(\Omega) * \widetilde{\mathbb{E}_Q(\Omega)})(g * \widetilde{g}) > 0$  for all  $0 \neq g \in C_c(\mathbb{R}^d)$ . In view of (30) and (34), our (claimed) almost sure limit  $\gamma_{\Lambda,Q}$  inherits translation boundedness from  $\gamma_\Lambda$ .

The (deterministic) measure  $\gamma_{\Lambda,Q}$  is positive definite, hence transformable by Lemma 1. Its Fourier transform has the form claimed as a result of the convolution theorem [13, Excs. 4.18]. The latter is applicable here because all expectation measures involved are finite measures, so that their Fourier transforms are represented by uniformly continuous functions on  $\mathbb{R}^d$ .

It remains to establish the limit property. Let us first assume that there is a (deterministic) compact set  $C$  so that  $\text{supp}(\Omega_x) \subset C$  almost surely. This implies  $\text{supp}(\zeta_{z,n}^{(\Omega)}) \subset C - C$  for all  $n$  and  $z$ , so that only terms from finitely many  $z \in \Lambda - \Lambda$  contribute to  $\gamma_{\Lambda,n;\text{mod}}^{(\Omega)}$  on any compact  $K \subset \mathbb{R}^d$ . In this case, we may use an elementary pointwise calculation to see that  $\gamma_{\Lambda,n;\text{mod}}^{(\Omega)}$  tends to the claimed limit, and Proposition 2 gives the assertion.

In the general case, this simple argument is not conclusive, and we need to estimate putative contributions from distant points  $z \in \Lambda - \Lambda$  to  $\gamma_{\Lambda,n;\text{mod}}^{(\Omega)}$ . For bounded  $K, B \ni 0$ , with

$\mu(\cdot) := \mathbb{E}(|\Omega|(\cdot))$  as above, we have

$$\begin{aligned}
 (35) \quad & \mathbb{E}_Q \left[ \left| \sum_{\substack{z \in \Lambda - \Lambda \\ z \notin B}} \frac{1}{\text{vol}(A_n)} \sum_{\substack{x \in A_n \\ x-z \in \Lambda}} (\Omega_x * \tilde{\Omega}_{x-z} * \delta_z)(K) \right| \right] \\
 & \leq \sum_{\substack{z \in \Lambda - \Lambda \\ z \notin B}} \frac{1}{\text{vol}(A_n)} \sum_{\substack{x \in A_n \\ x-z \in \Lambda}} \mathbb{E}_Q \left[ (|\Omega_x| * |\tilde{\Omega}_{x-z}| * \delta_z)(K) \right] = \sum_{\substack{z \in \Lambda - \Lambda \\ z \notin B}} \frac{1}{\text{vol}(A_n)} \sum_{\substack{x \in A_n \\ x-z \in \Lambda}} (\mu * \tilde{\mu} * \delta_z)(K) \\
 & = \left( \mu * \tilde{\mu} * \left( \frac{1}{\text{vol}(A_n)} \sum_{\substack{z \in \Lambda - \Lambda, x \in A_n \\ x-z \in \Lambda, z \notin B}} \delta_z \right) \right)(K) = \left( \mu * \tilde{\mu} * (\gamma_{\Lambda, n; \text{mod}}|_{\mathbb{R}^d \setminus B}) \right)(K) =: \phi_n(K, B).
 \end{aligned}$$

Since  $\{\gamma_{\Lambda, n; \text{mod}} \mid n \in \mathbb{N}\}$  are uniformly translation bounded by Lemma 3, we can choose a sequence of radii  $R_k \nearrow \infty$  according to Lemma 7 such that, for all compact  $K \subset \mathbb{R}^d$ ,

$$(36) \quad \sum_{k=1}^{\infty} \sup_{n \in \mathbb{N}} \phi_n(K, B_{R_k}(0)) < \infty.$$

We have proved above that, for each  $z \in \Lambda - \Lambda$ ,  $\zeta_{z,n}^{(\Omega)} \xrightarrow{n \rightarrow \infty} \zeta_{z,\infty}^{(\Omega)}$ , almost surely in the vague topology, where  $\zeta_{0,\infty}^{(\Omega)} = \text{dens}(\Lambda) \cdot \mathbb{E}_Q(\Omega * \tilde{\Omega})$  and  $\zeta_{z,\infty}^{(\Omega)} = \eta(z) \cdot \mathbb{E}_Q(\Omega) * \widetilde{\mathbb{E}_Q(\Omega)}$  for  $z \neq 0$ . Possibly after passing to another subsequence, we may now assume that the convergence is so fast that, for any  $g \in C_c(\mathbb{R}^d)$ ,

$$(37) \quad \left| \sum_{\substack{z \in \Lambda - \Lambda \\ |z| < R_n}} \left( (\zeta_{z,n}^{(\Omega)} - \zeta_{z,\infty}^{(\Omega)}) * \delta_z \right)(g) \right| \xrightarrow{n \rightarrow \infty} 0 \quad (\text{a.s.}).$$

Using Markov's inequality and (35), we conclude for any bounded  $B$  and  $\varepsilon > 0$  that

$$(38) \quad \mathbb{P} \left( \sum_{\substack{z \in \Lambda - \Lambda \\ |z| \geq R_n}} |\zeta_{z,n}^{(\Omega)} * \delta_z|(K) \geq \varepsilon \right) \leq \frac{\phi_n(K, B_{R_n}(0))}{\varepsilon},$$

which is summable by (36). Hence, by Borel-Cantelli,

$$(39) \quad \sum_{\substack{z \in \Lambda - \Lambda \\ |z| \geq R_n}} \zeta_{z,n}^{(\Omega)} * \delta_z \xrightarrow{n \rightarrow \infty} 0 \quad (\text{a.s.}).$$

Combining this with (37) shows that the limit is the expected one (from the pointwise calculation) also in this case, which yields the claim.  $\square$

Note that our argument is based on the potential selection of a subsequence of the original (deterministic)  $\mathcal{A}$ . However, it also shows that the limit derived in Theorem 2 is the only point of accumulation along any deterministic subsequence of  $\mathcal{A}$ .

**Remark 9.** RANDOMISATION OF MEYER SETS. A particularly relevant class of point sets in the theory of aperiodic order are *Meyer sets*, which are relatively dense sets  $\Lambda$  such that  $\Lambda - \Lambda$  is uniformly discrete. Such sets always have a diffraction measure with a non-trivial pure point part, with a relatively dense supporting set [59, 2], despite the fact that Meyer sets

can have entropy<sup>2</sup>. If modified by a family of random measures according to Theorem 2, the resulting diffraction still shows the original diffraction with its non-trivial pure point component, modulated by the function  $|\widehat{\mathbb{E}_Q(\Omega)}|^2$ , in addition to the diffuse background originating from the added randomness.  $\diamond$

Let us look at consequences of Theorem 2 in terms of some examples.

**Example 6. DETERMINISTIC CLUSTERS.** Let  $S \subset \mathbb{R}^d$  be a finite point set, and consider  $\Omega \equiv \delta_S = \sum_{x \in S} \delta_x$ . Clearly, this completely deterministic case gives  $\mathbb{E}_Q(|\Omega|) = \mathbb{E}_Q(\Omega) = \delta_S$  and  $\mathbb{E}_Q(\Omega * \widetilde{\Omega}) = \delta_S * \widetilde{\delta_S}$ , so that Theorem 2 gives  $\gamma_A^{(\Omega)} = (\delta_S * \widetilde{\delta_S}) * \gamma_A$  and  $\widehat{\gamma}_A^{(\Omega)} = |\widehat{\delta_S}|^2 \cdot \widehat{\gamma}_A$ , which is always true (rather than almost always) in this case. A particularly simple instance of this emerges from  $S = \{a\}$ , which effectively means a global translation by  $a$ . This leads to the relations  $\gamma_A^{(\Omega)} = \gamma_A$  and  $\widehat{\gamma}_A^{(\Omega)} = \widehat{\gamma}_A$ , as it must.  $\diamond$

**Example 7. RANDOM WEIGHT MODEL.** Here, we consider  $\Omega = H\delta_0$ , where  $H$  is a complex-valued random variable with a law  $\mu$  that satisfies  $\mathbb{E}_\mu(|H|^2) < \infty$  (hence also  $\mathbb{E}_\mu(|H|) < \infty$ ). Clearly, this gives  $\mathbb{E}_Q(\Omega) = \mathbb{E}_\mu(H) \delta_0$  and  $\mathbb{E}_Q(\Omega * \widetilde{\Omega}) = \mathbb{E}_\mu(|H|^2) \delta_0$ , so that Theorem 2 results in the diffraction formula

$$\widehat{\gamma}_A^{(\Omega)} = |\mathbb{E}_\mu(H)|^2 \cdot \widehat{\gamma}_A + \text{dens}(A) (\mathbb{E}_\mu(|H|^2) - |\mathbb{E}_\mu(H)|^2) \cdot \lambda \quad (\text{a.s.}).$$

The autocorrelation is clear from Theorem 2.  $\diamond$

**Remark 10. INTERPRETATION AS PARTICLE GAS.** A widely used special case of Example 7 is the random occupation model, or ‘A-gas’. Here,  $\Omega$  may take the value  $\delta_0$  (with probability  $p$ , for ‘occupied’) or 0 (with probability  $1 - p$ , for ‘empty’). This gives the diffraction

$$\widehat{\gamma}_A^{(\Omega)} = p^2 \cdot \widehat{\gamma}_A + \text{dens}(A) \cdot p(1 - p) \cdot \lambda \quad (\text{a.s.}),$$

which was derived in a similar setting in [7], and later generalised to Bernoulli and Markov systems [4], to systems with finite range Gibbs measures [11], and beyond [40, 41].  $\diamond$

The results of Examples 6 and 7 can be extended in many ways, some of which will be met later on. One further possibility consists in replacing a point by a ‘profile’, as described by an integrable function, or by a finite collection of such profiles, which could represent different types of atoms. The corresponding formulas for the autocorrelation and the diffraction are then easy analogues of the ones given so far.

**Example 8. RANDOM DISPLACEMENT MODEL.** Consider the random measure  $\Omega = \delta_X$ , where  $X$  is an  $\mathbb{R}^d$ -valued random variable with law  $\nu$ . If  $A \subset \mathbb{R}^d$  is a Borel set, one has

$$(\mathbb{E}_Q(\Omega))(A) = \int_{\mathbb{R}^d} \delta_x(A) d\nu(x) = \int_{\mathbb{R}^d} \mathbf{1}_A(x) d\nu(x) = \nu(A),$$

which shows that  $\mathbb{E}_Q(\Omega) = \nu$ . One also finds  $\mathbb{E}_Q(\Omega * \widetilde{\Omega}) = \nu(\mathbb{R}^d) \delta_0 = \delta_0$ . Theorem 2 now results in the equations

$$\begin{aligned} \gamma_A^{(\Omega)} &= (\nu * \widetilde{\nu}) * \gamma_A + \text{dens}(A) (\delta_0 - \nu * \widetilde{\nu}) & (\text{a.s.}), \\ \widehat{\gamma}_A^{(\Omega)} &= |\widehat{\nu}|^2 \cdot \widehat{\gamma}_A + \text{dens}(A) (1 - |\widehat{\nu}|^2) \cdot \lambda & (\text{a.s.}), \end{aligned}$$

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<sup>2</sup>The binary random tilings of Example 5 produce Meyer sets whenever  $b/a \in \mathbb{Q}$ .

which recovers Hof's result on the diffraction at high temperature [33].  $\diamond$

In comparison, Hof's approach to the random displacement model [33] also uses the SLLN, but does not require the FLC property. Instead, he needs an ergodicity assumption on the underlying point set; compare also [42].

**Remark 11.** EXTENSION OF THEOREM 2. The argument above is shown for FLC sets in a pointwise fashion, to make the result more transparent. However, it is clear that one does not need the FLC property itself. Indeed, it is sufficient to assume that the fixed point set  $\Lambda$ , relative to a chosen van Hove averaging sequence  $\mathcal{A}$ , possesses an autocorrelation that is a pure point measure of the form  $\gamma = \sum_{z \in F} \eta(z) \delta_z$  with  $F$  a locally finite point set. An argument with local test functions will then still connect to the SLLN and thus avoid the need for ergodic assumptions on the underlying set  $\Lambda$ .  $\diamond$

With hindsight, it is rather clear that the formulas of Theorem 2 are robust, and should also hold for other point sets, such as those coming from a homogeneous Poisson process or from a model set based particle gas, as introduced in [7]. So, to complement our approach of this section, let us now consider ergodic point processes instead, meaning that also the set  $\Lambda$  becomes part of the random structure.

## 5. ARBITRARY DIMENSIONS: POINT PROCESS APPROACH

Here, we are interested in the diffraction of certain random subsets of  $\mathbb{R}^d$ , where we restrict ourselves to the situation that these subsets are self-averaging in a suitable way. This will be guaranteed by the ergodicity of the underlying stochastic process. One further benefit is that we are freed from details of the averaging sequence and the potential selection of subsequences thereof. It is convenient to start by putting ourselves in the context of random counting measures, which we now summarise in a way that is tailored to diffraction theory.

**5.1. Random measures and point processes.** Let  $\mathcal{M}^+$  denote the set of all locally finite, positive measures  $\phi$  on  $\mathbb{R}^d$  (where we mean to include the 0 measure). That  $\phi$  is locally finite (some authors say  $\phi$  is boundedly finite or that  $\phi$  is a Radon measure) means that, for all bounded Borel sets  $A$ ,  $\phi(A) < \infty$ . The space  $\mathcal{M}^+$  is closed in the topology of vague convergence of measures (in fact,  $\mathcal{M}^+$  is a complete separable metric space, see [19, A 2.6]<sup>3</sup>). We let  $\Sigma_{\mathcal{M}^+}$  denote the  $\sigma$ -algebra of Borel sets of  $\mathcal{M}^+$ . The latter can be described as the  $\sigma$ -algebra of subsets of  $\mathcal{M}^+$  generated by the requirement that, for all Borel sets  $A \subset \mathbb{R}^d$ , the mapping  $\phi \mapsto \phi(A)$  is measurable; compare [37, Chs. 1.1 and 1.2] for background.

A *random measure* on  $\mathbb{R}^d$  is a random variable  $\Phi$  from a probability space  $(\Theta, \mathcal{F}, \pi)$  into  $(\mathcal{M}^+, \Sigma_{\mathcal{M}^+})$ . Let us write  $\mathcal{P}(\mathcal{M}^+)$  for the convex set of probability measures on  $\mathcal{M}^+$ . The *distribution* of a random measure  $\Phi$  is the probability measure  $P = P_\Phi \in \mathcal{P}(\mathcal{M}^+)$ , defined by  $P = \pi \circ \Phi^{-1}$ . In other words,  $P$  is the *law* of  $\Phi$ , written as  $\mathcal{L}(\Phi) = P$ . Note that, as soon as  $P$  is given or determined, one can usually ignore the underlying probability space.

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<sup>3</sup>We refer to the second edition of this work throughout, which comes in two volumes [19, 20]. All results we need are also contained in the original one volume edition [18], sometimes with a slightly different numbering.

For each  $t \in \mathbb{R}^d$ , let  $T_t$  denote the translation operator on  $\mathbb{R}^d$ , as defined by the mapping  $x \mapsto t + x$ . Clearly, one has  $T_t T_s = T_{t+s}$ , and the inverse of  $T_t$  is given by  $T_t^{-1} = T_{-t}$ . For functions  $f$  on  $\mathbb{R}^d$ , the corresponding translation action is defined via  $T_t f = f \circ T_{-t}$ , so that  $T_t f(x) = f(x - t)$ . Similarly, for  $\phi \in \mathcal{M}^+$ , let  $T_x \phi := \phi \circ T_{-x}$  be the image measure under the translation, so that  $(T_x \phi)(A) = \phi(T_{-x}(A)) = \phi(A - x)$  for any measurable subset  $A \subset \mathbb{R}^d$ , and  $(T_x \phi)(f) = \int_{\mathbb{R}^d} f(y) d(T_x \phi)(y) = \int_{\mathbb{R}^d} f(x+z) d\phi(z) = \phi(T_{-x} f)$  for functions. This means that there is a translation action of  $\mathbb{R}^d$  on  $\mathcal{M}^+$ . Finally, we also have a translation action on  $\mathcal{P}(\mathcal{M}^+)$ , via  $(T_x Q)(A) = Q(T_{-x} A)$  for any measurable  $A \subset \mathcal{M}^+$ .

Our primary interest is in random counting measures. A measure  $\phi$  on  $\mathbb{R}^d$  is called a *counting measure* if  $\phi(A) \in \mathbb{N}_0$  for all bounded Borel sets  $A$ . These are positive, integer-valued measures of the form  $\phi = \sum_{i \in I} \delta_{x_i}$ , where the index set  $I$  is (at most) countable and the support of  $\phi$  is a locally finite subset of  $\mathbb{R}^d$ . The (positive) counting measures form a subset  $\mathcal{N}^+ \subset \mathcal{M}^+$ . We can repeat the above discussion of  $\mathcal{M}^+$  by restricting everything to  $\mathcal{N}^+$ . The vague topology on  $\mathcal{N}^+$  is the same as its topology inherited from  $\mathcal{M}^+$ , and its  $\sigma$ -algebra of Borel sets  $\Sigma_{\mathcal{N}^+}$  consists of the intersections of the elements of  $\Sigma_{\mathcal{M}^+}$  with  $\mathcal{N}^+$ . The concepts of the law of a random measure and the translation action by  $\mathbb{R}^d$  carry over. In particular, for  $x \in \text{supp}(\phi)$  with  $\phi \in \mathcal{N}^+$ ,  $T_{-x} \phi$  corresponds to the counting measure obtained from  $\phi$  by translating its *support* so that  $x$  is shifted to the origin.

A *point process* on  $\mathbb{R}^d$  is a random variable  $\Phi$  from a probability space  $(\Theta, \mathcal{F}, \pi)$  into  $(\mathcal{N}^+, \Sigma_{\mathcal{N}^+})$ . Alternatively, a point process is a random measure for which  $\pi$ -almost all  $\theta \in \Theta$  are counting measures. Furthermore, it is called *simple* when, for  $\pi$ -almost all  $\theta \in \Theta$ , the atoms of  $\phi = \Phi(\theta)$  have weight (or multiplicity) 1.

In many instances, the point processes we are dealing with are *simple*. Whenever this happens, we feel free to identify point measures with their supports. In this case, the measures almost surely are Dirac combs of the form  $\phi = \delta_S$  with  $S \subset \mathbb{R}^d$  locally finite. Later on, we create compound processes in which an underlying point process is decorated with a random finite measure, and this will take us from  $\mathcal{N}^+$  to  $\mathcal{M}^+$ , which is also the reason why we introduced random measures above.

For a random measure (or a point process)  $\Phi$  with law  $P$ , the expectation measure  $\mathbb{E}_P(\Phi)$  is defined by

$$(40) \quad (\mathbb{E}_P(\Phi))(A) = \mathbb{E}_P(\Phi(A)) = \int_{\mathcal{N}^+} \phi(A) dP(\phi), \quad \text{for } A \subset \mathbb{R}^d \text{ Borel.}$$

It is a measure on  $\mathbb{R}^d$  which gives the *expected* mass (or number of points) that  $\Phi$  has in  $A$ . More precisely, in terms of the underlying probability space  $(\Theta, \mathcal{F}, \pi)$ , one writes

$$\mathbb{E}_P(\Phi(A)) = \int_{\Theta} \Phi(\theta)(A) d\pi(\theta) = \int_{\mathcal{N}^+} \phi(A) dP(\Phi).$$

It is common in the probability literature (and we adopt this slight abuse of notation here, too) to suppress the explicit dependence on  $(\Theta, \mathcal{F}, \pi)$  by simply writing  $\Phi$  for the general instance  $\Phi(\theta)$  of the process  $\Phi$ . The latter is called *stationary* when its law  $P$  is translation invariant, which means that  $T_t P = P \circ T_{-t} = P$  holds for all  $t \in \mathbb{R}^d$ .

**Remark 12.** INTENSITY OF A PROCESS. If  $P$  is stationary, we have  $T_t \mathbb{E}_P(\Phi) = \mathbb{E}_P(\Phi)$  for all  $t \in \mathbb{R}^d$ , whence  $\mathbb{E}_P(\Phi)$  must be a multiple of Lebesgue measure (the latter being Haar measure on  $\mathbb{R}^d$ ). Consequently,

$$I_P(\Phi) = \mathbb{E}_P(\Phi) = \rho \lambda,$$

where  $\rho \in [0, \infty]$  is usually called the *intensity* of  $P$ . Unfortunately, this term is already in use for the positive weights of Bragg peaks in diffraction theory. In the setting of simple point processes,  $\rho$  also has the meaning of a point density, averaged over all realisations of the process. In the ergodic case (see below for a definition), it is then almost surely the density of a given realisation in the usual sense. We thus often prefer to call  $\rho$  the *point density* of the simple point process or the *density* of the random measure.  $\diamond$

From now on, we always assume that  $\rho$  is *finite*. Let  $\Phi : (\Theta, \mathcal{F}, \pi) \longrightarrow (\mathcal{X}, \Sigma_{\mathcal{X}})$  be a stationary random measure (where  $\mathcal{X} = \mathcal{M}^+$ ) or point process ( $\mathcal{X} = \mathcal{N}^+$ ), with law  $P$ . Then,  $(\mathcal{X}, \Sigma_{\mathcal{X}}, P)$  is a probability space with translation invariant probability measure  $P$ . In fact, we will usually simply assume that  $(\mathcal{X}, \Sigma_{\mathcal{X}}, P)$  is itself the probability space (or basic process) we are dealing with. In general, there will be several different spaces, and to keep track of the processes, we use the law of the basic process as an index.

Let us recall that the random measure or point process  $\Phi$  is called *ergodic* when  $(\mathcal{X}, \Sigma_{\mathcal{X}}, P)$  is ergodic as a dynamical system [62] under the translation action of  $\mathbb{R}^d$ , see below for more. In particular, we do not refer to strict ergodicity this way.

**5.2. Palm distribution and autocorrelation.** Let  $P \in \mathcal{P}(\mathcal{N}^+)$  be stationary with finite density  $0 < \rho < \infty$ . The assumption  $\rho > 0$  is no restriction, since it is easy to see that a realisation of a stationary point process with intensity  $\rho = 0$  almost surely is the zero measure.

Let  $\mathbf{1}_{\mathcal{B}}$ , as usual, denote the characteristic function of the set  $\mathcal{B} \subset \mathcal{N}^+$ , and choose a Borel set  $A \subset \mathbb{R}^d$  with  $0 < \lambda(A) < \infty$ . The *Palm distribution*  $P_0$  is the probability measure on  $\mathcal{N}^+$  that satisfies

$$(41) \quad P_0(\mathcal{B}) = \frac{1}{\mathbb{E}_P(\Phi(A))} \int_{\mathcal{N}^+} \sum_{x \in A \cap \text{supp}(\Phi)} \Phi(\{x\}) \mathbf{1}_{\mathcal{B}}(T_{-x}\Phi) \, dP(\Phi)$$

for any  $\mathcal{B} \in \Sigma_{\mathcal{N}^+}$ , compare [57, Ch. 4.4] or [38, Ch. 3] for background. Due to stationarity, Remark 12 applies to  $\mathbb{E}_P(\Phi(A))$ , whence the prefactor simplifies to  $(\rho \lambda(A))^{-1}$ . Note that the sum under the integral runs over at most countably many points. Moreover, the definition does *not* depend on the actual choice of  $A$ . Intuitively,  $P_0$  describes the configuration  $\Phi$  as seen from a typical point in  $\text{supp}(\Phi)$ , with that point translated to the origin. Alternatively, in the case of simple point processes, one can think of  $P_0$  as the distribution of  $\Phi$ , conditioned on having a point measure at 0. This actually amounts to condition properly on an event of probability 0, which might need some further explanation.

The first point of view can be made precise, at least in the ergodic case, as a limit, via sampling points in  $\Phi$  over larger and larger balls, see [38, Thm. 3.6.6] or [20, Prop. 13.4.I and Prop. 13.4.IV] as well as Eq. (43) below. The second interpretation can be corroborated by conditioning  $\Phi$  to have a point in a small ball around 0 and then again taking a limit, see [20, Thm. 13.3.IV]. In more precise terms,  $P_0$  would be called the *Palm distribution with respect*

to  $0 \in \mathbb{R}^d$ , compare [36, Ch. 10] or [20, Ch. 13.1]. Since we will mostly be dealing with the stationary scenario, we refrain from spelling out the full name.

There is an alternative approach to the Palm distribution, which also applies to the random measure case, compare [20, Chs. 13.1 and 13.2]. Let  $\Phi : (\Theta, \mathcal{F}, \pi) \rightarrow (\mathcal{M}^+, \Sigma_{\mathcal{M}^+})$  be a stationary random measure with law  $P$  and finite mean density  $\rho < \infty$ . Then, the *Palm distribution* is the unique probability measure  $P_0$  on  $\mathcal{M}^+$  that satisfies

$$(42) \quad \mathbb{E}_P \left( \int_{\mathbb{R}^d} g(x, \Phi) d\Phi(x) \right) = \rho \int_{\mathbb{R}^d} \int_{\mathcal{M}^+} g(x, T_x \psi) dP_0(\psi) dx$$

for all non-negative functions  $g$  on  $\mathbb{R}^d \times \mathcal{M}^+$  for which  $\int_{\mathbb{R}^d} \int_{\mathcal{M}^+} g(x, \phi) d\phi(x) dP(\phi)$  is finite. When dealing with point processes, all this reduces to  $\mathcal{N}^+$  by simply replacing  $\mathcal{M}^+$  with  $\mathcal{N}^+$  throughout Eq. (42), compare [20, Ch. 13.2 and Thm. 13.2.III].

If  $\Phi$  is an ergodic, stationary random measure, an application of the ergodic theorem implies

$$(43) \quad \frac{1}{\lambda(B_n)} \int_{B_n} F(T_{-x}\Phi) d\Phi(x) \xrightarrow{n \rightarrow \infty} \rho \int_{\mathcal{M}^+} F(\Psi) dP_0(\Psi) \quad (\text{a.s.}),$$

for any non-negative measurable function  $F: \mathcal{M}^+ \rightarrow \mathbb{R}$ , see [20, Prop. 13.4.I] or the proof of [38, Thm. 3.6.6]. Here and below, we write  $B_n$  for  $B_n(0)$  and  $\lambda(B_n)$  for  $\text{vol}(B_n(0))$ .

In the literature, the probability measure  $P_0$  is usually called the *Palm distribution* of  $P$  (with respect to 0), while the term *Palm measure* is also in use for the unnormalised version  $\rho P_0$ , a convention we adopt here. The first moment measure of the latter coincides with the *autocorrelation measure* of the underlying *process* and is denoted by  $\gamma_P$ . This is motivated by the following result on the autocorrelation  $\gamma_P^{(\Phi)}$  of a given *realisation*, which is somewhat implicit in the literature. Its importance in our present context was first emphasised by Guéré in [28]; see also [43] for complementary aspects.

**Theorem 3.** *Let  $\Phi$  be a stationary, ergodic, positive random measure with distribution  $P$ . Assume that  $P$  has finite density  $\rho$ , and that  $P$  has locally finite second moments in the sense that  $\mathbb{E}_P(\Phi(A)^2) < \infty$  for any bounded  $A \subset \mathbb{R}^d$  (this follows for instance from the condition  $\mathbb{E}_P(\Phi(B_r(x))^2) < \infty$  for all  $x \in \mathbb{R}^d$  and some fixed radius  $r$ ). Let  $\Phi_n := \Phi|_{B_n(0)}$  denote the restriction of  $\Phi$  to the centred ball of radius  $n$ . Then, the natural autocorrelation  $\gamma_P^{(\Phi)}$  of  $\Phi$ , defined via an averaging sequence of centred nested balls, almost surely exists and satisfies*

$$\gamma_P^{(\Phi)} := \lim_{n \rightarrow \infty} \frac{\Phi_n * \widetilde{\Phi}_n}{\text{vol}(B_n(0))} = \lim_{n \rightarrow \infty} \frac{\Phi_n * \widetilde{\Phi}}{\text{vol}(B_n(0))} = \rho I_{P_0} = \gamma_P,$$

where the limit refers to the vague topology on  $\mathcal{M}^+$ . Here,  $I_{P_0}$  is the first moment measure of the Palm distribution,

$$I_{P_0}(A) = \int_{\mathcal{M}^+} \Psi(A) dP_0(\Psi), \quad \text{for } A \subset \mathbb{R}^d \text{ Borel.}$$

*Proof.* As test function, fix a non-negative continuous function  $g: \mathbb{R}^d \rightarrow [0, \infty)$  with compact support. With  $B_n^c := \mathbb{R}^d \setminus B_n$ , we have

$$\begin{aligned} \frac{1}{\lambda(B_n)} \int_{\mathbb{R}^d} g(x) d(\Phi_n * \widetilde{\Phi}_n)(x) &= \frac{1}{\lambda(B_n)} \int_{B_n \times B_n} g(x-y) d\Phi(x) d\Phi(y) \\ &= \frac{1}{\lambda(B_n)} \int_{B_n} \left( \int_{\mathbb{R}^d} g(x-y) d\Phi(y) - \int_{B_n^c} g(x-y) d\Phi(y) \right) d\Phi(x) \\ &= \frac{1}{\lambda(B_n)} \int_{B_n} F_g(T_{-x}\Phi) d\Phi(x) - R_n(g) \end{aligned}$$

(note that both integrals inside the big brackets in the second line are finite because  $g$  has compact support), where  $\phi \mapsto F_g(\phi) = \int_{\mathbb{R}^d} g(-z) d\phi(z)$  defines a measurable function, and the remainder is given by

$$R_n(g) = \frac{1}{\lambda(B_n)} \int_{B_n} \int_{B_n^c} g(x-y) d\Phi(y) d\Phi(x).$$

Note that  $R_n$ , which is a random measure, is precisely the difference between the elements of the two approximating sequences of random measures in the claim. In view of (43), it thus remains to show that  $\lim_{n \rightarrow \infty} R_n = 0$  almost surely. Choose  $k$  so that  $g(x) = 0$  for  $|x| > k$ , and fix some  $\varepsilon > 0$ . We then have, for  $n > k/\varepsilon$ ,

$$R_n(g) \leq \frac{\|g\|_\infty}{\lambda(B_n)} \int_{B_n} \Phi(B_n^c \cap (x + B_k)) d\Phi(x) \leq \frac{\|g\|_\infty}{\lambda(B_n)} \int_{B_n \setminus B_{(1-\varepsilon)n}} \Phi(x + B_k) d\Phi(x),$$

where  $\phi \mapsto G(\phi) := \phi(B_k)$  is again measurable. Hence we obtain

$$\begin{aligned} R_n(g) &\leq \frac{\|g\|_\infty}{\lambda(B_n)} \int_{B_n} G(T_{-x}\Phi) d\Phi(x) - \frac{\lambda(B_{(1-\varepsilon)n})}{\lambda(B_n)} \frac{\|g\|_\infty}{\lambda(B_{(1-\varepsilon)n})} \int_{B_{(1-\varepsilon)n}} G(T_{-y}\Phi) d\Phi(y) \\ &\xrightarrow{n \rightarrow \infty} (1 - (1-\varepsilon)^d) \|g\|_\infty \rho \int_{\mathcal{M}^+} G(\Psi) dP_0(\Psi) = (1 - (1-\varepsilon)^d) \|g\|_\infty \rho I_{P_0}(B_k), \end{aligned}$$

almost surely by (43). Now take  $\varepsilon \searrow 0$  to conclude.  $\square$

Continuing with the hypotheses of Theorem 3, our assumptions guarantee that the *second moment measure*  $\mu^{(2)}$  of  $P$ , defined on cylinder sets  $A \times A' \subset \mathbb{R}^d \times \mathbb{R}^d$  via  $\mu^{(2)}(A \times A') = \int_{\mathcal{N}^+} \Phi(A) \Phi(A') dP(\Phi)$ , is locally finite. This is a necessary and sufficient condition for the existence of the first moment measure of the Palm distribution (as a locally finite measure). In fact, in the stationary scenario, the autocorrelation of the process, denoted by  $\gamma_P$ , satisfies  $\gamma_P = \mu_{\text{red}}^{(2)}$ , where  $\mu_{\text{red}}^{(2)}$  is the so-called *reduced second moment measure* of  $P$ , and this, in turn, is the same as the intensity of the Palm measure. We offer a brief explanation of this (for more details, see [20, Prop. 13.2.VI] or [57, Ch. 4.5]).

First,  $\mu_{\text{red}}^{(2)}$  is obtained from  $\mu^{(2)}$  by disintegration, via factoring out the translation invariance. More precisely, following [20],  $\mu_{\text{red}}^{(2)}$  is the unique positive measure on  $\mathbb{R}^d$  such that

$$(44) \quad \int_{\mathbb{R}^d \times \mathbb{R}^d} h(x, y) d\mu^{(2)}(x, y) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} h(u, u+v) d\mu_{\text{red}}^{(2)}(v) d\lambda(u),$$

for all (real) functions  $h \in C_c(\mathbb{R}^d \times \mathbb{R}^d)$ . When  $h = f \otimes g$  is a product function (meaning that  $h(x, y) := f(x)g(y)$ ), one finds the relation

$$(45) \quad \mu^{(2)}(f \otimes g) = \mu_{\text{red}}^{(2)}(\tilde{f} * g)$$

via Fubini's theorem. Choosing  $g = f$ , it is clear that the measure  $\mu_{\text{red}}^{(2)}$  is positive definite. More generally, when dealing with complex-valued functions, one has to consider

$$\mu^{(2)}(\bar{f} \otimes g) = \mu_{\text{red}}^{(2)}(\tilde{f} * g),$$

which leads to some technical complications later on. Since we consider real-valued component processes only, we can stick to the simpler case of real-valued functions.

The connection of the reduced second moment to the intensity measure of the Palm measure comes through applying (42) to a function on  $\mathbb{R}^d \times \mathcal{M}^+$  defined by

$$(46) \quad (x, \phi) \mapsto g(x) \int_{\mathbb{R}^d} T_x h(y) d\phi(y),$$

where  $g, h$  are arbitrary, but fixed, non-negative measurable functions on  $\mathbb{R}^d$ . The left hand side of (42) then reads

$$\begin{aligned} \mathbb{E}_P \left( \int_{\mathbb{R}^d} g(x) \int_{\mathbb{R}^d} h(y-x) d\Phi(y) d\Phi(x) \right) &= \mathbb{E}_P \left( \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} g(x) h(y-x) d\Phi(y) d\Phi(x) \right) \\ &= \int_{\mathbb{R}^d \times \mathbb{R}^d} g(x) h(y-x) d\mu^{(2)}(x, y) = \lambda(g) \cdot \mu_{\text{red}}^{(2)}(h), \end{aligned}$$

where we employed Fubini's theorem and (44), while the right hand side reads

$$\begin{aligned} \rho \int_{\mathbb{R}^d} \int_{\mathcal{M}^+} g(x) \int_{\mathbb{R}^d} (T_x h)(y) d(T_x \phi)(y) dP_0(\phi) d\lambda(x) \\ = \rho \int_{\mathbb{R}^d} \int_{\mathcal{M}^+} g(x) \int_{\mathbb{R}^d} h(y) d\phi(y) dP_0(\phi) d\lambda(x) = \lambda(g) \cdot \rho I_{P_0}(h). \end{aligned}$$

Here, we used the notation of the intensity of the Palm measure for its first moment. Comparing these two calculations gives

$$(47) \quad \mu_{\text{red}}^{(2)} = \rho I_{P_0} = \gamma_P.$$

**Remark 13.** EQUIVALENT DEFINITIONS OF  $\mu_{\text{red}}^{(2)}$ . There are several different ways to define a reduced measure via disintegration. In particular, one could employ  $h(u, u \pm v)$  as well as  $h(u \pm v, u)$  in Eq. (44). Using translation invariance of Lebesgue measure, this boils down to just two different possibilities, the one with  $h(u, u + v)$  introduced above and the one with  $h(u, u - v)$ , which is used in [37, Prop. I.60]. Observing the relation

$$\widetilde{f * \tilde{g}} = \tilde{f} * g$$

together with  $\widetilde{\mu^{(2)}} = \mu^{(2)}$ , one can check that both versions define the same measure, as the process is restricted to positive (and thus real) random measures, so that no complex conjugation shows up in the  $\widetilde{\cdot}$ -operation. Alternatively, one can use commutativity of the convolution together with the symmetry of  $\mu^{(2)}$ , which implies  $\mu^{(2)}(f \otimes g) = \mu^{(2)}(g \otimes f)$ .  $\diamond$

**Remark 14.** RENEWAL PROCESS REVISITED. Consider a stationary renewal process  $\Phi$  on the real line, viewed as a random counting measure, with inter-arrival law  $\varrho$ . The latter is assumed to be a probability measure on  $\mathbb{R}_+$ , with expectation  $\int_{\mathbb{R}_+} x d\varrho(x) = 1$ . It is well known (compare [20, Thm. 13.3.I and Ex. 13.3(a)]) that the Palm distribution  $P_0$ , with respect to the origin, of (the law of)  $\Phi$  equals the law of

$$(48) \quad \delta_0 + \sum_{i \in \mathbb{Z}} \delta_{S_i},$$

where  $S_0 = 0$ ,

$$S_i = \begin{cases} X_1 + \dots + X_i, & \text{if } i \geq 1, \\ X_{i+1} + \dots + X_0, & \text{if } i \leq -1, \end{cases}$$

and  $(X_i)_{i \in \mathbb{Z}}$  are i.i.d. with law  $\varrho$ . We see immediately from (48) that the first moment measure  $I_{P_0}$  of  $P_0$  is given by formula (11), see also Proposition 1, and thus recover Theorem 1 as well as Remark 3 by specialising Theorem 3 to the case of a renewal process on the line.

With the interpretation as a random counting measure, we are no longer restricted to laws  $\varrho$  on  $\mathbb{R}_+$ . Indeed, when  $\varrho$  is any probability measure on  $\mathbb{R}$  with expectation 1 (which prevents the process from being recurrent), the random counting measure almost surely leads to the autocorrelation given in (11), and thus avoids the complications mentioned after Lemma 4; see also [25, Ch. XI.9] for a systematic exposition, and [19, Ex. 8.2(b)] for comparison.  $\diamond$

**Remark 15.** BARTLETT SPECTRUM. The diffraction measure  $\widehat{\gamma}$  of a stationary random measure  $\Phi$  (with  $\mathbb{E}(|\Phi(A)|^2) < \infty$  for all compact  $A \subset \mathbb{R}^d$ , say) is closely related to the so-called *Bartlett spectrum*  $\Gamma := (c_{\text{red}}^{(2)})^\wedge$  of  $\Phi$  as follows; compare [19, Ch. 8.2].

Recall that the *covariance measure*  $c^{(2)}$  is defined on cylinder sets via

$$\begin{aligned} c^{(2)}(A \times A') &= \int \Phi(A) \Phi(A') dP(\Phi) - \int \Phi(A) dP(\Phi) \int \Phi(A') dP(\Phi) \\ &= \mu^{(2)}(A \times A') - \rho^2(\lambda \otimes \lambda)(A \times A'), \end{aligned}$$

where  $\rho$  is the density of  $\Phi$ , compare [20, Eq. (9.5.12)]. Consequently, the relation between the reduced second moment measure  $\mu_{\text{red}}^{(2)}$  and the reduced covariance measure  $c_{\text{red}}^{(2)}$  of  $\Phi$  is

$$(49) \quad c_{\text{red}}^{(2)} = \mu_{\text{red}}^{(2)} - \rho^2 \lambda.$$

Since  $\widehat{\gamma}$  is the Fourier transform of  $\mu_{\text{red}}^{(2)}$  and  $\Gamma$  that of  $c_{\text{red}}^{(2)}$ , Eq. (49) translates into

$$(50) \quad \Gamma = \widehat{\gamma} - \rho^2 \delta_0.$$

From our perspective, the positive definite autocorrelation measure  $\gamma$  is a slightly more natural and universal object than the inverse Fourier transform  $\widetilde{\Gamma}$  of  $\Gamma$ , since  $\widehat{\gamma}$  corresponds to a physically observable quantity, namely diffraction. Eq. (50) gives  $\widetilde{\Gamma} = \gamma - \rho^2 \lambda$ , which is no longer positive definite. Rather, it is tailored to situations where 0 supports the only atom of  $\widehat{\gamma}$ , as in the homogeneous Poisson process; compare Example 9 below and the discussion in [19, p. 305], and see [19, Sec. 8.2] for further explicitly computable examples.  $\diamond$

To formulate the standard Poisson process in this setting, let us start with an intuitive picture. Imagine independently putting single points on the sites of  $\varepsilon\mathbb{Z}^d \subset \mathbb{R}^d$ , each with probability  $\rho\varepsilon^d$ , and imagine a process that arises from this construction in the limit  $\varepsilon \rightarrow 0$ . For a rigorous construction, one can start from a tiling of  $\mathbb{R}^d$  with translates of  $[0, 1]^d$  and then proceed, independently for each cell, as follows: Put a Poisson- $(\rho)$  distributed number of points in the cell, with their locations independently and uniformly distributed over the given cell, see [57, Sec. 2.4.1] for details. Such a more elaborate approach is needed when  $d > 1$ , as there is no analogue of the renewal process we used for  $d = 1$ .

**Example 9. HOMOGENEOUS POISSON PROCESS.** This process on  $\mathbb{R}^d$ , with (point) density  $\rho$  (compare Remark 4), is a random counting measure  $\Phi$  (with distribution  $P$ ) such that  $\Phi(A)$  is Poisson- $(\rho\lambda(A))$ -distributed for any measurable  $A \subset \mathbb{R}^d$  and that the random variables  $\Phi(A_1), \dots, \Phi(A_m)$  are independent for any collection of pairwise disjoint  $A_1, \dots, A_m \subset \mathbb{R}^d$ . With this setting, the expectation measure of the process is given by  $\mathbb{E}_P(\Phi) = \rho\lambda$ .

It is well-known that, under the Palm distribution, a Poisson process looks like the same Poisson process augmented by an additional point at 0, so that

$$(51) \quad P_0(\mathcal{B}) = \int \mathbf{1}_{\mathcal{B}}(\Phi + \delta_0) dP(\Phi), \quad \text{for } \mathcal{B} \subset \mathcal{N}$$

(alternatively, write  $\mathcal{L}(\Phi + \delta_0) = P_0$ , or  $P * \delta_{\delta_0} = P_0$ ), by a theorem of Slivnyak, compare [57, Ex. 4.3]. This is intuitively obvious from the approximation via independent coin flips on  $\varepsilon\mathbb{Z}^d$  and the idea of obtaining the Palm distribution via conditioning on the presence of a point at 0. Here, this gives in  $I_{P_0} = \delta_0 + I_P = \delta_0 + \rho\lambda$ . Since homogeneous Poisson processes are stationary and ergodic for the translation action of  $\mathbb{R}^d$ , we can now apply Theorem 3.

Consequently, for almost all realisations  $\Phi$  of a homogeneous Poisson process of density  $\rho$ , the autocorrelation measure and the diffraction measure are given by

$$(52) \quad \gamma_P = \rho\delta_0 + \rho^2\lambda \quad \text{and} \quad \widehat{\gamma}_P = \rho^2\delta_0 + \rho\lambda,$$

by an application of Eq. (4). This also extends Example 1 to arbitrary finite values of the point density  $\rho$ ; compare also [19, Ex. 8.2(a)].  $\diamond$

**Remark 16. MATÉRN'S HARD-CORE PROCESS.** One drawback of the Poisson process (of point density  $\rho > 0$  in  $\mathbb{R}^d$ , with  $d \geq 2$  say) for applications in physics is the missing uniform discreteness. The latter was introduced by Matérn through a hard-core condition, realised via a local thinning process applied to each realisation, see [58] and references therein. Informally, for some fixed radius  $R > 0$ , his construction works as follows. Each point of a realisation of a Poisson process is equipped with an independent mark that is drawn uniformly at random from  $[0, 1]$  (technically, this is a marked Poisson process). Then, only those points  $x$  are kept for which there exists no point with a smaller mark in  $B_R(x)$ . The autocorrelation of the modified process is still radially symmetric. Moreover, if  $R$  is the radius of the hard-core condition and  $B_R = B_R(0)$  as before, the autocorrelation for distances  $r \geq 2R$  is that of a homogeneous stationary Poisson process with a new, effective point density

$$\rho_{\text{eff}} = \frac{1 - e^{-\rho \text{vol}(B_R)}}{\text{vol}(B_R)}.$$

In fact, the new autocorrelation almost surely has the form  $\gamma = \rho_{\text{eff}} \delta_0 + \rho_{\text{eff}}^2 \lambda - \nu$ , where  $\nu$  is a radially symmetric measure with support  $\overline{B_{2R}}$ , as follows from [58, Thm. 1]. In fact,  $\nu$  is absolutely continuous with density  $\rho_{\text{eff}}^2 1_{B_R} - g$ , where  $g$  is a smooth function on  $B_{2R} \setminus \overline{B_R}$ . This density has a jump (with sign change) at  $|x| = R$ , but tends smoothly to 0 as  $|x| \nearrow 2R$ .

The diffraction of (this version of) the Matérn model is thus given by

$$\widehat{\gamma} = \rho_{\text{eff}}^2 \delta_0 + (\rho_{\text{eff}} - h) \lambda \quad (\text{a.s.}),$$

where  $h$  is a smooth (even analytic) function with  $h(k) = \mathcal{O}(|k|^{-(d+1)/2})$  as  $|k| \rightarrow \pm\infty$ . Indeed, one has

$$\widehat{1_{B_R}}(k) = \left(\frac{R}{|k|}\right)^{d/2} J_{d/2}(2\pi|k|R),$$

which is responsible for the above estimate via the exact asymptotic behaviour of the Bessel function  $J_{d/2}$  for large arguments. The remaining contribution, using the explicit expression of [58, Eq. (3.2)] and the reduction of the radially symmetric Fourier transform to a one-dimensional Hankel transform, gives another term. It can also be computed explicitly, though the resulting formula is too lengthy to write it down here. Its decay is as for the previous term, by an application of the (refined) Paley-Wiener theorem.  $\diamond$

At this point, let us recall from [7] one process that is of particular interest in the study of aperiodic solids. Unlike the Poisson process and most of its siblings, it shows a substantial amount of point spectrum. It is related to Remark 10, but based on the cut and project formalism, for which we refer the reader to [44].

**Example 10.** MODEL SET BASED PARTICLE GAS. Let  $\Lambda \subset \mathbb{R}^d$  be a regular model set, for simplicity with internal space  $\mathbb{R}^m$ . Let  $\mathcal{L}$  be the lattice in  $\mathbb{R}^d \times \mathbb{R}^m$  that is needed for the cut and project scheme, with projection image  $L$  in  $\mathbb{R}^d$ . We denote the corresponding star map (from  $L$  into  $\mathbb{R}^m$ ) by  $\star$ , so that (up to a translation)

$$\Lambda = \{x \in L \mid x^\star \in W\},$$

where  $W \subset \mathbb{R}^m$  is the window of  $\Lambda$ . The latter is assumed to be a compact set with non-empty interior and a boundary of measure 0. This guarantees  $\Lambda$  to be a Meyer set. Let  $f$  now be a continuous function on  $W$ , and consider the weighted Dirac comb

$$\omega = \sum_{x \in \Lambda} f(x^\star) \delta_x,$$

which is pure point diffractive by the model set theorem [32, 55, 7] with diffraction measure

$$\widehat{\gamma}_\omega = \sum_{k \in M} |\widehat{f}(-k^\star)|^2 \delta_k.$$

Here,  $M$  is the projection of the dual lattice  $\mathcal{L}^*$  into  $\mathbb{R}^d$ , on which the star map is well-defined, too. It is known as the *Fourier module* of the model set  $\Lambda$ .

Assume now that  $0 \leq f(y) \leq 1$  on  $W$ , and define a family of independent binary random variables  $(U_x)_{x \in \Lambda}$ , each taking values 0 or 1 with  $\mathbb{P}(\{U_x = 1\}) = f(x^\star)$ . The stochastic counterpart  $\omega_s$  of  $\omega$  is

$$\omega_s = \sum_{x \in \Lambda} U_x \delta_x,$$

which can be interpreted as a particle gas on  $\Lambda$ . By [7, Thm. 2 and Eq. (58)], it almost surely has diffraction

$$\widehat{\gamma}_{\omega_s} = \widehat{\gamma}_\omega + \text{dens}(\Lambda) \overline{V} \lambda.$$

Here,  $\overline{V} = \frac{1}{\text{vol}(W)} \int_W f(y)(1-f(y)) \, dy$  is the mean variance of the random variables, averaged over  $\Lambda$ , which is a consequence of the uniform distribution result for model sets [54, 45]. One can also calculate the entropy of this system [7]. Moreover, it is not difficult to restrict the process to produce Meyer sets – one simply has to choose a function  $f$  that is 1 on a subset of  $W$  with non-empty interior.  $\diamond$

**5.3. Compound processes.** Let us now go one step further by adding random clusters to the picture. To this end, let a stationary, ergodic, point process  $\Phi$  be given, with law  $P$ , density  $\rho$ , and locally finite expectation measure  $\mathbb{E}_P(\Phi)$ . This is called the *centre process* from now on. Moreover, let  $\Psi \in \mathcal{M}_{\text{bd}}^+$  be a positive random measure with law  $Q$ , subject to the condition that both its expected total mass,  $m := \mathbb{E}_Q(\Psi(\mathbb{R}^d)) > 0$ , and the second moment of the total mass,  $\mathbb{E}_Q((\Psi(\mathbb{R}^d))^2)$ , are finite. This is the *component process*. We will also consider signed component processes  $\Psi$  with values in  $\mathcal{M}_{\text{bd}}$ , in which case we assume that the second moment of the total variation measure is finite; see the appendix for some details on the required notions and modifications for signed measures.

A combined cluster process, or *cluster process* for short, is a combination of a centre process and a component process of cluster type, and is obtained by replacing each point  $x \in \text{supp}(\Phi)$  by an independent copy of  $\Psi$ , translated to that point  $x$ . We denote such a process by the pair  $(\Phi_P, \Psi_Q)$ . As before, we restrict ourselves to finite clusters here. Formally, let  $\Psi_1, \Psi_2, \dots$  be independent copies of  $\Psi$  (these are the individual *clusters*). When  $\Phi = \sum_i \delta_{X_i}$ , we put

$$\Phi_{\text{cl}} := \sum_i T_{X_i} \Psi_i = \sum_i \delta_{X_i} * \Psi_i,$$

and denote the resulting law by  $P_{\text{cl}} = R$ . Note that, when  $\Psi \equiv \delta_0$  is deterministic and concentrated to one point, we simply obtain  $\mathcal{L}(\Phi_P, \Psi_Q) = \mathcal{L}(\Phi)$ , and the cluster process coincides with the centre process.

If  $\Psi$  is a counting measure, the cluster process  $(\Phi_P, \Psi_Q)$  is again stationary and ergodic, and its expected density is given by  $m\rho$ , by [20, Prop. 12.3.IX]. This property actually holds in larger generality, which we need later on.

**Proposition 3.** *Let  $\Phi$  be a stationary and ergodic point process with law  $P$ , finite density  $\rho$  and locally finite second moments. Let  $\Psi$  be a (signed) random measure with law  $Q$ , finite mean and finite second moment. Then, the combined cluster process, which is a random measure, is again ergodic.*

*Sketch of proof.* If the component process is a (positive) point process as well, this result is stated and proved in [20, Sec. 12.3]. The necessary modifications for an extension to a (possibly signed) random measure as component process, which seem to be well-known but which we could not explicitly trace in the literature, are provided in the appendix.  $\square$

The second moment measures of the three processes are connected in a way that permits an explicit calculation of the autocorrelation  $\gamma_R$  in terms of  $\gamma_P$  and various expectation measures

of the component process with law  $Q$ . To employ this powerful connection, we recall another disintegration formula, this time for any random variable  $\Xi$  of the cluster process,

$$(53) \quad \mathbb{E}_R(\Xi) = \mathbb{E}_R(\mathbb{E}_R(\Xi \mid \text{given the centres})) = \mathbb{E}_P(\mathbb{E}_Q(\Xi \mid \text{given the centres})),$$

which (with obvious meaning) follows from the standard theorems on conditional expectation.

We are now in the position to use Eq. (44) in conjunction with Theorem 3 and Eq. (53) to calculate  $\mu_{\text{cl}}^{(2)} = \mu_R^{(2)}$ , and thus the autocorrelation of almost all realisations of the cluster process, where we first concentrate on positive random measures. The extension to signed measures follows in Section 5.4. We first need some technical results.

**Lemma 8.** *Let  $\lambda$  be Lebesgue measure on  $\mathbb{R}^d$ , as before, and  $\mu$  a finite Borel measure. Then, one has  $\mu * \lambda = c\lambda$  with  $c = \mu(\mathbb{R}^d)$ .*

*Proof.* Let  $g$  be an arbitrary continuous function on  $\mathbb{R}^d$ , with compact support. For all  $x \in \mathbb{R}^d$ , we have  $\lambda(T_{-x}g) = (T_x\lambda)(g) = \lambda(g)$  due to translation invariance of  $\lambda$ . The convolution  $\mu * \lambda$  is well-defined as  $\mu$  is finite while  $\lambda$  is translation bounded [13, Prop. 1.13]. One thus has

$$\begin{aligned} (\mu * \lambda)(g) &= \int_{\mathbb{R}^d \times \mathbb{R}^d} g(x+y) \, d\lambda(y) \, d\mu(x) = \int_{\mathbb{R}^d} \lambda(T_{-x}g) \, d\mu(x) \\ &= \int_{\mathbb{R}^d} \lambda(g) \, d\mu(x) = \mu(\mathbb{R}^d) \lambda(g). \end{aligned}$$

Since  $g$  was arbitrary, the claim follows.  $\square$

Given a measure  $\mu \in \mathcal{M}^+$  and a continuous function  $g$  on  $\mathbb{R}^d$  with compact support (possibly complex-valued), we define a new function  $g_\mu$  on  $\mathbb{R}^d$  via

$$(54) \quad g_\mu(x) := (T_x\mu)(g),$$

which is certainly measurable. It is easy to check that  $g_\mu$  satisfies

$$(55) \quad \tilde{g}_\mu = \tilde{g}\tilde{\mu}.$$

**Lemma 9.** *Let  $\mu \in \mathcal{M}_{\text{bd}}^+$  and let  $\gamma$  be a positive, translation bounded measure on  $\mathbb{R}^d$ . For arbitrary (possibly complex-valued)  $f, g \in C_c(\mathbb{R}^d)$ , one has the identity*

$$(\mu * \tilde{\mu} * \gamma)(f * \tilde{g}) = \gamma(f_\mu * \tilde{g}_\mu).$$

*This identity also holds when both  $\mu$  and  $\gamma$  are signed measures.*

*Proof.* Let  $f$  and  $g$  be  $\mu$ -measurable functions such that  $f * \tilde{g}$  is a continuous function with compact support, which includes the situation of the claim. One then finds, with Fubini, that

$$\begin{aligned} (\mu * \tilde{\mu} * \gamma)(f * \tilde{g}) &= \int \int \left( \int f(x+z+\xi) \, d\mu(x) \right) \left( \int \tilde{g}(y-\xi) \, d\tilde{\mu}(y) \right) \, d\lambda(\xi) \, d\gamma(z) \\ &= \int \int (T_{z+\xi}\mu)(f) (T_{-\xi}\tilde{\mu})(\tilde{g}) \, d\lambda(\xi) \, d\gamma(z) \\ &= \int \int f_\mu(z+\xi) \tilde{g}_{\tilde{\mu}}(-\xi) \, d\lambda(\xi) \, d\gamma(z) = \gamma(f_\mu * \tilde{g}_\mu), \end{aligned}$$

where all integrals are over  $\mathbb{R}^d$  and (55) was used in the last step.  $\square$

Specialising Lemma 9 to  $\gamma = \delta_0$  gives the relation

$$(56) \quad (\mu * \tilde{\mu})(f * \tilde{g}) = (f_\mu * \tilde{g}_\mu)(0) = \lambda(f_\mu \overline{g_\mu}),$$

which simplifies our further discussion.

**Remark 17.** TEST FUNCTIONS FOR MEASURES. Recall that two measures  $\mu, \nu \in \mathcal{M}(\mathbb{R}^d)$  are equal when  $\mu(h) = \nu(h)$  for all  $h \in C_c(\mathbb{R}^d)$ . When the measures are positive or signed (but not complex), real-valued functions suffice. In the latter case, it will be particularly helpful to restrict to functions of the form  $h = f * g$  with  $f, g \in C_c(\mathbb{R}^d)$ . Since the space  $C_c(\mathbb{R}^d)$  contains an approximate unit for convolution, the linear combinations of such functions are dense in  $C_c(\mathbb{R}^d)$ , so that they suffice to assess equality of measures.  $\diamond$

**Lemma 10.** *Under our general assumptions on the component process, one has*

$$\begin{aligned} (\mathbb{E}_Q(\Psi * \tilde{\Psi}))(f * \tilde{g}) &= \lambda(\mathbb{E}_Q(f_\Psi \overline{g_\Psi})) \quad \text{and} \\ (\mathbb{E}_Q(\Psi) * \widehat{\mathbb{E}_Q(\Psi)})(f * \tilde{g}) &= \lambda(f_{\mathbb{E}_Q(\Psi)} \overline{g_{\mathbb{E}_Q(\Psi)}}), \end{aligned}$$

where  $f, g \in C_c(\mathbb{R}^d)$ , possibly complex-valued.

*Proof.* Let  $f$  and  $g$  be chosen as in the previous proof, with complex-valued functions permitted. For the first claim, observing that each realisation of  $\Psi$  is a finite measure, a direct calculation with Eq. (56) gives

$$(\mathbb{E}_Q(\Psi * \tilde{\Psi}))(f * \tilde{g}) = \mathbb{E}_Q((\Psi * \tilde{\Psi})(f * \tilde{g})) = \mathbb{E}_Q(\lambda(f_\Psi \overline{g_\Psi})) = \lambda(\mathbb{E}_Q(f_\Psi \overline{g_\Psi})),$$

while the second identity simply is Eq. (56) with  $\mu = \mathbb{E}_Q(\Psi)$ , which is a finite measure by assumption.  $\square$

Recall that the *covariance* of two real-valued random variables  $X$  and  $Y$  related to the law  $Q$  (using our general notation as explained above) is defined as

$$(57) \quad \text{cov}_Q(X, Y) := \mathbb{E}_Q(XY) - \mathbb{E}_Q(X)\mathbb{E}_Q(Y),$$

where the index  $Q$  highlights the reference to the underlying law  $Q$ .

**Proposition 4.** *Let  $(\Xi, R)$  be a combined cluster process with stationary centre point process  $(\Phi, P)$  and real component process  $(\Psi, Q)$ , both with the usual assumptions on means and second moments as used above. Then,  $\Xi$  is locally square integrable in the sense that  $\mathbb{E}_R[(\Xi(B))^2] < \infty$  for any bounded Borel set  $B$ , and we have the reduction formula*

$$\mu_R^{(2)}(f \otimes g) = \mu_P^{(2)}(f_{\mathbb{E}_Q(\Psi)} \otimes g_{\mathbb{E}_Q(\Psi)}) + \rho \lambda(\text{cov}_Q(f_\Psi, g_\Psi)),$$

where  $\rho$  is the density of the centre process,  $f, g$  are continuous with compact support, and the covariance is defined as in (57).

*Proof.* In order to check that  $\mathbb{E}_R[(\Xi(B))^2] < \infty$  for bounded, Borel measurable  $B \subset \mathbb{R}^d$ , one can trace through the steps below, replacing  $f$  and  $g$  by  $\mathbf{1}_B$  (the corresponding integrals then

involve only positive terms and are finite by Fubini's theorem). In general, by assumption and the disintegration formula (53), one finds

$$\begin{aligned}\mu_R^{(2)}(f \otimes g) &= \int_{\mathcal{M}^+} \Xi(f) \Xi(g) dR(\Xi) \\ &= \int_{\mathcal{N}^+} \mathbb{E}_Q \left( \sum_{x,y \in \text{supp}(\Phi)} \Psi_x(T_{-x}f) \Psi_y(T_{-y}g) \right) dP(\Phi),\end{aligned}$$

where  $\Psi_x$  denotes the random measure at centre  $x$ . Since  $\Psi_x$  and  $\Psi_y$  are independent for  $x \neq y$ , the double sum over the support is split into a sum over the diagonal ( $x = y$ ) and a sum over all remaining terms ( $x \neq y$ ). Using the linearity of the expectation operator, the integrand can now be rewritten as a sum over two contributions, namely

$$\begin{aligned}&\sum_{x,y} \mathbb{E}_Q(\Psi(T_{-x}f)) \mathbb{E}_Q(\Psi(T_{-y}g)) \quad \text{and} \\ &\sum_x \left( \mathbb{E}_Q(\Psi(T_{-x}f) \Psi(T_{-x}g)) - \mathbb{E}_Q(\Psi(T_{-x}f)) \mathbb{E}_Q(\Psi(T_{-x}g)) \right).\end{aligned}$$

Inserting the first term into the previous calculation leads to the contribution

$$\mu_P^{(2)}(\mathbb{E}_Q(f_\Psi) \otimes \mathbb{E}_Q(g_\Psi)) = \mu_P^{(2)}(f_{\mathbb{E}_Q(\Psi)} \otimes g_{\mathbb{E}_Q(\Psi)}),$$

while the second results in

$$\mathbb{E}_P(\Phi)(\text{cov}_Q(f_\Psi, g_\Psi)) = \rho \lambda(\text{cov}_Q(f_\Psi, g_\Psi)),$$

where the last step follows from the stationarity of  $(\Phi, P)$ .  $\square$

**Theorem 4.** *Let  $\Phi$  be a stationary and ergodic point process with law  $P$ , finite density  $\rho$  and locally finite second moments. Let  $\Psi$  be a random measure with law  $Q$ , finite expectation measure and finite second moments. If  $(\Xi, R)$  denotes the combined cluster process built from the centre process  $(\Phi, P)$  and the component process  $(\Psi, Q)$ , it is also stationary and ergodic.*

*Moreover, the autocorrelation of the combined process satisfies*

$$\gamma_R = (\mathbb{E}_Q(\Psi) * \widetilde{\mathbb{E}_Q(\Psi)}) * \gamma_P + \rho (\mathbb{E}_Q(\Psi * \tilde{\Psi}) - \mathbb{E}_Q(\Psi) * \widetilde{\mathbb{E}_Q(\Psi)}),$$

*and this is almost surely the natural autocorrelation of a given realisation of the cluster process.*

*Proof.* Choose two measurable functions  $f$  and  $g$  such that  $f * \tilde{g} \in C_c(\mathbb{R}^d)$ . Then, in line with Remark 17 and Eq. (45), one finds via Proposition 4 that

$$\begin{aligned}\gamma_R(f * \tilde{g}) &= \mu_R^{(2)}(f \otimes g) = \mu_P^{(2)}(f_{\mathbb{E}_Q(\Psi)} \otimes g_{\mathbb{E}_Q(\Psi)}) + \rho \lambda(\text{cov}_Q(f_\Psi, g_\Psi)) \\ &= \gamma_P(f_{\mathbb{E}_Q(\Psi)} * \widetilde{g_{\mathbb{E}_Q(\Psi)}}) + \rho (\mathbb{E}_Q(\Psi * \tilde{\Psi}) - \mathbb{E}_Q(\Psi) * \widetilde{\mathbb{E}_Q(\Psi)})(f * \tilde{g}),\end{aligned}$$

where  $\mathbb{E}_P(\Phi) = \rho \lambda$  due to stationarity of  $(\Phi, P)$ . The second step makes use of Lemma 10.

The formula for the autocorrelation now follows from the observation that

$$\gamma_P(f_{\mathbb{E}_Q(\Psi)} * \widetilde{g_{\mathbb{E}_Q(\Psi)}}) = (\mathbb{E}_Q(\Psi) * \widetilde{\mathbb{E}_Q(\Psi)} * \gamma_P)(f * \tilde{g}),$$

which is an application of Lemma 9. The remaining claims are clear due to the assumed ergodicity, via an application of Proposition 3.  $\square$

An application of the convolution theorem gives the following consequence, where also the identity  $\widehat{\mathbb{E}_Q(\Psi)} = \mathbb{E}_Q(\widehat{\Psi})$  was used to highlight the structure of the result.

**Corollary 1.** *Under the assumption of Theorem 4, the diffraction measure of the combined cluster process is given by*

$$\widehat{\gamma}_R = |\mathbb{E}_Q(\widehat{\Psi})|^2 \cdot \widehat{\gamma}_P + \rho (\mathbb{E}_Q(|\widehat{\Psi}|^2) - |\mathbb{E}_Q(\widehat{\Psi})|^2) \lambda$$

which is then almost surely also the diffraction measure of a given realisation.  $\square$

The result parallels our previous formulas, as was to be expected. Nevertheless, it does not follow from Theorem 2 in general, because realisations of stationary point processes in  $\mathbb{R}^d$  generically fail to be FLC sets. Before we discuss possible generalisations beyond the case of positive random measures, let us look at some examples.

**Example 11.** POISSON CLUSTER PROCESS. An important special case emerges when the centre process is the homogeneous Poisson process of Example 9, with point density  $\rho$ . Let  $\gamma_P$  and  $\widehat{\gamma}_P$  be the corresponding measures. If we couple a cluster component process  $\Psi$  to it, with law  $Q$  and  $m := \mathbb{E}_Q(\Psi)(\mathbb{R}^d)$  its expected number of points, our general formula for the compound process  $(\Phi_P, \Psi_Q)$  applies. With Lemma 8, the convolution formula can be simplified, and the result reads as follows.

For almost all realisations of a Poisson cluster process  $(\Phi_P, \Psi_Q)$ , the natural autocorrelation measure exists and is given by

$$\gamma_R = \gamma_{P,Q} = (m\rho)^2 \lambda + \rho \mathbb{E}_Q(\Psi * \widetilde{\Psi}),$$

where  $\mathbb{E}_Q(\Psi * \widetilde{\Psi})$  is a finite positive measure (of expected total mass  $\geq m^2$ ), due to our general assumption that  $\mathbb{E}_Q((\Psi(\mathbb{R}^d))^2)$  is finite. Consequently, the diffraction measure is almost surely given by

$$\widehat{\gamma}_R = (m\rho)^2 \delta_0 + \rho (\mathbb{E}_Q(\Psi * \widetilde{\Psi}))^\wedge \cdot \lambda,$$

where  $(\mathbb{E}_Q(\Psi * \widetilde{\Psi}))^\wedge$  is a uniformly continuous Radon-Nikodym density for Lebesgue measure. These formulas include the case of deterministic clusters; compare Example 6.  $\diamond$

**Remark 18.** RANDOM DISPLACEMENT OF POISSON PROCESSES. An interesting pair of processes is the combination of the homogeneous Poisson process from Example 9 with Hof's random displacement model from Example 8. A simple calculation shows that

$$\gamma_R = \gamma_P^{(\nu)} = \gamma_P \quad \text{and} \quad \widehat{\gamma}_R = \widehat{\gamma}_P$$

in this case (and, in fact,  $R$  and  $P$  have the same law here). From a physical point of view, this is in line with the behaviour of an ideal gas at high temperatures. When the Poisson process is a good model for the gas, and random displacement one for the disorder due to high temperature, compare the discussion in [33], the combination should still be an ideal gas – and this is precisely what happens, as reflected by the two identities.  $\diamond$

**Remark 19.** PARTICLE GAS CLUSTER PROCESS. It is clear that the particle gas of Example 10 satisfies all requirements for a centre process, so that we can apply the cluster process machinery to it, too. This produces physically interesting and relevant examples with

a substantial amount of point spectrum. This observation remains true for more complicated particle gas models with interactions, under certain conditions on the potential of the underlying Gibbs measure, say; compare [11] for further details and examples.  $\diamond$

**Example 12.** NEYMAN-SCOTT PROCESSES. Let  $K$  be a non-negative random integer with law  $\mathcal{L}(K) = \mu$ , mean  $m := \mathbb{E}_\mu(K)$  and finite second moment,  $\mathbb{E}_\mu(K^2) < \infty$ . Now, let  $Y_1, Y_2, \dots$  be a family of  $\mathbb{R}^d$ -valued i.i.d. random variables with common distribution  $\nu$ , and independent of  $K$ . Define the cluster distribution via

$$\Psi := \sum_{j=1}^K \delta_{Y_j},$$

i.e., a cluster has a random size  $K$ , while the positions of its atoms are independently drawn from the probability distribution  $\nu$ . The induced distribution for  $\Psi$  is again called  $Q$ . With a calculation similar to the one in Example 8, one finds

$$\mathbb{E}_Q(\Psi)(A) = \mathbb{E}_Q\left(\sum_{i=1}^K \mathbf{1}_A(X_i)\right) = \mathbb{E}_\mu\left(\sum_{i=1}^K \int_{\mathbb{R}^d} \mathbf{1}_A(x_i) d\nu(x_i)\right) = \mathbb{E}_\mu(K \cdot \nu(A)) = m\nu(A)$$

for  $A \subset \mathbb{R}^d$  Borel, so that  $\mathbb{E}_Q(\Psi) = m\nu$  and  $\mathbb{E}_Q(\Psi) * \mathbb{E}_Q(\tilde{\Psi}) = m^2(\nu * \tilde{\nu})$ . Moreover, one has

$$\mathbb{E}_Q(\Psi * \tilde{\Psi})(A) = \mathbb{E}_Q\left(\sum_{k,\ell=1}^K \mathbf{1}_A(X_k - X_\ell)\right) = m\delta_0(A) + \mathbb{E}_\mu(K(K-1))(\nu * \tilde{\nu})(A),$$

which gives  $\mathbb{E}_Q(\Psi * \tilde{\Psi}) = m\delta_0 + \mathbb{E}_\mu(K(K-1))(\nu * \tilde{\nu})$ , so that the general formulas from Theorem 2 can now be applied again. Note that  $\mathbb{E}_\mu(K(K-1)) = \mathbb{E}_\mu(K^2) - m$ .

If the centre process is once more the homogeneous Poisson process with mean (point) density  $\rho$ , Lemma 8 gives similar simplifications as in Example 11. Consequently, for the resulting law  $R$ , the autocorrelation is almost surely given by

$$\gamma_R = (m\rho)^2 \lambda + m\rho\delta_0 + \rho(\mathbb{E}_\mu(K^2) - m)(\nu * \tilde{\nu}),$$

whence the corresponding diffraction measure is given by

$$\hat{\gamma}_R = (m\rho)^2 \delta_0 + \rho(m + (\mathbb{E}_\mu(K^2) - m)|\hat{\nu}|^2) \lambda,$$

which is an interesting extension of the Poisson process; compare [19, Ex. 8.2(f)] for a circularly symmetric case in  $\mathbb{R}^2$ .  $\diamond$

**5.4. Autocorrelation for signed (ergodic) processes.** It is intuitively clear that the results of this section are not really restricted to point processes or positive measures for the clusters. Here, we sketch how they can be adapted to the situation of signed random measures. Consider a stationary, possibly signed, random measure  $\Psi$  (with law  $Q$  and ‘finite second moments’, meaning that  $\mathbb{E}_Q((|\Psi|(A))^2) < \infty$  holds for any bounded  $A \subset \mathbb{R}^d$ ), with second moment measure  $\mu^{(2)}$ , defined as before via bounded  $f$  of compact support as

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} f(x, y) d\mu^{(2)}(x, y) = \mathbb{E}_Q\left(\int_{\mathbb{R}^d \times \mathbb{R}^d} f(x, y) d\Psi(x) d\Psi(y)\right).$$

The reduced second moment measure  $\mu_{\text{red}}^{(2)}$  on  $\mathbb{R}^d$  with the property

$$(58) \quad \mu_{\text{red}}^{(2)}(f * \tilde{g}) = \mu^{(2)}(f \otimes g)$$

is defined in complete analogy to the positive case. The analogue of Theorem 3 is:

**Theorem 5.** *Let  $\Phi$  be a stationary and ergodic, random, signed measure with distribution  $P$ . Assume that  $\Phi$  has finite second moments in the sense that  $\mathbb{E}_P((|\Phi|(A))^2) < \infty$  for any bounded measurable set  $A \subset \mathbb{R}^d$  (which follows, for example, from  $\mathbb{E}_P((|\Phi|(B_r(x)))^2) < \infty$  for all  $x \in \mathbb{R}^d$  and some open ball  $B_r$ ). Let  $\Phi_n := \Phi|_{B_n}$  denote the restriction of  $\Phi$  to the ball of radius  $n$  around 0. Then, the natural autocorrelation of  $\Phi$ , which is defined with an averaging sequence of nested, centred balls, almost surely exists and satisfies*

$$\gamma_P^{(\Phi)} := \lim_{n \rightarrow \infty} \frac{\Phi_n * \widetilde{\Phi}_n}{\lambda(B_n)} = \lim_{n \rightarrow \infty} \frac{\Phi_n * \tilde{\Phi}}{\lambda(B_n)} = \mu_{\text{red}}^{(2)} = \gamma_P,$$

where the limit refers to the vague topology on  $\mathcal{N}$ . Here,  $\mu_{\text{red}}^{(2)}$  is the reduced second moment measure of  $P$  according to (58).

*Proof.* The proof is a variation of that of Theorem 3. Fix a continuous function  $h : \mathbb{R}^d \rightarrow \mathbb{R}$  with compact support. We have to check that

$$(59) \quad \frac{1}{\lambda(B_n)} (\Phi_n * \widetilde{\Phi}_n)(h) \xrightarrow{n \rightarrow \infty} \mu_{\text{red}}^{(2)}(h) \quad (\text{a.s.}).$$

Let  $\Phi$  be an ergodic, random, signed measure as above and  $F$  an ergodic random function on  $\mathbb{R}^d$ , the latter with the property that

$$(60) \quad \mathbb{E}_P \left( \int_A |F(x)| \, d|\Phi|(x) \right) < \infty$$

for any bounded measurable  $A \subset \mathbb{R}^d$ . We can then define an additive covariant spatial process  $X_A$  in the sense of [46], indexed by bounded measurable subsets  $A$ , via

$$X_A := \int_A F(x) \, d\Phi(x).$$

Note that ergodicity of  $\Phi$  and  $F$  implies that  $(X_A, \mathbb{R}^d)$  is again ergodic, meaning that the shift-invariant  $\sigma$ -field is trivial. Now, [46, Cor. 4.9] yields

$$\lim_{n \rightarrow \infty} \frac{1}{\lambda(B_n)} X_{B_n} = \mathbb{E}_P \left( \frac{1}{\lambda(B_1)} X_{B_1} \right) \quad (\text{a.s.}).$$

Applying this to  $\Phi$  as in the theorem, together with  $F(x) := \int_{\mathbb{R}^d} h(x-y) \, d\Phi(y)$ , yields

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{\lambda(B_n)} \int_{B_n} F(x) \, d\Phi(x) &= \lim_{n \rightarrow \infty} \frac{1}{\lambda(B_n)} (\Phi_n * \tilde{\Phi})(h) \\ &= \mathbb{E}_P \left( \frac{1}{\lambda(B_1)} \int_{B_1} \int_{\mathbb{R}^d} h(x-y) \, d\Phi(y) \, d\Phi(x) \right) \\ &= \frac{1}{\lambda(B_1)} \int_{\mathbb{R}^d \times \mathbb{R}^d} \mathbf{1}_{B_1}(x) h(x-y) \, d\mu^{(2)}(x, y) \\ &= \frac{1}{\lambda(B_1)} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \mathbf{1}_{B_1}(x) h(z) \, d\mu_{\text{red}}^{(2)}(z) \, dx = \int_{\mathbb{R}^d} h \, d\mu_{\text{red}}^{(2)} \end{aligned}$$

almost surely, which is almost the claim. The difference between  $\Phi_n * \tilde{\Phi}$  and  $\Phi_n * \tilde{\Phi}_n$  can be treated as in the proof of Theorem 3.  $\square$

Combining Proposition 5 and Theorem 5, and observing that the calculations in the proof of Proposition 4 carry over literally to the signed case, we obtain

**Corollary 2.** *The statements of Theorem 4 and Corollary 1 remain true for cluster processes with signed clusters.*  $\square$

**Example 13.** SIGNED POISSON PROCESS. If we combine the homogeneous Poisson process of Example 9 with the random weight model of Example 7, and chose weights 1 and  $-1$  with equal probability, Corollary 2 implies the almost sure diffraction

$$\hat{\gamma} = \rho\lambda.$$

In particular, one has  $\hat{\gamma} = \lambda$  for density  $\rho = 1$ , which makes this signed Poisson point set, on the level of the 2-point correlations, indistinguishable from the signed Bernoulli sequence (or process) on  $\mathbb{Z}^d$ . This is remarkable in view of the rather different geometric structure and demonstrates the intrinsic difficulty of the corresponding inverse problem.  $\diamond$

**5.5. Equilibria of critical branching Brownian motions in  $d \geq 3$ .** Consider a system of particles performing independent Brownian motions in  $\mathbb{R}^d$ ,  $d \geq 3$  (for ease of comparison with the cited literature, we assume that the variance parameter is  $\sigma^2 = 2$ ).

Additionally, each particle, after an exponentially distributed lifetime with parameter  $V$ , either doubles or dies, where each possibility occurs with probability  $1/2$ . In the situation of a birth event, the daughter particles appear at the position of the mother. Note that if we start with a finite number of particles, the expected number of particles is preserved for all time, as the expected number of offspring equals 1. This is what ‘critical’ in the name refers to. Imagine we start such a system from a homogeneous Poisson process of density  $\rho$ , denote by  $\Phi_t$  the random configuration observed at time  $t \geq 0$ , and its distribution by  $P_t$ . Here,  $P_t$  is stationary with density  $\rho$ , see [31] and the references given there for background.

It follows from [31, Thm. 2.3] that the first moment measure of the Palm distribution of  $P_t$  is given by

$$(61) \quad I_{(P_t)_0} = \delta_0 + (\rho + f_t)\lambda,$$

where

$$f_t(x) = V \int_0^t \int_{\mathbb{R}^d} p_s(0, y) p_s(y, x) dy ds = \frac{V}{2} \int_0^{2t} p_u(0, x) du,$$

with  $p_t(x, y) = (4\pi t)^{-d/2} \exp(-|x - y|^2/(4t))$  the  $d$ -dimensional Brownian transition density (with variance parameter 2). As explained in [31], there is a genealogical interpretation behind (61): In view of the interpretation of the Palm distribution as the configuration around a typical individual,  $\delta_0$  is the contribution of this individual,  $f_t \lambda$  that from its relatives in the family decomposition of the branching process, and  $\rho \lambda$  is the contribution from unrelated individuals.

Furthermore, by [31, Thm. 2.2],  $P_t$  converges (vaguely) towards  $P_\infty$ , which is the unique, ergodic, equilibrium distribution of density  $\rho$  (cf. [15] for uniqueness), and the limit  $t \rightarrow \infty$

can be taken in (61) to obtain

$$I_{(P_\infty)_0} = \delta_0 + (\rho + f_\infty)\lambda,$$

where

$$f_\infty(x) = \frac{V}{2} \int_0^\infty p_u(0, x) \, du = \frac{V}{2} \frac{\Gamma(\frac{d-2}{2})}{4\pi^{d/2}} \frac{1}{|x|^{d-2}}$$

is (up to the prefactor  $V/2$ ) the Green function of Brownian motion. Thus, using Lemma 2, we have

**Theorem 6.** *Let  $\Phi_\infty$  be a realisation of the critical branching Brownian motion, from the equilibrium distribution  $P_\infty$ . The autocorrelation is then almost surely given by*

$$\gamma = \rho\delta_0 + \rho(\rho + f_\infty)\lambda,$$

while

$$\hat{\gamma} = \rho^2\delta_0 + \rho\left(1 + \frac{V}{2} \frac{1}{4\pi^2|k|^2}\right)\lambda.$$

is the corresponding diffraction measure.  $\square$

**Remark 20.** EXTENSION OF THEOREM 6. One can also consider the scenario where, instead of Brownian motion, particles move during their lifetime according to a symmetric, stable process of index  $\alpha \in (0, 2]$  in  $\mathbb{R}^d$  ( $\alpha = 2$  corresponds to Brownian motion). Such processes have discontinuous paths, and their transition density  $p_t^{(\alpha)}(x, y) = p_t^{(\alpha)}(0, y - x)$  satisfies

$$\int_{\mathbb{R}^d} e^{ik \cdot x} p_t^{(\alpha)}(0, x) \, dx = \exp(-t|k|^\alpha)$$

(in general, no explicit form of  $p_t^{(\alpha)}$  is known). By [31, Thm. 2.2], non-trivial equilibria exist if the spatial dimension  $d$  satisfies  $d > \alpha$ . In this case, a reasoning analogous to that above yields the following: The autocorrelation of a realisation  $\Phi_\infty^{(\alpha)}$  of the equilibrium of a system of critical, branching, symmetric  $\alpha$ -stable processes (with density  $\rho$ ) is almost surely given by

$$\gamma = \rho\delta_0 + \rho(\rho + f_\infty^{(\alpha)})\lambda,$$

where

$$f_\infty^{(\alpha)}(x) = \frac{V}{2} \int_0^\infty p_u^{(\alpha)}(0, x) \, du = \frac{V}{2} \frac{\Gamma((d-\alpha)/2)}{2^\alpha \pi^{d/2} \Gamma(\alpha/2)} \frac{1}{|x|^{d-\alpha}}$$

(for the form of the Green function of the symmetric  $\alpha$ -stable process, see [14, Ex. 1.7]). Hence, the diffraction measure is almost surely given by

$$\hat{\gamma} = \rho^2\delta_0 + \rho\left(1 + \frac{V}{2} \frac{1}{(2\pi)^\alpha |k|^\alpha}\right)\lambda,$$

by another application of Lemma 2.  $\diamond$

Note that, due to the independence properties of the branching mechanism, these equilibria can also be considered as Poisson cluster processes. In contrast to the scenario considered above, clusters in  $\Phi_\infty$  are infinite, and the spatial correlation decays only algebraically (without being integrable).

## 6. OUTLOOK

This article demonstrates that various aspects of mathematical diffraction theory for random point sets and measures can be approached systematically with methods from point process theory, as was originally suggested in [28]. At the same time, the approach is sufficiently concrete to allow for many explicitly computable examples, several of which were presented above. They comprise many formulas from the somewhat scattered literature on this subject in a unified setting. There are, of course, many more examples, but we hope that the probabilistic platform advertised here will prove useful for them as well.

The next step in this development needs to consider point processes and random measures with interactions, such as those governed by Gibbs measures. First steps are contained in [33, 4, 28, 40, 41, 10, 21, 11] and indicate that both qualitative and quantitative results are possible, though some further development of the theory is needed.

A continuation along this path would also make the results more suitable for real applications in physics and crystallography, though it is largely unclear at the moment what surprises the corresponding inverse problem might have to offer here.

### APPENDIX: ERGODICITY FOR CLUSTER PROCESSES WITH SIGNED RANDOM MEASURES

Let  $\mathcal{M} = \mathcal{M}(\mathbb{R}^d)$  be the space of (locally finite) real or signed measures on  $\mathbb{R}^d$ , equipped with the topology of vague convergence, with  $\mathcal{M}^+ = \mathcal{M}^+(\mathbb{R}^d)$  denoting the subspace of positive measures. Let  $\Sigma_{\mathcal{M}}$  denote the Borel  $\sigma$ -algebra of  $\mathbb{R}^d$ . Note that the latter is also generated by the mappings  $\mathcal{M} \ni \mu \mapsto \mu(A)$ , for bounded and measurable sets  $A \subset \mathbb{R}^d$ . Recall that any  $\mu \in \mathcal{M}$  admits a unique Hahn-Jordan decomposition

$$\mu = \mu_+ - \mu_-, \quad \text{with } \mu_+, \mu_- \in \mathcal{M}^+ \text{ mutually singular.}$$

The mappings  $\mu \mapsto \mu_+$  and  $\mu \mapsto \mu_-$  are  $\Sigma_{\mathcal{M}}$ -measurable. We write  $|\mu| := \mu_+ + \mu_- \in \mathcal{M}^+$  for the total variation measure of  $\mu$ . A *random signed measure*  $\Phi$  is a random variable with values in  $(\mathcal{M}, \Sigma_{\mathcal{M}})$ . In the context of signed random measures, it is convenient to work with the *characteristic functional*

$$(62) \quad \varphi_{\Phi}(h) := \mathbb{E} \left[ \exp \left( i \int h \, d\Phi \right) \right],$$

which is defined for any  $h: \mathbb{R}^d \rightarrow \mathbb{R}$  that is bounded and measurable with compact support. Here and below, we suppress  $\mathbb{R}^d$  as the integration region. In analogy to the Laplace functional for positive random measures, the distribution of  $\Phi$  is determined by  $\varphi_{\Phi}$ .

Here, we are interested in signed cluster processes: Let  $\Phi$  be a stationary counting process with finite intensity  $\rho$ , and  $\Psi_j$  (with  $j \in \mathbb{N}$ ) independent (and independent from  $\Phi$ ), identically distributed, random, signed measures such that  $\mathbb{E}[|\Psi_1|]$  is a *finite* measure. Then, given a realisation  $\Phi = \sum_j \delta_{X_j}$ , where  $X_j$  are the positions of the atoms of  $\Phi$  (in some enumeration), the cluster process is defined as

$$(63) \quad \Xi := \sum_j T_{X_j} \Psi_j.$$

Note that for any bounded  $B \subset \mathbb{R}^d$ ,

$$\mathbb{E}[|\Xi(B)|] \leq \mathbb{E}\left[\sum_j |\Psi_j|(B - X_j)\right] = \rho \int_{\mathbb{R}^d} \int_{B-x} d\mathbb{E}[|\Psi_1|] dx = \rho(\mathbb{E}[|\Psi_1|] * \lambda)(B) < \infty,$$

so that (63) is indeed well-defined.

**Lemma 11.** *Let  $\Psi$  be a signed random measure on  $\mathbb{R}^d$ . The following are equivalent:*

- (1)  $\Psi$  is ergodic.
- (2) For any  $U, V \in \Sigma_{\mathcal{M}}$ ,

$$\lim_{n \rightarrow \infty} \frac{1}{\lambda(B_n)} \int_{B_n} (\mathbb{P}(\Psi \in U \cap T_x V) - \mathbb{P}(\Psi \in U) \mathbb{P}(\Psi \in V)) dx = 0.$$

- (3) For any  $g, h: \mathbb{R}^d \rightarrow \mathbb{R}$  measurable with compact support,

$$\lim_{n \rightarrow \infty} \frac{1}{\lambda(B_n)} \int_{B_n} (\varphi_{\Psi}(g + T_x h) - \varphi_{\Psi}(g) \varphi_{\Psi}(h)) dx = 0.$$

Furthermore, it suffices to restrict to  $U, V$  to a semiring which generates  $\Sigma_{\mathcal{M}}$  in (2), and it suffices to restrict to continuous  $g, h$  with compact support in (3).

*Proof.* This is a straightforward adaptation of the proofs of Propositions 12.3.III and 12.3.VI and Lemma 12.3.II of [20] to the signed case.  $\square$

The following result is an analogue [20, Prop. 12.3.IX] for the signed measure case. Since we have not been able to find a proof in the literature, we provide a sketch.

**Proposition 5.** *Let  $\Phi, \Psi_j$ , and  $\Xi := \sum_j T_{X_j} \Psi_j$  be as above. If  $\Phi$  is ergodic, then  $\Xi$  is ergodic as well.*

*Sketch of proof.* We verify condition (3) from Lemma 11. Observe that for any  $f: \mathbb{R}^d \rightarrow \mathbb{R}$  with compact support and any  $\varepsilon > 0$ , we can find  $R < \infty$  such that

$$(64) \quad \mathbb{P}\left(\sum_{j: |X_j| \geq R} \left| \int f d(T_{X_j} \Psi_j) \right| \geq \varepsilon\right) \leq \varepsilon.$$

To check (64), let  $R'$  be large enough so that  $\text{supp}(f) \subset [-R', R']^d$ , and note that for  $R > R'$ , the left-hand side of (64) is bounded by

$$\mathbb{P}\left(\sum_{j: |X_j|_{\infty} \geq R} |\Psi_j|([-R', R']^d + X_j) \geq \frac{\varepsilon}{\|f\|_{\infty}}\right) \leq \frac{\|f\|_{\infty}}{\varepsilon} \mathbb{E}\left[\sum_{j: |X_j|_{\infty} \geq R} |T_{X_j} \Psi_j|([-R', R']^d)\right].$$

The expectation on the right-hand side above equals

$$\begin{aligned} & \rho \int_{\mathbb{R}^d \setminus [-R, R]^d} \int_{\mathbb{R}^d} \mathbf{1}_{[-R', R']^d}(x - y) d\mathbb{E}[|\Psi_1|](y) dx \\ & \leq \rho(2R')^d \mathbb{E}[|\Psi_1|](\mathbb{R}^d \setminus [-(R - R'), (R - R')]^d), \end{aligned}$$

which converges to 0 as  $R \rightarrow \infty$  because  $\mathbb{E}[|\Psi_1|]$  is a finite measure.

Let  $g, h: \mathbb{R}^d \rightarrow \mathbb{R}$  continuous with compact support and define

$$G(\Phi) := \mathbb{E} \left[ \exp(i \int g d\Xi) \mid \Phi \right], \quad H(\Phi) := \mathbb{E} \left[ \exp(i \int h d\Xi) \mid \Phi \right].$$

Decompose

$$\begin{aligned} \int (g + T_x h) d\Xi &= \sum_{j: X_j \in [-R, R]^d} \int T_{X_j} g d\Psi_j + \sum_{j: X_j \notin [-R, R]^d} \int T_{X_j} g d\Psi_j \\ &+ \sum_{j: X_j \in [-R, R]^{d-x}} \int T_{X_j+x} h d\Psi_j + \sum_{j: X_j \notin [-R, R]^{d-x}} \int T_{X_j+x} h d\Psi_j, \end{aligned}$$

and choose  $R$  so large that (64) is fulfilled for  $f = g$  and  $f = h$ . Recall that, for any real-valued random variables  $X, Y$  with  $\mathbb{P}(|Y| \geq \varepsilon) \leq \varepsilon$ , we have

$$\left| \mathbb{E} e^{i(X+Y)} - \mathbb{E} e^{iX} \right| \leq \mathbb{E} \left| e^{i(X+Y)} - e^{iX} \right| \leq \mathbb{E} \left[ |e^{iX}| |e^{iY} - 1| \right] \leq \varepsilon + \mathbb{P}(|Y| \geq \varepsilon) \leq 2\varepsilon.$$

For  $A \subset \mathbb{R}^d$ , write  $\Xi_A := \sum_{j: X_j \in A} T_{X_j} \Psi_j$  for the random measure which consists of clusters with centres in  $A$ . For  $x \in \mathbb{R}^d \setminus [-2R, 2R]^d$ , we then have

$$\begin{aligned} &\left| \mathbb{E} \left[ \exp(i \int (g + T_x h) d\Xi) \right] - \mathbb{E} [G(\Phi)H(T_x \Phi)] \right| \\ &\leq \left| \mathbb{E} \left[ \exp(i \int (g + T_x h) d\Xi) \right] - \mathbb{E} \left[ \exp(i \int g d\Xi_{[-R, R]^d} + i \int T_x h d\Xi_{[-R, R]^{d-x}}) \right] \right| \\ &\quad + \left| \mathbb{E} \left[ \mathbb{E} \left[ \exp(i \int g d\Xi_{[-R, R]^d} + i \int T_x h d\Xi_{[-R, R]^{d-x}}) \mid \Phi \right] \right] - \mathbb{E} [G(\Phi)H(T_x \Phi)] \right|. \end{aligned}$$

The first term on the right-hand side is bounded by  $2\varepsilon$ . Observing that the conditional expectation in the second term is in fact a product because clusters with centres in disjoint regions are (conditionally) independent, we can bound the second term from above by

$$\begin{aligned} &\left| \mathbb{E} \left[ \mathbb{E} \left[ \exp(i \int g d\Xi_{[-R, R]^d}) \mid \Phi \right] \left( \mathbb{E} \left[ \exp(i \int T_x h d\Xi_{[-R, R]^{d-x}}) \mid \Phi \right] - H(T_x \Phi) \right) \right] \right| \\ &\quad + \left| \mathbb{E} \left[ \left( \mathbb{E} \left[ \exp(i \int g d\Xi_{[-R, R]^d}) \mid \Phi \right] - G(\Phi) \right) H(T_x \Phi) \right] \right| \\ &\leq \mathbb{E} \left| \exp(i \int T_x h d\Xi_{[-R, R]^{d-x}}) - \exp(i \int T_x h d\Xi) \right| \\ &\quad + \mathbb{E} \left| \exp(i \int g d\Xi_{[-R, R]^d}) - \exp(i \int g d\Xi) \right|, \end{aligned}$$

which is not more than  $2\varepsilon$ .

Thus, using the relation  $\mathbb{E} \left[ \mathbb{E} \left[ \exp(i \int (g + T_x h) d\Xi) \mid \Phi \right] \right] = \varphi_{\Xi}(g + T_x h)$  together with  $\mathbb{E} [G(\Phi)] = \varphi_{\Xi}(g)$  and  $\mathbb{E} [H(\Phi)] = \mathbb{E} [H(T_x \Phi)] = \varphi_{\Xi}(h)$ , we obtain

$$\begin{aligned} (65) \quad &\limsup_{n \rightarrow \infty} \frac{1}{\lambda(B_n)} \left| \int_{B_n} (\varphi_{\Psi}(g + T_x h) - \varphi_{\Psi}(g) \varphi_{\Psi}(h)) dx \right| \\ &\leq \limsup_{n \rightarrow \infty} \frac{1}{\lambda(B_n)} \left| \int_{B_n} (\mathbb{E} [G(\Phi)H(T_x \Phi)] - \mathbb{E} [G(\Phi)] \mathbb{E} [H(\Phi)]) dx \right| + 4\varepsilon = 4\varepsilon \end{aligned}$$

by ergodicity of  $\Phi$  (in order to deduce this literally from statement (2) in Lemma 11, one can for instance discretise the support of  $g$  and  $h$  and approximate  $G(\Phi)$ ,  $H(\Phi)$  with functions depending only on the random vector  $(\Phi(c_i))_{1 \leq i \leq N}$ , where  $\{c_i \mid 1 \leq i \leq N\}$  is a collection of disjoint (small) cubes). Finally, take  $\varepsilon \rightarrow 0$  to conclude.  $\square$

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