

# STOCHASTIC TAMED 3D NAVIER-STOKES EQUATIONS: EXISTENCE, UNIQUENESS AND ERGODICITY

MICHAEL RÖCKNER<sup>1,2</sup>, XICHENG ZHANG<sup>1,3,4</sup>

<sup>1</sup>*Fakultät für Mathematik, Universität Bielefeld  
Postfach 100131, D-33501 Bielefeld, Germany*

<sup>2</sup>*Departments of Mathematics and Statistics, Purdue University  
W. Laffayette, IN, 47907, USA*

<sup>3</sup>*School of Mathematics and Statistics  
The University of New South Wales, Sydney, 2052, Australia*

<sup>4</sup>*Department of Mathematics, Huazhong University of Science and Technology  
Wuhan, Hubei 430074, P.R.China*

*M. Röckner: roeckner@math.uni-bielefeld.de  
X. Zhang: XichengZhang@gmail.com*

ABSTRACT. In this paper, we prove the existence of a unique strong solution to a stochastic tamed 3D Navier-Stokes equation in the whole space as well as in the periodic boundary case. Then, we also study the Feller property of solutions, and prove the existence of invariant measures for the corresponding Feller semigroup in the case of periodic conditions. Moreover, in the case of periodic boundary and degenerated additive noise, using the notion of asymptotic strong Feller property proposed by Hairer and Mattingly [14], we prove the uniqueness of invariant measures for the corresponding transition semigroup.

## CONTENTS

1. Introduction	1
2. Preliminaries	4
2.1. Notations and Assumptions	4
2.2. Estimates on $A$ and $B$	5
2.3. Tightness Criterion	8
3. Existence and Uniqueness of Strong Solution	10
3.1. Weak and Strong Solution	10
3.2. Pathwise Uniqueness	12
3.3. Existence of Martingale Solution	13
4. Feller Property and Invariant Measure	21
5. Ergodicity: Uniqueness of Invariant Measure	24
5.1. Asymptotic Strong Feller Property	25
5.2. A Support Property of Invariant Measure	31
6. Appendix	34
6.1. Two Basic Estimates	34
6.2. Proof of the Derivative Flow Equation	36
References	39

---

*Key words and phrases.* Navier-Stokes Equation, Invariant measure, Ergodicity, Asymptotic strong Feller property.

*AMS Classification(2000):* 60H15, 37A25.

## 1. INTRODUCTION

The classical 3D Navier-Stokes equations (NSE) describe the time evolution of an incompressible fluid and are given by

$$\partial_t \mathbf{u}(t) = \nu \Delta \mathbf{u}(t) - (\mathbf{u}(t) \cdot \nabla) \mathbf{u}(t) + \nabla p(t) + \mathbf{f}(t),$$

and

$$\operatorname{div} \mathbf{u}(t) = 0,$$

where  $\mathbf{u}(t, x) = (u^1(t, x), u^2(t, x), u^3(t, x))$  represents the velocity field,  $\nu$  is the viscosity constant,  $p(t, x)$  denotes the pressure, and  $\mathbf{f}$  is an external force field acting on the fluid. In [18], Leray initially constructed a weak solution for the Cauchy problem of NSE in the whole space, since then, it is still not known whether there exists a smooth solution existing for all times. In [24], we analyzed the following tamed scheme for the classical 3D NSE:

$$\partial_t \mathbf{u}(t) = \nu \Delta \mathbf{u}(t) - (\mathbf{u}(t) \cdot \nabla) \mathbf{u}(t) + \nabla p(t) - g_N(|\mathbf{u}(t)|^2) \mathbf{u}(t) + \mathbf{f}(t),$$

where the taming function  $g_N : \mathbb{R}_+ \mapsto \mathbb{R}_+$  is smooth and satisfies for some  $N > 0$

$$\begin{cases} g_N(r) = 0, & \text{if } r \leq N, \\ g_N(r) = (r - N)/\nu, & \text{if } r \geq N + 1, \\ g'_N(r) \geq 0, & r \geq 0. \end{cases} \quad (1.1)$$

Therein, we prove the existence of smooth solutions to this tamed equation when  $\mathbf{f}$  and the initial velocity are smooth. The main feature of this tamed equation is that if there is a bounded smooth solution to the classical 3D NSE, then this smooth solution must satisfy our tamed equation for some  $N$  large enough.

Following the above tamed scheme, in the present paper we shall study the stochastic tamed 3D NSE. Let us now describe our model equation. Let  $\mathbb{D} := \mathbb{R}^3$  or  $\mathbb{T}^3$  (in the periodic case), where  $\mathbb{T} = [0, 1)$  is the unit circle. Note that any function from  $\mathbb{T}^3$  to  $\mathbb{T}^3$  can be identified with a periodic function from  $\mathbb{R}^3$  to  $\mathbb{R}^3$ . We consider the following stochastic tamed 3D Navier-Stokes equation with  $\nu = 1$  in  $\mathbb{D}$ :

$$\begin{aligned} d\mathbf{u}(t) &= \left[ \Delta \mathbf{u}(t) - (\mathbf{u}(t) \cdot \nabla) \mathbf{u}(t) + \nabla p(t) - g_N(|\mathbf{u}(t)|^2) \mathbf{u}(t) + \mathbf{f}(t, \mathbf{u}(t)) \right] dt \\ &+ \sum_{k=1}^{\infty} \left[ (\sigma_k(t) \cdot \nabla) \mathbf{u}(t) + \nabla \tilde{p}_k(t) + \mathbf{h}_k(t, \mathbf{u}(t)) \right] dW_t^k \end{aligned} \quad (1.2)$$

subject to the incompressibility condition

$$\operatorname{div} \mathbf{u}(t) = 0, \quad (1.3)$$

and the initial condition

$$\mathbf{u}(0) = \mathbf{u}_0, \quad (1.4)$$

where  $p(t, x)$  and  $\tilde{p}(t, x)$  are two unknown scalar functions,  $N > 0$  and the taming function  $g_N : \mathbb{R}^+ \mapsto \mathbb{R}^+$  as above satisfies (1.1), and  $\{W_t^k; t \geq 0, k = 1, 2, \dots\}$  is a family of independent one dimensional standard Brownian motions on some complete filtration probability space  $(\Omega, \mathcal{F}, P; (\mathcal{F}_t)_{t \geq 0})$ . The stochastic integral is understood as Itô's integral. The entries of the coefficients are given as follows:

$$\begin{aligned} \mathbb{R}_+ \times \mathbb{D} \times \mathbb{R}^3 \ni (t, x, \mathbf{u}) &\mapsto \mathbf{f}(t, x, \mathbf{u}) \in \mathbb{R}^3 \\ \mathbb{R}_+ \times \mathbb{D} \ni (t, x) &\mapsto \sigma(t, x) \in \mathbb{R}^3 \times l^2 \\ \mathbb{R}_+ \times \mathbb{D} \times \mathbb{R}^3 \ni (t, x, \mathbf{u}) &\mapsto \mathbf{h}(t, x, \mathbf{u}) \in \mathbb{R}^3 \times l^2, \end{aligned}$$

where  $l^2$  denotes the Hilbert space consisted of all sequences of square summable real numbers with standard norm  $\|\cdot\|_{l^2}$ . In the following,  $\mathbf{f}, \sigma$  and  $\mathbf{h}$  are always assumed to be measurable with respect to all their variables.

The study of stochastic Navier-Stokes equations (SNSE) began with the work of Bensoussan -Temam in [1]. Using Galerkin's approximation and compactness method, Flandoli and Gatarek in [9] proved the existence of martingale solutions and stationary solutions for any dimensional stochastic Navier-Stokes equations in a bounded domain. In particular, when the transition semigroup is well defined, the stationary martingale solutions will yield the existence of invariant measures. We remark that their results can not be used in the case of whole space because of the absence of compact Sobolev embeddings. Recently, Mikulevicius and Rozovskii in [22] proved the existence of martingale solutions to SNSE in  $\mathbb{R}^d (d \geq 2)$  under less assumptions on the coefficients. To avoid the use of compact Sobolev embeddings, they used the approach of mollifying and cutting off the coefficients. In the case of two dimension, they also obtained the existence and pathwise uniqueness of  $L^2$ -continuous adapted solutions.

On the other hand, the ergodicity of invariant measure for 2D stochastic Navier-Stokes equations has been studied extensively (cf. [10, 21, 6, 14] and reference therein). Especially, Hairer and Mattingly [14] recently developed two important tools: the asymptotic strong Feller property and an approximative integration by parts formula in the Malliavin calculus, and then used them to derive an optimal ergodicity result for 2D SNSE in the sense that the random forces only has two modes. As pointed out in [14], the asymptotic strong Feller property is much weaker than the usual strong Feller property since many degenerated equations have the former property rather than the later one.

Up to now, to the best of our knowledge, most of the well known results about the stochastic Navier-Stokes equations such as the existence of invariant measures and the ergodicity under different conditions on the noise are for 2D SNSE. As for the three dimensional case, there are only a few results(cf. [3, 4, 11]), of course, because of the absence of uniqueness. But we also refer to the recent results on the existence of Markov selections in [3, 4, 11].

In the present paper, we shall prove the existence of a unique strong solution to our stochastic tamed 3D Navier-Stokes equation (1.2) under some assumptions on  $\mathbf{f}, \sigma$  and  $\mathbf{h}$ . Here, the word "strong" means "strong" both in the sense of the theory of stochastic differential equations and the theory of partial differential equations. Let  $\mathbb{H}^m$  denote the Sobolev space of divergence free vector fields(see (2.2) below). Instead of working on the evolutional triple  $\mathbb{H}^1 \subset \mathbb{H}^0 \subset \mathbb{H}^{-1}$ , we shall work on the evolution triple  $\mathbb{H}^2 \subset \mathbb{H}^1 \subset \mathbb{H}^0$ . This will enable us to obtain the "strong" solution in the sense of partial differential equation. For the "strong" solution in the sense of stochastic differential equation, we shall use the famous Yamada-Watanabe theorem: the existence of martingale solutions plus pathwise uniqueness implies the existence of a unique strong solution. Different from the method in [22], we still use the classical Galerkin approximation to prove our existence of strong solutions. To overcome the absence of compact Sobolev embeddings, we shall use localization method to prove tightness. We think that it is of interest in itself and can be used in other cases.

After obtaining the existence of a unique strong solution to Eq. (1.2), we turn to the study of uniqueness of invariant measure in the case of periodic boundary conditions and degenerated additive noise. As a first step, we need to prove the Feller property and the existence of an invariant measure. Then, using the asymptotic strong Feller property and approximative integration by parts formula in [14], we can prove the uniqueness of

invariant measure. As said above, since we shall work in the first order Sobolev space  $\mathbb{H}^1$ , all of our discussions will take place in  $\mathbb{H}^1$ . This requires some delicate analysis and calculations, and the special form of  $g_N$  plays an important role throughout this paper. It should be emphasized that the optimal results in [14] seem to depend strongly on the structure of 2D Navier-Stokes equations, we can not develop a similar non-adapted analysis along their lines to obtain some optimal result for our tamed 3D SNSE.

This paper is organized as follows: in Section 2, we give some preliminaries, that include some necessary estimates and a tightness result for later use. In Section 3, we shall prove the existence and uniqueness result by Galerkin's approximation. In Section 4, we study the Feller property of the solutions to Eq. (1.2) and the existence of invariant measures for the Feller semigroup in the case of periodic boundary conditions. In Section 5, we study the ergodicity of invariant measure. In the Appendix, for the reader's convenience, two necessary basic estimates are given, and the derivative flow equation is proved.

## 2. PRELIMINARIES

**2.1. Notations and Assumptions.** Let  $C_0^\infty(\mathbb{D}; \mathbb{R}^3)$  denote the set of all smooth functions from  $\mathbb{D}$  to  $\mathbb{R}^3$  with compact supports. When  $\mathbb{D} = \mathbb{T}^3$ , a function  $f \in C_0^\infty(\mathbb{D}; \mathbb{R}^3)$  means that it is a smooth periodic function from  $\mathbb{R}^3$  to  $\mathbb{R}^3$ . For  $p \geq 1$ , let  $L^p(\mathbb{D}; \mathbb{R}^3)$  be the vector valued  $L^p$ -space in which the norm is denoted by  $\|\cdot\|_{L^p}$ . For  $m \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ , let  $H^m$  be the usual Sobolev space on  $\mathbb{D}$  with values in  $\mathbb{R}^3$ , i.e., the closure of  $C_0^\infty(\mathbb{D}; \mathbb{R}^3)$  with respect to the norm:

$$\|\mathbf{u}\|_{H^m} = \left( \int_{\mathbb{D}} |(I - \Delta)^{m/2} \mathbf{u}|^2 dx \right)^{1/2}.$$

Here as usual,  $(I - \Delta)^{m/2}$  is defined by Fourier transformation. For two separable Hilbert spaces  $\mathbb{K}$  and  $\mathbb{H}$ ,  $L_2(\mathbb{K}; \mathbb{H})$  will denote the space of all Hilbert-Schmidt operators from  $\mathbb{K}$  to  $\mathbb{H}$  with norm  $\|\cdot\|_{L_2(\mathbb{K}; \mathbb{H})}$ .

The following Gagliardo-Nirenberg interpolation inequality will be used frequently. It plays an essential role in the study of Navier-Stokes equation(cf. [29]). Let  $q \in [1, \infty]$  and  $m \in \mathbb{N}$ . If

$$\frac{1}{q} = \frac{1}{2} - \frac{m\alpha}{3}, \quad 0 \leq \alpha \leq 1,$$

then for any  $\mathbf{u} \in H^m$

$$\|\mathbf{u}\|_{L^q} \leq C_{m,q} \|\mathbf{u}\|_{H^m}^\alpha \|\mathbf{u}\|_{L^2}^{1-\alpha}. \quad (2.1)$$

Set for  $m \in \mathbb{N}_0$

$$\mathbb{H}^m := \{\mathbf{u} \in H^m : \operatorname{div}(\mathbf{u}) = 0\}. \quad (2.2)$$

Then  $(\mathbb{H}^m, \|\cdot\|_{H^m})$  is a separable Hilbert space. We shall denote the norm  $\|\cdot\|_{H^m}$  in  $\mathbb{H}^m$  by  $\|\cdot\|_{\mathbb{H}^m}$ . We remark that  $\mathbb{H}^0$  is a closed linear subspace of the Hilbert space  $L^2(\mathbb{D}; \mathbb{R}^3) = H^0$ .

Let  $\mathcal{P}$  be the orthogonal projection from  $L^2(\mathbb{D}; \mathbb{R}^3)$  to  $\mathbb{H}^0$  (cf. [7] [19]). It is well known that  $\mathcal{P}$  commutes with the derivative operators, and that  $\mathcal{P}$  can be restricted to a bounded linear operator from  $H^m$  to  $\mathbb{H}^m$ . For any  $\mathbf{u} \in \mathbb{H}^0$  and  $\mathbf{v} \in L^2(\mathbb{D}; \mathbb{R}^3)$ , we have

$$\langle \mathbf{u}, \mathbf{v} \rangle_{\mathbb{H}^0} := \langle \mathbf{u}, \mathcal{P}\mathbf{v} \rangle_{\mathbb{H}^0} = \langle \mathbf{u}, \mathbf{v} \rangle_{L^2}.$$

Let  $\mathcal{V}$  be defined by

$$\mathcal{V} := \{\mathbf{u} : \mathbf{u} \in C_0^\infty(\mathbb{D}; \mathbb{R}^3), \operatorname{div}(\mathbf{u}) = 0\}.$$

We have the following density result(cf. [24]).

**Lemma 2.1.**  $\mathcal{V}$  is dense in  $\mathbb{H}^m$  for any  $m \in \mathbb{N}$ .

We now introduce the following assumptions on the coefficients  $\mathbf{f}$ ,  $\sigma$  and  $\mathbf{h}$ :

**(H1)** For any  $T > 0$ , there exist a constant  $C_{T,\mathbf{f}} > 0$  and a function  $H_{\mathbf{f}}(t, x) \in L^1([0, T] \times \mathbb{D})$  such that for any  $t \in [0, T]$ ,  $x \in \mathbb{D}$ ,  $\mathbf{u} \in \mathbb{R}^3$  and  $j = 1, 2, 3$

$$\begin{aligned} |\partial_{x^j} \mathbf{f}(t, x, \mathbf{u})|^2 + |\mathbf{f}(t, x, \mathbf{u})|^2 &\leq C_{T,\mathbf{f}} \cdot |\mathbf{u}|^2 + H_{\mathbf{f}}(t, x) \\ |\partial_{u^j} \mathbf{f}(t, x, \mathbf{u})| &\leq C_{T,\mathbf{f}}. \end{aligned}$$

**(H2)** For any  $T > 0$ , there exists a constant  $c_{\sigma,T} > 0$  such that

$$\sup_{t \in [0, T], x \in \mathbb{D}} \|\partial_{x^j} \sigma(t, x)\|_{l^2} \leq c_{\sigma,T}, \quad j = 1, 2, 3,$$

and

$$\sup_{t \in \mathbb{R}_+, x \in \mathbb{D}} \|\sigma(t, x)\|_{l^2}^2 \leq 1/2. \quad (2.3)$$

**(H3)** For any  $T > 0$ , there exist a constant  $C_{T,\mathbf{h}} > 0$  and a function  $H_{\mathbf{h}}(t, x) \in L^1([0, T] \times \mathbb{D})$  such that for any  $t \in [0, T]$ ,  $x \in \mathbb{D}$ ,  $\mathbf{u} \in \mathbb{R}^3$  and  $j = 1, 2, 3$

$$\begin{aligned} \|\partial_{x^j} \mathbf{h}(t, x, \mathbf{u})\|_{l^2}^2 + \|\mathbf{h}(t, x, \mathbf{u})\|_{l^2}^2 &\leq C_{T,\mathbf{h}} \cdot |\mathbf{u}|^2 + H_{\mathbf{h}}(t, x) \\ \|\partial_{u^j} \mathbf{h}(t, x, \mathbf{u})\|_{l^2} &\leq C_{T,\mathbf{h}}. \end{aligned}$$

For any  $\mathbf{u} \in \mathbb{H}^2$ , define

$$A(\mathbf{u}) := \mathcal{P} \Delta \mathbf{u} - \mathcal{P}((\mathbf{u} \cdot \nabla) \mathbf{u}) - \mathcal{P}(g_N(|\mathbf{u}|^2) \mathbf{u}), \quad (2.4)$$

and for any  $\mathbf{v} \in \mathbb{H}^2$ , we write

$$\begin{aligned} \llbracket A(\mathbf{u}), \mathbf{v} \rrbracket &:= \langle A(\mathbf{u}), (I - \Delta) \mathbf{v} \rangle_{\mathbb{H}^0} \\ &= A_1(\mathbf{u}, \mathbf{v}) + A_2(\mathbf{u}, \mathbf{v}) + A_3(\mathbf{u}, \mathbf{v}), \end{aligned}$$

where

$$\begin{aligned} A_1(\mathbf{u}, \mathbf{v}) &:= \langle \Delta \mathbf{u}, (I - \Delta) \mathbf{v} \rangle_{\mathbb{H}^0} \\ A_2(\mathbf{u}, \mathbf{v}) &:= -\langle (\mathbf{u} \cdot \nabla) \mathbf{u}, (I - \Delta) \mathbf{v} \rangle_{\mathbb{H}^0} \\ A_3(\mathbf{u}, \mathbf{v}) &:= -\langle g_N(|\mathbf{u}|^2) \mathbf{u}, (I - \Delta) \mathbf{v} \rangle_{\mathbb{H}^0}. \end{aligned}$$

Below, for the sake of simplicity, the variable “ $x$ ” in the coefficients will be dropped. Define for  $k \in \mathbb{N}$

$$B_k(t, \mathbf{u}) := \mathcal{P}((\sigma_k(t) \cdot \nabla) \mathbf{u}) + \mathcal{P} \mathbf{h}_k(t, \mathbf{u}). \quad (2.5)$$

Letting the operator  $P$  act on both sides of equation (1.2), we can and shall consider the following equivalent abstract stochastic evolution equation in the sequel :

$$\begin{cases} d\mathbf{u}(t) &= \left[ A(\mathbf{u}(t)) + \mathcal{P} \mathbf{f}(t, \mathbf{u}(t)) \right] dt + \sum_{k=1}^{\infty} B_k(t, \mathbf{u}(t)) dW_t^k, \\ \mathbf{u}(0) &= \mathbf{u}_0 \in \mathbb{H}^1. \end{cases} \quad (2.6)$$

**2.2. Estimates on  $A$  and  $B$ .** We now prepare several important estimates for later use. In the sequel, we shall use the following convention: The letter  $C$  with subscripts will denote a constant depending on its subscripts and the coefficients. The letter  $C$  without subscripts will denote an absolute constant, i.e., its value does not depend on any data. All the constants may have different values in different places.

**Lemma 2.2.** *For any  $\mathbf{u} \in \mathbb{H}^2$ , we have*

$$\|A(\mathbf{u})\|_{\mathbb{H}^0} \leq C(1 + \|\mathbf{u}\|_{\mathbb{H}^2}^2), \quad (2.7)$$

$$\langle A(\mathbf{u}), \mathbf{u} \rangle_{\mathbb{H}^0} = -\|\nabla \mathbf{u}\|_{\mathbb{H}^0}^2 - \|\sqrt{g_N(|\mathbf{u}|^2)} \cdot |\mathbf{u}|\|_{L^2}^2 \quad (2.8)$$

$$\leq -\|\nabla \mathbf{u}\|_{\mathbb{H}^0}^2 - \|\mathbf{u}\|_{L^4}^4 + N\|\mathbf{u}\|_{\mathbb{H}^0}^2, \quad (2.9)$$

$$\llbracket A(\mathbf{u}), \mathbf{u} \rrbracket \leq -\frac{1}{2}\|\mathbf{u}\|_{\mathbb{H}^2}^2 - \frac{1}{2}\|\mathbf{u}\| \cdot \|\nabla \mathbf{u}\|_{L^2}^2 + (1 + N)\|\nabla \mathbf{u}\|_{\mathbb{H}^0}^2 + \|\mathbf{u}\|_{\mathbb{H}^0}^2. \quad (2.10)$$

*Proof.* The estimate (2.7) is direct from (2.4) and the Sobolev inequality (2.1). The estimate (2.9) follows from

$$\langle (\mathbf{u} \cdot \nabla) \mathbf{u}, \mathbf{u} \rangle_{\mathbb{H}^0} = 0,$$

and  $g_N \geq 0$ .

For the inequality (2.10), we have

$$\begin{aligned} A_1(\mathbf{u}, \mathbf{u}) &= -\|(I - \Delta)\mathbf{u}\|_{\mathbb{H}^0}^2 + \langle \mathbf{u}, (I - \Delta)\mathbf{u} \rangle_{\mathbb{H}^0} \\ &= -\|\mathbf{u}\|_{\mathbb{H}^2}^2 + \|\nabla \mathbf{u}\|_{\mathbb{H}^0}^2 + \|\mathbf{u}\|_{\mathbb{H}^0}^2, \end{aligned}$$

and by Young's inequality

$$\begin{aligned} A_2(\mathbf{u}, \mathbf{u}) &\leq \frac{1}{2}\|(I - \Delta)\mathbf{u}\|_{\mathbb{H}^0}^2 + \frac{1}{2}\|(\mathbf{u} \cdot \nabla)\mathbf{u}\|_{\mathbb{H}^0}^2 \\ &\leq \frac{1}{2}\|\mathbf{u}\|_{\mathbb{H}^2}^2 + \frac{1}{2}\|\mathbf{u}\| \cdot \|\nabla \mathbf{u}\|_{\mathbb{H}^0}^2, \end{aligned}$$

where

$$|\mathbf{u}|^2 = \sum_{k=1}^3 |u^k|^2, \quad |\nabla \mathbf{u}|^2 = \sum_{k,i=1}^3 |\partial_i u^k|^2.$$

From (1.1), we also have

$$\begin{aligned} A_3(\mathbf{u}, \mathbf{u}) &= -\langle \nabla(g_N(|\mathbf{u}|^2)\mathbf{u}), \nabla \mathbf{u} \rangle_{\mathbb{H}^0} - \langle g_N(|\mathbf{u}|^2)\mathbf{u}, \mathbf{u} \rangle_{\mathbb{H}^0} \\ &= -\sum_{k,i=1}^3 \int_{\mathbb{D}} \partial_i u^k \cdot \partial_i (g_N(|\mathbf{u}|^2)u^k) dx - \int_{\mathbb{D}} |\mathbf{u}|^2 \cdot g_N(|\mathbf{u}|^2) dx \\ &\leq -\sum_{k,i=1}^3 \int_{\mathbb{D}} \partial_i u^k \cdot (g_N(|\mathbf{u}|^2) \cdot \partial_i u^k - g'_N(|\mathbf{u}|^2) \partial_i |\mathbf{u}|^2 \cdot u^k) dx \\ &= -\int_{\mathbb{D}} |\nabla \mathbf{u}|^2 \cdot g_N(|\mathbf{u}|^2) dx - \frac{1}{2} \int_{\mathbb{D}} g'_N(|\mathbf{u}|^2) |\nabla |\mathbf{u}|^2|^2 dx \\ &\leq -\int_{\mathbb{D}} |\nabla \mathbf{u}|^2 \cdot |\mathbf{u}|^2 dx + N\|\nabla \mathbf{u}\|_{\mathbb{H}^0}^2. \end{aligned}$$

Combining the above calculations yields (2.10).  $\square$

**Lemma 2.3.** *Let  $\mathbf{v} \in \mathcal{V}$ , and let the support of  $\mathbf{v}$  be contained in  $\mathcal{O} := \{x \in \mathbb{D}, |x| \leq m\}$  for some  $m \in \mathbb{N}$ . Let  $T > 0$ . For any  $\mathbf{u}, \mathbf{u}' \in \mathbb{H}^2$  and  $t \in [0, T]$ , we have*

$$|[A(\mathbf{u}), \mathbf{v}]| \leq C_{\mathbf{v}} \cdot (1 + \|\mathbf{u}\|_{L^3(\mathcal{O})}^3), \quad (2.11)$$

$$\|\langle B.(t, \mathbf{u}), \mathbf{v} \rangle_{\mathbb{H}^1}\|_{l^2}^2 \leq C_{\mathbf{v}, T} \cdot (1 + \|H_{\mathbf{h}}(t)\|_{L^1(\mathbb{D})} + \|\mathbf{u}\|_{L^2(\mathcal{O})}^2) \quad (2.12)$$

and

$$|[A(\mathbf{u}) - A(\mathbf{u}'), \mathbf{v}]| \leq C_{\mathbf{v}} \cdot \|\mathbf{u} - \mathbf{u}'\|_{L^2(\mathcal{O})} \cdot (\|\mathbf{u}\|_{\mathbb{H}^1}^2 + \|\mathbf{u}'\|_{\mathbb{H}^1}^2). \quad (2.13)$$

*Proof.* For the estimate (2.11), we have

$$A_1(\mathbf{u}, \mathbf{v}) = \langle \mathbf{u}, (I - \Delta)\Delta\mathbf{v} \rangle_{\mathbb{H}^0} \leq C\|\mathbf{u}\|_{L^2(\mathcal{O})} \cdot \|\mathbf{v}\|_{\mathbb{H}^4}$$

$$A_2(\mathbf{u}, \mathbf{v}) = \langle \mathbf{u}^* \cdot \mathbf{u}, \nabla(I - \Delta)\mathbf{v} \rangle_{\mathbb{H}^0} \leq C\|\mathbf{u}\|_{L^2(\mathcal{O})}^2 \cdot \sup_{x \in \mathbb{D}} |\nabla(I - \Delta)\mathbf{v}(x)|,$$

where  $\mathbf{u}^*$  denotes the transposition of the row vector  $\mathbf{u}$ , and

$$A_3(\mathbf{u}, \mathbf{v}) \leq \|\mathbf{u}\|_{L^3(\mathcal{O})}^3 \cdot \sup_{x \in \mathbb{D}} |(I - \Delta)\mathbf{v}(x)|.$$

Combining them gives (2.11).

For the estimate (2.12), by **(H2)** and **(H3)** we have

$$\begin{aligned} \|\langle B.(t, \mathbf{u}), \mathbf{v} \rangle_{\mathbb{H}^1}\|_{l^2}^2 &\leq C \sup_{t \in \mathbb{R}_+, x \in \mathbb{D}} \|\sigma(t, x)\|_{l^2}^2 \cdot \sup_{x \in \mathbb{D}} |\nabla(I - \Delta)\mathbf{v}(x)|^2 \cdot \|\mathbf{u}\|_{L^2(\mathcal{O})}^2 \\ &\quad + C \sup_{x \in \mathbb{D}} \|\nabla_x \sigma(t, x)\|_{l^2}^2 \cdot \sup_{x \in \mathbb{D}} |(I - \Delta)\mathbf{v}(x)|^2 \cdot \|\mathbf{u}\|_{L^2(\mathcal{O})}^2 \\ &\quad + C \sup_{x \in \mathbb{D}} |(I - \Delta)\mathbf{v}(x)|^2 \cdot (C_{T, \mathbf{h}} \cdot \|\mathbf{u}\|_{L^2(\mathcal{O})}^2 + \|H_{\mathbf{h}}(t)\|_{L^1(\mathbb{D})}) \\ &\leq C_{\mathbf{v}, T} \cdot (1 + \|H_{\mathbf{h}}(t)\|_{L^1(\mathbb{D})} + \|\mathbf{u}\|_{L^2(\mathcal{O})}^2). \end{aligned}$$

We now look at (2.13). For  $A_1$ , we clearly have

$$|A_1(\mathbf{u}, \mathbf{v}) - A_1(\mathbf{u}', \mathbf{v})| = |\langle (\mathbf{u} - \mathbf{u}') \cdot \mathbf{1}_{\mathcal{O}}, (I - \Delta)\Delta\mathbf{v} \rangle_{\mathbb{H}^0}| \leq C_{\mathbf{v}} \cdot \|\mathbf{u} - \mathbf{u}'\|_{L^2(\mathcal{O})}.$$

For  $A_2$ , we have

$$\begin{aligned} &|A_2(\mathbf{u}, \mathbf{v}) - A_2(\mathbf{u}', \mathbf{v})| \\ &= |\langle \mathbf{u}^* \cdot \mathbf{u} - \mathbf{u}'^* \cdot \mathbf{u}', \nabla(I - \Delta)\mathbf{v} \rangle_{\mathbb{H}^0}| \\ &\leq C_{\mathbf{v}} \cdot \|\mathbf{u} - \mathbf{u}'\|_{L^2(\mathcal{O})} \cdot (\|\mathbf{u}\|_{\mathbb{H}^0} + \|\mathbf{u}'\|_{\mathbb{H}^0}). \end{aligned}$$

For  $A_3$ , by the Sobolev inequality (2.1) we similarly have

$$\begin{aligned} |A_3(\mathbf{u}, \mathbf{v}) - A_3(\mathbf{u}', \mathbf{v})| &\leq C_{\mathbf{v}} \cdot \|\mathbf{u} - \mathbf{u}'\|_{L^2(\mathcal{O})} \cdot (\|\mathbf{u}\|_{L^4}^2 + \|\mathbf{u}'\|_{L^4}^2) \\ &\leq C_{\mathbf{v}} \cdot \|\mathbf{u} - \mathbf{u}'\|_{L^2(\mathcal{O})} \cdot (\|\mathbf{u}\|_{\mathbb{H}^1}^2 + \|\mathbf{u}'\|_{\mathbb{H}^1}^2). \end{aligned}$$

□

**Lemma 2.4.** *For any  $T > 0$  and  $\mathbf{u} \in \mathbb{H}^2$*

$$\|B(t, \mathbf{u})\|_{L_2(l^2; \mathbb{H}^0)}^2 \leq \frac{1}{2}\|\mathbf{u}\|_{\mathbb{H}^1}^2 + C_T\|\mathbf{u}\|_{\mathbb{H}^0}^2 + \|H_{\mathbf{h}}(t)\|_{L^1(\mathbb{D})}, \quad (2.14)$$

$$\|B(t, \mathbf{u})\|_{L_2(l^2; \mathbb{H}^1)}^2 \leq \frac{1}{2}\|\mathbf{u}\|_{\mathbb{H}^2}^2 + C_T\|\mathbf{u}\|_{\mathbb{H}^1}^2 + C\|H_{\mathbf{h}}(t)\|_{L^1(\mathbb{D})}. \quad (2.15)$$

*Proof.* First of all, by **(H2)** and **(H3)** we have

$$\begin{aligned}
\|B(t, \mathbf{u})\|_{L_2(l^2; \mathbb{H}^0)}^2 &= \sum_{k=1}^{\infty} \int_{\mathbb{D}} |B_k(t, x, \mathbf{u}(x))|^2 dx \\
&\leq \int_{\mathbb{D}} \|\sigma(t, x)\|_{l^2}^2 \cdot |\nabla \mathbf{u}(x)|^2 dx + \int_{\mathbb{D}} (C_{T, \mathbf{h}} |\mathbf{u}(x)|^2 + H_{\mathbf{h}}(t, x)) dx \\
&\leq \frac{1}{2} \|\mathbf{u}\|_{\mathbb{H}^1}^2 + C_T \|\mathbf{u}\|_{\mathbb{H}^0}^2 + \|H_{\mathbf{h}}(t)\|_{L^1(\mathbb{D})}.
\end{aligned}$$

Secondly, noting that

$$\|B(t, \mathbf{u})\|_{L_2(l^2; \mathbb{H}^1)}^2 = \|B(t, \mathbf{u})\|_{L_2(l^2; \mathbb{H}^0)}^2 + \|\nabla B(t, \mathbf{u})\|_{L_2(l^2; \mathbb{H}^0)}^2$$

and

$$\begin{aligned}
\partial_{x^j} B_k(t, \mathbf{u}) &= \mathcal{P} \partial_{x^j} ((\sigma_k(t) \cdot \nabla) \mathbf{u}) + \mathcal{P} \partial_{x^j} \mathbf{h}_k(t, \mathbf{u}) \\
&= \mathcal{P} \left( (\partial_{x^j} \sigma_k(t) \cdot \nabla) \mathbf{u} + (\sigma_k(t) \cdot \nabla) \partial_{x^j} \mathbf{u} \right) \\
&\quad + \mathcal{P} \left( (\partial_{x^j} \mathbf{h}_k)(t, \mathbf{u}) + \sum_{i=1}^3 \partial_{u^i} \mathbf{h}_k(t, \mathbf{u}) \cdot \partial_{x^j} u^i \right),
\end{aligned}$$

by **(H2)** and **(H3)** we have

$$\|B(t, \mathbf{u})\|_{L_2(l^2; \mathbb{H}^1)}^2 \leq \frac{1}{2} \|\mathbf{u}\|_{\mathbb{H}^2}^2 + C_T \|\mathbf{u}\|_{\mathbb{H}^1}^2 + C \|H_{\mathbf{h}}(t)\|_{L^1(\mathbb{D})}.$$

□

**2.3. Tightness Criterion.** In the following, we only give a tightness criterion in the case of  $\mathbb{D} = \mathbb{R}^3$ . When  $\mathbb{D} = \mathbb{T}^3$ , since  $\mathbb{H}^1$  is compactly embedded in  $\mathbb{H}^0$ , the corresponding result is simple and well known.

By  $\mathbb{H}_{loc}^0$  we denote the space of all locally  $L^2$ -integrable and divergence free vector fields endowed with the Fréchet metric: for  $\mathbf{u}, \mathbf{v} \in \mathbb{H}_{loc}^0$

$$\rho(\mathbf{u}, \mathbf{v}) := \sum_{m=1}^{\infty} 2^{-m} \left( \left[ \int_{|x| \leq m} |\mathbf{u}(x) - \mathbf{v}(x)|^2 dx \right]^{1/2} \wedge 1 \right).$$

Thus,  $(\mathbb{H}_{loc}^0, \rho)$  is a Polish space and  $\mathbb{H}^0 \subset \mathbb{H}_{loc}^0$ .

Let  $\mathbb{X} := C(\mathbb{R}_+; \mathbb{H}_{loc}^0)$  denote the space of all continuous functions from  $\mathbb{R}_+$  to  $(\mathbb{H}_{loc}^0, \rho)$  equipped with the metric

$$\rho_{\mathbb{X}}(\mathbf{u}, \mathbf{v}) := \sum_{m=1}^{\infty} 2^{-m} \left( \sup_{t \in [0, m]} \rho(\mathbf{u}(t), \mathbf{v}(t)) \wedge 1 \right).$$

In the following, we shall fix a complete orthonormal basis  $\mathcal{E} := \{\mathbf{e}_k, k \in \mathbb{N}\} \subset \mathcal{V}$  of  $\mathbb{H}^1$  such that  $\text{span}\{\mathcal{E}\}$  is a dense subset of  $\mathbb{H}^3$  and, in the case of periodic boundary conditions, we also require that  $\mathcal{E}$  is an orthogonal basis of  $\mathbb{H}^0$ . Moreover, for  $\mathbf{u} \in \mathbb{H}^0$  and  $\mathbf{v} \in \mathbb{H}^2$ , the inner product  $\langle \mathbf{u}, \mathbf{v} \rangle_{\mathbb{H}^1}$  is taken in the generalized sense, i.e.,

$$\langle \mathbf{u}, \mathbf{v} \rangle_{\mathbb{H}^1} = \langle \mathbf{u}, (I - \Delta) \mathbf{v} \rangle_{\mathbb{H}^0}.$$

We need the following relative compactness result.

**Lemma 2.5.** *Let  $K \subset \mathbb{X}$ . If for any  $T > 0$*

$$(1^\circ) \sup_{\mathbf{u} \in K} \sup_{s \in [0, T]} \|\mathbf{u}(s)\|_{\mathbb{H}^1} < +\infty,$$

(2°)  $\lim_{\delta \rightarrow 0} \sup_{\mathbf{u} \in K} \sup_{t, s \in [0, T], |t-s| < \delta} |\langle \mathbf{u}(t) - \mathbf{u}(s), \mathbf{e} \rangle_{\mathbb{H}^1}| = 0$  for any  $\mathbf{e} \in \mathcal{E}$ ,  
then  $K$  is relatively compact in  $\mathbb{X}$ .

*Proof.* We only need to prove that  $K$  is relatively compact in  $C([0, T]; \mathbb{H}_{loc}^0)$  for any  $T > 0$ .

Let  $\{\mathbf{u}_n, n \in \mathbb{N}\} \subset K$  be any sequence of  $K$ . Define for  $\mathbf{e} \in \mathcal{E}$

$$G_n^{\mathbf{e}}(t) := \langle \mathbf{u}_n(t), \mathbf{e} \rangle_{\mathbb{H}^1}.$$

Then, by (1°) and (2°), the sequence  $\{t \mapsto G_n^{\mathbf{e}}(t), n \in \mathbb{N}\}$  is uniformly bounded and equi-continuous on  $[0, T]$ . Hence, by Ascoli-Arzelà's lemma, there exist a subsequence  $n_l$  (depending on  $\mathbf{e}$ ) and a continuous function  $G^{\mathbf{e}}(t)$  such that  $G_{n_l}^{\mathbf{e}}(t)$  uniformly converges to  $G^{\mathbf{e}}(t)$  on  $[0, T]$ . Since  $\mathcal{E}$  is countable, by a diagonalization method, we may further find a common subsequence (still denoted by  $n$ ) such that for any  $\mathbf{e} \in \mathcal{E}$

$$\lim_{n \rightarrow \infty} \sup_{t \in [0, T]} |G_n^{\mathbf{e}}(t) - G^{\mathbf{e}}(t)| = 0.$$

Thus, by the weak compactness of closed balls in  $\mathbb{H}^1$ , there is a  $\mathbf{u} \in L^\infty(0, T; \mathbb{H}^1)$  such that for any  $\mathbf{e} \in \mathcal{E}$

$$\lim_{n \rightarrow \infty} \sup_{t \in [0, T]} |\langle \mathbf{u}_n(t) - \mathbf{u}(t), \mathbf{e} \rangle_{\mathbb{H}^1}| = 0.$$

By a simple approximation we further have for any  $\mathbf{v} \in \mathbb{H}^1$

$$\lim_{n \rightarrow \infty} \sup_{t \in [0, T]} |\langle \mathbf{u}_n(t) - \mathbf{u}(t), \mathbf{v} \rangle_{\mathbb{H}^1}| = 0.$$

Note that  $(I - \Delta)^{-1} \mathbf{v} \in \mathbb{H}^2$  for any  $\mathbf{v} \in \mathbb{H}^0$ . Hence we also have for any  $\mathbf{v} \in \mathbb{H}^0$

$$\lim_{n \rightarrow \infty} \sup_{t \in [0, T]} |\langle \mathbf{u}_n(t) - \mathbf{u}(t), \mathbf{v} \rangle_{\mathbb{H}^0}| = 0.$$

Hence, by Helmholtz-Weyl's decomposition (cf. [28, 12])

$$\lim_{n \rightarrow \infty} \sup_{t \in [0, T]} |\langle \mathbf{u}_n(t) - \mathbf{u}(t), \mathbf{v} \rangle| = 0 \tag{2.16}$$

for any  $\mathbf{v} \in L^2(\mathbb{R}^3; \mathbb{R}^3)$ .

We now show that

$$\lim_{n \rightarrow \infty} \sup_{t \in [0, T]} \rho(\mathbf{u}_n(t), \mathbf{u}(t)) = 0.$$

It suffices to prove that for any  $m \in \mathbb{N}$

$$\lim_{n \rightarrow \infty} \sup_{t \in [0, T]} \int_{|x| \leq m} |\mathbf{u}_n(t, x) - \mathbf{u}(t, x)|^2 dx = 0,$$

which follows from (1°), (2.16) and the following Friedrichs inequality (cf. [17, p.176]): Let  $\mathcal{O} \subset \mathbb{R}^3$  be any bounded domain. For any  $\epsilon > 0$ , there exist  $N_\epsilon \in \mathbb{N}$  and functions  $h_i \in L^2(\mathcal{O}), i = 1, \dots, N_\epsilon$  such that for any  $w \in W_0^{1,2}(\mathcal{O})$

$$\int_{\mathcal{O}} |w(x)|^2 dx \leq \sum_{i=1}^{N_\epsilon} \left( \int_{\mathcal{O}} w(x) h_i(x) dx \right)^2 + \epsilon \int_{\mathcal{O}} |\nabla w(x)|^2 dx.$$

□

**Lemma 2.6.** Let  $\mu_n$  be a family of probability measures on  $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$ . Assume that

(1°) For each  $\mathbf{e} \in \mathcal{E}$  and any  $\epsilon, T > 0$

$$\limsup_{\delta \downarrow 0} \sup_n \mu_n \left\{ \mathbf{u} \in \mathbb{X} : \sup_{s, t \in [0, T], |s-t| \leq \delta} |\langle \mathbf{u}(t) - \mathbf{u}(s), \mathbf{e} \rangle_{\mathbb{H}^1}| > \epsilon \right\} = 0.$$

(2°) For any  $T > 0$

$$\limsup_{R \rightarrow \infty} \sup_n \mu_n \left\{ \mathbf{u} \in \mathbb{X} : \sup_{s \in [0, T]} \|\mathbf{u}(s)\|_{\mathbb{H}^1} > R \right\} = 0.$$

Then  $\{\mu_n, n \in \mathbb{N}\}$  is tight on  $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$ .

*Proof.* Fix  $\eta > 0$ . For any  $l \in \mathbb{N}$ , by (2°) one can choose  $R_l$  sufficiently large such that

$$\sup_n \mu_n \left\{ \mathbf{u} \in \mathbb{X} : \sup_{s \in [0, l]} \|\mathbf{u}(s)\|_{\mathbb{H}^1} > R_l \right\} \leq \frac{\eta}{2^l}. \quad (2.17)$$

For  $k, l \in \mathbb{N}$  and  $\mathbf{e}_i \in \mathcal{E}$ , by (1°) one may choose  $\delta_{k, i, l} > 0$  small enough such that

$$\sup_n \mu_n \left\{ \mathbf{u} \in \mathbb{X} : \sup_{s, t \in [0, l], |s-t| \leq \delta_{k, i, l}} |\langle \mathbf{u}(t) - \mathbf{u}(s), \mathbf{e}_i \rangle_{\mathbb{H}^1}| > \frac{1}{k} \right\} \leq \frac{\eta}{2^{k+i+l}}. \quad (2.18)$$

Now let us define

$$K_1 := \bigcap_{k, l \in \mathbb{N}, \mathbf{e}_i \in \mathcal{E}} \left\{ \mathbf{u} \in \mathbb{X} : \sup_{s, t \in [0, l], |s-t| \leq \delta_{k, i, l}} |\langle \mathbf{u}(t) - \mathbf{u}(s), \mathbf{e}_i \rangle_{\mathbb{H}^1}| \leq \frac{1}{k} \right\}$$

$$K_2 := \bigcap_{l \in \mathbb{N}} \left\{ \mathbf{u} \in \mathbb{X} : \sup_{s \in [0, l]} \|\mathbf{u}(s)\|_{\mathbb{H}^1} \leq R_l \right\}.$$

By Lemma 2.5,  $K_1 \cap K_2$  is a relatively compact set in  $\mathbb{X}$ . By (2.17) and (2.18), we also have

$$\sup_n \mu_n (K_1^c \cup K_2^c) \leq 2\eta.$$

In view of the arbitrariness of  $\eta$ ,  $\{\mu_n, n \in \mathbb{N}\}$  is tight on  $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$ .  $\square$

### 3. EXISTENCE AND UNIQUENESS OF STRONG SOLUTION

**3.1. Weak and Strong Solution.** For a metric space  $\mathbb{U}$ , we use  $\mathcal{P}(\mathbb{U})$  to denote the total of all probability measures on  $\mathbb{U}$ . We first introduce the following notion of weak solutions to Eq. (2.6).

**Definition 3.1.** We say that Eq. (2.6) has a weak solution with initial law  $\nu \in \mathcal{P}(\mathbb{H}^1)$  if there exist a stochastic basis  $(\Omega, \mathcal{F}, P; (\mathcal{F}_t)_{t \geq 0})$ , an  $\mathbb{H}^1$ -valued continuous  $(\mathcal{F}_t)$ -adapted stochastic process  $\mathbf{u}$  and an infinite sequence of independent standard  $(\mathcal{F}_t)$ -Brownian motions  $\{W^k(t), t \geq 0, k \in \mathbb{N}\}$  such that

- (i)  $\mathbf{u}(0)$  has law  $\nu$  in  $\mathbb{H}^1$ ;
- (ii) for almost all  $\omega \in \Omega$  and any  $T > 0$ ,  $\mathbf{u}(\cdot, \omega) \in L^2([0, T]; \mathbb{H}^2)$ ;
- (iii) it holds that in  $\mathbb{H}^0$

$$\mathbf{u}(t) = \mathbf{u}_0 + \int_0^t \left[ A(\mathbf{u}(s)) + \mathcal{P}\mathbf{f}(s, \mathbf{u}(s)) \right] ds + \sum_{k=1}^{\infty} \int_0^t B_k(s, \mathbf{u}(s)) dW_s^k,$$

for all  $t \geq 0$ ,  $P$ -a.s..

This solution is denoted by  $(\Omega, \mathcal{F}, P; (\mathcal{F}_t)_{t \geq 0}; W; \mathbf{u})$ .

**Remark 3.2.** Under **(H1)**-**(H3)**, by (2.7) the above integrals are meaningful.

**Definition 3.3.** (*Pathwise Uniqueness*) Given two weak solutions of Eq. (2.6) defined on the same probability space together with the same Brownian motion

$$\begin{aligned} &(\Omega, \mathcal{F}, P; (\mathcal{F}_t)_{t \geq 0}; W, \mathbf{u}) \\ &(\Omega, \mathcal{F}, P; (\mathcal{F}_t)_{t \geq 0}; W, \tilde{\mathbf{u}}), \end{aligned}$$

if  $P\{\mathbf{u}(0) = \tilde{\mathbf{u}}(0)\} = 1$ , then  $P\{\omega : \mathbf{u}(t, \omega) = \tilde{\mathbf{u}}(t, \omega), \forall t \geq 0\} = 1$ .

We have the following martingale characterization for the weak solution (cf. [27, 25]).

**Proposition 3.4.** Let  $\mathcal{E}$  be given in Subsection 2.3. For  $\nu \in \mathcal{P}(\mathbb{H}^1)$ , both of the following statements are equivalent:

- (i) Eq. (2.6) has a weak solution with initial law  $\nu$ .
- (ii) There exists a probability measure  $P_\nu \in \mathcal{P}(\mathbb{X})$  such that for  $P_\nu$ -almost all  $\mathbf{u} \in \mathbb{X}$  and any  $T > 0$

$$\mathbf{u} \in L^\infty([0, T]; \mathbb{H}^1) \cap L^2([0, T]; \mathbb{H}^2), \quad (3.1)$$

and for any  $h \in C_0^\infty(\mathbb{R})$ , i.e., any smooth function with compact support, and any  $\mathbf{e} \in \mathcal{E}$

$$\begin{aligned} M_{\mathbf{e}}^h(t, \mathbf{u}) &:= h(\langle \mathbf{u}(t), \mathbf{e} \rangle_{\mathbb{H}^1}) - h(\langle \mathbf{u}(0), \mathbf{e} \rangle_{\mathbb{H}^1}) \\ &\quad - \int_0^t h'(\langle \mathbf{u}(s), \mathbf{e} \rangle_{\mathbb{H}^1}) \cdot \llbracket A(\mathbf{u}(s)), \mathbf{e} \rrbracket ds \\ &\quad - \int_0^t h'(\langle \mathbf{u}(s), \mathbf{e} \rangle_{\mathbb{H}^1}) \cdot \langle \mathbf{f}(s, \mathbf{u}(s)), \mathbf{e} \rangle_{\mathbb{H}^1} ds \\ &\quad - \frac{1}{2} \int_0^t h''(\langle \mathbf{u}(s), \mathbf{e} \rangle_{\mathbb{H}^1}) \cdot \|\langle B(s, \mathbf{u}(s)), \mathbf{e} \rangle_{\mathbb{H}^1}\|_{l_2}^2 ds \end{aligned}$$

is a continuous local martingale under  $P_\nu$  with respect to  $\mathcal{B}_t(\mathbb{X})$ . Here and below,  $\mathcal{B}_t(\mathbb{X})$  denotes the sub  $\sigma$ -algebra of  $\mathbb{X}$  up to time  $t$ .

**Remark 3.5.** Condition (3.1) guarantees that  $P_\nu$  concentrates on  $C([0, \infty), \mathbb{H}^1)$  by Ito's formula (cf. [26, 23]). Indeed, if we define for each  $\mathbf{e}_j \in \mathcal{E}$

$$M_{\mathbf{e}_j}(t, \mathbf{u}) := \langle \mathbf{u}(t) - \mathbf{u}(0), \mathbf{e}_j \rangle_{\mathbb{H}^1} - \int_0^t \llbracket A(\mathbf{u}(s)), \mathbf{e}_j \rrbracket ds - \int_0^t \langle \mathbf{f}(s, \mathbf{u}(s)), \mathbf{e}_j \rangle_{\mathbb{H}^1} ds$$

and

$$M(t, \mathbf{u}) := \sum_{j=1}^{\infty} M_{\mathbf{e}_j}(t, \mathbf{u}) \mathbf{e}_j,$$

then we can show  $t \mapsto M(t, \mathbf{u})$  is an  $\mathbb{H}^1$ -valued continuous square integrable martingale under  $P_\nu$  with respect to  $\mathcal{B}_t(\mathbb{X})$  (cf. [13]). Thus, the following equality holds in  $\mathbb{H}^0$

$$\mathbf{u}(t) = \mathbf{u}(0) + \int_0^t A(\mathbf{u}(s)) ds + \int_0^t \mathbf{f}(s, \mathbf{u}(s)) ds + M(t, \mathbf{u}), \quad P_\nu - a.s..$$

By Itô's formula (cf. [26, 23]), one obtains  $P_\nu(C([0, \infty), \mathbb{H}^1)) = 1$ .

In order to introduce the notion of strong solutions to Eq. (2.6), we need a canonical realization of an infinite sequence of independent standard Brownian motions on a Polish space.

Let  $C(\mathbb{R}_+; \mathbb{R})$  denote the space of all continuous functions defined on  $\mathbb{R}_+$ , which is equipped with the metric

$$\tilde{\rho}(w, w') = \sum_{k=1}^{\infty} 2^{-k} \left( \sup_{t \in [0, k]} |w(t) - w'(t)| \wedge 1 \right).$$

Define the product space  $\mathbb{W} := \prod_{j=1}^{\infty} C(\mathbb{R}_+; \mathbb{R})$ , which is endowed with the metric:

$$\rho_{\mathbb{W}}(w, w') = \sum_{j=1}^{\infty} 2^{-j} (\tilde{\rho}(w^j, w'^j) \wedge 1), \quad w = (w^1, w^2, \dots), w' = (w'^1, w'^2, \dots).$$

Then  $(\mathbb{W}, \rho_{\mathbb{W}})$  is a Polish space. Let  $\mathcal{B}_t(\mathbb{W}) \subset \mathcal{B}(\mathbb{W})$  be the  $\sigma$ -algebra up to time  $t$ . We endow  $(\mathbb{W}, \mathcal{B}(\mathbb{W}))$  with the Wiener measure  $\mathbb{P}$  such that the coordinate process

$$w(t) := (w^1(t), w^2(t), \dots)$$

is an infinite sequence of independent standard  $\mathcal{B}_t(\mathbb{W})$ -Brownian motions on  $(\mathbb{W}, \mathcal{B}(\mathbb{W}), \mathbb{P})$ .

Let  $\mathbb{B} := C(\mathbb{R}_+; \mathbb{H}^1)$  denote the space of all continuous functions from  $\mathbb{R}_+$  to  $\mathbb{H}^1$ , which is endowed with the metric

$$\rho_{\mathbb{B}}(\mathbf{u}, \mathbf{v}) := \sum_{k=1}^{\infty} 2^{-k} \left( \sup_{t \in [0, k]} \|\mathbf{u}(t) - \mathbf{v}(t)\|_{\mathbb{H}^1} \wedge 1 \right).$$

In the following,  $\mathcal{B}_t(\mathbb{B})$  denotes the sub  $\sigma$ -algebra of  $\mathbb{B}$  up to time  $t$ . For a measure space  $(S, \mathcal{S}, \lambda)$ ,  $\overline{\mathcal{S}}^\lambda$  will denote the completion of  $\mathcal{S}$  with respect to  $\lambda$ .

**Definition 3.6.** Let  $(\Omega, \mathcal{F}, P; (\mathcal{F}_t)_{t \geq 0}; W, \mathbf{u})$  be a weak solution of Eq. (2.6) with initial distribution  $\nu \in \mathcal{P}(\mathbb{H}^1)$ . If there exists a  $\overline{\mathcal{B}(\mathbb{H}^1) \times \mathcal{B}(\mathbb{W})}^{\nu \times \mathbb{P}} / \mathcal{B}(\mathbb{B})$ -measurable functional  $F_\nu : \mathbb{H}^1 \times \mathbb{W} \mapsto \mathbb{B}$ , with the property that for every  $t > 0$

$$F_\nu \in \hat{\mathcal{B}}_t / \mathcal{B}_t(\mathbb{B}); \quad \hat{\mathcal{B}}_t := \overline{\mathcal{B}(\mathbb{H}^1) \times \mathcal{B}_t(\mathbb{W})}^{\nu \times \mathbb{P}}, \quad (3.2)$$

and such that

$$\mathbf{u}(\cdot) = F_\nu(\mathbf{u}(0), W(\cdot)), \quad P - a.s.,$$

we call  $\mathbf{u}$  together with  $W$  a strong solution.

We shall say that Eq. (2.6) has a unique strong solution associated with  $\nu \in \mathcal{P}(\mathbb{H}^1)$  if there exists a functional  $F_\nu : \mathbb{H}^1 \times \mathbb{W} \mapsto \mathbb{B}$  with the same properties as above such that

- (i) for any infinite sequence of independent standard  $(\mathcal{F}_t)$ -Brownian motions  $\{W(t), t \geq 0\}$  on stochastic basis  $(\Omega, \mathcal{F}, P; (\mathcal{F}_t)_{t \geq 0})$ , and any  $\mathbb{H}^1$ -valued random variable  $\mathbf{u}_0 \in \mathcal{F}_0$  with distribution  $\nu$ ,

$$(\Omega, \mathcal{F}, P; (\mathcal{F}_t)_{t \geq 0}; W, F_\nu(\mathbf{u}_0, W(\cdot))) \text{ is a weak solution of Eq. (2.6);}$$

- (ii) for any weak solution  $(\Omega, \mathcal{F}, P; (\mathcal{F}_t)_{t \geq 0}; W; \mathbf{u})$  of Eq. (2.6) with initial law  $\nu$ ,

$$\mathbf{u}(\cdot) = F_\nu(\mathbf{u}(0), W(\cdot)), \quad P - a.s..$$

The following Yamada-Watanabe theorem holds in this case (cf. [25]).

**Theorem 3.7.** Existence of weak solutions plus pathwise uniqueness implies the existence of a unique strong solution.

**3.2. Pathwise Uniqueness.** We first prove the following pathwise uniqueness result.

**Theorem 3.8.** *Under (H1)-(H3), pathwise uniqueness holds for Eq. (2.6).*

*Proof.* Let  $\mathbf{u}$  and  $\tilde{\mathbf{u}}$  be two weak solutions of Eq. (2.6) defined on the same probability space together with the same Brownian motion, and starting from the same initial value  $\mathbf{u}_0$ . For any  $T > 0$  and  $R > 0$ , define stopping time

$$\tau_R := \inf\{t \in [0, T] : \|\mathbf{u}(t)\|_{\mathbb{H}^1} \vee \|\tilde{\mathbf{u}}(t)\|_{\mathbb{H}^1} \geq R\}.$$

By the definition of weak solutions, one knows that  $\tau_R \uparrow \infty$  as  $R \uparrow \infty$ .

Set

$$\mathbf{w}(t) := \mathbf{u}(t) - \tilde{\mathbf{u}}(t).$$

Then by Ito's formula, we have

$$\begin{aligned} \|\mathbf{w}(t)\|_{\mathbb{H}^0}^2 &= 2 \int_0^t \langle A(\mathbf{u}(s)) - A(\tilde{\mathbf{u}}(s)), \mathbf{w}(s) \rangle_{\mathbb{H}^0} ds \\ &\quad + 2 \int_0^t \langle \mathbf{f}(s, \mathbf{u}(s)) - \mathbf{f}(s, \tilde{\mathbf{u}}(s)), \mathbf{w}(s) \rangle_{\mathbb{H}^0} ds \\ &\quad + 2 \sum_{k=1}^{\infty} \int_0^t \langle B_k(s, \mathbf{u}(s)) - B_k(s, \tilde{\mathbf{u}}(s)), \mathbf{w}(s) \rangle_{\mathbb{H}^0} dW_s^k \\ &\quad + \sum_{k=1}^{\infty} \int_0^t \|B_k(s, \mathbf{u}(s)) - B_k(s, \tilde{\mathbf{u}}(s))\|_{\mathbb{H}^0}^2 ds \\ &=: I_1(t) + I_2(t) + I_3(t) + I_4(t). \end{aligned} \tag{3.3}$$

By  $|g_N(r) - g_N(r')| \leq |r - r'|$  and a simple calculation, it is easy to see that

$$\begin{aligned} I_1(t) &= -2 \int_0^t \|\nabla \mathbf{w}(s)\|_{\mathbb{H}^0}^2 ds + 2 \int_0^t \langle \nabla \mathbf{w}(s), (\mathbf{u}^*(s) \cdot \mathbf{u}(s) - \tilde{\mathbf{u}}^*(s) \cdot \tilde{\mathbf{u}}(s)) \rangle_{\mathbb{H}^0} ds \\ &\quad - 2 \int_0^t \langle g_N(|\mathbf{u}(s)|^2) \mathbf{u}(s) - g_N(|\tilde{\mathbf{u}}(s)|^2) \tilde{\mathbf{u}}(s), \mathbf{w}(s) \rangle_{\mathbb{H}^0} ds \\ &\leq - \int_0^t \|\nabla \mathbf{w}(s)\|_{\mathbb{H}^0}^2 ds + \int_0^t \|\mathbf{u}^*(s) \cdot \mathbf{u}(s) - \tilde{\mathbf{u}}^*(s) \cdot \tilde{\mathbf{u}}(s)\|_{\mathbb{H}^0}^2 ds \\ &\quad + 8 \int_0^t \|\mathbf{w}(s)\| \cdot (|\mathbf{u}(s)| + |\tilde{\mathbf{u}}(s)|) \| \mathbf{w}(s) \|_{\mathbb{H}^0}^2 ds. \end{aligned}$$

Noting that by the Sobolev inequality (2.1)

$$\begin{aligned} &\|\mathbf{u}^*(s) \cdot \mathbf{u}(s) - \tilde{\mathbf{u}}^*(s) \cdot \tilde{\mathbf{u}}(s)\|_{\mathbb{H}^0}^2 \\ &\leq \|\mathbf{w}(s)\| (|\mathbf{u}(s)| + |\tilde{\mathbf{u}}(s)|) \| \mathbf{w}(s) \|_{\mathbb{H}^0} \\ &\leq 2 \|\mathbf{w}(s)\|_{L^4}^2 (\|\mathbf{u}(s)\|_{L^4}^2 + \|\tilde{\mathbf{u}}(s)\|_{L^4}^2) \end{aligned} \tag{3.4}$$

$$\leq 2C_{1,4}^2 \cdot \|\mathbf{w}(s)\|_{\mathbb{H}^1}^{3/2} \|\mathbf{w}(s)\|_{\mathbb{H}^0}^{1/2} (\|\mathbf{u}(s)\|_{\mathbb{H}^1}^2 + \|\tilde{\mathbf{u}}(s)\|_{\mathbb{H}^1}^2), \tag{3.5}$$

we have by Young's inequality

$$\begin{aligned} I_1(t \wedge \tau_R) &\leq - \int_0^{t \wedge \tau_R} \|\nabla \mathbf{w}(s)\|_{\mathbb{H}^0}^2 ds + C_R \int_0^{t \wedge \tau_R} \|\mathbf{w}(s)\|_{\mathbb{H}^1}^{3/2} \|\mathbf{w}(s)\|_{\mathbb{H}^0}^{1/2} ds \\ &\leq - \frac{1}{2} \int_0^{t \wedge \tau_R} \|\nabla \mathbf{w}(s)\|_{\mathbb{H}^0}^2 ds + C_R \int_0^{t \wedge \tau_R} \|\mathbf{w}(s)\|_{\mathbb{H}^0}^2 ds. \end{aligned}$$

Moreover, it is clear that

$$I_2(t \wedge \tau_R) \leq C_T \int_0^{t \wedge \tau_R} \|\mathbf{w}(s)\|_{\mathbb{H}^0}^2 ds$$

and

$$I_4(t \wedge \tau_R) \leq \sup_{t \geq 0, x \in \mathbb{D}} \|\sigma(t, x)\|_{l^2} \cdot \int_0^{t \wedge \tau_R} \|\nabla \mathbf{w}(s)\|_{\mathbb{H}^0}^2 ds + C_T \int_0^{t \wedge \tau_R} \|\mathbf{w}(s)\|_{\mathbb{H}^0}^2 ds$$

Taking expectations for (3.3) and combining the above calculations as well as (2.3) give that for any  $t \in [0, T]$

$$\begin{aligned} \mathbb{E}\|\mathbf{w}(t \wedge \tau_R)\|_{\mathbb{H}^0}^2 &\leq C_{R,T} \cdot \mathbb{E} \left( \int_0^{t \wedge \tau_R} \|\mathbf{w}(s)\|_{\mathbb{H}^0}^2 ds \right) \\ &\leq C_{R,T} \int_0^t \mathbb{E}\|\mathbf{w}(s \wedge \tau_R)\|_{\mathbb{H}^0}^2 ds. \end{aligned}$$

By Gronwall's inequality, we get for any  $t \in [0, T]$

$$\mathbb{E}\|\mathbf{w}(t \wedge \tau_R)\|_{\mathbb{H}^0}^2 = 0.$$

Now the uniqueness follows by letting  $R \uparrow \infty$  and Fatou's lemma.  $\square$

**3.3. Existence of Martingale Solution.** We now prove the existence of a weak solution to Eq. (2.6).

**Theorem 3.9.** *Under (H1)-(H3), for any initial law  $\nu \in \mathcal{P}(\mathbb{H}^1)$ , there exists a weak solution  $\mathbf{u}$  to Eq. (2.6) in the sense of Definition 3.1.*

We shall use Galerkin's approximation to prove this theorem. In the following, we fix a stochastic basis  $(\Omega, \mathcal{F}, P; (\mathcal{F}_t)_{t \geq 0})$ , and an infinite sequence of independent standard  $(\mathcal{F}_t)$ -Brownian motions  $\{W^k(t), t \geq 0, k \in \mathbb{N}\}$ , as well as an  $\mathcal{F}_0$ -measurable random variable  $\mathbf{u}_0$  having law  $\nu$ .

Recall that  $\mathcal{E} = \{\mathbf{e}_k, k \in \mathbb{N}\} \subset \mathcal{V}$  is a complete orthonormal basis of  $\mathbb{H}^1$ . Fix  $n \in \mathbb{N}$ . For  $\mathbf{y} = (y^1, \dots, y^n) \in \mathbb{R}^n$ , set

$$\begin{aligned} \mathbf{y} \cdot \mathbf{e} &:= \sum_{i=1}^n y^i \mathbf{e}_i \in \mathcal{V}, \\ A_n(\mathbf{y}) &:= ([A(\mathbf{y} \cdot \mathbf{e}), \mathbf{e}_1], \dots, [A(\mathbf{y} \cdot \mathbf{e}), \mathbf{e}_n]), \\ \mathbf{f}_n(t, \mathbf{y}) &:= (\langle \mathbf{f}(t, \mathbf{y} \cdot \mathbf{e}), \mathbf{e}_1 \rangle_{\mathbb{H}^1}, \dots, \langle \mathbf{f}(t, \mathbf{y} \cdot \mathbf{e}), \mathbf{e}_n \rangle_{\mathbb{H}^1}), \\ B_k^n(t, \mathbf{y}) &:= (\langle B_k(t, \mathbf{y} \cdot \mathbf{e}), \mathbf{e}_1 \rangle_{\mathbb{H}^1}, \dots, \langle B_k(t, \mathbf{y} \cdot \mathbf{e}), \mathbf{e}_n \rangle_{\mathbb{H}^1}). \end{aligned}$$

Consider the following stochastic ordinary differential equation

$$\begin{cases} d\mathbf{y}_n(t) &= [A_n(\mathbf{y}_n(t)) + \mathbf{f}_n(t, \mathbf{y}_n(t))]dt + \sum_k B_k^n(t, \mathbf{y}_n(t))dW_t^k, \\ \mathbf{y}_n(0) &= (\langle \mathbf{u}_0, \mathbf{e}_1 \rangle_{\mathbb{H}^1}, \dots, \langle \mathbf{u}_0, \mathbf{e}_n \rangle_{\mathbb{H}^1}). \end{cases}$$

By Lemmas 2.2 and 2.4, we have for some  $C_{n,N} > 0$

$$\begin{aligned} \langle \mathbf{y}, A_n(\mathbf{y}) + \mathbf{f}_n(t, \mathbf{y}) \rangle_{\mathbb{R}^n} &\leq C_{n,N}(|\mathbf{y}|^2 + 1) \\ \|B^n(t, \mathbf{y})\|_{l^2 \otimes \mathbb{R}^n} &\leq C_{n,N}(|\mathbf{y}|^2 + 1). \end{aligned}$$

Moreover, by (H1)-(H3) it is easy to see that

$$\mathbb{R}^n \ni \mathbf{y} \mapsto A_n(\mathbf{y}) + \mathbf{f}_n(t, \mathbf{y}) \in \mathbb{R}^n$$

and

$$\mathbb{R}^n \ni \mathbf{y} \mapsto B^n(t, \mathbf{y}) \in l^2 \times \mathbb{R}^n$$

are locally Lipschitz continuous. Hence, by the theory of SDE(cf. [16, 23]), there is a unique continuous adapted process  $\mathbf{y}_n(t)$  satisfying

$$\mathbf{y}_n(t) = \mathbf{y}_n(0) + \int_0^t A_n(\mathbf{y}_n(s))ds + \int_0^t \mathbf{f}_n(s, \mathbf{y}_n(s))ds + \sum_{k=1}^{\infty} \int_0^t B_k^n(s, \mathbf{y}_n(s))dW_s^k.$$

Set

$$\begin{aligned} \mathbf{u}_n(t) &:= \mathbf{y}_n(t) \cdot \mathbf{e} = \sum_{i=1}^n y_n^i(t) \mathbf{e}_i, \\ \Pi_n A(\mathbf{u}_n(t)) &:= \sum_{i=1}^n \llbracket A(\mathbf{u}_n(t)), \mathbf{e}_i \rrbracket \mathbf{e}_i, \\ \Pi_n \mathbf{f}(t, \mathbf{u}_n(t)) &:= \sum_{i=1}^n \langle \mathbf{f}(t, \mathbf{y}_n(t)), \mathbf{e}_i \rangle_{\mathbb{H}^1} \mathbf{e}_i, \\ \Pi_n B_k(t, \mathbf{u}_n(t)) &:= \sum_{i=1}^n \langle B_k(t, \mathbf{y}_n(t)), \mathbf{e}_i \rangle_{\mathbb{H}^1} \mathbf{e}_i. \end{aligned}$$

Then

$$\begin{aligned} \mathbf{u}_n(t) &= \mathbf{u}_n(0) + \int_0^t \Pi_n A(\mathbf{u}_n(s))ds + \int_0^t \Pi_n \mathbf{f}(s, \mathbf{u}_n(s))ds \\ &\quad + \sum_{k=1}^{\infty} \int_0^t \Pi_n B_k(s, \mathbf{u}_n(s))dW_s^k, \end{aligned} \tag{3.6}$$

and for any  $n \geq i$

$$\begin{aligned} \langle \mathbf{u}_n(t), \mathbf{e}_i \rangle_{\mathbb{H}^1} &= \langle \mathbf{u}_0, \mathbf{e}_i \rangle_{\mathbb{H}^1} + \int_0^t \llbracket A(\mathbf{u}_n(s)), \mathbf{e}_i \rrbracket ds + \int_0^t \langle \mathbf{f}(s, \mathbf{u}_n(s)), \mathbf{e}_i \rangle_{\mathbb{H}^1} ds \\ &\quad + \sum_{k=1}^{\infty} \int_0^t \langle B_k(s, \mathbf{u}_n(s)), \mathbf{e}_i \rangle_{\mathbb{H}^1} dW_s^k. \end{aligned} \tag{3.7}$$

We now prove a series of lemmas.

**Lemma 3.10.** *For any  $T > 0$ , there exists a positive constant  $C_{T,N} > 0$  such that for any  $n \in \mathbb{N}$*

$$\mathbb{E} \left( \sup_{t \in [0, T]} \|\mathbf{u}_n(t)\|_{\mathbb{H}^1}^2 \right) + \int_0^T \mathbb{E} \|\mathbf{u}_n(s)\|_{\mathbb{H}^2}^2 ds + \int_0^T \mathbb{E} \|\nabla |\mathbf{u}_n(s)|^2\|_{L^2}^2 ds \leq C_{T,N}, \tag{3.8}$$

and also in the periodic case

$$\int_0^T \mathbb{E} \|\mathbf{u}_n(s)\|_{L^4}^4 ds \leq C_{T,N}. \tag{3.9}$$

*Proof.* We have by Ito's formula and Lemmas 2.2 and 2.4

$$\|\mathbf{u}_n(t)\|_{\mathbb{H}^1}^2 = \|\mathbf{u}_0\|_{\mathbb{H}^1}^2 + 2 \int_0^t \llbracket A(\mathbf{u}_n(s)), \mathbf{u}_n(s) \rrbracket ds + 2 \int_0^t \langle \mathbf{f}(s, \mathbf{u}_n(s)), \mathbf{u}_n(s) \rangle_{\mathbb{H}^1} ds$$

$$\begin{aligned}
& +M(t) + \int_0^t \|B(s, \mathbf{u}_n(s))\|_{L_2(l^2; \mathbb{H}^1)}^2 ds \\
\leq & \|\mathbf{u}_0\|_{\mathbb{H}^1}^2 - \int_0^t \|\mathbf{u}_n(s)\|_{\mathbb{H}^2}^2 ds - \int_0^t \|\mathbf{u}_n(s)| \cdot |\nabla \mathbf{u}_n(s)\|_{L^2}^2 ds \\
& + 2(1+N) \int_0^t \|\nabla \mathbf{u}_n(s)\|_{\mathbb{H}^0}^2 ds + 2 \int_0^t \|\mathbf{u}_n(s)\|_{\mathbb{H}^0}^2 ds \\
& + 2 \int_0^t \|\mathbf{f}(s, \mathbf{u}_n(s))\|_{\mathbb{H}^0} \cdot \|\mathbf{u}_n(s)\|_{\mathbb{H}^2} ds + M(t) \\
& + \int_0^t \left( \frac{1}{2} \|\mathbf{u}_n(s)\|_{\mathbb{H}^2}^2 + C_T \|\mathbf{u}_n(s)\|_{\mathbb{H}^1}^2 + C \|H_{\mathbf{h}}(s)\|_{L^1(\mathbb{D})} \right) ds,
\end{aligned} \tag{3.10}$$

where  $M(t)$  is a continuous martingale defined by

$$M(t) := 2 \sum_{k=1}^{\infty} \int_0^t \langle B_k(s, \mathbf{u}_n(s)), \mathbf{u}_n(s) \rangle_{\mathbb{H}^1} dW_s^k.$$

Taking expectations and by Young's inequality, one finds that for any  $t \in [0, T]$

$$\begin{aligned}
\mathbb{E} \|\mathbf{u}_n(t)\|_{\mathbb{H}^1}^2 & \leq \mathbb{E} \|\mathbf{u}_0\|_{\mathbb{H}^1}^2 - \frac{1}{4} \int_0^t \mathbb{E} \|\mathbf{u}_n(s)\|_{\mathbb{H}^2}^2 ds - \int_0^t \mathbb{E} \|\mathbf{u}_n(s)| \cdot |\nabla \mathbf{u}_n(s)\|_{L^2}^2 ds \\
& + 2(1+N) \int_0^t \mathbb{E} \|\nabla \mathbf{u}_n(s)\|_{\mathbb{H}^0}^2 ds + C_T \int_0^t \mathbb{E} \|\mathbf{u}_n(s)\|_{\mathbb{H}^0}^2 ds \\
& + C_T \int_0^t \left( \|H_{\mathbf{f}}(s)\|_{L^1(\mathbb{D})} + \|H_{\mathbf{h}}(s)\|_{L^1(\mathbb{D})} \right) ds.
\end{aligned}$$

Hence, by Gronwall's inequality we have for any  $T > 0$

$$\sup_{t \in [0, T]} \mathbb{E} \|\mathbf{u}_n(t)\|_{\mathbb{H}^1}^2 + \int_0^T \mathbb{E} \|\mathbf{u}_n(s)\|_{\mathbb{H}^2}^2 ds + \int_0^T \mathbb{E} \|\nabla |\mathbf{u}_n(s)|^2\|_{L^2}^2 ds \leq C_{T, N}. \tag{3.11}$$

Here, the constant  $C_{T, N}$  is independent of  $n$ , and we have used that  $|\nabla |\mathbf{u}|^2| \leq C |\mathbf{u}| \cdot |\nabla \mathbf{u}|$ .

Furthermore, from (3.10) and using Burkholder's inequality, Young's inequality, Lemma 2.4 and (3.11), we have for any  $T > 0$  and  $\epsilon > 0$

$$\begin{aligned}
\mathbb{E} \left( \sup_{t \in [0, T]} \|\mathbf{u}_n(t)\|_{\mathbb{H}^1}^2 \right) & \leq C_{T, N} + C \mathbb{E} \left( \int_0^T \|B(s, \mathbf{u}_n(s))\|_{L_2(l^2; \mathbb{H}^0)}^2 \cdot \|\mathbf{u}_n(s)\|_{\mathbb{H}^2}^2 ds \right)^{1/2} \\
& \leq C_{T, N} + \epsilon \cdot \mathbb{E} \left( \sup_{t \in [0, T]} \|B(s, \mathbf{u}_n(s))\|_{L_2(l^2; \mathbb{H}^0)}^2 \right) \\
& \quad + C_{\epsilon} \int_0^T \mathbb{E} \|\mathbf{u}_n(s)\|_{\mathbb{H}^2}^2 ds \\
& \leq C_{T, N, \epsilon} + \epsilon \cdot C_T \mathbb{E} \left( \sup_{t \in [0, T]} \|\mathbf{u}_n(t)\|_{\mathbb{H}^1}^2 \right).
\end{aligned}$$

Choosing  $\epsilon$  small enough, we get

$$\mathbb{E} \left( \sup_{t \in [0, T]} \|\mathbf{u}_n(t)\|_{\mathbb{H}^1}^2 \right) \leq C_{T, N}.$$

In the periodic case, since  $\mathcal{E}$  is also orthogonal in  $\mathbb{H}^0$ , we have by (2.9) and **(H1)**-**(H3)**

$$\begin{aligned}
\mathbb{E}\|\mathbf{u}_n(t)\|_{\mathbb{H}^0}^2 &= \mathbb{E}\|\mathbf{u}_0\|_{\mathbb{H}^0}^2 + 2 \int_0^t \mathbb{E}\langle A(\mathbf{u}_n(s)), \mathbf{u}_n(s) \rangle_{\mathbb{H}^0} ds \\
&\quad + 2 \int_0^t \mathbb{E}\langle \mathbf{f}(s, \mathbf{u}_n(s)), \mathbf{u}_n(s) \rangle_{\mathbb{H}^0} ds + \int_0^t \mathbb{E}\|B(s, \mathbf{u}_n(s))\|_{L_2(l^2; \mathbb{H}^0)}^2 ds \\
&\leq \mathbb{E}\|\mathbf{u}_0\|_{\mathbb{H}^1}^2 - 2 \int_0^t \mathbb{E}\|\nabla \mathbf{u}_n(s)\|_{\mathbb{H}^0}^2 ds - 2 \int_0^t \mathbb{E}\|\mathbf{u}_n(s)\|_{L^4}^4 ds \\
&\quad + C_N \int_0^t \mathbb{E}\|\mathbf{u}_n(s)\|_{\mathbb{H}^0}^2 ds + C_T,
\end{aligned}$$

which yields (3.9) by Gronwall's lemma.  $\square$

**Lemma 3.11.** *Let  $\mu_n$  be the law of  $\mathbf{u}_n$  in  $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$ . Then the family of probability measures  $\{\mu_n, n \in \mathbb{N}\}$  is tight on  $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$ .*

*Proof.* Set for  $R > 0$

$$\tau_R^n := \inf\{t \geq 0 : \|\mathbf{u}_n(t)\|_{\mathbb{H}^1} \geq R\}.$$

Then, by (3.8) we have for any  $T > 0$

$$\sup_n P(\tau_R^n < T) = \sup_n P\left(\sup_{t \in [0, T]} \|\mathbf{u}_n(t)\|_{\mathbb{H}^1} > R\right) \leq \frac{C_{T, N}}{R^2}. \quad (3.12)$$

On the other hand, from (3.7) and using (2.11), Lemma 2.4 and Burkholder's inequality, we have for any  $q \geq 2$  and  $s, t \in [0, T]$

$$\begin{aligned}
&\mathbb{E}|\langle \mathbf{u}_n(t \wedge \tau_R^n) - \mathbf{u}_n(s \wedge \tau_R^n), \mathbf{e}_i \rangle_{\mathbb{H}^1}|^q \\
&\leq C \mathbb{E} \left| \int_{s \wedge \tau_R^n}^{t \wedge \tau_R^n} \llbracket A(\mathbf{u}_n(s)), \mathbf{e}_i \rrbracket ds \right|^q + C \mathbb{E} \left| \int_{s \wedge \tau_R^n}^{t \wedge \tau_R^n} \langle \mathbf{f}(s, \mathbf{u}_n(s)), \mathbf{e}_i \rangle_{\mathbb{H}^1} ds \right|^q \\
&\quad + C \mathbb{E} \left| \int_{s \wedge \tau_R^n}^{t \wedge \tau_R^n} \langle B_k(s, \mathbf{u}_n(s)), \mathbf{e}_i \rangle_{\mathbb{H}^1} dW_s^k \right|^q \\
&\leq C_{\mathbf{e}_i} \cdot \mathbb{E} \left| \int_{s \wedge \tau_R^n}^{t \wedge \tau_R^n} (1 + \|\mathbf{u}_n(s)\|_{\mathbb{H}^1}^3) ds \right|^q + C_{\mathbf{e}_i} \cdot \mathbb{E} \left| \int_{s \wedge \tau_R^n}^{t \wedge \tau_R^n} \|\mathbf{f}(s, \mathbf{u}_n(s))\|_{\mathbb{H}^0} ds \right|^q \\
&\quad + C_{\mathbf{e}_i} \cdot \mathbb{E} \left| \int_{s \wedge \tau_R^n}^{t \wedge \tau_R^n} \|B(s, \mathbf{u}_n(s))\|_{L_2(l^2; \mathbb{H}^0)}^2 ds \right|^{q/2} \leq C_{\mathbf{e}_i, R, T} \cdot |t - s|^{q/2}.
\end{aligned}$$

By Kolomogorov's criterion(cf. [15]), for any  $T > 0$  and  $0 < \alpha < \frac{1}{2}$  we get

$$\mathbb{E} \left( \sup_{s, t \in [0, T], |t-s| \leq \delta} |\langle \mathbf{u}_n(t \wedge \tau_R^n) - \mathbf{u}_n(s \wedge \tau_R^n), \mathbf{e}_i \rangle_{\mathbb{H}^1}| \right) \leq C_{\mathbf{e}_i, R, T} \cdot \delta^\alpha.$$

So, for any  $\epsilon > 0$  and  $R > 0$

$$\begin{aligned}
&\sup_n P \left\{ \sup_{s, t \in [0, T], |t-s| \leq \delta} |\langle \mathbf{u}_n(t) - \mathbf{u}_n(s), \mathbf{e}_i \rangle_{\mathbb{H}^1}| > \epsilon \right\} \\
&\leq \sup_n P \left\{ \sup_{s, t \in [0, T], |t-s| \leq \delta} |\langle \mathbf{u}_n(t) - \mathbf{u}_n(s), \mathbf{e}_i \rangle_{\mathbb{H}^1}| > \epsilon; \tau_R^n \geq T \right\}
\end{aligned}$$

$$+ \sup_n P \{ \tau_R^n < T \} \leq \frac{C_{\mathbf{e}_i, R, T} \cdot \delta}{\epsilon} + \frac{C_{T, N}}{R^2},$$

which then gives that

$$\limsup_{\delta \downarrow 0} \sup_n P \left\{ \sup_{s, t \in [0, T], |t-s| \leq \delta} |\langle \mathbf{u}_n(t) - \mathbf{u}_n(s), \mathbf{e}_i \rangle_{\mathbb{H}^1}| > \epsilon \right\} = 0. \quad (3.13)$$

The tightness of  $\{\mu_n, n \in \mathbb{N}\}$  now follows from (3.12), (3.13) and Lemma 2.6.  $\square$

In the sequel, without loss of generality, we assume that  $\mu_n$  weakly converges to  $\mu \in \mathcal{P}(\mathbb{X})$ . By Skorohod's embedding theorem (cf. [15]), there exist a probability space  $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$  and  $\mathbb{X}$ -valued random variables  $\tilde{\mathbf{u}}^n$  and  $\tilde{\mathbf{u}}$  such that

(I)  $\tilde{\mathbf{u}}^n$  has the same law as  $\mathbf{u}^n$  in  $\mathbb{X}$  for each  $n \in \mathbb{N}$ ;

(II)  $\tilde{\mathbf{u}}^n \rightarrow \tilde{\mathbf{u}}$  in  $\mathbb{X}$ ,  $\tilde{P}$ -a.e., and  $\tilde{\mathbf{u}}$  has law  $\mu$ .

Moreover, by (3.8) and Fatou's lemma, we have for any  $T > 0$

$$\mathbb{E}^\mu \left( \sup_{t \in [0, T]} \|\mathbf{u}(t)\|_{\mathbb{H}^1}^2 \right) = \mathbb{E}^{\tilde{P}} \left( \sup_{t \in [0, T]} \|\tilde{\mathbf{u}}(t)\|_{\mathbb{H}^1}^2 \right) < +\infty \quad (3.14)$$

$$\int_0^T \mathbb{E}^\mu \|\mathbf{u}(s)\|_{\mathbb{H}^2}^2 ds = \int_0^T \mathbb{E}^{\tilde{P}} \|\tilde{\mathbf{u}}(s)\|_{\mathbb{H}^2}^2 ds < +\infty \quad (3.15)$$

$$\int_0^T \mathbb{E}^\mu \|\nabla |\mathbf{u}(s)|\|_{L^2}^2 ds = \int_0^T \mathbb{E}^{\tilde{P}} \|\nabla |\tilde{\mathbf{u}}(s)|\|_{L^2}^2 ds < +\infty, \quad (3.16)$$

and also in the periodic case

$$\int_0^T \mathbb{E}^\mu \|\mathbf{u}(s)\|_{L^4}^4 ds = \int_0^T \mathbb{E}^{\tilde{P}} \|\tilde{\mathbf{u}}(s)\|_{L^4}^4 ds < +\infty. \quad (3.17)$$

Let  $h \in C_0^\infty(\mathbb{R})$  and  $\mathbf{e}_i \in \mathcal{E}$ . Define for any  $t \geq 0$  and  $\mathbf{u} \in \mathbb{X}$

$$M_{\mathbf{e}_i}^h(t, \mathbf{u}) := I_1^h(t, \mathbf{u}) - I_2^h(t, \mathbf{u}) - I_3^h(t, \mathbf{u}) - I_4^h(t, \mathbf{u}) - I_5^h(t, \mathbf{u})$$

where

$$I_1^h(t, \mathbf{u}) := h(\langle \mathbf{u}(t), \mathbf{e}_i \rangle_{\mathbb{H}^1})$$

$$I_2^h(t, \mathbf{u}) := h(\langle \mathbf{u}(0), \mathbf{e}_i \rangle_{\mathbb{H}^1})$$

$$I_3^h(t, \mathbf{u}) := \int_0^t h'(\langle \mathbf{u}(s), \mathbf{e}_i \rangle_{\mathbb{H}^1}) \cdot \llbracket A(\mathbf{u}(s)), \mathbf{e}_i \rrbracket ds$$

$$I_4^h(t, \mathbf{u}) := \int_0^t h'(\langle \mathbf{u}(s), \mathbf{e}_i \rangle_{\mathbb{H}^1}) \cdot \langle \mathbf{f}(s, \mathbf{u}(s)), \mathbf{e}_i \rangle_{\mathbb{H}^1} ds$$

$$I_5^h(t, \mathbf{u}) := \frac{1}{2} \int_0^t h''(\langle \mathbf{u}(s), \mathbf{e}_i \rangle_{\mathbb{H}^1}) \cdot \|\langle B(s, \mathbf{u}(s)), \mathbf{e}_i \rangle_{\mathbb{H}^1}\|_{l^2}^2 ds.$$

**Lemma 3.12.** *We have*

$$\sup_n \mathbb{E}^{\tilde{P}} |M_{\mathbf{e}_i}^h(t, \tilde{\mathbf{u}}_n)|^{4/3} + \mathbb{E}^{\tilde{P}} |M_{\mathbf{e}_i}^h(t, \tilde{\mathbf{u}})|^{4/3} < +\infty. \quad (3.18)$$

*Proof.* It is clear that  $I_1^h(t, \tilde{\mathbf{u}}_n)$  and  $I_2^h(t, \tilde{\mathbf{u}}_n)$  are bounded by some constant  $C_h$ .

Let the support of  $\mathbf{e}_i$  be contained in  $\mathcal{O} := \{x \in \mathbb{R}^3, |x| \leq m\}$  for some  $m \in \mathbb{N}$ . For  $I_3^h$ , noting that in the whole space case,

$$\|\mathbf{u}\|_{L^6} \leq C \|\nabla \mathbf{u}\|_{L^2},$$

by (2.11) and (3.8) we have

$$\begin{aligned}
\mathbb{E}^{\tilde{P}} |I_3^h(t, \tilde{\mathbf{u}}_n)|^{4/3} &\leq C_h \int_0^T \mathbb{E}^{\tilde{P}} |[[A(\tilde{\mathbf{u}}_n(s)), \mathbf{e}_i]]|^{4/3} ds \\
&\leq C_{h, \mathbf{e}_i} \int_0^T \mathbb{E}^{\tilde{P}} \left(1 + \|\tilde{\mathbf{u}}_n(s)\|_{L^3(\mathcal{O})}^4\right) ds \\
&\leq C_{h, \mathbf{e}_i} \int_0^T \mathbb{E}^{\tilde{P}} \left(1 + \|\tilde{\mathbf{u}}_n(s)\|_{L^{12}(\mathcal{O})}^4\right) ds \\
&= C_{h, \mathbf{e}_i} \int_0^T \mathbb{E}^{\tilde{P}} \left(1 + \|\tilde{\mathbf{u}}_n(s)\|_{L^6}^2\right) ds \\
&\leq C_{h, \mathbf{e}_i} \int_0^T \mathbb{E}^{\tilde{P}} \left(1 + \|\nabla |\tilde{\mathbf{u}}_n(s)|\|_{L^2}^2\right) ds \\
&\leq C_{h, \mathbf{e}_i, T, N}.
\end{aligned}$$

In the periodic case, by (2.11) and (3.9) we have

$$\begin{aligned}
\mathbb{E}^{\tilde{P}} |I_3^h(t, \tilde{\mathbf{u}}_n)|^{4/3} &\leq C_h \int_0^T \mathbb{E}^{\tilde{P}} \left(1 + \|\tilde{\mathbf{u}}_n(s)\|_{L^3(\mathbb{T}^3)}^4\right) ds \\
&\leq C_{h, \mathbf{e}_i} \int_0^T \mathbb{E}^{\tilde{P}} \left(1 + \|\tilde{\mathbf{u}}_n(s)\|_{L^4(\mathbb{T}^3)}^4\right) ds \\
&\leq C_{h, \mathbf{e}_i, T, N}.
\end{aligned}$$

For  $I_5^h$ , by (2.12) we similarly have

$$\mathbb{E}^{\tilde{P}} |I_5^h(t, \tilde{\mathbf{u}}_n)|^2 \leq C_{h, \mathbf{e}_i, T, N}.$$

For  $I_4^h$ , it is clear that

$$\mathbb{E}^{\tilde{P}} |I_4^h(t, \tilde{\mathbf{u}}_n)|^2 \leq C_{h, \mathbf{e}_i, T, N}.$$

Moreover, by (3.14)-(3.17) we also have

$$\mathbb{E}^{\tilde{P}} |M_{\mathbf{e}_i}^h(t, \tilde{\mathbf{u}})|^{4/3} < +\infty.$$

The proof is thus complete.  $\square$

**Lemma 3.13.** *For any  $t > 0$  and  $\epsilon > 0$ ,*

$$\lim_{n \rightarrow \infty} \tilde{P}(|M_{\mathbf{e}_i}^h(t, \tilde{\mathbf{u}}_n) - M_{\mathbf{e}_i}^h(t, \tilde{\mathbf{u}})| > \epsilon) = 0. \quad (3.19)$$

*That is,  $M_{\mathbf{e}_i}^h(t, \tilde{\mathbf{u}}_n)$  converges to  $M_{\mathbf{e}_i}^h(t, \tilde{\mathbf{u}})$  in probability  $\tilde{P}$  as  $n \rightarrow \infty$ .*

*Proof.* Let the support of  $\mathbf{e}_i$  be contained in  $\mathcal{O} := \{x \in \mathbb{R}^3, |x| \leq m\}$  for some  $m \in \mathbb{N}$ . By (II) and the dominated convergence theorem

$$\begin{aligned}
\lim_{n \rightarrow \infty} \mathbb{E}^{\tilde{P}} |I_1^h(t, \tilde{\mathbf{u}}_n) - I_1^h(t, \tilde{\mathbf{u}})| &= 0 \\
\lim_{n \rightarrow \infty} \mathbb{E}^{\tilde{P}} |I_2^h(t, \tilde{\mathbf{u}}_n) - I_2^h(t, \tilde{\mathbf{u}})| &= 0.
\end{aligned}$$

For  $I_3^h$ , define for any  $R > 0$

$$\tilde{\tau}_R^n := \inf\{t \geq 0 : \|\tilde{\mathbf{u}}_n(t)\|_{\mathbb{H}^1} \geq R\}.$$

Then, by (I) and (3.8), for any  $T > 0$  we have

$$\sup_n \tilde{P}(\tilde{\tau}_R^n \leq T) \leq \frac{C_{T,N}}{R^2}.$$

Thus, by the dominated convergence theorem and Lemma 2.3, we have from the proof of Lemma 3.12

$$\begin{aligned} & \lim_{n \rightarrow \infty} \tilde{P}(|I_3^h(t, \tilde{\mathbf{u}}_n) - I_3^h(t, \tilde{\mathbf{u}})| > \epsilon) \\ & \leq \lim_{R \rightarrow \infty} \lim_{n \rightarrow \infty} \tilde{P}(|I_3^h(t, \tilde{\mathbf{u}}_n) - I_3^h(t, \tilde{\mathbf{u}})| > \epsilon; \tilde{\tau}_R^n > t) + \lim_{R \rightarrow \infty} \sup_n \tilde{P}(\tilde{\tau}_R^n \leq T) \\ & \leq \lim_{R \rightarrow \infty} \lim_{n \rightarrow \infty} \mathbb{E}^{\tilde{P}}(1_{\{\tilde{\tau}_R^n > t\}} \cdot |I_3^h(t, \tilde{\mathbf{u}}_n) - I_3^h(t, \tilde{\mathbf{u}})|) / \epsilon \\ & \leq \lim_{R \rightarrow \infty} \mathbb{E}^{\tilde{P}} \left( \int_0^t \overline{\lim}_{n \rightarrow \infty} \left( 1_{\{\tilde{\tau}_R^n > t\}} \cdot \left| h'(\langle \tilde{\mathbf{u}}_n(s), \mathbf{e}_i \rangle_{\mathbb{H}^1}) \cdot \llbracket A(\tilde{\mathbf{u}}_n(s)), \mathbf{e}_i \rrbracket \right. \right. \right. \\ & \quad \left. \left. \left. - h'(\langle \tilde{\mathbf{u}}(s), \mathbf{e}_i \rangle_{\mathbb{H}^1}) \cdot \llbracket A(\tilde{\mathbf{u}}(s)), \mathbf{e}_i \rrbracket \right| \right) ds \right) / \epsilon = 0. \end{aligned}$$

Similarly, we also have

$$\begin{aligned} \lim_{n \rightarrow \infty} \tilde{P}(|I_4^h(t, \tilde{\mathbf{u}}_n) - I_4^h(t, \tilde{\mathbf{u}})| > \epsilon) &= 0 \\ \lim_{n \rightarrow \infty} \tilde{P}(|I_5^h(t, \tilde{\mathbf{u}}_n) - I_5^h(t, \tilde{\mathbf{u}})| > \epsilon) &= 0. \end{aligned}$$

Combining the above calculations yields (3.19).  $\square$

We can now give the proof of Theorem 3.9.

*Proof of Theorem 3.9:* Now let  $t > s$  and  $G$  be any bounded and real valued  $\mathcal{B}_s(\mathbb{X})$ -measurable continuous function on  $\mathbb{X}$ . Then by (3.18) and (3.19), we have

$$\begin{aligned} & \mathbb{E}^\mu \left( (M_{\mathbf{e}_i}^h(t, \mathbf{u}) - M_{\mathbf{e}_i}^h(s, \mathbf{u})) \cdot G(\mathbf{u}) \right) \\ &= \mathbb{E}^{\tilde{P}} \left( (M_{\mathbf{e}_i}^h(t, \tilde{\mathbf{u}}) - M_{\mathbf{e}_i}^h(s, \tilde{\mathbf{u}})) \cdot G(\tilde{\mathbf{u}}) \right) \\ &= \lim_{n \rightarrow \infty} \mathbb{E}^{\tilde{P}} \left( (M_{\mathbf{e}_i}^h(t, \tilde{\mathbf{u}}_n) - M_{\mathbf{e}_i}^h(s, \tilde{\mathbf{u}}_n)) \cdot G(\tilde{\mathbf{u}}_n) \right) \\ &= \lim_{n \rightarrow \infty} \mathbb{E}^P \left( (M_{\mathbf{e}_i}^h(t, \mathbf{u}_n) - M_{\mathbf{e}_i}^h(s, \mathbf{u}_n)) \cdot G(\mathbf{u}_n) \right) = 0, \end{aligned}$$

where the last step is due to the martingale property of  $M_{\mathbf{e}_i}^h(t, \mathbf{u}_n)$  on  $(\Omega, \mathcal{F}, P; (\mathcal{F}_t)_{t \geq 0})$  and  $G(\mathbf{u}_n) \in \mathcal{F}_s$ . This means that  $\{M_{\mathbf{e}_i}^h(t, \mathbf{u}), t \geq 0\}$  is a  $\mathcal{B}_t(\mathbb{X})$ -martingale. The existence of a weak solution to Eq. (2.6) now follows from Proposition 3.4.

Summarizing Theorems 3.8, 3.9 and 3.7, we have the following main result in the present paper.

**Theorem 3.14.** *Under (H1)-(H3), for any  $\mathbf{u}_0 \in \mathbb{H}^1$ , there exists a unique  $\mathbf{u}(t, x)$  such that*

(1 $^\circ$ )  $\mathbf{u} \in L^2(\Omega, P; C([0, T], \mathbb{H}^1)) \cap L^2(\Omega, P; L^2([0, T], \mathbb{H}^2))$  for any  $T > 0$ , and

$$\mathbb{E} \left( \sup_{t \in [0, T]} \|\mathbf{u}(t)\|_{\mathbb{H}^1}^2 \right) + \int_0^T \mathbb{E} \|\mathbf{u}(s)\|_{\mathbb{H}^2}^2 ds \leq C_T (1 + \|\mathbf{u}_0\|_{\mathbb{H}^1}^2) (1 + N); \quad (3.20)$$

(2°) it holds that in  $\mathbb{H}^0$

$$\mathbf{u}(t) = \mathbf{u}_0 + \int_0^t \left[ A(\mathbf{u}(s)) + \mathcal{P}\mathbf{f}(s, \mathbf{u}(s)) \right] ds + \sum_{k=1}^{\infty} \int_0^t B_k(s, \mathbf{u}(s)) dW_s^k,$$

for all  $t \geq 0$ ,  $P$ -a.s..

*Proof.* We only need to prove the estimate (3.20). By Ito's formula, (2.8) and Lemma 2.4, we have

$$\begin{aligned} \mathbb{E}\|\mathbf{u}(t)\|_{\mathbb{H}^0}^2 &= \|\mathbf{u}_0\|_{\mathbb{H}^0}^2 + 2 \int_0^t \mathbb{E}\langle A(\mathbf{u}(s)), \mathbf{u}(s) \rangle_{\mathbb{H}^0} ds \\ &\quad + 2 \int_0^t \mathbb{E}\langle \mathbf{f}(s, \mathbf{u}(s)), \mathbf{u}(s) \rangle_{\mathbb{H}^0} ds + \int_0^t \mathbb{E}\|B(s, \mathbf{u}(s))\|_{L_2(l^2; \mathbb{H}^0)}^2 ds \\ &\leq \|\mathbf{u}_0\|_{\mathbb{H}^0}^2 + C - \frac{1}{2} \int_0^t \mathbb{E}\|\mathbf{u}(s)\|_{\mathbb{H}^1}^2 ds + C \int_0^t \mathbb{E}\|\mathbf{u}(s)\|_{\mathbb{H}^0}^2 ds. \end{aligned}$$

By Gronwall's inequality, we obtain

$$\sup_{t \in [0, T]} \mathbb{E}\|\mathbf{u}(t)\|_{\mathbb{H}^0}^2 + \int_0^T \mathbb{E}\|\mathbf{u}(s)\|_{\mathbb{H}^1}^2 ds \leq C_T (\|\mathbf{u}_0\|_{\mathbb{H}^0}^2 + 1).$$

Using this estimate, as in in the proof of (3.8), we obtain (3.20).  $\square$

#### 4. FELLER PROPERTY AND INVARIANT MEASURE

In the following, we consider the time homogenous case, i.e., the coefficients  $\mathbf{f}$ ,  $\sigma$  and  $\mathbf{h}$  are independent of  $t$ , and assume a stronger assumption than **(H3)**, namely:

**(H3')** There exist a constant  $C_{\mathbf{h}} > 0$  and a function  $H_{\mathbf{h}}(x) \in L^1(\mathbb{D})$  such that for any  $x \in \mathbb{D}$ ,  $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$  and  $j = 1, 2, 3$

$$\begin{aligned} \|\partial_{x^j} \mathbf{h}(x, \mathbf{u})\|_{l^2}^2 + \|\mathbf{h}(x, \mathbf{u})\|_{l^2}^2 &\leq C_{\mathbf{h}} \cdot |\mathbf{u}|^2 + H_{\mathbf{h}}(x), \\ \|\partial_{x^j} \mathbf{h}(x, \mathbf{u}) - \partial_{x^j} \mathbf{h}(x, \mathbf{v})\|_{l^2} &\leq C_{\mathbf{h}} \cdot |\mathbf{u} - \mathbf{v}|, \\ \|\partial_{u^j} \mathbf{h}(x, \mathbf{u})\|_{l^2} &\leq C_{\mathbf{h}}, \\ \|\partial_{u^j} \mathbf{h}(x, \mathbf{u}) - \partial_{u^j} \mathbf{h}(x, \mathbf{v})\|_{l^2} &\leq C_{\mathbf{h}} \cdot |\mathbf{u} - \mathbf{v}|. \end{aligned}$$

For fixed initial value  $\mathbf{u}_0 = \mathbf{v} \in \mathbb{H}^1$ , we denote the unique solution in Theorem 3.14 by  $\mathbf{u}(t; \mathbf{v})$ . Then  $\{\mathbf{u}(t; \mathbf{v}) : \mathbf{v} \in \mathbb{H}^1, t \geq 0\}$  forms a strong Markov process with state space  $\mathbb{H}^1$ . We have

**Lemma 4.1.** For  $\mathbf{v}, \mathbf{v}' \in \mathbb{H}^1$  and  $R > 0$ , define

$$\tau_R^{\mathbf{v}} := \inf \{t \geq 0 : \|\mathbf{u}(t; \mathbf{v})\|_{\mathbb{H}^1} > R\}$$

and

$$\tau_R^{\mathbf{v}, \mathbf{v}'} := \tau_R^{\mathbf{v}} \wedge \tau_R^{\mathbf{v}'}$$

Assume **(H1)**, **(H2)** and **(H3')**, then

$$\mathbb{E}\|\mathbf{u}(t \wedge \tau_R^{\mathbf{v}, \mathbf{v}'}; \mathbf{v}) - \mathbf{u}(t \wedge \tau_R^{\mathbf{v}, \mathbf{v}'}; \mathbf{v}')\|_{\mathbb{H}^1}^2 \leq C_{t, R} \cdot \|\mathbf{v} - \mathbf{v}'\|_{\mathbb{H}^1}^2.$$

*Proof.* Write  $\mathbf{u}(t) := \mathbf{u}(t; \mathbf{v})$ ,  $\tilde{\mathbf{u}}(t) := \mathbf{u}(t; \mathbf{v}')$  and

$$\mathbf{w}(t) := \mathbf{u}(t) - \tilde{\mathbf{u}}(t).$$

Set  $t_R := \tau_R^{\mathbf{v}, \mathbf{v}'} \wedge t$ . By Ito's formula([26, 23]), we have

$$\begin{aligned}
\|\mathbf{w}(t_R)\|_{\mathbb{H}^1}^2 &= \|\mathbf{w}(0)\|_{\mathbb{H}^1}^2 + 2 \int_0^{t_R} \langle A(\mathbf{u}(s)) - A(\tilde{\mathbf{u}}(s)), \mathbf{w}(s) \rangle_{\mathbb{H}^1} ds \\
&\quad + 2 \int_0^{t_R} \langle \mathbf{f}(s, \mathbf{u}(s)) - \mathbf{f}(s, \tilde{\mathbf{u}}(s)), \mathbf{w}(s) \rangle_{\mathbb{H}^1} ds \\
&\quad + 2 \sum_{k=1}^{\infty} \int_0^{t_R} \langle B_k(s, \mathbf{u}(s)) - B_k(s, \tilde{\mathbf{u}}(s)), \mathbf{w}(s) \rangle_{\mathbb{H}^1} dW_s^k \\
&\quad + \sum_{k=1}^{\infty} \int_0^{t_R} \|B_k(s, \mathbf{u}(s)) - B_k(s, \tilde{\mathbf{u}}(s))\|_{\mathbb{H}^1}^2 ds \\
&=: \|\mathbf{w}(0)\|_{\mathbb{H}^1}^2 + I_1(t_R) + I_2(t_R) + I_3(t_R) + I_4(t_R).
\end{aligned}$$

By  $|g_N(r) - g_N(r')| \leq |r - r'|$  and Young's inequality, it is easy to see that

$$\begin{aligned}
I_1(t_R) &= -2 \int_0^{t_R} \|\mathbf{w}(s)\|_{\mathbb{H}^2}^2 ds + 2 \int_0^{t_R} \|\mathbf{w}(s)\|_{\mathbb{H}^1}^2 ds \\
&\quad + 2 \int_0^{t_R} \langle ((\mathbf{u}(s) \cdot \nabla) \mathbf{u}(s) - (\tilde{\mathbf{u}}(s) \cdot \nabla) \tilde{\mathbf{u}}(s)), (I - \Delta) \mathbf{w}(s) \rangle_{\mathbb{H}^0} ds \\
&\quad - 2 \int_0^{t_R} \langle g_N(|\mathbf{u}(s)|^2) \mathbf{u}(s) - g_N(|\tilde{\mathbf{u}}(s)|^2) \tilde{\mathbf{u}}(s), (I - \Delta) \mathbf{w}(s) \rangle_{\mathbb{H}^0} ds \\
&\leq - \int_0^{t_R} \|\mathbf{w}(s)\|_{\mathbb{H}^2}^2 ds + 2 \int_0^{t_R} \|\mathbf{w}(s)\|_{\mathbb{H}^1}^2 ds \\
&\quad + C \int_0^{t_R} \|(\mathbf{w}(s) \cdot \nabla) \mathbf{u}(s)\|_{L^2}^2 ds + C \int_0^{t_R} \|(\tilde{\mathbf{u}}(s) \cdot \nabla) \mathbf{w}(s)\|_{L^2}^2 ds \\
&\quad + C \int_0^{t_R} \| |\mathbf{w}(s)| \cdot (|\mathbf{u}(s)|^2 + |\tilde{\mathbf{u}}(s)|^2) \|_{L^2}^2 ds.
\end{aligned}$$

By Hölder's inequality and the Sobolev inequality (2.1), we further have

$$\begin{aligned}
I_1(t_R) &\leq - \int_0^{t_R} \|\mathbf{w}(s)\|_{\mathbb{H}^2}^2 ds + 2 \int_0^{t_R} \|\mathbf{w}(s)\|_{\mathbb{H}^1}^2 ds \\
&\quad + C_R \int_0^{t_R} \|\mathbf{w}(s)\|_{L^\infty}^2 ds + C \int_0^{t_R} \|\tilde{\mathbf{u}}(s)\|_{L^6}^2 \cdot \|\nabla \mathbf{w}(s)\|_{L^3}^2 ds \\
&\quad + C \int_0^{t_R} \|\mathbf{w}(s)\|_{L^6}^2 \cdot (\|\mathbf{u}(s)\|_{L^6}^2 + \|\tilde{\mathbf{u}}(s)\|_{L^6}^2) ds \\
&\leq - \int_0^{t_R} \|\mathbf{w}(s)\|_{\mathbb{H}^2}^2 ds + C_R \int_0^{t_R} \|\mathbf{w}(s)\|_{\mathbb{H}^1}^2 ds \\
&\quad + C_R \int_0^{t_R} \|\mathbf{w}(s)\|_{\mathbb{H}^2}^{3/2} \cdot \|\mathbf{w}(s)\|_{\mathbb{H}^0}^{1/2} ds + C_R \int_0^{t_R} \|\mathbf{w}(s)\|_{\mathbb{H}^2} \cdot \|\mathbf{w}(s)\|_{\mathbb{H}^1} ds \\
&\leq - \frac{3}{4} \int_0^{t_R} \|\mathbf{w}(s)\|_{\mathbb{H}^2}^2 ds + C_R \int_0^{t_R} \|\mathbf{w}(s)\|_{\mathbb{H}^1}^2 ds.
\end{aligned}$$

By **(H1)**, **(H2)** and **(H3')**, we similarly have

$$\begin{aligned} I_2(t_R) &\leq \frac{1}{4} \int_0^{t_R} \|\mathbf{w}(s)\|_{\mathbb{H}^2}^2 ds + C_R \int_0^{t_R} \|\mathbf{w}(s)\|_{\mathbb{H}^0}^2 ds \\ I_4(t_R) &\leq \frac{1}{2} \int_0^{t_R} \|\mathbf{w}(s)\|_{\mathbb{H}^2}^2 ds + C_R \int_0^{t_R} \|\mathbf{w}(s)\|_{\mathbb{H}^1}^2 ds. \end{aligned}$$

So,

$$\begin{aligned} \mathbb{E}\|\mathbf{w}(t \wedge \tau_R)\|_{\mathbb{H}^1}^2 &\leq \|\mathbf{w}(0)\|_{\mathbb{H}^1}^2 + C_R \int_0^{t_R} \|\mathbf{w}(s)\|_{\mathbb{H}^1}^2 ds \\ &\leq \|\mathbf{v} - \mathbf{v}'\|_{\mathbb{H}^1}^2 + C_R \int_0^t \|\mathbf{w}(s \wedge \tau_R)\|_{\mathbb{H}^1}^2 ds. \end{aligned}$$

By Gronwall's inequality, we get the desired estimate.  $\square$

Let  $C_b^{loc}(\mathbb{H}^1)$  denote the set of all bounded and locally uniformly continuous functions on  $\mathbb{H}^1$ . Then  $C_b^{loc}(\mathbb{H}^1)$  is clearly a Banach space under the sup norm

$$\|\phi\|_\infty := \sup_{\mathbf{u} \in \mathbb{H}^1} |\phi(\mathbf{u})|.$$

For  $t > 0$ , we define the semigroup  $\mathbf{T}_t$  associated with  $\{\mathbf{u}(t; \mathbf{v}) : \mathbf{v} \in \mathbb{H}^1, t \geq 0\}$  by

$$\mathbf{T}_t \phi(\mathbf{v}) := \mathbb{E}(\phi(\mathbf{u}(t; \mathbf{v}))), \quad \phi \in C_b^{loc}(\mathbb{H}^1).$$

We have:

**Theorem 4.2.** *Under **(H1)**, **(H2)** and **(H3')**, for every  $t > 0$ ,  $\mathbf{T}_t$  maps  $C_b^{loc}(\mathbb{H}^1)$  into  $C_b^{loc}(\mathbb{H}^1)$ . That is,  $(\mathbf{T}_t)_{t \geq 0}$  is a Feller semigroup on  $C_b^{loc}(\mathbb{H}^1)$ .*

*Proof.* Let  $\phi \in C_b^{loc}(\mathbb{H}^1)$  be given. We want to prove that for any  $t > 0$  and  $m \in \mathbb{N}$

$$\lim_{\delta \rightarrow 0} \sup_{\mathbf{v}, \mathbf{v}' \in \mathbb{B}_m, \|\mathbf{v} - \mathbf{v}'\|_{\mathbb{H}^1} \leq \delta} |\mathbf{T}_t \phi(\mathbf{v}) - \mathbf{T}_t \phi(\mathbf{v}')| = 0, \quad (4.1)$$

where  $\mathbb{B}_m := \{\mathbf{v} \in \mathbb{H}^1 : \|\mathbf{v}\|_{\mathbb{H}^1} \leq m\}$  denotes the ball in  $\mathbb{H}^1$ .

For any  $\mathbf{v}, \mathbf{v}' \in \mathbb{B}_m$  and  $R > m$ , as in Lemma 4.1, define

$$\tau_R^{\mathbf{v}} := \{t \geq 0 : \|\mathbf{u}(t; \mathbf{v})\|_{\mathbb{H}^1} > R\}$$

and

$$\tau_R^{\mathbf{v}, \mathbf{v}'} := \tau_R^{\mathbf{v}} \wedge \tau_R^{\mathbf{v}'}$$

By (3.20), we have

$$\begin{aligned} &|\mathbb{E}[\phi(\mathbf{u}(t; \mathbf{v})) - \phi(\mathbf{u}(t \wedge \tau_R^{\mathbf{v}, \mathbf{v}'}; \mathbf{v}))]| \\ &\leq 2\|\phi\|_\infty \cdot P(\tau_R^{\mathbf{v}, \mathbf{v}'} < t) \\ &\leq 2\|\phi\|_\infty \cdot \sup_{\mathbf{v} \in \mathbb{B}_m} \mathbb{E} \left( \sup_{s \in [0, t]} \|\mathbf{u}(s; \mathbf{v})\|_{\mathbb{H}^1}^2 \right) / R^2 \\ &\leq 2\|\phi\|_\infty \cdot C_{t, m, N} / R^2. \end{aligned}$$

For any  $\epsilon > 0$ , choose  $R > m$  sufficiently large such that for any  $\mathbf{v}, \mathbf{v}' \in \mathbb{B}_m$

$$\mathbb{E}|\phi(\mathbf{u}(t; \mathbf{v})) - \phi(\mathbf{u}(t \wedge \tau_R^{\mathbf{v}, \mathbf{v}'}; \mathbf{v}))| \leq \epsilon \quad (4.2)$$

$$\mathbb{E}|\phi(\mathbf{u}(t; \mathbf{v}')) - \phi(\mathbf{u}(t \wedge \tau_R^{\mathbf{v}, \mathbf{v}'}; \mathbf{v}'))| \leq \epsilon. \quad (4.3)$$

For this  $R$ , since  $\phi$  is uniformly continuous on  $\mathbb{B}_R$ , one may choose  $\eta > 0$  such that for any  $\mathbf{u}, \mathbf{u}' \in \mathbb{B}_R$  with  $\|\mathbf{u} - \mathbf{u}'\|_{\mathbb{H}^1} \leq \eta$

$$|\phi(\mathbf{u}) - \phi(\mathbf{u}')| \leq \epsilon.$$

Thus, for any  $\mathbf{v}, \mathbf{v}' \in \mathbb{B}_m$  with  $\|\mathbf{v} - \mathbf{v}'\|_{\mathbb{H}^1} \leq \frac{\sqrt{\epsilon} \cdot \eta}{\sqrt{2C_\phi \cdot C_{t,R}}}$ , by Lemma 4.1 we have

$$\begin{aligned} & \mathbb{E}|\phi(\mathbf{u}(t \wedge \tau_R^{\mathbf{v}, \mathbf{v}'}; \mathbf{v})) - \phi(\mathbf{u}(t \wedge \tau_R^{\mathbf{v}', \mathbf{v}'}; \mathbf{v}'))| \\ & \leq \epsilon + 2C_\phi \cdot P(\|\mathbf{u}(t \wedge \tau_R^{\mathbf{v}, \mathbf{v}'}; \mathbf{v}) - \mathbf{u}(t \wedge \tau_R^{\mathbf{v}', \mathbf{v}'}; \mathbf{v}')\|_{\mathbb{H}^1} > \eta) \leq 2\epsilon. \end{aligned} \quad (4.4)$$

Combining (4.2) (4.3) and (4.4), we get (4.1).  $\square$

In the periodic case, we have the following existence of invariant measures associated to  $(\mathbf{T}_t)_{t \geq 0}$ .

**Theorem 4.3.** *Under (H1), (H2) and (H3'), in the periodic case, there is an invariant measure  $\nu$  associated to the semigroup  $(\mathbf{T}_t)_{t \geq 0}$  such that for any  $t \geq 0$  and  $\phi \in C_b^{loc}(\mathbb{H}^1)$*

$$\int_{\mathbb{H}^1} \mathbf{T}_t \phi(\mathbf{u}) \nu(d\mathbf{u}) = \int_{\mathbb{H}^1} \phi(\mathbf{u}) \nu(d\mathbf{u}).$$

*Proof.* In the following, we assume that  $\mathbf{u}_0 = 0$ . Using Itô's formula, we have by (2.9) and (2.14)

$$\begin{aligned} \mathbb{E}\|\mathbf{u}(t)\|_{\mathbb{H}^0}^2 &= 2 \int_0^t \mathbb{E}\langle A(\mathbf{u}(s)), \mathbf{u}(s) \rangle_{\mathbb{H}^0} ds + 2 \int_0^t \mathbb{E}\langle \mathbf{f}(\mathbf{u}(s)), \mathbf{u}(s) \rangle_{\mathbb{H}^0} ds \\ &\quad + \int_0^t \mathbb{E}\|B(\mathbf{u}(s))\|_{L_2(l^2; \mathbb{H}^0)}^2 ds \\ &\leq -\frac{3}{2} \int_0^t \mathbb{E}\|\nabla \mathbf{u}(s)\|_{\mathbb{H}^0}^2 ds - 2 \int_0^t \mathbb{E}\|\mathbf{u}(s)\|_{L^4}^4 ds \\ &\quad + 2N \int_0^t \mathbb{E}\|\mathbf{u}(s)\|_{\mathbb{H}^0}^2 ds + C_{\mathbf{h}, \mathbf{f}} \int_0^t \mathbb{E}\|\mathbf{u}(s)\|_{\mathbb{H}^0}^2 ds + C_{\mathbf{h}, \mathbf{f}} t \\ &\leq -\frac{3}{2} \int_0^t \mathbb{E}\|\mathbf{u}(s)\|_{\mathbb{H}^1}^2 ds - 2 \int_0^t \mathbb{E}\|\mathbf{u}(s)\|_{L^4}^4 ds \\ &\quad + C_{\mathbf{h}, \mathbf{f}, N} \int_0^t \mathbb{E}\|\mathbf{u}(s)\|_{\mathbb{H}^0}^2 ds + C_{\mathbf{h}, \mathbf{f}} t. \end{aligned}$$

In the periodic case, noting that for any  $\epsilon > 0$

$$\|\mathbf{u}\|_{\mathbb{H}^0}^2 \leq C\|\mathbf{u}\|_{L^4}^2 \leq \epsilon\|\mathbf{u}\|_{L^4}^4 + C_\epsilon,$$

we further have

$$\mathbb{E}\|\mathbf{u}(t)\|_{\mathbb{H}^0}^2 \leq -\frac{3}{2} \int_0^t \mathbb{E}\|\mathbf{u}(s)\|_{\mathbb{H}^1}^2 ds - \int_0^t \mathbb{E}\|\mathbf{u}(s)\|_{L^4}^4 ds + C_{\mathbf{h}, \mathbf{f}, N} \cdot t.$$

Hence, for any  $t \geq 0$

$$\mathbb{E}\|\mathbf{u}(t)\|_{\mathbb{H}^0}^2 + \int_0^t \mathbb{E}\|\mathbf{u}(s)\|_{\mathbb{H}^1}^2 ds + \int_0^t \mathbb{E}\|\mathbf{u}(s)\|_{L^4}^4 ds \leq C_{\mathbf{h}, \mathbf{f}, N} \cdot t. \quad (4.5)$$

On the other hand, by Ito's formula again and (2.10), (2.15), as above we have

$$\mathbb{E}\|\mathbf{u}(t)\|_{\mathbb{H}^1}^2 = 2 \int_0^t \mathbb{E}[A(\mathbf{u}(s)), \mathbf{u}(s)] ds + 2 \int_0^t \mathbb{E}\langle \mathbf{f}(\mathbf{u}(s)), \mathbf{u}(s) \rangle_{\mathbb{H}^1} ds$$

$$\begin{aligned}
& + \int_0^t \mathbb{E} \|B(\mathbf{u}(s))\|_{L_2(\ell^2; \mathbb{H}^1)}^2 ds \\
& \leq -\frac{1}{4} \int_0^t \mathbb{E} \|\mathbf{u}(s)\|_{\mathbb{H}^2}^2 ds + C_{\mathbf{h}, \mathbf{f}, N} \cdot \int_0^t \mathbb{E} \|\mathbf{u}(s)\|_{\mathbb{H}^1}^2 ds + C_{\mathbf{h}, \mathbf{f}, N} \cdot t \\
& \leq -\frac{1}{4} \int_0^t \mathbb{E} \|\mathbf{u}(s)\|_{\mathbb{H}^2}^2 ds + C_{\mathbf{h}, \mathbf{f}, N} \cdot t.
\end{aligned}$$

Therefore, for any  $t \geq 0$

$$\frac{1}{t} \int_0^t \mathbb{E} \|\mathbf{u}(s)\|_{\mathbb{H}^2}^2 ds \leq C_{\mathbf{h}, \mathbf{f}, N}$$

In the periodic case, since  $\mathbb{H}^2$  is compactly embedded into  $\mathbb{H}^1$ , the existence of an invariant measure  $\nu$  now follows from the classical Krylov-Bogoliubov method (cf. [2]).  $\square$

## 5. ERGODICITY: UNIQUENESS OF INVARIANT MEASURE

In the following, we shall work in the case  $\mathbb{D} = \mathbb{T}^3$ , and suppose that for  $\mathbf{f} \in \mathbb{H}^0$ , the mean value of  $f$  on  $\mathbb{T}^3$  vanishes, i.e.,

$$\int_{\mathbb{T}^3} \mathbf{f}(x) dx = 0.$$

In this case, we assume that the orthonormal basis  $\mathcal{E}$  of  $\mathbb{H}^1$  consists of the eigenvectors of  $\mathcal{P}\Delta$ , i.e.,

$$\mathcal{P}\Delta \mathbf{e}_i = -\lambda_i \mathbf{e}_i, \quad \langle \mathbf{e}_i, \mathbf{e}_i \rangle_{\mathbb{H}^1} = 1, \quad i = 1, 2, \dots,$$

where  $0 < \lambda_1 \leq \dots \leq \lambda_n \uparrow \infty$ . Recalling that the following Poincaré inequality holds:

$$\|\mathbf{u}\|_{\mathbb{H}^0}^2 \leq 1/\lambda_1 \|\nabla \mathbf{u}\|_{\mathbb{H}^0}^2, \quad (5.1)$$

two equivalent norms in  $\mathbb{H}^1$  and  $\mathbb{H}^2$  are given by

$$\|\mathbf{u}\|_{\mathbb{H}^1} := \|\nabla \mathbf{u}\|_{\mathbb{H}^0}, \quad \|\mathbf{u}\|_{\mathbb{H}^2} := \|\Delta \mathbf{u}\|_{\mathbb{H}^0}.$$

We shall use these two norms in what follows.

For  $m \in \mathbb{N}$ , let  $\Omega := C_0(\mathbb{R}_+; \mathbb{R}^m)$  denote the space of all continuous functions with initial values 0,  $P$  the standard Wiener measure on  $\mathcal{F} := \mathcal{B}(C_0(\mathbb{R}_+; \mathbb{R}^m))$ . Then, the coordinate process

$$W_t(\omega) := \omega(t), \quad \omega \in \Omega,$$

is a standard Wiener process on  $(\Omega, \mathcal{F}, P)$ .

Consider the following stochastic tamed 3D Navier-Stokes equation:

$$\begin{cases} d\mathbf{u}(t) &= A(\mathbf{u}(t))dt + d\mathbf{w}(t), \\ \mathbf{u}(0) &= \mathbf{u}_0 \in \mathbb{H}^1, \end{cases} \quad (5.2)$$

where  $\mathbf{w}(t) := QW_t$  is the noise, and the linear map  $Q : \mathbb{R}^m \mapsto \mathbb{H}^1$  is given by

$$Qe_i = q_i \mathbf{e}_i, \quad q_i > 0, \quad i = 1, \dots, m.$$

Here,  $\{e_i, i = 1, \dots, m\}$  is the canonical basis of  $\mathbb{R}^m$ .

Set

$$\mathcal{E}_0 := \sum_{i=1}^m q_i^2 / \lambda_i, \quad \mathcal{E}_1 := \sum_{i=1}^m q_i^2.$$

Then the quadratic variation of  $\mathbf{w}(t)$  in  $\mathbb{H}^0$  and  $\mathbb{H}^1$  are given respectively by

$$[\mathbf{w}]_{\mathbb{H}^0}(t) = \mathcal{E}_0 t, \quad [\mathbf{w}]_{\mathbb{H}^1}(t) = \mathcal{E}_1 t.$$

We remark that  $\mathcal{E}_0 \leq \mathcal{E}_1/\lambda_1$ .

Our main result in this section is the following:

**Theorem 5.1.** *Let  $(\mathbf{T}_t)_{t \geq 0}$  be the transition semigroup associated with (5.2). For any sufficiently large  $m_* = m_*(\mathcal{E}_1, \lambda_1, N) \in \mathbb{N}$ , there exists a unique invariant probability measure associated with  $(\mathbf{T}_t)_{t \geq 0}$ .*

We shall divide the proof into two parts. In the first part, we shall prove the asymptotic strong Feller property of  $(\mathbf{T}_t)_{t \geq 0}$  (cf. [14, Proposition 3.12]). In the second part, we shall prove a support property of the invariant measure, namely that the origin 0 is contained in the support of each invariant measure (cf. [5]). By [14, Proposition 3.12 and Corollary 3.17], these two parts will imply Theorem 5.1.

**5.1. Asymptotic Strong Feller Property.** Let  $\mathbf{u}(t, \omega; \mathbf{u}_0)$  be the unique solution of Eq. (5.2). For  $0 \leq s < t$ , let  $\mathcal{J}_{s,t}$  denote the derivative flow of  $\mathbf{u}(t, \omega; \mathbf{u}_0)$  between  $s$  and  $t$  with respect to the initial values  $\mathbf{u}_0$ , i.e., for every  $\mathbf{v} \in \mathbb{H}^1$ ,  $\mathcal{J}_{s,t}\mathbf{v} \in \mathbb{H}^1$  satisfies

$$\partial_t \mathcal{J}_{s,t}\mathbf{v} = \Delta \mathcal{J}_{s,t}\mathbf{v} + \mathcal{K}(\mathbf{u}(t, \omega; \mathbf{u}_0), \mathcal{J}_{s,t}\mathbf{v}), \quad \mathcal{J}_{s,s}\mathbf{v} = \mathbf{v}, \quad (5.3)$$

where  $\mathcal{K}$  is linear with respect to the second component and given by

$$\mathcal{K}(\mathbf{u}, \mathbf{v}) := -\mathcal{P}((\mathbf{v} \cdot \nabla)\mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{v}) - \mathcal{P}(g_N(|\mathbf{u}|^2)\mathbf{v} + 2g'_N(|\mathbf{u}|^2)\langle \mathbf{u}, \mathbf{v} \rangle_{\mathbb{R}^3}\mathbf{u}).$$

In the Appendix, we shall prove that for every  $\omega$

$$(\mathcal{J}_{0,t}\mathbf{v})(\omega) = \lim_{\epsilon \downarrow 0} \frac{\mathbf{u}(t, \omega; \mathbf{u}_0 + \epsilon\mathbf{v}) - \mathbf{u}(t, \omega; \mathbf{u}_0)}{\epsilon} \quad \text{in } \mathbb{H}^1. \quad (5.4)$$

Let us now consider the Malliavin derivative of  $\mathbf{u}(t, \omega; \mathbf{u}_0)$  with respect to  $\omega$ . Let  $\mathcal{H}$  be the Cameron-Martin space, i.e., all absolutely continuous functions from  $\mathbb{R}_+$  to  $\mathbb{R}^m$  with locally square integrable derivative. For any  $v \in \mathcal{H}$ , the Malliavin derivative is defined by

$$D^v \mathbf{u}(t, \omega; \mathbf{u}_0) := \lim_{\epsilon \rightarrow 0} \frac{\mathbf{u}(t, \omega + \epsilon v; \mathbf{u}_0) - \mathbf{u}(t, \omega; \mathbf{u}_0)}{\epsilon}, \quad P - a.s.. \quad (5.5)$$

Notice that  $v$  can be random and possibly nonadapted to the filtration generated by  $W$ . For the sake of simplicity, we write  $\mathcal{A}_t v := D^v \mathbf{u}(t, \omega; \mathbf{u}_0)$ . Then

$$\partial_t \mathcal{A}_t v = \Delta \mathcal{A}_t v + \mathcal{K}(\mathbf{u}(t, \omega; \mathbf{u}_0), \mathcal{A}_t v) + Q\dot{v}(t), \quad \mathcal{A}_0 v = 0, \quad (5.6)$$

where  $\dot{v}(s)$  is the derivative of  $v(s)$  with respect to  $s$ .

By the formula of variation of constants, it is easy to see that

$$\mathcal{A}_t v = \int_0^t \mathcal{J}_{s,t} Q \dot{v}(s) ds.$$

Moreover, for any  $\mathbf{v} \in \mathbb{H}^1$  and  $v \in \mathcal{H}$ , set

$$\mathbf{v}(t) := \mathcal{J}_{0,t}\mathbf{v} - \mathcal{A}_t v.$$

Then

$$\partial_t \mathbf{v}(t) = \Delta \mathbf{v}(t) + \mathcal{K}(\mathbf{u}(t), \mathbf{v}(t)) - Q\dot{v}(t), \quad \mathbf{v}(0) = \mathbf{v}. \quad (5.7)$$

As done in [14], our main aim is to construct a suitable  $v$  such that  $\mathbf{v}(t)$  exponentially decays to zero in some sense as  $t \rightarrow \infty$ . We first introduce some necessary notations and prove some preparing lemmas.

Let  $\mathbb{H}_\ell^1$  denote the following finite dimensional subspace of  $\mathbb{H}^1$  (called low mode space)

$$\mathbb{H}_\ell^1 := \text{span}\{\mathbf{e}_1, \dots, \mathbf{e}_m\}.$$

Then we have the following direct sum decomposition:

$$\mathbb{H}^1 = \mathbb{H}_\ell^1 \oplus \mathbb{H}_h^1,$$

and for any  $\mathbf{u} \in \mathbb{H}^1$

$$\mathbf{u} = \mathbf{u}_\ell + \mathbf{u}_h, \quad \mathbf{u}_\ell \in \mathbb{H}_\ell^1, \quad \mathbf{u}_h \in \mathbb{H}_h^1.$$

The co-dimensional space  $\mathbb{H}_h^1$  is also called high mode space. For any  $\mathbf{v} \in \mathbb{H}^0$ , we define

$$\Pi_\ell \mathbf{v} := \sum_{i=1}^m \langle (-\Delta) \mathbf{e}_i, \mathbf{v} \rangle_{\mathbb{H}^0} \mathbf{e}_i \in \mathbb{H}_\ell^1$$

and

$$\Pi_h \mathbf{v} := \mathbf{v} - \Pi_\ell \mathbf{v} \in \mathbb{H}^0.$$

In what follows, we shall always write  $\mathbf{v}_\ell = \Pi_\ell \mathbf{v}$  and  $\mathbf{v}_h = \Pi_h \mathbf{v}$ .

The following lemma is immediate.

**Lemma 5.2.** *For any  $\mathbf{u} \in \mathbb{H}^2$*

$$\|\Delta \Pi_h \mathbf{u}\|_{\mathbb{H}^0}^2 \geq \lambda_m \|\nabla \Pi_h \mathbf{u}\|_{\mathbb{H}^0}^2.$$

We also need the following lemma. Recall that  $\|\mathbf{u}\|_{\mathbb{H}^2} = \|\Delta \mathbf{u}\|_{\mathbb{H}^0}$ .

**Lemma 5.3.** *For any  $\mathbf{u}, \mathbf{v} \in \mathbb{H}^2$ , set*

$$\mathcal{N}(\mathbf{u}) := \|\mathbf{u}\|_{\mathbb{H}^2}^2 + \|\mathbf{u}\| \cdot \|\nabla \mathbf{u}\|_{L^2}^2. \quad (5.8)$$

*Then*

$$\langle \mathbf{v}_h, \mathcal{K}(\mathbf{u}, \mathbf{v}) \rangle_{\mathbb{H}^1} \leq \frac{1}{2} \|\Delta \mathbf{v}_h\|_{\mathbb{H}^0}^2 + C_N \cdot \mathcal{N}(\mathbf{u}) \cdot (\|\mathbf{v}_h\|_{\mathbb{H}^1}^2 + \|\mathbf{v}_\ell\|_{\mathbb{H}^1}^2)$$

*and*

$$\|\Pi_\ell \mathcal{K}(\mathbf{u}, \mathbf{v})\|_{\mathbb{H}^1}^2 \leq C_m \|\mathbf{v}\|_{\mathbb{H}^0} \cdot (1 + \|\mathbf{u}\|_{L^4}^2),$$

*where the constant  $C_N$  (resp.  $C_m$ ) only depends on  $N$  (resp.  $m$ ).*

*Proof.* For the first, we write

$$\langle \mathbf{v}_h, \mathcal{K}(\mathbf{u}, \mathbf{v}) \rangle_{\mathbb{H}^1} = I_1 + I_2 + I_3 + I_4,$$

where

$$\begin{aligned} I_1 &:= -\langle \mathbf{v}_h, \mathcal{P}((\mathbf{v} \cdot \nabla) \mathbf{u}) \rangle_{\mathbb{H}^1}, \\ I_2 &:= -\langle \mathbf{v}_h, \mathcal{P}((\mathbf{u} \cdot \nabla) \mathbf{v}) \rangle_{\mathbb{H}^1}, \\ I_3 &:= -\langle \mathbf{v}_h, \mathcal{P}(g_N(|\mathbf{u}|^2) \mathbf{v}) \rangle_{\mathbb{H}^1}, \\ I_4 &:= -2 \langle \mathbf{v}_h, \mathcal{P}(g'_N(|\mathbf{u}|^2) \langle \mathbf{u}, \mathbf{v} \rangle_{\mathbb{R}^3} \mathbf{u}) \rangle_{\mathbb{H}^1}. \end{aligned}$$

For  $I_1$ , by Young's inequality and the Sobolev inequality (2.1) we have

$$\begin{aligned} I_1 &\leq \frac{1}{8} \|\Delta \mathbf{v}_h\|_{\mathbb{H}^0}^2 + 2 \|\mathbf{v}\| \cdot \|\nabla \mathbf{u}\|_{L^2}^2 \\ &\leq \frac{1}{8} \|\Delta \mathbf{v}_h\|_{\mathbb{H}^0}^2 + 2 \|\mathbf{v}\|_{L^6}^2 \cdot \|\nabla \mathbf{u}\|_{L^3}^2 \\ &\leq \frac{1}{8} \|\Delta \mathbf{v}_h\|_{\mathbb{H}^0}^2 + C \|\mathbf{v}_h\|_{\mathbb{H}^1}^2 \cdot \|\mathbf{u}\|_{\mathbb{H}^2}^2 + C \|\mathbf{v}_\ell\|_{\mathbb{H}^1}^2 \cdot \|\mathbf{u}\|_{\mathbb{H}^2}^2. \end{aligned}$$

For  $I_2$ , we have

$$I_2 \leq \frac{1}{8} \|\Delta \mathbf{v}_h\|_{\mathbb{H}^0}^2 + 2 \|\nabla \mathbf{v}\| \cdot \|\mathbf{u}\|_{L^2}^2$$

$$\begin{aligned}
&\leq \frac{1}{8} \|\Delta \mathbf{v}_h\|_{\mathbb{H}^0}^2 + 2 \|\mathbf{v}\|_{\mathbb{H}^1}^2 \cdot \|\mathbf{u}\|_{L^\infty}^2 \\
&\leq \frac{1}{8} \|\Delta \mathbf{v}_h\|_{\mathbb{H}^0}^2 + C \|\mathbf{v}_h\|_{\mathbb{H}^1}^2 \cdot \|\mathbf{u}\|_{\mathbb{H}^2}^2 + C \|\mathbf{v}_\ell\|_{\mathbb{H}^1}^2 \cdot \|\mathbf{u}\|_{\mathbb{H}^2}^2.
\end{aligned}$$

For  $I_3$ , we have

$$\begin{aligned}
I_3 &= -\langle \nabla \mathbf{v}_h, g_N(|\mathbf{u}|^2) \nabla \mathbf{v} \rangle_{\mathbb{H}^0} - \langle \nabla \mathbf{v}_h, g'_N(|\mathbf{u}|^2) \nabla |\mathbf{u}|^2 \mathbf{v} \rangle_{\mathbb{H}^0} \\
&\leq \|\mathbf{u}\|_{L^\infty}^2 \cdot \|\nabla \mathbf{v}_h\| \cdot \|\nabla \mathbf{v}\|_{L^1} + \|\nabla \mathbf{v}_h\|_{L^6} \cdot \|\mathbf{v}\|_{L^3} \cdot \|\nabla |\mathbf{u}|^2\|_{L^2} \\
&\leq C \|\mathbf{u}\|_{\mathbb{H}^2}^2 \cdot (\|\mathbf{v}_h\|_{\mathbb{H}^1}^2 + \|\mathbf{v}_\ell\|_{\mathbb{H}^1}^2) + \frac{1}{8} \|\mathbf{v}_h\|_{\mathbb{H}^2}^2 + C \|\mathbf{v}\|_{\mathbb{H}^1}^2 \cdot \|\mathbf{u}\| \cdot \|\nabla \mathbf{u}\|_{L^2}^2.
\end{aligned}$$

For  $I_4$ , noting that

$$|g''_N(r)| \leq C \cdot 1_{\{N < r < N+1\}},$$

we similarly have

$$I_4 \leq C \|\mathbf{u}\|_{\mathbb{H}^2}^2 \cdot (\|\mathbf{v}_h\|_{\mathbb{H}^1}^2 + \|\mathbf{v}_\ell\|_{\mathbb{H}^1}^2) + \frac{1}{8} \|\mathbf{v}_h\|_{\mathbb{H}^2}^2 + C_N \|\mathbf{v}\|_{\mathbb{H}^1}^2 \cdot \|\mathbf{u}\| \cdot \|\nabla \mathbf{u}\|_{L^2}^2.$$

Combining the above calculations, we obtain the first estimate.

As for the second one, we may write

$$\|\Pi_\ell \mathcal{K}(\mathbf{u}, \mathbf{v})\|_{\mathbb{H}^1}^2 = \sum_{i=1}^m \langle \mathbf{e}_i, \mathcal{K}(\mathbf{u}, \mathbf{v}) \rangle_{\mathbb{H}^1}^2 = \sum_{i=1}^m \sum_{j=1}^4 J_{ij},$$

where

$$\begin{aligned}
J_{i1} &:= -\langle \mathbf{e}_i, \mathcal{P}((\mathbf{v} \cdot \nabla) \mathbf{u}) \rangle_{\mathbb{H}^1} \\
J_{i2} &:= -\langle \mathbf{e}_i, \mathcal{P}((\mathbf{u} \cdot \nabla) \mathbf{v}) \rangle_{\mathbb{H}^1} \\
J_{i3} &:= -\langle \mathbf{e}_i, \mathcal{P}(g_N(|\mathbf{u}|^2) \mathbf{v}) \rangle_{\mathbb{H}^1} \\
J_{i4} &:= -2 \langle \mathbf{e}_i, \mathcal{P}(g'_N(|\mathbf{u}|^2) (\mathbf{u} \cdot \mathbf{v}) \mathbf{u}) \rangle_{\mathbb{H}^1}.
\end{aligned}$$

For  $J_{i1}$ , we have

$$\begin{aligned}
J_{i1} &= \langle \Delta \mathbf{e}_i, (\mathbf{v} \cdot \nabla) \mathbf{u} \rangle_{\mathbb{H}^0} = -\langle \nabla \Delta \mathbf{e}_i, \mathbf{v} \otimes \mathbf{u} \rangle_{\mathbb{H}^0} \\
&\leq \|\nabla \Delta \mathbf{e}_i\|_{L^\infty} \|\mathbf{v}\| \cdot \|\mathbf{u}\|_{L^1} \leq C_{\mathbf{e}_i} \|\mathbf{v}\|_{\mathbb{H}^0} \|\mathbf{u}\|_{\mathbb{H}^0}.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
J_{i2} &\leq C_{\mathbf{e}_i} \|\mathbf{v}\|_{\mathbb{H}^0} \|\mathbf{u}\|_{\mathbb{H}^0} \\
J_{i3} &\leq C_{\mathbf{e}_i} \|\mathbf{v}\|_{\mathbb{H}^0} \|\mathbf{u}\|_{L^4}^2 \\
J_{i4} &\leq C_{\mathbf{e}_i} \|\mathbf{v}\|_{\mathbb{H}^0} \|\mathbf{u}\|_{L^4}^2.
\end{aligned}$$

Summarizing the above calculations, we obtain the second estimate.  $\square$

We now prove the following crucial estimate about the solution  $\mathbf{u}(t)$ .

**Lemma 5.4.** (i) For any  $\eta > 0$ , there exist constants  $C_\eta, C_{\mathcal{E}_1, \lambda_1, N, \eta} > 0$  such that for any  $t > 0$  and  $\mathbf{u}_0 \in \mathbb{H}_1$

$$\mathbb{E} \exp \left\{ \eta \int_0^t \mathcal{N}(\mathbf{u}(s; \mathbf{u}_0)) ds \right\} \leq \exp \{ C_\eta \|\mathbf{u}_0\|_{\mathbb{H}^1}^2 + C_{\mathcal{E}_1, \lambda_1, N, \eta} t \},$$

where  $\mathcal{N}(\mathbf{u})$  is defined by (5.8).

(ii) There exist constants  $C_N, C_{\mathcal{E}_1, \lambda_1, N} > 0$  such that for any  $t > 0$  and  $\mathbf{u}_0 \in \mathbb{H}_1$

$$\mathbb{E} \|\mathbf{u}(t; \mathbf{u}_0)\|_{\mathbb{H}^1}^2 \leq \|\mathbf{u}_0\|_{\mathbb{H}^1}^2 (C_N \cdot t + 1) e^{-t/2} + C_{\mathcal{E}_1, \lambda_1, N}.$$

*Proof.* By Itô's formula, we have

$$d\|\mathbf{u}(t)\|_{\mathbb{H}^0}^2 = 2\langle \mathbf{u}(t), A(\mathbf{u}(t)) \rangle_{\mathbb{H}^0} dt + 2\langle \mathbf{u}(t), d\mathbf{w}(t) \rangle_{\mathbb{H}^0} + \mathcal{E}_0 dt. \quad (5.9)$$

By (2.9) and Young's inequality, we know

$$\begin{aligned} \langle \mathbf{u}(t), A(\mathbf{u}(t)) \rangle_{\mathbb{H}^0} &\leq -\|\nabla \mathbf{u}(t)\|_{\mathbb{H}^0}^2 - \|\mathbf{u}(t)\|_{L^4}^4 + N\|\mathbf{u}(t)\|_{\mathbb{H}^0}^2 \\ &\leq -\|\mathbf{u}(t)\|_{\mathbb{H}^1}^2 - \frac{1}{2}\|\mathbf{u}(t)\|_{\mathbb{H}^0}^4 + \frac{N^2}{2}. \end{aligned} \quad (5.10)$$

Using Lemma 6.2 in the Appendix, we get for any  $t, \eta > 0$

$$\mathbb{E} \exp \left\{ \eta \int_0^t \|\mathbf{u}(s)\|_{\mathbb{H}^1}^2 ds \right\} \leq \exp \{ \eta \|\mathbf{u}_0\|_{\mathbb{H}^0}^2 + C_{\mathcal{E}_0, N, \eta} t \}. \quad (5.11)$$

Again, by Itô's formula and (2.10), we have

$$\begin{aligned} d\|\mathbf{u}(t)\|_{\mathbb{H}^1}^2 &= 2[\mathbf{u}(t), A(\mathbf{u}(t))] dt + 2\langle \mathbf{u}(t), d\mathbf{w}(t) \rangle_{\mathbb{H}^1} + \mathcal{E}_1 dt \\ &\leq (-\mathcal{N}(\mathbf{u}(t)) + C_N \|\mathbf{u}(t)\|_{\mathbb{H}^1}^2) dt + 2\langle \mathbf{u}(t), d\mathbf{w}(t) \rangle_{\mathbb{H}^1} + \mathcal{E}_1 dt. \end{aligned}$$

As in the proof of Lemma 6.2 in the Appendix, using (5.11) and exponential martingales, we then get the first estimate.

On the other hand, from (5.9) and (5.10), we have

$$d\|\mathbf{u}(t)\|_{\mathbb{H}^0}^2 \leq -\frac{1}{2}\|\mathbf{u}(t)\|_{\mathbb{H}^0}^2 dt + 2\langle \mathbf{u}(t), d\mathbf{w}(t) \rangle_{\mathbb{H}^0} + (\mathcal{E}_0 + N^2 + \frac{1}{2}) dt.$$

It is direct by Gronwall's inequality that

$$\mathbb{E} \|\mathbf{u}(t; \mathbf{u}_0)\|_{\mathbb{H}^0}^2 \leq \|\mathbf{u}_0\|_{\mathbb{H}^0}^2 e^{-t/2} + 2(\mathcal{E}_0 + N^2 + 1).$$

Thus, thanks to

$$\|\mathbf{u}(t)\|_{\mathbb{H}^1}^2 \leq \|\mathbf{u}(t)\|_{\mathbb{H}^2} \|\mathbf{u}(t)\|_{\mathbb{H}^0}, \quad (5.12)$$

we obtain

$$\begin{aligned} d(e^{t/2} \|\mathbf{u}(t)\|_{\mathbb{H}^1}^2) &= e^{t/2} (2[\mathbf{u}(t), A(\mathbf{u}(t))] + \mathcal{E}_1 + \frac{1}{2} \|\mathbf{u}(t)\|_{\mathbb{H}^1}^2) dt + 2e^{t/2} \langle \mathbf{u}(t), d\mathbf{w}(t) \rangle_{\mathbb{H}^1} \\ &\leq e^{t/2} (-\|\mathbf{u}(t)\|_{\mathbb{H}^2}^2 + C_N \|\mathbf{u}(t)\|_{\mathbb{H}^1}^2 + \mathcal{E}_1) dt + 2e^{t/2} \langle \mathbf{u}(t), d\mathbf{w}(t) \rangle_{\mathbb{H}^1} \\ &\leq e^{t/2} (C_N \|\mathbf{u}(t)\|_{\mathbb{H}^0}^2 + \mathcal{E}_1) dt + 2e^{t/2} \langle \mathbf{u}(t), d\mathbf{w}(t) \rangle_{\mathbb{H}^1}. \end{aligned}$$

Therefore,

$$\begin{aligned} &e^{t/2} \mathbb{E} \|\mathbf{u}(t; \mathbf{u}_0)\|_{\mathbb{H}^1}^2 \\ &\leq \|\mathbf{u}_0\|_{\mathbb{H}^1}^2 + C_N \int_0^t e^{s/2} \mathbb{E} \|\mathbf{u}(s; \mathbf{u}_0)\|_{\mathbb{H}^0}^2 ds + 2\mathcal{E}_1 e^{t/2} \\ &\leq \|\mathbf{u}_0\|_{\mathbb{H}^1}^2 + C_N \int_0^t (\|\mathbf{u}_0\|_{\mathbb{H}^0}^2 + 2e^{s/2} (\mathcal{E}_0 + N^2 + 1)) ds + 2\mathcal{E}_1 e^{t/2} \\ &\leq \|\mathbf{u}_0\|_{\mathbb{H}^1}^2 + C_N \|\mathbf{u}_0\|_{\mathbb{H}^0}^2 t + e^{t/2} (C_N (\mathcal{E}_0 + N^2 + 1) + 2\mathcal{E}_1), \end{aligned}$$

which then gives the second estimate.  $\square$

Based on the previous discussions and lemmas, we can now prove the following proposition, which will imply the asymptotic strong Feller property of  $(\mathbf{T}_t)_{t \geq 0}$  according to [14, Proposition 3.12].

**Proposition 5.5.** *Let  $(\mathbf{T}_t)_{t \geq 0}$  be the semigroup associated with (5.2). There exist a constant  $m_* := m_*(\mathcal{E}_1, N) \in \mathbb{N}$  and constants  $C_0, C_1, \gamma > 0$  such that for any  $t > 0$ ,  $\mathbf{u}_0 \in \mathbb{H}^1$ , and any Fréchet differentiable function  $\varphi$  on  $\mathbb{H}^1$  with  $\|\varphi\|_\infty, \|\nabla\varphi\|_\infty < +\infty$*

$$\|\nabla\mathbf{T}_t\varphi(\mathbf{u}_0)\|_{\mathbb{H}^1} \leq C_0 \cdot \exp\{C_1\|\mathbf{u}_0\|_{\mathbb{H}^1}^2\} \cdot (\|\varphi\|_\infty + e^{-\gamma t}\|\nabla\varphi\|_\infty).$$

*Proof.* For any  $\mathbf{v} \in \mathbb{H}^1$  with  $\|\mathbf{v}\|_{\mathbb{H}^1} = 1$ , define

$$\mathbf{v}_\ell(t) := \begin{cases} \mathbf{v}_\ell \cdot (1 - t/(2\|\mathbf{v}_\ell\|_{\mathbb{H}^1})), & t \in [0, 2\|\mathbf{v}_\ell\|_{\mathbb{H}^1}] \\ 0, & t \in (2\|\mathbf{v}_\ell\|_{\mathbb{H}^1}, \infty). \end{cases}$$

Let  $\mathbf{v}_h(t)$  solve the following linear evolution equation:

$$\partial_t \mathbf{v}_h(t) = \Delta \Pi_h \mathbf{v}_h(t) + \Pi_h \mathcal{K}(\mathbf{u}(t), \mathbf{v}_h(t) + \mathbf{v}_\ell(t)), \quad \mathbf{v}_h(0) = \mathbf{v}_h.$$

Set

$$\mathbf{v}(t) := \mathbf{v}_\ell(t) + \mathbf{v}_h(t)$$

and

$$\dot{v}(t) := Q^{-1} \left( \frac{\mathbf{v}_\ell \cdot 1_{\{t < 2\|\mathbf{v}_\ell\|_{\mathbb{H}^1}\}}}{2\|\mathbf{v}_\ell\|_{\mathbb{H}^1}} + \Delta \mathbf{v}_\ell(t) + \Pi_\ell \mathcal{K}(\mathbf{u}(t), \mathbf{v}(t)) \right).$$

Then  $v(t) \in \mathcal{H}$  is a continuous adapted process. From the construction, one finds that  $\mathbf{v}(t)$  together with  $v(t)$  solves the equation (5.7).

Thus, we have

$$\begin{aligned} \langle \nabla \mathbf{T}_t \varphi(\mathbf{u}_0), \mathbf{v} \rangle_{\mathbb{H}^1} &= \mathbb{E} \langle (\nabla \varphi)(\mathbf{u}(t; \mathbf{u}_0)), \mathcal{I}_{0,t} \mathbf{v} \rangle_{\mathbb{H}^1} \\ &= \mathbb{E} \langle (\nabla \varphi)(\mathbf{u}(t; \mathbf{u}_0)), \mathcal{A}_t v(t) \rangle_{\mathbb{H}^1} + \mathbb{E} \langle (\nabla \varphi)(\mathbf{u}(t; \mathbf{u}_0)), \mathbf{v}(t) \rangle_{\mathbb{H}^1} \\ &= \mathbb{E} (D^v(\varphi(\mathbf{u}(t; \mathbf{u}_0)))) + \mathbb{E} \langle (\nabla \varphi)(\mathbf{u}(t; \mathbf{u}_0)), \mathbf{v}(t) \rangle_{\mathbb{H}^1} \\ &= \mathbb{E} \left( \varphi(\mathbf{u}(t; \mathbf{u}_0)) \cdot \int_0^t \dot{v}(s) dW_s \right) + \mathbb{E} \langle (\nabla \varphi)(\mathbf{u}(t; \mathbf{u}_0)), \mathbf{v}(t) \rangle_{\mathbb{H}^1} \\ &\leq \|\varphi\|_\infty \int_0^t \mathbb{E} |\dot{v}(s)|^2 ds + \|\nabla \varphi\|_\infty \mathbb{E} \|\mathbf{v}(t)\|_{\mathbb{H}^1}, \end{aligned} \tag{5.13}$$

where the last equality is due to the integration by parts formula in the Malliavin calculus (cf. [20]).

By the chain rule, we have by Lemmas 5.3 and 5.2

$$\begin{aligned} \partial_t \|\mathbf{v}_h(t)\|_{\mathbb{H}^1}^2 &= -2\|\Delta \Pi_h \mathbf{v}_h(t)\|_{\mathbb{H}^0}^2 + 2\langle \mathbf{v}_h(t), \Pi_h \mathcal{K}(\mathbf{u}(t), \mathbf{v}(t)) \rangle_{\mathbb{H}^1} \\ &\leq -\|\Delta \Pi_h \mathbf{v}_h(t)\|_{\mathbb{H}^0}^2 + C_N \cdot \mathcal{N}(\mathbf{u}(t)) \cdot (\|\mathbf{v}_h(t)\|_{\mathbb{H}^1}^2 + \|\mathbf{v}_\ell(t)\|_{\mathbb{H}^1}^2) \\ &\leq (-\lambda_m + C_N \cdot \mathcal{N}(\mathbf{u}(t))) \cdot \|\mathbf{v}_h(t)\|_{\mathbb{H}^1}^2 + C_N \cdot \mathcal{N}(\mathbf{u}(t)) \cdot \|\mathbf{v}_\ell(t)\|_{\mathbb{H}^1}^2. \end{aligned}$$

Noting that  $v(t) = 0$  for  $t \geq 2$ , by Gronwall's inequality we get

$$\begin{aligned} \|\mathbf{v}_h(t)\|_{\mathbb{H}^1}^2 &\leq \|\mathbf{v}_h(0)\|_{\mathbb{H}^1}^2 \exp \left\{ -\lambda_m t + C_N \int_0^t \mathcal{N}(\mathbf{u}(s)) ds \right\} \\ &\quad + \exp \left\{ -\lambda_m(t-2) + C_N \int_0^t \mathcal{N}(\mathbf{u}(s)) ds \right\} \int_0^2 \|\mathbf{v}_\ell(s)\|_{\mathbb{H}^1}^2 ds. \end{aligned}$$

By (i) of Lemma 5.4, since  $\lambda_m \uparrow \infty$  as  $m \rightarrow \infty$ , there exist constants  $\gamma > 0$  and  $m_* = m_*(\mathcal{E}_1, \lambda_1, N) \in \mathbb{N}$  such that for all  $t \geq 0$

$$\mathbb{E} \|\mathbf{v}_h(t)\|_{\mathbb{H}^1}^2 \leq C_{\mathcal{E}_1, \lambda_1, N} \cdot e^{C_N \|\mathbf{u}_0\|_{\mathbb{H}^1}^2 - \gamma t}.$$

Hence, for any  $t \geq 2$

$$\mathbb{E}\|\mathbf{v}(t)\|_{\mathbb{H}^1} \leq C_{\varepsilon_1, \lambda_1, N} \cdot e^{C_N \|\mathbf{u}_0\|_{\mathbb{H}^1}^2 - \gamma t}. \quad (5.14)$$

On the other hand, by Lemma 5.3 we have

$$\begin{aligned} \mathbb{E}|\dot{v}(t)|^2 &\leq C_m \left(1 + \mathbb{E}(\|\mathbf{v}(t)\|_{\mathbb{H}^0} (1 + \|\mathbf{u}(t)\|_{L^4}^2))\right) \\ &\leq C_m \left(1 + (\mathbb{E}\|\mathbf{v}(t)\|_{\mathbb{H}^0}^2)^{1/2} (1 + \mathbb{E}\|\mathbf{u}(t)\|_{L^4}^4)^{1/2}\right) \\ &\leq C_m \left(1 + (\mathbb{E}\|\mathbf{v}(t)\|_{\mathbb{H}^0}^2)^{1/2} (1 + \mathbb{E}\|\mathbf{u}(t)\|_{L^4}^4)\right). \end{aligned}$$

In view of  $f(t) := \int_0^t \mathbb{E}\|\mathbf{u}(s)\|_{L^4}^4 ds \leq C_N(1+t)$  (see (4.5)), integrating the above inequality and using (5.14), we obtain

$$\begin{aligned} \int_0^\infty \mathbb{E}|\dot{v}(t)|^2 dt &\leq C_{m, \varepsilon_1, \lambda_1, N, \gamma} \cdot e^{C_N \|\mathbf{u}_0\|_{\mathbb{H}^1}^2} \cdot \left(1 + \int_0^\infty e^{-\gamma t} df(t)\right) \\ &\leq C_{m, \varepsilon_1, \lambda_1, N, \gamma} \cdot e^{C_N \|\mathbf{u}_0\|_{\mathbb{H}^1}^2}. \end{aligned} \quad (5.15)$$

The proof is thus complete by combining (5.13)-(5.15).  $\square$

## 5.2. A Support Property of Invariant Measure.

**Proposition 5.6.** *The point 0 belongs to the support of any invariant measure of  $(\mathbf{T}_t)_{t \geq 0}$ .*

For the proof we need the following lemma, whose proof in turn is inspired by [5].

**Lemma 5.7.** *For any  $r_1, r_2 > 0$ , there exists a  $T > 0$  such that*

$$\inf_{\|\mathbf{u}_0\|_{\mathbb{H}^1} \leq r_1} P\{\omega : \|\mathbf{u}(T, \omega; \mathbf{u}_0)\|_{\mathbb{H}^1} \leq r_2\} > 0.$$

*Proof.* Set

$$\mathbf{v}(t) := \mathbf{u}(t) - \mathbf{w}(t).$$

Then

$$\mathbf{v}'(t) = A(\mathbf{v}(t) + \mathbf{w}(t)), \quad \mathbf{v}(0) = \mathbf{u}_0.$$

Let  $T > 0$  and  $\epsilon \in (0, 1)$ , to be determined below. We assume that

$$\sup_{t \in [0, T]} \|\mathbf{w}(t)\|_{\mathbb{H}^6} < \epsilon. \quad (5.16)$$

First of all, by the chain rule, we have

$$\frac{d}{dt} \|\mathbf{v}(t)\|_{\mathbb{H}^0}^2 = J_1 + J_2 + J_3 + J_4,$$

where

$$\begin{aligned} J_1 &:= -2\|\nabla \mathbf{v}(t)\|_{\mathbb{H}^0}^2 + 2\langle \Delta \mathbf{w}(t), \mathbf{v}(t) \rangle_{\mathbb{H}^0} \\ J_2 &:= -2\langle \mathbf{v}(t), ((\mathbf{v}(t) + \mathbf{w}(t)) \cdot \nabla)(\mathbf{v}(t) + \mathbf{w}(t)) \rangle_{\mathbb{H}^0} \\ J_3 &:= -2\langle \mathbf{v}(t) + \mathbf{w}(t), g_N(|\mathbf{v}(t) + \mathbf{w}(t)|^2)(\mathbf{v}(t) + \mathbf{w}(t)) \rangle_{\mathbb{H}^0} \\ J_4 &:= 2\langle \mathbf{w}(t), g_N(|\mathbf{v}(t) + \mathbf{w}(t)|^2)(\mathbf{v}(t) + \mathbf{w}(t)) \rangle_{\mathbb{H}^0}. \end{aligned}$$

For  $J_1$ , by (5.16) we have

$$J_1 \leq -2\|\nabla \mathbf{v}(t)\|_{\mathbb{H}^0}^2 + C\epsilon \|\mathbf{v}(t)\|_{\mathbb{H}^0}.$$

Here and below,  $C$  denotes an absolute constant.

For  $J_2$ , by the Sobolev inequality (2.1) and (5.16) we have

$$J_2 = -2\langle \mathbf{w}(t), ((\mathbf{v}(t) + \mathbf{w}(t)) \cdot \nabla)(\mathbf{v}(t) + \mathbf{w}(t)) \rangle_{\mathbb{H}^0}$$

$$\begin{aligned}
&\leq 2\|\nabla \mathbf{w}(t)\|_{L^\infty} \|\mathbf{v}(t) + \mathbf{w}(t)\|_{\mathbb{H}^0}^2 \\
&\leq C\epsilon \cdot \|\mathbf{v}(t)\|_{\mathbb{H}^0}^2 + C\epsilon.
\end{aligned}$$

For  $J_3$ , we obviously have

$$J_3 \leq 0.$$

For  $J_4$ , by (2.1) and Young's inequality we have

$$\begin{aligned}
J_4 &\leq 2\|\mathbf{w}(t)\|_{L^\infty} \cdot \|\mathbf{v}(t) + \mathbf{w}(t)\|_{L^3}^3 \\
&\leq C\epsilon \cdot \|\mathbf{v}(t)\|_{L^3}^3 + C\epsilon^4 \\
&\leq C\epsilon \cdot \|\nabla \mathbf{v}(t)\|_{\mathbb{H}^0}^{3/2} \|\mathbf{v}(t)\|_{\mathbb{H}^0}^{3/2} + C\epsilon^4 \\
&\leq \|\nabla \mathbf{v}(t)\|_{\mathbb{H}^0}^2 + C\epsilon^4 \cdot \|\mathbf{v}(t)\|_{\mathbb{H}^0}^6 + C\epsilon^4.
\end{aligned}$$

Combing the above calculations gives that

$$\begin{aligned}
\frac{d}{dt} \|\mathbf{v}(t)\|_{\mathbb{H}^0}^2 &\leq -\|\nabla \mathbf{v}(t)\|_{\mathbb{H}^0}^2 + C\epsilon \cdot \|\mathbf{v}(t)\|_{\mathbb{H}^0}^6 + C\epsilon \\
&\leq -\frac{1}{\lambda_1} \|\mathbf{v}(t)\|_{\mathbb{H}^0}^2 + C\epsilon \cdot \|\mathbf{v}(t)\|_{\mathbb{H}^0}^6 + C\epsilon,
\end{aligned}$$

where the second step is due to the Poincare inequality (5.1).

Note that  $\|\mathbf{v}(t)\|_{\mathbb{H}^0}^2$  depends on  $\epsilon$  through (5.16). By Lemma 6.1 in the Appendix, for any  $\delta, h > 0$ , we may choose a  $T_0 > 0$  sufficiently large and an  $\epsilon$  small enough such that

$$\sup_{t \in [0, T_0]} \|\mathbf{v}(t)\|_{\mathbb{H}^0} \leq 2r_1, \quad (5.17)$$

and

$$\sup_{t \in [T_0, T_0+h]} \|\mathbf{v}(t)\|_{\mathbb{H}^0} < \delta. \quad (5.18)$$

Let us now turn to the estimate of the first order Sobolev norm of  $\mathbf{v}(t)$ . By the chain rule again, we have

$$\frac{d}{dt} \|\mathbf{v}(t)\|_{\mathbb{H}^1}^2 = 2[\mathbf{v}(t), A(\mathbf{v}(t) + \mathbf{w}(t))] = I_1 + I_2 + I_3 + I_4,$$

where

$$\begin{aligned}
I_1 &:= 2[\mathbf{v}(t) + \mathbf{w}(t), A(\mathbf{v}(t) + \mathbf{w}(t))] \\
I_2 &:= -2\langle \Delta^2 \mathbf{w}(t), \mathbf{v}(t) + \mathbf{w}(t) \rangle_{\mathbb{H}^0} \\
I_3 &:= 2\langle \Delta \mathbf{w}(t), ((\mathbf{v}(t) + \mathbf{w}(t)) \cdot \nabla)(\mathbf{v}(t) + \mathbf{w}(t)) \rangle_{\mathbb{H}^0} \\
I_4 &:= 2\langle \Delta \mathbf{w}(t), g_N(|\mathbf{v}(t) + \mathbf{w}(t)|^2)(\mathbf{v}(t) + \mathbf{w}(t)) \rangle_{\mathbb{H}^0}.
\end{aligned}$$

For  $I_1$ , by (2.10) and (5.12) we have

$$\begin{aligned}
I_1 &\leq -\|\mathbf{v}(t) + \mathbf{w}(t)\|_{\mathbb{H}^2}^2 + C_N \|\mathbf{v}(t) + \mathbf{w}(t)\|_{\mathbb{H}^1}^2 \\
&\leq -\frac{1}{2} \|\mathbf{v}(t)\|_{\mathbb{H}^2}^2 + 4N \|\mathbf{v}(t)\|_{\mathbb{H}^1}^2 + C_N \epsilon \\
&\leq -\frac{1}{4} \|\mathbf{v}(t)\|_{\mathbb{H}^2}^2 + C_N \|\mathbf{v}(t)\|_{\mathbb{H}^0}^2 + C_N \epsilon.
\end{aligned}$$

Here and below,  $C_N$  denotes a constant only depending on  $N$ .

For  $I_2$ , we have

$$I_2 \leq C\epsilon + C\epsilon \|\mathbf{v}(t)\|_{\mathbb{H}^0}.$$

For  $I_3$ , we have

$$\begin{aligned} I_3 &\leq 2\|\nabla\Delta\mathbf{w}(t)\|_{L^\infty}\|\mathbf{v}(t) + \mathbf{w}(t)\|_{\mathbb{H}^0}^2 \\ &\leq C\epsilon + C\epsilon\|\mathbf{v}(t)\|_{\mathbb{H}^0}^2. \end{aligned}$$

For  $I_4$ , we have

$$I_4 \leq C\epsilon + C\epsilon\|\mathbf{v}(t)\|_{L^3}^3 \leq \frac{1}{8}\|\mathbf{v}(t)\|_{\mathbb{H}^2}^2 + C\epsilon + C\epsilon\|\mathbf{v}(t)\|_{\mathbb{H}^0}^6.$$

Combing the above calculations gives that

$$\begin{aligned} \frac{d}{dt}\|\mathbf{v}(t)\|_{\mathbb{H}^1}^2 &\leq -\frac{1}{8}\|\mathbf{v}(t)\|_{\mathbb{H}^2}^2 + C_N \cdot \|\mathbf{v}(t)\|_{\mathbb{H}^0}^2 + C\epsilon \cdot \|\mathbf{v}(t)\|_{\mathbb{H}^0}^6 + C\epsilon \\ &\leq -C_0\|\mathbf{v}(t)\|_{\mathbb{H}^1}^2 + C_N \cdot \|\mathbf{v}(t)\|_{\mathbb{H}^0}^2 + C\epsilon\|\mathbf{v}(t)\|_{\mathbb{H}^0}^6 + C\epsilon. \end{aligned}$$

By Gronwall's inequality, for any  $0 < t_1 < t_2$  we have

$$\begin{aligned} \|\mathbf{v}(t_2)\|_{\mathbb{H}^1}^2 &\leq e^{-C_0(t_2-t_1)}\|\mathbf{v}(t_1)\|_{\mathbb{H}^1}^2 + \frac{1}{C_0} \left( C_N \cdot \sup_{t \in [t_1, t_2]} \|\mathbf{v}(t)\|_{\mathbb{H}^0}^2 \right. \\ &\quad \left. + C\epsilon \cdot \sup_{t \in [t_1, t_2]} \|\mathbf{v}(t)\|_{\mathbb{H}^0}^6 + C\epsilon \right). \end{aligned}$$

Firstly, letting  $t_1 = 0$  and  $t_2 = T_0$  by (5.17) gives

$$\begin{aligned} \|\mathbf{v}(T_0)\|_{\mathbb{H}^1}^2 &\leq r_1^2 + \frac{1}{C_0} \left( C_N \cdot \sup_{t \in [0, T_0]} \|\mathbf{v}(t)\|_{\mathbb{H}^0}^2 \right. \\ &\quad \left. + C\epsilon \cdot \sup_{t \in [0, T_0]} \|\mathbf{v}(t)\|_{\mathbb{H}^0}^6 + C\epsilon \right) \\ &\leq C_{N, C_0}(r_1^6 + 1). \end{aligned}$$

Secondly, letting  $t_1 = T_0$  and  $t_2 = T_0 + h$  yields

$$\begin{aligned} \|\mathbf{v}(T_0 + h)\|_{\mathbb{H}^1}^2 &\leq e^{-C_0 h} C_{N, C_0}(r_1^6 + 1) + \frac{1}{C_0} \left( C_N \cdot \sup_{t \in [T_0, T_0+h]} \|\mathbf{v}(t)\|_{\mathbb{H}^0}^2 \right. \\ &\quad \left. + C\epsilon \cdot \sup_{t \in [T_0, T_0+h]} \|\mathbf{v}(t)\|_{\mathbb{H}^0}^6 + C\epsilon \right), \end{aligned}$$

which together with (5.18) implies that for some  $T$  large enough and  $\epsilon > 0$  small enough

$$\|\mathbf{v}(T)\|_{\mathbb{H}^1} \leq r_2/2.$$

Therefore, there exist a  $T$  sufficiently large and an  $\epsilon$  small enough such that for any  $\|\mathbf{u}_0\|_{\mathbb{H}^1} \leq r_1$

$$\|\mathbf{u}(T, \omega; \mathbf{u}_0)\|_{\mathbb{H}^1} \leq r_2.$$

That is, if we set

$$\Omega_\epsilon := \left\{ \omega : \sup_{t \in [0, T]} \|\mathbf{w}(t, \omega)\|_{\mathbb{H}^6} < \epsilon \right\},$$

then

$$\Omega_\epsilon \subset \cap_{\|\mathbf{u}_0\|_{\mathbb{H}^1} \leq r_1} \{\omega : \|\mathbf{u}(T, \omega; \mathbf{u}_0)\|_{\mathbb{H}^1} \leq r_2\}.$$

The desired estimate now follows from the fact that  $\Omega_\epsilon$  is an open subset of  $\Omega$  and  $P(\Omega_\epsilon) > 0$ .  $\square$

*Proof of Proposition 5.6:* For  $r > 0$ , let  $\mathbb{B}_r := \{\mathbf{u}_0 \in \mathbb{H}^1 : \|\mathbf{u}_0\|_{\mathbb{H}^1} \leq r\}$  be the ball in  $\mathbb{H}^1$ . For each invariant measure  $\mu$ , we can choose some  $r_1 > 0$  such that

$$\mu(\mathbb{B}_{r_1}) \geq 1/2.$$

By Lemma 5.7, we further have for any  $r_2 > 0$

$$\begin{aligned} \mu(\mathbb{B}_{r_2}) &= \int_{\mathbb{H}^1} (\mathbf{T}_T 1_{\mathbb{B}_{r_2}})(\mathbf{u}_0) \mu(d\mathbf{u}_0) \\ &\geq \int_{\mathbb{B}_{r_1}} (\mathbf{T}_T 1_{\mathbb{B}_{r_2}})(\mathbf{u}_0) \mu(d\mathbf{u}_0) \\ &\geq \inf_{\mathbf{u}_0 \in \mathbb{B}_{r_1}} (\mathbf{T}_T 1_{\mathbb{B}_{r_2}})(\mathbf{u}_0) \cdot \mu(\mathbb{B}_{r_1}) > 0, \end{aligned}$$

which means that 0 belongs to the support of  $\mu$ .

*Proof of Theorem 5.1:* The assertion follows from Propositions 5.5 and 5.6 due to [14, Proposition 3.12, Corollary 3.17].

## 6. APPENDIX

**6.1. Two Basic Estimates.** In this subsection, we prove two basic estimates used in Section 5.

**Lemma 6.1.** *Let  $\{\varphi_\epsilon(\cdot, r_0), \epsilon \in (0, 1), r_0 \geq 0\}$  be a family of positive real functions on  $\mathbb{R}_+$  with  $\varphi_\epsilon(0, r_0) = r_0$ . Suppose that for some  $p > 1$ ,  $C_0, C_1, C_2 > 0$ ,  $C_3 \geq 0$  and any  $\epsilon \in (0, 1)$  and  $t \geq 0$*

$$\varphi'_\epsilon(t, r_0) \leq -C_0 \varphi_\epsilon(t, r_0) + C_1 \epsilon \cdot \varphi_\epsilon(t, r_0)^p + C_2 \epsilon + C_3.$$

*Then: (i) For any  $T > 0$  and  $R > 0$ , there exists  $\epsilon_0 > 0$  such that*

$$\sup_{t \in [0, T], \epsilon \in [0, \epsilon_0], r_0 \in [0, R]} \varphi_\epsilon(t, r_0) \leq 2R + 2C_3/C_0.$$

*(ii) If  $C_3 = 0$ , then for any  $\delta > 0$  and  $R, h > 0$ , there exists  $T > 0$  and  $\epsilon_0 > 0$  such that*

$$\sup_{t \in [T, T+h], \epsilon \in [0, \epsilon_0], r_0 \in [0, R]} \varphi_\epsilon(t, r_0) \leq \delta.$$

*Proof.* Let  $C_4^\epsilon := (C_2 \epsilon + C_3)/C_0$  and set

$$\phi(t) := e^{C_0 t} (\varphi_\epsilon(t, r_0) - C_4^\epsilon).$$

Then for fixed  $T > 0$  and any  $t \in [0, T]$

$$\phi'(t) \leq C_1 e^{C_0 t} \epsilon \cdot \varphi_\epsilon(t, r_0)^p \leq C_1 \epsilon \cdot (\phi(t) + C_4^\epsilon \cdot e^{C_0 T})^p.$$

Solving this differential inequality gives that

$$\phi(T) \leq \left[ (\phi(0) + C_4^\epsilon \cdot e^{C_0 T})^{1-p} + C_1(1-p)\epsilon T \right]^{\frac{1}{1-p}} - C_4^\epsilon \cdot e^{C_0 T}.$$

Hence,

$$\begin{aligned} \varphi_\epsilon(T, r_0) &\leq e^{-C_0 T} \left[ (r_0 + C_4^\epsilon \cdot (e^{C_0 T} - 1))^{1-p} + C_1(1-p)\epsilon T \right]^{\frac{1}{1-p}} \\ &\leq \left[ (e^{-C_0 T} R + C_4^\epsilon)^{1-p} + C_1(1-p)\epsilon T e^{(p-1)C_0 T} \right]^{\frac{1}{1-p}}. \end{aligned}$$

Now the assertions easily follow by suitable choices of  $\epsilon$  and  $T$ . □

We now prove the following exponential estimate.

**Lemma 6.2.** *Let  $X_t$  be a positive Itô process of the form*

$$X_t = x_0 + \int_0^t M_s dW_s + \int_0^t N_s ds, \quad (6.1)$$

where  $s \mapsto M_s, N_s$  are two measurable adapted processes. Suppose that there exist a positive process  $Y_s$  and some  $\alpha > 1$  and  $C_0, C_1, C_2, C_3 > 0$  such that for any  $s \geq 0$

$$N_s \leq -C_0 X_s^\alpha - Y_s + C_1, \quad |M_s|^2 \leq C_2 X_s + C_3. \quad (6.2)$$

Then for any  $t, \eta > 0$

$$\mathbb{E}e^{\eta X_t} \leq C_{\alpha, \eta} \cdot \exp\{e^{-C_0 t/2} \eta x_0\}, \quad (6.3)$$

and

$$\mathbb{E} \exp \left\{ \eta \int_0^t \left( \frac{C_0}{2} X_s^\alpha + Y_s \right) ds \right\} \leq \exp\{\eta x_0 + C_{\alpha, \eta} t\}. \quad (6.4)$$

*Proof.* Let us first prove that for any  $t, \eta > 0$

$$\mathbb{E}e^{\eta X_t} < +\infty. \quad (6.5)$$

Set for  $R > 0$

$$\tau_R := \inf\{t \geq 0 : |X_t| \geq R\}.$$

By Itô's formula, (6.2) and Young's inequality, we have

$$\begin{aligned} de^{\eta X_t} &= \eta e^{\eta X_t} M_t dW_t + \eta e^{\eta X_t} N_t dt + \frac{\eta^2}{2} e^{\eta X_t} |M_t|^2 dt \\ &\leq \eta e^{\eta X_t} M_t dW_t + \eta e^{\eta X_t} (-C_0 X_t^\alpha + C_1 + \frac{\eta}{2}(C_2 X_t + C_3)) dt \\ &\leq \eta e^{\eta X_t} M_t dW_t + \eta e^{\eta X_t} \left(-\frac{C_0}{2} X_t^\alpha + C_{\alpha, \eta}\right) dt \\ &\leq \eta e^{\eta X_t} M_t dW_t + \eta e^{\eta X_t} \left(-\frac{C_0}{2} X_t + C_{\alpha, \eta}\right) dt. \end{aligned}$$

Set

$$f_R(t) := \mathbb{E}e^{\eta X_{t \wedge \tau_R}}.$$

Then

$$f'_R(t) \leq C_{\alpha, \eta} f_R(t).$$

Hence

$$f_R(t) = \mathbb{E}e^{\eta X_{t \wedge \tau_R}} \leq e^{\eta x_0} e^{C_{\alpha, \eta} t}.$$

By Fatou's lemma, we obtain (6.5).

We now set

$$f(t) := \mathbb{E}e^{\eta X_t}.$$

Then by Jensen's inequality, we obtain

$$f'(t) \leq -\frac{C_0}{2} f(t) \log f(t) + C_{\alpha, \eta} f(t).$$

Solving this differential equality gives the first estimate (6.3).

On the other hand, for any  $t, \eta > 0$ , we have by (6.1) and (6.2)

$$\eta \int_0^t \left( \frac{C_0}{2} X_s^\alpha + Y_s \right) ds \leq \eta x_0 + \eta \int_0^t M_s dW_s + \int_0^t \left( -\frac{\eta C_0}{2} X_s^\alpha + C_1 \eta \right) ds.$$

Noting that by (6.2) and (6.3)

$$\mathbb{E} \exp \left\{ \frac{\eta^2}{2} \int_0^t |M_s|^2 ds \right\} \leq \frac{1}{t} \int_0^t \mathbb{E} \exp \{ t \eta^2 |M_s|^2 / 2 \} ds < +\infty,$$

we know by Novikov's criterion that

$$t \mapsto \exp \left\{ \eta \int_0^t M_s dW_s - \frac{\eta^2}{2} \int_0^t |M_s|^2 ds \right\} =: \mathcal{E}(M)(t)$$

is an exponential martingale. Moreover, by (6.2) and Young's inequality

$$\frac{\eta^2}{2} |M_s|^2 - \frac{\eta C_0}{2} X_s^\alpha + C_1 \eta \leq C_{\alpha, \eta}.$$

Therefore,

$$\begin{aligned} \mathbb{E} \exp \left\{ \eta \int_0^t \left( \frac{C_0}{2} X_s^\alpha + Y_s \right) ds \right\} &\leq e^{\eta x_0} \mathbb{E} (\mathcal{E}(M)(t) \cdot \exp \{ C_{\alpha, \eta} t \}) \\ &= e^{\eta x_0} \cdot \exp \{ C_{\alpha, \eta} t \}. \end{aligned}$$

The proof is thus complete.  $\square$

**6.2. Proof of the Derivative Flow Equation.** In this subsection, we prove (5.3). Note that (5.5) can be proved similarly.

**Lemma 6.3.** *For any  $T > 0$ , there exists a constant  $C_{N, T} > 0$  such that for each  $\omega$  and  $\mathbf{u}_0 \in \mathbb{H}^1$*

$$\sup_{t \in [0, T]} \|\mathbf{u}(t, \omega)\|_{\mathbb{H}^1}^2 + \int_0^T \|\mathbf{u}(t, \omega)\|_{\mathbb{H}^2}^2 \leq C_{N, T} \left( 1 + \|\mathbf{u}_0\|_{\mathbb{H}^1}^6 + \sup_{t \in [0, T]} \|\mathbf{w}(t, \omega)\|_{\mathbb{H}^5}^{12} \right).$$

*Proof.* Following the proof of Lemma 5.7, let us give different estimates for  $J_i, i = 1, 2, 3, 4$ .

For  $J_1$ , by (5.16) we have

$$J_1 \leq -2 \|\nabla \mathbf{v}(t)\|_{\mathbb{H}^0}^2 + 2 \|\Delta \mathbf{w}(t)\|_{\mathbb{H}^0} \cdot \|\mathbf{v}(t)\|_{\mathbb{H}^0}.$$

For  $J_2$ , by the Sobolev inequality (2.1) and Young's inequality we have

$$\begin{aligned} J_2 &= -2 \langle \mathbf{w}(t), ((\mathbf{v}(t) + \mathbf{w}(t)) \cdot \nabla)(\mathbf{v}(t) + \mathbf{w}(t)) \rangle_{\mathbb{H}^0} \\ &= 2 \langle \nabla \mathbf{w}(t), (\mathbf{v}(t) + \mathbf{w}(t)) \otimes (\mathbf{v}(t) + \mathbf{w}(t)) \rangle_{\mathbb{H}^0} \\ &\leq 2 \|\nabla \mathbf{w}(t)\|_{L^\infty} \|\mathbf{v}(t) + \mathbf{w}(t)\|_{\mathbb{H}^0}^2 \\ &\leq C \|\nabla \mathbf{w}(t)\|_{L^\infty} \|\mathbf{v}(t) + \mathbf{w}(t)\|_{L^4}^2 \\ &\leq \|\mathbf{v}(t) + \mathbf{w}(t)\|_{L^4}^4 + C \|\nabla \mathbf{w}(t)\|_{L^\infty}^2. \end{aligned}$$

For  $J_3$ , we have

$$J_3 \leq -2 \|\mathbf{v}(t) + \mathbf{w}(t)\|_{L^4}^4 + N \|\mathbf{v}(t) + \mathbf{w}(t)\|_{\mathbb{H}^0}^2.$$

For  $J_4$ , by (2.1) and Young's inequality we have

$$\begin{aligned} J_4 &\leq 2 \|\mathbf{w}(t)\|_{L^\infty} \cdot \|\mathbf{v}(t) + \mathbf{w}(t)\|_{L^3}^3 \\ &\leq 2 \|\mathbf{w}(t)\|_{L^\infty} \cdot \|\mathbf{v}(t) + \mathbf{w}(t)\|_{L^4}^3 \\ &\leq -\|\mathbf{v}(t) + \mathbf{w}(t)\|_{L^4}^4 + C \|\mathbf{w}(t)\|_{L^\infty}^4. \end{aligned}$$

Hence

$$\frac{d}{dt} \|\mathbf{v}(t)\|_{\mathbb{H}^0}^2 \leq C_N \|\mathbf{v}(t)\|_{\mathbb{H}^0}^2 + C (\|\mathbf{w}(t)\|_{\mathbb{H}^2}^4 + \|\mathbf{w}(t)\|_{\mathbb{H}^3}^2).$$

By Gronwall's inequality, we get

$$\sup_{t \in [0, T]} \|\mathbf{v}(t)\|_{\mathbb{H}^0}^2 \leq C_{N, T} \left( \|\mathbf{v}_0\|_{\mathbb{H}^0}^2 + \sup_{t \in [0, T]} (\|\mathbf{w}(t)\|_{\mathbb{H}^2}^4 + \|\mathbf{w}(t)\|_{\mathbb{H}^3}^2) \right). \quad (6.6)$$

Using the similar calculations as in the proof of Lemma 5.7, one finds that

$$\frac{d}{dt} \|\mathbf{v}(t)\|_{\mathbb{H}^1}^2 \leq -\frac{1}{8} \|\mathbf{v}(t)\|_{\mathbb{H}^2}^2 + C_N (1 + \|\mathbf{w}(t)\|_{\mathbb{H}^5}^4) \cdot (1 + \|\mathbf{v}(t)\|_{\mathbb{H}^0}^6),$$

which together with (6.6) gives the desired estimate.  $\square$

For  $\mathbf{v} \in \mathbb{H}^1$ , let us consider a small perturbation of the initial values given by  $\mathbf{u}_\epsilon(0) = \mathbf{u}_0 + \epsilon \mathbf{v}$ . The corresponding solution of Eq. (5.2) is denoted by  $\mathbf{u}_\epsilon(t)$ .

Set

$$\mathbf{v}_\epsilon(t) := (\mathbf{u}_\epsilon(t) - \mathbf{u}(t))/\epsilon.$$

Then  $\mathbf{v}_\epsilon(t)$  satisfies

$$\begin{aligned} \mathbf{v}'_\epsilon(t) &= \Delta \mathbf{v}_\epsilon(t) - (\mathbf{u}_\epsilon(t) \cdot \nabla) \mathbf{v}_\epsilon(t) - (\mathbf{v}_\epsilon(t) \cdot \nabla) \mathbf{u}(t) \\ &\quad - g_N(|\mathbf{u}_\epsilon(t)|^2) \mathbf{v}_\epsilon(t) - (g_N(|\mathbf{u}_\epsilon(t)|^2) - g_N(|\mathbf{u}(t)|^2))/\epsilon \cdot \mathbf{u}(t), \end{aligned}$$

with initial value  $\mathbf{v}_\epsilon(0) = \mathbf{v}$ .

We have:

**Lemma 6.4.** *For any  $T > 0$ , there is a  $C_{N, T} > 0$  such that for any  $\epsilon \in (0, 1)$*

$$\sup_{t \in [0, T]} \|\mathbf{v}_\epsilon(t)\|_{\mathbb{H}^1}^2 + \int_0^T \|\mathbf{v}_\epsilon(t)\|_{\mathbb{H}^2}^2 dt \leq C_{T, N}.$$

*Proof.* As in the proof of Lemma 4.1, we have

$$\begin{aligned} \frac{d}{dt} \|\mathbf{v}_\epsilon(t)\|_{\mathbb{H}^1}^2 &\leq -\|\mathbf{v}_\epsilon(t)\|_{\mathbb{H}^2}^2 + 2\|\mathbf{v}_\epsilon(t)\|_{\mathbb{H}^1}^2 + C\|\mathbf{u}_\epsilon(t)\|_{L^6}^2 \cdot \|\nabla \mathbf{v}_\epsilon(t)\|_{L^3}^2 \\ &\quad + C\|\mathbf{v}_\epsilon(t)\|_{L^\infty}^2 \cdot \|\mathbf{u}(t)\|_{\mathbb{H}^1}^2 + C\|\mathbf{v}_\epsilon(t)\|_{L^6}^2 \cdot (\|\mathbf{u}_\epsilon(t)\|_{L^6}^4 + \|\mathbf{u}(t)\|_{L^6}^4) \\ &\leq -\frac{1}{2} \|\mathbf{v}_\epsilon(t)\|_{\mathbb{H}^2}^2 + C(\|\mathbf{u}_\epsilon(t)\|_{\mathbb{H}^1}^4 + \|\mathbf{u}(t)\|_{\mathbb{H}^1}^4 + 1) \cdot \|\mathbf{v}_\epsilon(t)\|_{\mathbb{H}^1}^2, \end{aligned}$$

which together with Lemma 6.3 gives the desired estimate.  $\square$

We are now in a position to prove (5.4). Set

$$\mathbf{j}_\epsilon(t) := \mathbf{v}_\epsilon(t) - \mathcal{J}_{0, t} \mathbf{v},$$

where  $\mathcal{J}_{0, t} \mathbf{v}$  satisfies (5.3).

By Taylor's formula, we have

$$\begin{aligned} &g_N(|\mathbf{u}_\epsilon(t)|^2) - g_N(|\mathbf{u}(t)|^2) \\ &= g'_N(|\mathbf{u}(t)|^2)(|\mathbf{u}_\epsilon(t)|^2 - |\mathbf{u}(t)|^2) + g''_N(\theta)(|\mathbf{u}_\epsilon(t)|^2 - |\mathbf{u}(t)|^2)^2/2 \\ &= \epsilon^2 \cdot g'_N(|\mathbf{u}(t)|^2) |\mathbf{v}_\epsilon(t)|^2 + 2g'_N(|\mathbf{u}(t)|^2) \langle \mathbf{v}_\epsilon(t), \mathbf{u}(t) \rangle_{\mathbb{R}^3} \\ &\quad + g''_N(\theta)(|\mathbf{u}_\epsilon(t)|^2 - |\mathbf{u}(t)|^2)^2/2, \end{aligned}$$

where  $\theta$  takes some value between  $|\mathbf{u}(t)|^2$  and  $|\mathbf{u}_\epsilon(t)|^2$ .

Thus, it is not hard to see that  $\mathbf{j}_\epsilon(t)$  satisfies

$$\mathbf{j}'_\epsilon(t) = \Delta \mathbf{j}_\epsilon(t) - \sum_{i=1}^8 J_i(t),$$

where

$$\begin{aligned}
J_1(t) &:= \epsilon \cdot (\mathbf{v}_\epsilon(t) \cdot \nabla) \mathbf{v}_\epsilon(t) \\
J_2(t) &:= (\mathbf{u}(t) \cdot \nabla) \mathbf{j}_\epsilon(t) \\
J_3(t) &:= (\mathbf{j}_\epsilon(t) \cdot \nabla) \mathbf{u}(t) \\
J_4(t) &:= (g_N(|\mathbf{u}_\epsilon(t)|^2) - g_N(|\mathbf{u}(t)|^2)) \cdot \mathbf{v}_\epsilon(t) \\
J_5(t) &:= g_N(|\mathbf{u}(t)|^2) \cdot \mathbf{j}_\epsilon(t) \\
J_6(t) &:= \epsilon \cdot g'_N(|\mathbf{u}(t)|^2) |\mathbf{v}_\epsilon(t)|^2 \cdot \mathbf{u}(t) \\
J_7(t) &:= 2g'_N(|\mathbf{u}(t)|^2) \langle \mathbf{j}_\epsilon(t), \mathbf{u}(t) \rangle_{\mathbb{R}^3} \cdot \mathbf{u}(t) \\
J_8(t) &:= g''_N(\theta) (|\mathbf{u}_\epsilon(t)|^2 - |\mathbf{u}(t)|^2)^2 \cdot \mathbf{u}(t) / \epsilon.
\end{aligned}$$

By the chain rule and Young's inequality, we have

$$\frac{d}{dt} \|\mathbf{j}_\epsilon(t)\|_{\mathbb{H}^1}^2 \leq -\|\mathbf{j}_\epsilon(t)\|_{\mathbb{H}^2}^2 + 2\|\mathbf{j}_\epsilon(t)\|_{\mathbb{H}^0}^2 + C \sum_{i=1}^8 \|J_i(t)\|_{\mathbb{H}^0}^2.$$

For  $J_1(t)$ , we have

$$\|J_1(t)\|_{\mathbb{H}^0}^2 \leq C\epsilon^2 \cdot \|\mathbf{v}_\epsilon(t)\|_{\mathbb{H}^1}^2 \cdot \|\mathbf{v}_\epsilon(t)\|_{\mathbb{H}^2}^2.$$

Here and below, the constant  $C$  is independent of  $\epsilon$ .

For  $J_2(t)$ , we have

$$\|J_2(t)\|_{\mathbb{H}^0}^2 \leq C \|\mathbf{u}(t)\|_{\mathbb{H}^2}^2 \cdot \|\mathbf{j}_\epsilon(t)\|_{\mathbb{H}^1}^2.$$

For  $J_3(t)$ , we have

$$\begin{aligned}
\|J_3(t)\|_{\mathbb{H}^0}^2 &\leq \|\mathbf{u}(t)\|_{\mathbb{H}^1}^2 \cdot \|\mathbf{j}_\epsilon(t)\|_{L^\infty}^2 \leq C \|\mathbf{u}(t)\|_{\mathbb{H}^1}^2 \cdot \|\mathbf{j}_\epsilon(t)\|_{\mathbb{H}^1} \cdot \|\mathbf{j}_\epsilon(t)\|_{\mathbb{H}^2} \\
&\leq C \|\mathbf{u}(t)\|_{\mathbb{H}^1}^4 \cdot \|\mathbf{j}_\epsilon(t)\|_{\mathbb{H}^1}^2 + \frac{1}{4} \|\mathbf{j}_\epsilon(t)\|_{\mathbb{H}^2}^2.
\end{aligned}$$

For  $J_4(t)$ , we have

$$\begin{aligned}
\|J_4(t)\|_{\mathbb{H}^0}^2 &\leq C\epsilon^2 \cdot \|\mathbf{v}_\epsilon(t)\|_{L^6}^4 \cdot (\|\mathbf{u}_\epsilon(t)\|_{L^6}^2 + \|\mathbf{u}(t)\|_{L^6}^2) \\
&\leq C\epsilon^2 \cdot \|\mathbf{v}_\epsilon(t)\|_{\mathbb{H}^1}^4 \cdot (\|\mathbf{u}_\epsilon(t)\|_{\mathbb{H}^1}^2 + \|\mathbf{u}(t)\|_{\mathbb{H}^1}^2).
\end{aligned}$$

For  $J_5(t)$ , we have

$$\|J_5(t)\|_{\mathbb{H}^0}^2 \leq C \|\mathbf{u}(t)\|_{L^6}^4 \cdot \|\mathbf{j}_\epsilon(t)\|_{L^6}^2 \leq C \|\mathbf{u}(t)\|_{\mathbb{H}^1}^4 \cdot \|\mathbf{j}_\epsilon(t)\|_{\mathbb{H}^1}^2.$$

For  $J_6(t)$ , we have

$$\|J_6(t)\|_{\mathbb{H}^0}^2 \leq C\epsilon^2 \cdot \|\mathbf{v}_\epsilon(t)\|_{\mathbb{H}^1}^2 \cdot \|\mathbf{u}(t)\|_{\mathbb{H}^1}^2.$$

For  $J_7(t)$ , we have

$$\|J_7(t)\|_{\mathbb{H}^0}^2 \leq C \cdot \|\mathbf{j}_\epsilon(t)\|_{\mathbb{H}^1}^2 \cdot \|\mathbf{u}(t)\|_{\mathbb{H}^1}^4.$$

For  $J_8(t)$ , we have

$$\begin{aligned}
\|J_8(t)\|_{\mathbb{H}^0}^2 &\leq C\epsilon^2 \cdot \|\mathbf{v}_\epsilon(t)\|^2 \cdot (|\mathbf{u}_\epsilon(t)|^2 + |\mathbf{u}(t)|^2)_{\mathbb{H}^0}^2 \\
&\leq C\epsilon^2 \cdot \|\mathbf{v}_\epsilon(t)\|_{L^4}^4 \cdot (\|\mathbf{u}_\epsilon(t)\|_{L^4}^4 + \|\mathbf{u}(t)\|_{L^4}^4) \\
&\leq C\epsilon^2 \cdot \|\mathbf{v}_\epsilon(t)\|_{\mathbb{H}^1}^4 \cdot (\|\mathbf{u}_\epsilon(t)\|_{\mathbb{H}^1}^4 + \|\mathbf{u}(t)\|_{\mathbb{H}^1}^4).
\end{aligned}$$

Combining the above calculations and Lemmas 6.3 and 6.4 yields that

$$\frac{d}{dt} \|\mathbf{j}_\epsilon(t)\|_{\mathbb{H}^1}^2 \leq C\epsilon^2 (1 + \|\mathbf{v}_\epsilon(t)\|_{\mathbb{H}^2}^2) + C(1 + \|\mathbf{u}(t)\|_{\mathbb{H}^2}^2) \cdot \|\mathbf{j}_\epsilon(t)\|_{\mathbb{H}^1}^2.$$

By Gronwall's inequality, we get

$$\|\mathbf{j}_\epsilon(t)\|_{\mathbb{H}^1}^2 \leq C\epsilon^2 \left( 1 + \int_0^t \|\mathbf{v}_\epsilon(s)\|_{\mathbb{H}^2}^2 ds \right) \cdot \exp \left\{ C + C \int_0^t \|\mathbf{u}(s)\|_{\mathbb{H}^2}^2 ds \right\},$$

which together with Lemmas 6.3 and 6.4 clearly gives (5.4).

### Acknowledgements:

This work was done while the second named author was a fellow of Alexander-Humboldt Foundation in Bielefeld University. He is very grateful to the generous support of Alexander-Humboldt Foundation. Financial supports of the DFG through SFB 701 and ARC Discovery grant DP0663153 of Australia are also gratefully acknowledged.

### REFERENCES

- [1] A. Bensoussan and R. Temam: Equations stochastiques de type Navier-Stokes. *J. Funct. Anal.*, 13, 195-222, (1973).
- [2] G. Da Prato and J. Zabczyk: *Stochastic equations in infinite dimensions*. Cambridge: Cambridge University Press, 1992.
- [3] G. Da Prato and A. Debussche: Ergodicity for the 3D stochastic Navier-Stokes equations. *J. Math. Pures Appl.* (9) 82 (2003), no. 8, 877-947.
- [4] A. Debussche and C. Odasso: Markov solutions for the 3D stochastic Navier-Stokes equations with state dependent noise. *J. Evol. Equ.* 6 (2006), no. 2, 305-324.
- [5] W. E and J. C. Mattingly: Ergodicity for the Navier-Stokes equation with degenerate random forcing: finite-dimensional approximation. *Comm. Pure Appl. Math.* 54 (2001), no. 11, 1386-1402.
- [6] W. E, J.C. Mattingly and Ya. Sinai: Gibbsian dynamics and ergodicity for the stochastically forced Navier-Stokes equation. *Comm. Math. Phys.* 224 (2001), no. 1, 83-106.
- [7] E. B. Fabes, B. F. Jones, N. M. Riviere: The initial value problem for the Navier-Stokes equations with data in  $L^p$ . *Arch. Rational Mech. Anal.*, 45 (1972), 222-240.
- [8] F. Flandoli: Dissipativity and invariant measures for stochastic Navier-Stokes equations. *NoDEA* 1, 403-426,(1994).
- [9] F. Flandoli and D. Gatarek: Martingale and stationary solutions for stochastic Navier-Stokes equations. *Probab. Theory Related Fields*, 102, 367-391(1995).
- [10] F. Flandoli, B. Maslowski: Ergodicity of the 2-D Navier-Stokes equation under random perturbations. *Comm. Math. Phys.* 172 (1995), no. 1, 119-141.
- [11] F. Flandoli, M. Romito: Markov selections for the 3D stochastic Navier-Stokes equations. *Probab. Theory Related Fields*, 140:407-458(2008).
- [12] G.P. Galdi: An introduction to the Navier-Stokes initial-boundary value problem. *Fundamental directions in mathematical fluid mechanics*, 1-70, Adv. Math. Fluid Mech., Birkhäuser, Basel, 2000.
- [13] B. Goldys, M. Röckner and X. Zhang: Martingale Solutions and Markov Selections for Stochastic Evolution Equations. Preprint.
- [14] M. Hairer and J.C. Mattingly: Ergodicity of the 2D Navier-Stokes equations with degenerate stochastic forcing. *Annals of Mathematics*, 164(2006), 993-1032.
- [15] O. Kallenberg: *Foundations of Modern Probability*, Second Edition. Springer-Verlag, New-York, Berlin, 2001.
- [16] N. V. Krylov: A simple proof of the existence of a solution to the Itô equation with monotone coefficients. *Theory Probab. Appl.* 35 (1990), no. 3, 583-587.
- [17] O.A. Ladyzhenskaya: *The mathematical theory of viscous incompressible flow*. Second English edition, revised and enlarged. Translated from the Russian by Richard A. Silverman and John Chu. Mathematics and its Applications, Vol. 2 Gordon and Breach, Science Publishers, New York-London-Paris 1969 xviii+224.
- [18] J. Leray: Sur le mouvement d'un liquide visqueux emplissant l'espace, *Acta Math.*, 63(1934), 193-248.

- [19] P.L. Lions: *Mathematical Topics in Fluid Mechanics, Volume 1, Incompressible Models*. Oxford Lect. Series in Math. and its App. 3, 1996.
- [20] P. Malliavin: *Stochastic Analysis*, Springer-Verlag, Berlin-NewYork, 1995.
- [21] J.C. Mattingly: Exponential Convergence for the Stochastically Forced Navier-Stokes Equations and other Partially Dissipative Dynamics. *Commun. Math. Phys.*, 230,421-462,(2002).
- [22] R. Mikulevicius, B.L. Rozovskii: Global  $L_2$ -solution of Stochastic Navier-Stokes Equations, *Ann. of Prob.*, 2005, Vol.33, No.1, 137-176.
- [23] C. Prévôt and M. Röckner: *A concise course on stochastic partial differential equations*. Lecture Notes in Mathematics, 1905. Springer, Berlin, 2007. vi+144 pp.
- [24] M. Röckner, X. Zhang: Tamed 3D Navier-Stokes Equation: Existence, Uniqueness and Regularity. <http://arXiv:math/0703254>.
- [25] M. Röckner, B. Schmulland and X. Zhang: Yamada-Watanabe Theorem and Markov Property for Stochastic Evolution Equation. BiBos-Preprint, 07-12-269(2007).
- [26] B.L. Rozovskii: *Stochastic evolution systems. Linear theory and applications to nonlinear filtering*. Mathematics and its Applications (Soviet Series), 35, Kluwer Academic Publishers, 1990.
- [27] E.M. Stein: *Singular Integrals and Differentiability Properties of Functions*. Princeton University Press, Princeton, New Jersey, 1970.
- [28] R. Temam: *Navier-Stokes equations. Theory and numerical analysis*. Studies in Mathematics and its Applications, Vol. 2. North-Holland Publishing Co., Amsterdam-New York-Oxford, 1977. x+500 pp.
- [29] K. Taira: *Analytic Semigroups and Semilinear Initial Boundary Value Problems*, London Math. Society Lect. Note Series, 223, Cambridge University Press, 1995.