An integral characterization of random permutations. A point process approach

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Abstract

We consider random finite permutations and prove the following version of Thoma's theorem in [8]: Random finite permutations which are class functions satisfy a new integration by parts formula iff they are given by a certain *Ewens-Sütö process*. The main source of inspiration for the results in this note is the fundamental work of Andras Sütö [7], from which some results are reestablished here again in the present point process approach.

1 Introduction

We are interested in the construction of point processes realizing configurations of finite cyclic permutations which represent a cycle decomposition of a finite permutation. This means, we consider random finite permutations considered as a point process on a space of cycles. This point of view differs from the one taken for instance by Olshanski in [6] where also random permutations are constructed. But there $\mathcal{S}_n$, the symmetric group on $[1, n] = \{1, \dots, n\}$, is imbedded into $\mathcal{S}_\infty$, the infinite symmetric group, whereas here the groups $\mathcal{S}_n$ as well as $\mathcal{S}_f$, the collection of all finite permutations, are considered as a subset of $\mathcal{M}^{\cdot}_f(\mathcal{C}_f)$, the set of finite subsets of the set $\mathcal{C}_f$ of finite cyclic permutations of a subset of the natural numbers $\mathbb{N}$.

We'll start the construction with a special finite measure ρ on $\mathbb{N}$ and build with it a point process $\mathcal{E}_\rho$ on $\mathcal{S}_f$ which we call the *Ewens-Sütö process for* ρ. It is a special mixture of a sequence of point processes on $\mathcal{S}_n$, which had been discovered independently by Ewens [2] and Sütö [7] in completely different contexts. Our main result is its characterization by means of an integration by parts fomula in terms of its Campbell measure. This result can be viewed as a version of Thoma's theorem im [8].

We mention finally the important paper of Fichtner [3] where already random permutations of random point configurations had been constructed.

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2 The main lemma

We consider the following finite Poisson process P_ρ on the set of natural numbers $\mathbb{N}$. Its intensity measure ρ is defined by

$$\rho(j) = d(j) \cdot \frac{z^j}{j}, j \epsilon \mathbb{N}, \tag{2.1}$$

where $0 < z < 1$ and $d > 0$. We assume that ρ is a *finite* measure on $\mathbb{N}$, i.e. $\rho(\mathbb{N}) = \sum_{j=1}^{\infty} \rho(j) < \infty$. This is a condition on the function d. Condition (2.1) implies that P_ρ is a law on the collection $\mathcal{M}_{\ddot{f}}(\mathbb{N})$ of all finite point measures μ on $\mathbb{N}$.

Examples for ρ are: (1) d is a constant d given by some natural number; (2) $d(j) = C \cdot j^{-\frac{\nu}{2}}$ where $\nu \in \mathbb{N}$ and C is a positive constant.

We denote by $(\zeta_j)_{j \epsilon \mathbb{N}}$ the field variables $\zeta_j(\mu) = \mu(j)$. It is well known that these variables are independent if P_ρ is the underlying law; moreover ζ_j has a Poisson distribution with parameter $\rho(j)$. This implies immediately that

$$P_\rho(\mu) = \exp(-\rho(\mathbb{N})) \cdot z^{N(\mu)} \cdot d(\mu) \cdot q(\mu), \tag{2.2}$$

where

$$q(\mu) = \prod_{j \geq 1} \frac{1}{\mu(j)! \cdot j^{\mu(j)}}, \tag{2.3}$$

$$d(\mu) = \prod_{j \geq 1} d(j)^{\mu(j)} \, and \tag{2.4}$$

$$N(\mu) = \sum_{j \geq 1} j \cdot \mu(j). \tag{2.5}$$

We remark that here the products resp. the sum terminate after finitely many steps because μ is finite. The range of N therefore is $\mathbb{N}_0$, the collection of natural numbers augmented by 0.

Denoting by $\mathcal{M}$ the identity on $\mathcal{M}_{\ddot{f}}(\mathbb{N})$, we have

$$P_\rho\{\mathcal{M} = \mu, N = n\} = \exp(-\rho(\mathbb{N})) \cdot z^{N(\mu)} \cdot d(\mu) \cdot q(\mu) \cdot 1_{\{N=n\}}(\mu), n \geq 0. \tag{2.6}$$

Summing over all μ and n we obtain

$$exp(\rho(\mathbb{N})) = \sum_{n \geq 0} Q_n(d) \cdot z^n, \tag{2.7}$$

where

$$Q_n(d) = \sum_{\mu : N(\mu) = n} d(\mu) \cdot q(\mu), n \leq 0, Q_0(d) = 1,$$

denotes the socalled *canonical partition function of the ideal Bose gas* in quantum statistical mechanics, where it is the starting point of the investigations.

Comparing the power series on the right hand side of (2.7) with the one obtained by inserting the power series $\rho(\mathbb{N})$ into the exponential, we obtain as an aside the following representation of the canonical partition function which is of independent combinatorial interest.

Proposition 2.1 $Q_n(d) = \sum_{k\geq 0} \frac{1}{k!} \sum_{\lambda:|\lambda|=n} \prod_{l=1}^{k} \frac{d(\lambda(l))}{\lambda(l)}$.

Here the summation is taken over all point measures λ on the interval $[1,k]$, which do not vanish; and $|\lambda| = \lambda(\mathbb{N})$.

In case of example (1) $Q_n(d) = \binom{d+n-1}{n}$ which contains Cauchy's formula for $d = 1$.

Formula (2.6) implies that the random variable N is distributed according to the following version of the *negative binomial distribution:*

Proposition 2.2 $P_\rho\{N = n\} = \exp(-\rho(\mathbb{N})) \cdot z^n \cdot Q_n(d), n \geq 0$.

Corollary 2.3 $P_\rho\{\mathcal{M} = \mu | N = n\} = \frac{1}{Q_n(d)} \cdot d(\mu) \cdot q(\mu) \cdot 1_{\{N=n\}}(\mu), n \geq 0, \mu \in \mathcal{M}_{\ddot{f}}(\mathbb{N})$

We use the following notations in the sequel:

$$P_\rho^{(n)} = P_\rho(.|N = n), n \geq 1; \tag{2.8}$$
$$P_\rho^{(0)} = \delta_0; (\delta_j \, denotes \ the \ Dirac \ measure \ at \, j); \tag{2.9}$$
$$\mathcal{M}_{\ddot{(n)}} = \{N = n\}. \tag{2.10}$$

Observe that $P_\rho^{(n)}$ does no longer depend on z.

Our aim will be now to calculate the Campbell measure of $P_\rho^{(n)}$, which is defined as follows:

$$\mathcal{C}_{P_\rho^{(n)}}(h) = \sum_{\mu \in \mathcal{M}_{\ddot{f}}(\mathbb{N})} \sum_{j \in \mathbb{N}} h(j,\mu) \cdot \mu(j) \cdot P_\rho^{(n)}(\mu), h \in F_+.$$

Here F_+ denotes the set of all non negative real functions. Now we are in the position to state and prove the *main lemma* of this note.

Lemma 2.4 $\mathcal{C}_{P_\rho^{(n)}} = \sum_{\mu \in \mathcal{M}_f(\mathbb{N})} \sum_{j \in \mathbb{N}} h(j, \mu + \delta_j) \cdot \frac{Q_{n-j}(d)}{Q_n(d)} \cdot \frac{d(j)}{j} \cdot P_\rho^{(n-j)}(\mu), h \in F_+$.

Proof. By definition one has

$$\mathcal{C}_{P_\rho^{(n)}}(h) = \frac{1}{P_\rho\{N = n\}} \cdot \sum_{\mu,j} 1_{\mathcal{M}_{\ddot{(n)}}(\mathbb{N})}(\mu) \cdot h(j,\mu) \cdot \mu(j) \cdot P_\rho(\mu).$$

Mecke's characterization of the Poisson process in [4] implies that this equals

$$= \frac{1}{P_\rho\{N = n\}} \cdot \sum_{\mu,j} 1_{\mathcal{M}_{\ddot{(n)}}(\mathbb{N})}(\mu + \delta_j) \cdot h(j, \mu + \delta_j) \cdot \rho(j) \cdot P_\rho(\mu).$$

But $(\mu + \delta_j \in \mathcal{M}^{\cdot\cdot}_{(n)}(\mathbb{N})$ iff $\mu \in \mathcal{M}^{\cdot\cdot}_{(n-j)}(\mathbb{N}))$. Thus one gets

$$= \sum_{\mu \in \mathcal{M}_f(\mathbb{N})} \sum_{j \in \mathbb{N}} h(j, \mu + \delta_j) \cdot \frac{Q_{n-j}(d)}{Q_n(d)} \cdot \frac{d(j)}{j} \cdot P_\rho^{(n-j)}(\mu).\mathbf{qed}$$

A first immediate application of the lemma shows that its intensity measure $\nu_{P_\rho^{(n)}}$, defined as $\nu_{P_\rho^{(n)}}(j) = P_\rho^{(n)}(\zeta_j)$, is given by

$$\nu_{P_\rho^{(n)}}(j) = \frac{Q_{n-j}(d)}{Q_n(d)} \cdot \frac{d(j)}{j}, j \in [1, n]. \tag{2.11}$$

3 The Ewens-Sütö cycle process

Given $n \geq 1$, consider the symmetric group $\mathcal{S}_n$, acting on $[1, n]$. We use the following properties of $\mathcal{S}_n$ (see [1] e.g.) : Every permutation $\sigma \in \mathcal{S}_n$ can be decomposed in a unique way into disjoint cycles. Let $r_j(\sigma)$ be the number of cycles with length j. Then $\sum_j j \cdot r_j(\sigma) = n$. A *conjugacy class* consists of those permutations σ having the same decomposition into cycles, i.e. having the same $r_j(\sigma)$.

A permutation $\sigma \in \mathcal{S}_n$ is always considered as a simple point measure of disjoint cycles, including the trivial ones:

$$\sigma = \sum_{x \in \sigma} \delta_x.$$

Here the sum is taken over all cycles x of the cycle decomposition of σ. The neutral element of $\mathcal{S}_n$ decomposes into trivial cycles only. We consider $\mathcal{S}_n$ as a subset of $\mathcal{M}^{\cdot}_f(\mathcal{C}_f)$ where $\mathcal{C}_f$ denotes the collection of all finite cycles x. This means that x is a cyclic permutation of a finite subset $I \in \mathbb{N}$, $I \neq \emptyset$. (Recall that $\mathcal{M}^{\cdot}_f(\mathcal{C}_f)$ is the set of finite subsets of $\mathcal{C}_f$ considered as simple point measures.) For $n = 0$ $\mathcal{S}_0$ denotes the singleton $\{0\}$ consisting of the measure 0 on $\mathcal{C}_f$. As a consequence $\mathcal{S}_f = \bigcup_{n \geq 0} \mathcal{S}_n$, the set of all finite permutations of $\mathbb{N}$, is well defined subset of $\mathcal{M}^{\cdot}_f(\mathcal{C}_f)$.

The connection between the cycle configurations σ in $\mathcal{M}^{\cdot}_f(\mathcal{C}_f)$ and the configurations μ in $\mathcal{M}^{\cdot\cdot}_f(\mathbb{N})$ is given by the transformation

$$r : \sigma \mapsto \mu = r(\sigma),$$

where $\mu(j) = r_j(\sigma)$ is the number of cycles of length j in σ. The length of a cycle x is the cardinality of its domain, denoted by $N(x)$. Thus N is a counting variable defined on $\mathcal{C}_f$.

Given $\mu \in \mathcal{M}^{\cdot\cdot}_f(\mathbb{N})$ we denote by

$$\mathcal{K}_\mu = \mathcal{S}_{N(\mu)} \cap \{r = \mu\}$$

the conjugacy class of permutations defined by μ. It is well known [1] that $|\mathcal{K}_\mu| = N(\mu)! \cdot q(\mu)$.

We define the following law on $\mathcal{S}_n$

$$\mathcal{E}_\rho^{(n)}(\sigma) = \frac{1}{|\mathcal{K}_{r(\sigma)}|} \cdot P_\rho^{(n)}(r(\sigma)), \ \ \sigma \in \mathcal{S}_n. \tag{3.1}$$

These probabilities are trivially extended to probabilities on the whole space $\mathcal{M}^{\cdot}_f(\mathcal{C}_f)$. We call $\mathcal{E}^{(n)}_\rho$ the *Ewens-Sütö cycle process* for the parameters (n,ρ). It is a simple point process of cycles realizing a permutation σ of [1,n]. Observe that $\mathcal{E}^{(n)}_\rho$ is constant on conjugacy classes and thus a so-called *class function*. Explicitly this Ewens-Sütö cycle process is given by

$$\mathcal{E}^{(n)}_\rho(\sigma) = \frac{1}{n!Q_n(d)} \cdot d(\mu) \cdot 1_{\{N=n\}}(\mu), \;\; \sigma \in \mathcal{S}_n, \tag{3.2}$$

where $\mu = r(\sigma)$. We make the following useful observation that $P^{(n)}_\rho$ is the image of $\mathcal{E}^{(n)}_\rho$ under r, denoted by $r\mathcal{E}^{(n)}_\rho$. In the sequel we'll use the notation: $N(\sigma) := N(r(\sigma)),\;\sigma \in \mathcal{M}^{\cdot}_f(\mathcal{C}_f)$. Finally we observe that $\mathcal{E}^{(n)}_\rho$ is well defined also for $n = 0$: $\mathcal{E}^{(0)}_\rho = \delta_0$, 0 denoting measure 0 from $\mathcal{M}^{\cdot}_f(\mathcal{C}_f)$.

We now define the main object of this note. The *Ewens -Sütö cycle process for* ρ is the following mixture of the $(\mathcal{E}^{(n)}_\rho)_{n\geq 0}$ with respect to the negative binomial distribution of N under P_ρ:

$$\mathcal{E}_\rho = \exp(-\rho(\mathbb{N})) \cdot \sum_{n=0}^{\infty} z^n \cdot Q_n(d) \cdot \mathcal{E}^{(n)}_\rho. \tag{3.3}$$

$\mathcal{E}_\rho$ is a simple point process on the space $\mathcal{C}_f$ of finite cycles. To be more precise: According to the negative binomial distribution n is realized first, then a decomposition of some $\sigma \in \mathcal{S}_n$ into disjoint cycles is realized according to $\mathcal{E}^{(n)}_\rho$.

Observe that $r\mathcal{E}_\rho = P_\rho$ and thus in particular

$$\mathcal{E}_\rho\{N = n\} = \exp(-\rho(\mathbb{N})) \cdot z^n \cdot Q_n(d),\;\; n \geq 0.$$

This means that N, the length of a permutation, is distributed according to the negative binomial distribution for the Ewens -Sütö cycle process $\mathcal{E}_\rho$.

4 The distribution of cycle lengths in the Ewens-Sütö field.

Given (n,ρ) we consider the following random field, which we call the *Ewens-Süto field for* (n,ρ):

$$\Xi^{(n)}_\rho \equiv (\mathcal{S}_n, \mathcal{E}^{(n)}_\rho, (\xi_a)_{a\in[1,n]}) \tag{4.1}$$

Here $\xi_a(\sigma)$ denotes the length of the cycle in σ containing a.

Lemma 4.1 $\Xi^{(n)}_\rho$ *is identically distributed in the following sense. For any choice of distinct* $a_1,\dots,a_k \in [1,n]$ *the distribution of* $(\xi_{a_j})_{j\in[1,k]}$ *is the same.*

Proof. By definition for any $j_1,\dots,j_k \in [1,n]$

$$\mathcal{E}^{(n)}_\rho\{\xi_{a_1} = j_1,\dots,\xi_{a_k} = j_k\} = \sum_{\mu\in\mathcal{M}^{\cdot\cdot}_n(\mathbb{N})} \frac{1}{|\mathcal{K}_\mu|} \cdot P^{(n)}_\rho(\mu) \cdot \sum_{\sigma\in\mathcal{K}_\mu} 1_{\{\xi_{a_1}=j_1,\dots,\xi_{a_k}=j_k\}}(\sigma). \tag{4.2}$$

We have to show that the inner sum does not depend on the choice of distinct a_l. Let $b_1, \dots, b_k \in [1,n]$ be another choice of distinct elements in $[1,n]$. Then choose a permutation $\tau \in \mathcal{S}_n$ such that $\tau(a_l) = b_l$, $l = 1, \dots, k$ and consider the conjugation transformation $\sigma \mapsto \tau\sigma\tau^{-1} := \sum_{x\in\sigma} \delta_{\tau x \tau^{-1}}$. It is obvious that $(\xi_{a_1}(\sigma) = j_1, \dots, \xi_{a_k}(\sigma) = j_k$ iff $\xi_{b_1}(\tau\sigma\tau^{-1}) = j_1, \dots, \xi_{b_k}(\tau\sigma\tau^{-1}) = j_k)$ (see for example [1]). This implies that

$$\sum_{\sigma\in\mathcal{K}_\mu} 1_{\{\xi_{a_1}=j_1,\dots,\xi_{a_k}=j_k\}}(\sigma) = \sum_{\sigma\in\mathcal{K}_\mu} 1_{\{\xi_{b_1}=j_1,\dots,\xi_{b_k}=j_k\}}(\tau\sigma\tau^{-1}).\mathbf{qed} \tag{4.3}$$

Theorem 4.2 *(Sütö [7]). (1) For any choice of distinct $a_1, \dots, a_k \in [1,n]$ and distinct $j_1, \dots, j_k \in [1,n]$*

$$\mathcal{E}_\rho^{(n)}\{\xi_{a_1} = j_1, \dots, \xi_{a_k} = j_k\} = \frac{j_1 \cdots j_k}{n(n-1)\dots(n-k+1)} \mathcal{E}_\rho^{(n)}(\zeta_{j_1} \cdots \zeta_{j_k}). \tag{4.4}$$

(2) For any choice of distinct $j_1, \dots, j_k \in [1,n]$

$$\mathcal{E}_\rho^{(n)}(\zeta_{j_1} \cdots \zeta_{j_k}) = \prod_{l=1}^k \frac{d(j_l)}{j_l} \cdot \frac{Q_{n-(j_1+\dots+j_k)}(d)}{Q_n(d)}. \tag{4.5}$$

As an immediate consequence the distribution of the lengths of the cycles containing distinct $a_1, \dots, a_k \in [1,n]$ is given by

$$\mathcal{E}_\rho^{(n)}\{\xi_{a_1} = j_1, \dots, \xi_{a_k} = j_k\} = \frac{d(j_1)\cdots d(j_k)}{n(n-1)\cdots(n-k+1)} \frac{Q_{n-(j_1+\dots+j_k)}(d)}{Q_n(d)}, \tag{4.6}$$

provided that $j_1, \dots, j_k$ are distinct. We note that the distribution of $(\xi_{a_l})_{l=1,\dots,k}$ depends only on d but not on z.

Proof. (1) Given distinct $j_1, \dots, j_k \in [1,n]$ we have that

$$\begin{aligned} &\textstyle\sum^*_{a_1,\dots,a_k} \mathcal{E}_\rho^{(n)}\{\xi_{a_1} = j_1, \dots, \xi_{a_k} = j_k\} \\ &= \sum_{\sigma\in\mathcal{S}_n} \mathcal{E}_\rho^{(n)}(\sigma) \sum_{m_1,\dots,m_k} \textstyle\sum^*_{a_1,\dots,a_k} 1_{\{\xi_{a_l}=j_l, \zeta_{j_l}=m_l; l=1,\dots,k\}}(\sigma). \end{aligned}$$

Here the sum $\sum^*_{a_1,\dots,a_k}$ is taken over all distinct $a_1, \dots, a_k \in [1,n]$ and $\zeta_j(\sigma) = \zeta_j(r(\sigma))$, $j \in [1,n]$. Since the inner sum equals to

$$\prod_{l=1}^k j_l \cdot m_l \cdot 1_{\{\zeta_{j_l}=m_l, l=1,\dots,k\}}(\sigma),$$

we obtain that

$$\textstyle\sum^*_{a_1,\dots,a_k} \mathcal{E}_\rho^{(n)}\{\xi_{a_1} = j_1, \dots, \xi_{a_k} = j_k\} = j_1 \dots j_k \cdot \mathcal{E}_\rho^{(n)}(\zeta_{j_1} \dots \zeta_{j_k}).$$

Combining this with lemma 2 we get (4.4).

(2) To evaluate the moment measure $\mathcal{E}_\rho^{(n)}(\zeta_{j_1}\cdots\zeta_{j_k})$ we use lemma 1 and the fact that ζ_j are class functions, i.e. they depend only on $r(\sigma)$. Thus

$$\begin{aligned}\mathcal{E}_\rho^{(n)}(\zeta_{j_1}\cdots\zeta_{j_k}) &= P_\rho^{(n)}(\zeta_{j_1}\ldots\zeta_{j_k}) = \mathcal{C}_{P_\rho^{(n)}}(1_{\{j_1\}}\otimes(\zeta_{j_2}\ldots\zeta_{j_k}))\\ &= \frac{d(j_1)}{j_1}\cdot\frac{Q_{n-j_1}(d)}{Q_n(d)}\cdot P_\rho^{(n-j_1)} * \Delta_{j_1}(\zeta_{j_2}\cdots\zeta_{j_k}).\end{aligned}$$

Here $*\Delta_{j_1}$ denotes the convolution with respect to the point process $\delta_{\delta_{j_1}}$. Using then that $j_1,\ldots,j_k$ are distinct by assumption the Campbell measure of $P_\rho^{(n)}$ factorizes and one obtains

$$\mathcal{E}_\rho^{(n)}(\zeta_{j_1}\cdots\zeta_{j_k}) = \frac{d(j_1)}{j_1}\cdot\frac{Q_{n-j_1}(d)}{Q_n(d)}\cdot P_\rho^{(n-j_1)}(\zeta_{j_2}\cdots\zeta_{j_k}).$$

Iterating this procedure yields (4.5). **qed**

5 An integration by parts formula characterizing $\mathcal{E}_\rho$

In this section we derive an equation for $\mathcal{E}_\rho$ in terms of its Campbell measure. We use the following transformation:

$$N\otimes r : \mathcal{C}_f\times\mathcal{M}_f^{\cdot}(\mathcal{C}_f)\to\mathbb{N}_0\times\mathcal{M}_f^{\cdot\cdot}(\mathbb{N});\ \ (x,\sigma)\mapsto(N(x),r(\sigma)). \tag{5.1}$$

Observe that $\mathcal{C}_{P_\rho}$ is the image of $\mathcal{C}_{\mathcal{E}_\rho}$ under $N\otimes r$ because P_ρ is the image of $\mathcal{E}_\rho$ under r .

We now want to compute the Campbell measure of $\mathcal{E}_\rho$ for *class functions* $\tilde{h}$ of the type $\tilde{h} := h\circ(N\otimes r)$ with arbitrary $h:\mathbb{N}_0\times\mathcal{M}_f^{\cdot\cdot}(\mathbb{N})\to\mathbb{R}$. Thus $\tilde{h}:\mathcal{C}_f\times\mathcal{M}_f^{\cdot}(\mathcal{C}_f)\to\mathbb{R}$.

Using again Mecke's characterization of the Poisson process , we obtain that

$$\mathcal{C}_{\mathcal{E}_\rho}(\tilde{h}) = \mathcal{C}_{P_\rho}(h) = \sum_{\mu\in\mathcal{M}_f^{\cdot\cdot}(\mathbb{N})}\sum_{j\in\mathbb{N}} h(j,\mu+\delta_j)\rho(j)P_\rho(\mu). \tag{5.2}$$

Going then back to the level of cycles and permutations we find that

$$\mathcal{C}_{\mathcal{E}_\rho}(\tilde{h}) = \sum_{\sigma\in\mathcal{S}_f}\sum_{x\in\mathcal{C}_f} h(N(x),r(\sigma)+\delta_{N(x)})1_{\mathcal{S}_N}(\sigma+\delta_x)\frac{\rho(N(x))}{(N(x)-1)!}\mathcal{E}_\rho(\sigma). \tag{5.3}$$

Here we used the fact that the number of cycles in some fixed domain of length $N(x)$ is equal to $(N(x)-1)!$. Moreover ($\sigma+\delta_x\in\mathcal{S}_N$ iff $\sigma+\delta_x\in\mathcal{S}_{N(\sigma+\delta_x)}$). Since $r(\sigma)+\delta_{N(x)} = r(\sigma+\delta_x)$, setting $\tau_\rho(j) = \frac{1}{(j-1)!}\cdot\rho(j) = \frac{z^j}{j!}d(j),\ j\in\mathbb{N}$, we obtain that

$$\mathcal{C}_{\mathcal{E}_\rho}(\tilde{h}) = \sum_{\sigma\in\mathcal{S}_f}\sum_{x\in\mathcal{C}_f}\tilde{h}(x,\sigma+\delta_x)\cdot 1_{\mathcal{S}_N}(\sigma+\delta_x)\cdot\tau_\rho(N(x))\cdot\mathcal{E}_\rho(\sigma). \tag{5.4}$$

To summarize, we have

Theorem 5.1 *The Ewens-Sütö cycle process $\mathcal{E}_\rho$ is a simple point process Q on $\mathcal{C}_f$ which is concentrated on $\mathcal{S}_f$, constant on conjugacy classes and solves the following integration by parts formula*

$$\mathcal{C}_Q(\tilde{h}) = \sum_{\sigma\in\mathcal{S}_f}\sum_{x\in\mathcal{C}_f}\tilde{h}(x,\sigma+\delta_x)\cdot 1_{\mathcal{S}_N}(\sigma+\delta_x)\cdot\tau_\rho(N(x))\cdot Q(\sigma), \tag{$\sum_{\tau_\rho}^{\mathcal{S}_N}$}$$

provided that $\tilde{h}$ is a class function.

An immediate consequence is that the intensity measure of $\mathcal{E}_\rho$ is $\tau_\rho(N(.))$.This is the probabilistic significance of $\tau_\rho \circ N$ for the Ewens-Sütö process: $\tau_\rho \circ N(x)$ is the expected number of random permutations possessing x as a cyclic permutation.

Our next aim is to show *the converse of this theorem.* Let Q be an element of $\mathcal{PM}^{\cdot}_f(\mathcal{C}_f)$ i.e. a law on $\mathcal{M}^{\cdot}_f(\mathcal{C}_f)$ which is concentrated on $\mathcal{S}_f$ and is constant on conjugacy classes. We assume that Q is a solution of the equation $(\sum_{\tau_\rho}^{\mathcal{S}_N})$. Then there exists a function $P' : \mathcal{M}^{\cdot\cdot}_f(\mathbb{N}) \to \mathbb{R}_+$ which factorizes Q in the sense that $Q(\sigma) = P'(r(\sigma))$, $\sigma \in \mathcal{S}_f$. Consider then $P := rQ$. P is a point process on $\mathcal{M}^{\cdot}_f(\mathcal{C}_f)$ with $P(\mu) = |\mathcal{K}_\mu| \cdot P'(\mu)$, $\mu \in \mathcal{M}^{\cdot\cdot}_f(\mathbb{N})$. Moreover, given any $h \in F_+(\mathbb{N}_0 \times \mathcal{M}^{\cdot\cdot}_f(\mathbb{N}))$ and setting $\tilde{h} = h \circ (N \otimes r)$, with the help of $(\sum_{\tau}^{\mathcal{S}_N})$, we obtain then that $\mathcal{C}_P(h)$ equals

$$\begin{aligned}\mathcal{C}_Q(\tilde{h}) &= \sum_{\sigma \in \mathcal{S}_f} \sum_{x \in \mathcal{C}_f} h(N(x), r(\sigma) + \delta_{N(x)}) \cdot 1_{\mathcal{S}_{N(\sigma)+N(x)}}(\sigma + \delta_x) \cdot \tau_\rho(N(x)) \cdot Q(\sigma) \\ &= \sum_{\mu \in \mathcal{M}^{\cdot\cdot}_f(\mathbb{N})} P'(\mu) \sum_{j=1}^{\infty} \cdot h(j, \mu + \delta_j) \cdot \tau_\rho(j) \cdot \sum_{\sigma \in \mathcal{K}_\mu} \sum_{x : N(x) = j} 1_{\mathcal{S}_{N(\mu)+j}}(\sigma + \delta_x).\end{aligned}$$

It is obvious that the inner double sum factorizes and equals to $(j-1)! \cdot |\mathcal{K}_\mu|$. Thus we obtain

$$\mathcal{C}_P(h) = \sum_{\mu \in \mathcal{M}^{\cdot\cdot}_f(\mathbb{N})} \sum_{j=1}^{\infty} h(j, \mu + \delta_j) \cdot \rho(j) \cdot P(\mu).$$

for any $h \in F_+(\mathbb{N}_0 \times \mathcal{M}^{\cdot\cdot}_f(\mathbb{N}))$. This means that P solves the integration by parts formula characterizing the Poisson process P_ρ (see [4]). Hence $P = P_\rho$. Then it follows that

$$Q(\sigma) = \frac{1}{|\mathcal{K}_{r(\sigma)}|} \cdot P_\rho(r(\sigma)), \ \ \sigma \in \mathcal{S}_f.$$

Thus $Q = \mathcal{E}_\rho$. To summarize we have

Theorem 5.2 *Let ρ be as above and Q be a simple point process on $\mathcal{C}_f$ which is concentrated on $\mathcal{S}_f$ and constant on conjugacy classes. If Q is a solution of $(\sum_{\tau_\rho}^{\mathcal{S}_N})$ then Q is the Ewens-Sütö cycle process $\mathcal{E}_\rho$.*

Combining the last two theorems we see that random permutations of $\mathcal{S}_f$ whose distributions are class functions , are solutions of $(\sum_{\tau_\rho}^{\mathcal{S}_N})$ iff they are special mixtures of Ewens-Sütö cycle processes $(\mathcal{E}_\rho^{(n)})_{n \geq 0}$. This result can be viewed as a version of Thoma's theorem (see [8]).

Some *concluding remarks* are in order here: The integration by parts formula $(\sum_{\tau_\rho}^{\mathcal{S}_N})$ has to be compared with the corresponding characterization of Gibbs processes of abstract particles, interacting in the sense of a classical gas (see [5]). In the present situation we consider a system of cyclic permutations which interact strongly in the sense that their configuration represents the cycle decomposition of a permutation. Equation $(\sum_{\tau_\rho}^{\mathcal{S}_N})$ contains the precise expression of such an interaction: The cycles are *hard rods*, but moreover they are *glued together* such that they form a permutation. This therefore is a first step in finding characterizations in the spirit of statistical mechanics of other *random tesselations* like the Plancherel process (see [6]) or Delaunay and Voronoi tesselations.

References

[1] Börner,H.:Representations of groups, North Holland, Amsterdam (1963).

[2] Ewens, W.J.: The sampling theory of selectively neutral alleles. Theoretical biology 3, 87-112 (1972).

[3] Fichtner, K.H.: Random permutations of countable sets. Probab. Theory Related Fields 89 , no. 1, 35–60 (1991).

[4] Mecke, J.: Stationäre zufällige Maße auf lokalkompakten Abelschen Gruppen. Z. Wahrscheinlichkeitstheorie verw. Geb. 9, 36-58 (1967).

[5] Nguyen, X.X., Zessin, H.: Integral and Differential Characterizations of the Gibbs Process. Math. Nachr. 88. 105 - 115 (1979).

[6] Olshanski, G.: An introduction to harmonic analysis on the infinite symmetric group, 127-160. In Vershik, A. (ed.): Asymptotic combinatorics with applications to mathematical physics. Lecture notes in mathematics 1815 (2003)

[7] Sütö, A.: Percolation transition in the Bose gas. J. Phys. A: Math. Gen. 4689-4710 (1993).

[8] Thoma, E.: Die unzerlegbaren, positiv-definiten Klassenfunktionen der abzählbar unendlichen symmetrischen Gruppe. Math. Z. 85, 40-61 (1964).