

ON REGULARITY IN CODIMENSION ONE OF IRREDUCIBLE COMPONENTS OF MODULE VARIETIES

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ABSTRACT. Let Λ be a tame quasi-tilted algebra and \mathbf{d} the dimension vector of an indecomposable Λ -module. In the paper we prove that each irreducible component of the variety of Λ -modules of dimension vector \mathbf{d} is regular in codimension one.

Throughout the paper k denotes a fixed algebraically closed field. Unless otherwise stated all considered modules are finite dimensional ones. By \mathbb{Z} , \mathbb{N} , and \mathbb{N}_+ , we denote the sets of integers, nonnegative integers, and positive integers, respectively. For $i, j \in \mathbb{Z}$ we denote by $[i, j]$ the set of all $l \in \mathbb{Z}$ such that $i \leq l \leq j$.

INTRODUCTION AND THE MAIN RESULT

The class of finite dimensional algebras may be divided into two disjoint subclasses [20] (see also [18]). The first class consists of the tame algebras for which indecomposable modules occur in each dimension in a finite number of one-parameter families (1.1). The second class is formed by the wild algebras whose representation theory is as complicated as the study of finite dimensional vector spaces together with two (not necessarily commuting) endomorphisms. The solution of the latter problem would imply the classification of modules over arbitrary finite dimensional algebra, hence one may hope to solve classification problems only for tame algebras.

Given a finite dimensional algebra Λ and a nonnegative element \mathbf{d} of the Grothendieck group of the category of Λ -modules (such elements of the Grothendieck group are called dimension vectors), it is an interesting problem to study the variety $\text{mod}_\Lambda(\mathbf{d})$ of Λ -modules of dimension vector \mathbf{d} (2.1) possessing a natural action of a corresponding product $\text{GL}(\mathbf{d})$ of general linear groups (2.2) (see [5, 6, 8, 9, 11, 15, 19, 22–24, 37, 39] for some results in this direction). In particular, one may ask questions about irreducible components of $\text{mod}_\Lambda(\mathbf{d})$ and their properties. This problem is nontrivial only for algebras which are not hereditary, since $\text{mod}_\Lambda(\mathbf{d})$ is an affine space provided $\text{gldim } \Lambda \leq 1$. A natural class of algebras obtained by weakening homological constraints put on

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hereditary algebras is that of the quasi-tilted algebras (1.7). It seems to be natural to study how much varieties of modules over quasi-tilted algebras differ from varieties of modules over hereditary algebras.

The categories of modules over quasi-tilted algebras have been intensively studied (see for example [25–27, 33, 38]). In particular, the categories of modules over tame quasi-tilted algebras have been described (see [41], also [28, 35, 36]). Since some geometric properties of points of varieties of modules translate into properties of corresponding modules, it is appealing to use information about categories of modules in studying varieties of modules.

This strategy has been successfully applied in [5, 8, 9, 11]. In particular, the main results of [8, 9] imply that if \mathbf{d} is the dimension vector of an indecomposable module over a tame quasi-tilted algebra Λ , then $\text{mod}_\Lambda(\mathbf{d})$ has at most two irreducible components. Moreover, if $\text{mod}_\Lambda(\mathbf{d})$ is irreducible, then it is normal, hence its unique irreducible component is regular in codimension one (2.3). However, no further information about properties of the irreducible components of $\text{mod}_\Lambda(\mathbf{d})$ in the remaining cases was obtained at that time. A first step in order to fill this gap was made in [13] by proving that if X is an indecomposable directing module (1.8) over a tame algebra Λ (which may be assumed without loss of generality to be quasi-tilted) of dimension vector \mathbf{d} , then the closure of the $\text{GL}(\mathbf{d})$ -orbit of X (which is an irreducible component of $\text{mod}_\Lambda(\mathbf{d})$) is normal, hence again regular in codimension one. The main result of the paper extends the latter property to all irreducible components of varieties of modules over quasi-tilted algebras for dimension vectors of indecomposable modules.

Theorem 1. *Let Λ be a quasi-tilted algebra and \mathbf{d} a dimension vector of an indecomposable Λ -module. If \mathcal{V} is an irreducible component of $\text{mod}_\Lambda(\mathbf{d})$, then \mathcal{V} is regular in codimension one.*

We remark that the dimension vectors of the indecomposable modules over the tame quasi-tilted algebras can be characterized combinatorially (1.7).

An important direction of research is study of properties of closures of orbits in varieties of modules (see for example [4, 12, 16, 42, 48–51]). If X is a module of dimension vector \mathbf{d} over an algebra Λ with $\text{Ext}_\Lambda^1(X, X) = 0$, then one easily gets that the closure of the $\text{GL}(\mathbf{d})$ -orbit of X is an irreducible component of $\text{mod}_\Lambda(\mathbf{d})$ [29, II.2.7, Folgerung]. Thus we have the following immediate consequence of the main result of the paper.

Corollary 2. *Let X be an indecomposable module over a tame quasi-tilted algebra Λ of dimension vector \mathbf{d} . If $\text{Ext}_\Lambda^1(X, X) = 0$, then the closure of the $\text{GL}(\mathbf{d})$ -orbit of X is regular in codimension one.*

It is an open problem if the closure of the orbit of X is regular in codimension one for an arbitrary indecomposable module X over a

tame quasi-tilted algebra. In [47] Zwara presented an example showing that in general closures of orbits of (arbitrary) modules over tame quasi-tilted (even hereditary) algebras do not have to be regular in codimension one.

The paper is organized as follows. In Section 1 we present necessary information about quivers and quasi-tilted algebras, while in Section 2 we collect useful facts about varieties of modules. Next, in Section 3 we describe technical conditions under which a distinguished component of a module variety is regular in codimension one. This condition is used among others in Section 4 in order to prove the main result.

For a basic background on the representation theory of algebras (in particular, on tilting theory) we refer to [1]. Basic algebraic geometry used in the article can be found in [31].

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1. PRELIMINARIES ON QUIVERS AND QUASI-TILTED ALGEBRAS

1.1. A finite dimensional algebra Λ is called tame if for every $d \in \mathbb{N}_+$ there exist Λ - $k[T]$ -bimodules M_1, \dots, M_n , which are free of rank d as $k[T]$ -modules (in particular, in contrary to our usual assumption, they are not finite dimensional) such that for each indecomposable Λ -module X there exist $i \in [1, n]$ and $\lambda \in k$ such that $X \simeq M_i \otimes_{k[T]} (k[T]/(T - \lambda))$.

1.2. By a quiver Δ we mean a finite set Δ_0 of vertices and a finite set Δ_1 of arrows together with maps $s, t : \Delta_1 \rightarrow \Delta_0$ which assign to $\alpha \in \Delta_1$ the starting vertex s_α and the terminating vertex t_α . By a path of length $n \in \mathbb{N}_+$ we mean a sequence $\sigma = \alpha_1 \cdots \alpha_n$ with $\alpha_1, \dots, \alpha_n \in \Delta_1$ such that $s_{\alpha_i} = t_{\alpha_{i+1}}$ for each $i \in [1, n - 1]$. In the above situation we put $s_\sigma = s_{\alpha_n}$ and $t_\sigma = t_{\alpha_1}$. Additionally, for each $x \in \Delta_0$ we introduce the trivial path of length 0 starting and terminating at x and denoted by x . A path σ of positive length such that $s_\sigma = t_\sigma$ is called an oriented cycle. A subquiver Δ' of Δ (i.e. a pair (Δ'_0, Δ'_1) such that $\Delta'_0 \subseteq \Delta_0$, $\Delta'_1 \subseteq \Delta_1$, and $s_\alpha, t_\alpha \in \Delta'_0$ for each $\alpha \in \Delta'_1$) is called convex if for every path $\alpha_1 \cdots \alpha_n$ in Δ with $\alpha_1, \dots, \alpha_n \in \Delta_1$ and $s_{\alpha_n}, t_{\alpha_1} \in \Delta'_0$, $s_{\alpha_i} \in \Delta'_0$ for each $i \in [1, n - 1]$.

1.3. For a quiver Δ we denote by $k\Delta$ the path algebra of Δ defined as follows. The elements of $k\Delta$ are the formal linear combinations of paths in Δ and for two paths σ_1 and σ_2 the product of σ_1 and σ_2 is either the composition $\sigma_1\sigma_2$ of paths, if $s_{\sigma_1} = t_{\sigma_2}$, or 0, otherwise. Fix $x, y \in \Delta_0$ and let $\rho = \lambda_1\sigma_1 + \cdots + \lambda_n\sigma_n$ for $n \in \mathbb{N}_+$, $\lambda_1, \dots, \lambda_n \in k$, and paths $\sigma_1, \dots, \sigma_n$. If $s_{\sigma_i} = x$ and $t_{\sigma_i} = y$ for each $i \in [1, n]$, then we write $s_\rho = x$ and $t_\rho = y$. If additionally, the length of σ_i is at least (more than) 1 for each $i \in [1, n]$, then ρ is called a relation (an admissible relation, respectively). A set \mathfrak{R} of relations is called minimal if ρ does not belong to the ideal $\langle \mathfrak{R} \setminus \{\rho\} \rangle$ generated by $\mathfrak{R} \setminus \{\rho\}$ for each

$\rho \in \mathfrak{R}$. If \mathfrak{R} is a minimal set of admissible relations such that there exists $n \in \mathbb{N}_+$ with the property $\sigma \in \langle \mathfrak{R} \rangle$ for each path σ of length at least n , then the pair (Δ, \mathfrak{R}) is called a bound quiver. If (Δ, \mathfrak{R}) is a bound quiver, then $k\Delta/\langle \mathfrak{R} \rangle$ is called the path algebra of (Δ, \mathfrak{R}) .

Gabriel proved (see for example [1, Corollaries I.6.10 and II.3.7]) that for each finite dimensional algebra Λ there exists a bound quiver (Δ, \mathfrak{R}) such that the category $\text{mod } \Lambda$ of Λ -modules is equivalent to the category of modules over the path algebra of (Δ, \mathfrak{R}) . In addition, Δ is uniquely (up to isomorphism) determined by Λ and we call it the Gabriel quiver of Λ . Consequently, from now on we assume that all considered algebras are path algebras of bound quivers. In particular, all considered algebras will be finite dimensional.

1.4. Let Δ be a quiver. By a representation of Δ we mean a collection $M = (M_x, M_\alpha)_{x \in \Delta_0, \alpha \in \Delta_1}$ of finite dimensional vector spaces M_x , $x \in \Delta_0$, and linear maps $M_\alpha : M_{s_\alpha} \rightarrow M_{t_\alpha}$, $\alpha \in \Delta_1$. If M and N are representations of Δ , then the morphism space $\text{Hom}_\Delta(M, N)$ consists of the collections $f = (f_x)_{x \in \Delta_0}$ of linear maps $f_x : M_x \rightarrow N_x$, $x \in \Delta_0$, such that $N_\alpha f_{s_\alpha} = f_{t_\alpha} M_\alpha$ for each $\alpha \in \Delta_1$. If $\sigma = \alpha_1 \cdots \alpha_n$, for $n \in \mathbb{N}_+$ and $\alpha_1, \dots, \alpha_n \in \Delta_1$, is a path, then for a representation M of Δ we put $M_\sigma = M_{\alpha_1} \cdots M_{\alpha_n}$. Similarly, if $\rho = \lambda_1 \sigma_1 + \cdots + \lambda_n \sigma_n$, for $n \in \mathbb{N}_+$, $\lambda_1, \dots, \lambda_n \in k$, and paths $\sigma_1, \dots, \sigma_n$ in Δ , is a relation, then for a representation M of Δ we put $M_\rho = \lambda_1 M_{\sigma_1} + \cdots + \lambda_n M_{\sigma_n}$.

1.5. Let (Δ, \mathfrak{R}) be a bound quiver. By $\text{rep}(\Delta, \mathfrak{R})$ we denote the full subcategory of the category of representations of Δ consisting of the representations M such that $M_\rho = 0$ for each $\rho \in \mathfrak{R}$. If Λ is the path algebra of (Δ, \mathfrak{R}) , then the assignment which assigns to a Λ -module M the representation $(M_x, M_\alpha)_{x \in \Delta_0, \alpha \in \Delta_1}$, where $M_x = xM$ for $x \in \Delta_0$ and $M_\alpha(m) = \alpha m$ for $\alpha \in \Delta_1$ and $m \in M_{s_\alpha}$, induces an equivalence between $\text{mod } \Lambda$ and $\text{rep}(\Delta, \mathfrak{R})$ (see for example [1, Theorem III.1.6]). We will usually treat this equivalence as identification. With $M \in \text{rep}(\Delta, \mathfrak{R})$ (hence also with every Λ -module) we associate its dimension vector $\mathbf{dim} M \in \mathbb{Z}^{\Delta_0}$ defined by $(\mathbf{dim} M)_x = \dim_k M_x$ for $x \in \Delta_0$. Obviously, $\mathbf{dim} M \in \mathbb{N}^{\Delta_0}$. We call the elements of \mathbb{N}^{Δ_0} dimension vectors. By the support of a dimension vector \mathbf{d} we mean the full subquiver $\text{supp } \mathbf{d}$ of Δ with $\{x \in \Delta_0 \mid d_x \neq 0\}$ as the set of vertices (a subquiver Δ' of Δ is called full if Δ'_1 consists precisely of all $\alpha \in \Delta_1$ such that $s_\alpha, t_\alpha \in \Delta'_0$). A dimension vector \mathbf{d} is called sincere (connected) if $\text{supp } \mathbf{d} = \Delta_0$ ($\text{supp } \mathbf{d}$ is connected, respectively).

1.6. Let (Δ, \mathfrak{R}) be a bound quiver. With (Δ, \mathfrak{R}) we associate the bilinear form $\langle -, - \rangle_{\Delta, \mathfrak{R}} : \mathbb{Z}^{\Delta_0} \times \mathbb{Z}^{\Delta_0} \rightarrow \mathbb{Z}$ and the quadratic form $q_{\Delta, \mathfrak{R}} : \mathbb{Z}^{\Delta_0} \rightarrow \mathbb{Z}$ in the following way:

$$\langle \mathbf{d}', \mathbf{d}'' \rangle_{\Delta, \mathfrak{R}} = \sum_{x \in \Delta_0} d'_x d''_x - \sum_{x \in \Delta_1} d'_{s_\alpha} d''_{t_\alpha} + \sum_{\rho \in \mathfrak{R}} d'_{s_\rho} d''_{t_\rho}$$

for $\mathbf{d}', \mathbf{d}'' \in \mathbb{Z}^{\Delta_0}$ and $q_{\Delta, \mathfrak{R}}(\mathbf{d}) = \langle \mathbf{d}, \mathbf{d} \rangle_{\Delta, \mathfrak{R}}$ for $\mathbf{d} \in \mathbb{Z}^{\Delta_0}$. Let Λ be the path algebra of (Δ, \mathfrak{R}) . If Λ is triangular (i.e. there are no oriented cycles in Δ), then $\langle -, - \rangle_{\Delta', \mathfrak{R}'} = \langle -, - \rangle_{\Delta, R}$ for each bound quiver (Δ', \mathfrak{R}') such that $k\Delta'/\langle \mathfrak{R}' \rangle \simeq \Lambda$ [14, Proposition 1.2]. Thus in this case we may write $\langle -, - \rangle_{\Lambda}$ and q_{Λ} instead of $\langle -, - \rangle_{\Delta, \mathfrak{R}}$ and $q_{\Delta, \mathfrak{R}}$, respectively. Additionally, if $\text{gldim } \Lambda \leq 2$, then

$$\begin{aligned} & \langle \mathbf{dim } M, \mathbf{dim } N \rangle_{\Lambda} \\ &= \dim_k \text{Hom}_{\Lambda}(M, N) - \dim_k \text{Ext}_{\Lambda}^1(M, N) + \dim_k \text{Ext}_{\Lambda}^2(M, N) \end{aligned}$$

for all Λ -modules M and N [14, Proposition 2.2].

1.7. An algebra Λ is called quasi-tilted if $\text{gldim } \Lambda \leq 2$ and either $\text{pd}_{\Lambda} X \leq 1$ or $\text{id}_{\Lambda} X \leq 1$ for each indecomposable Λ -module X . Equivalently, an algebra Λ is quasi-tilted if and only if there exists a tilting object T in a hereditary category \mathcal{H} such that $\Lambda \simeq \text{End}_{\mathcal{H}}(T)^{\text{op}}$ [26, Theorem II.2.3]. The quasi-tilted algebras are triangular [26, Proposition III.1.1 (b)], hence the forms $\langle -, - \rangle_{\Lambda}$ and q_{Λ} are defined for a quasi-tilted algebra Λ . Moreover, a quasi-tilted algebra Λ is tame if and only if $q_{\Lambda}(\mathbf{d}) \geq 0$ for each dimension vector \mathbf{d} [41, Theorem A]. Additionally, if \mathbf{d} is a dimension vector, then there exists an indecomposable Λ -module of dimension vector \mathbf{d} if and only if \mathbf{d} is connected and $q_{\Lambda}(\mathbf{d}) \in \{0, 1\}$. Finally, if \mathbf{d} is a connected dimension vector and $q_{\Lambda}(\mathbf{d}) = 1$, then there exists a unique (up to isomorphism) indecomposable Λ -module of dimension vector \mathbf{d} .

1.8. An important class of quasi-tilted algebra is that of the tilted algebras. An algebra Λ is called tilted if there exists a tilting module T over a hereditary algebra H such that $\Lambda \simeq \text{End}_H(T)^{\text{op}}$ (see for example [1, Chapter VIII] for more on tilted algebras). Recall that a module T over an algebra Λ is called tilting if $\text{pd}_{\Lambda} T \leq 1$, $\text{Ext}_{\Lambda}^1(T, T) = 0$, and T is a direct sum of n pairwise nonisomorphic indecomposable Λ -modules, where n is the number of the vertices in the Gabriel quiver of Λ . For a tilting module T over an algebra Λ we define two full subcategories $\mathcal{F}(T)$ and $\mathcal{T}(T)$ of $\text{mod } \Lambda$ by

$$\mathcal{F}(T) = \{M \in \text{mod } \Lambda \mid \text{Hom}_{\Lambda}(T, M) = 0\}$$

and

$$\mathcal{T}(T) = \{N \in \text{mod } \Lambda \mid \text{Ext}_{\Lambda}^1(T, N) = 0\}.$$

Bakke proved [2, Theorem] that an algebra Λ is tilted if and only if there exists a directing tilting Λ -module T . Here, a module T over an algebra Λ is called directing if and only if there exists no sequence

$$X_0 \xrightarrow{f_1} X_1 \rightarrow \cdots \rightarrow X_{n-1} \xrightarrow{f_n} X_n$$

of nonzero nonisomorphism f_1, \dots, f_n between indecomposable Λ -modules X_0, \dots, X_n such that X_0 and X_n are direct summands of

T and $X_{i-1} \simeq \tau_\Lambda X_{i+1}$ for some $i \in [1, n-1]$, where τ_Λ denotes the Auslander–Reiten translation of $\text{mod } \Lambda$ (see [1, Section IV.2] for a definition and basic properties of the Auslander–Reiten translation). The following properties of directing tilting modules will be useful for us (see [1, Chapter VIII; 2]): if T is a directing tilting module over an algebra Λ , then $\text{mod } \Lambda = \mathcal{F}(T) \vee \mathcal{T}(T)$, $\text{pd}_\Lambda M \leq 1$ for each $M \in \mathcal{F}(T)$, $\text{id}_\Lambda N \leq 1$ for each $N \in \mathcal{T}(T)$, and $\text{Hom}_\Lambda(N, M) = 0$ and $\text{Ext}_\Lambda^1(M, N) = 0$ for all $M \in \mathcal{F}(T)$ and $N \in \mathcal{T}(T)$. In the paper, if \mathcal{X} and \mathcal{Y} are two full subcategories of $\text{mod } \Lambda$ for an algebra Λ , then we denote by $\mathcal{X} \vee \mathcal{Y}$ the additive closure of their union.

1.9. Another important class of quasi-tilted algebras is formed by the concealed canonical algebras introduced in [32] — we refer to [32] for the definition of concealed canonical algebras and proofs of their properties listed below. If Λ is a concealed canonical algebra, then it possesses a sincere separating tubular family \mathcal{R} (in the sense of [38], see also [34, 40]). Moreover, if

$$\mathcal{P} = \{M \in \text{mod } \Lambda \mid \text{Hom}_\Lambda(R, M) = 0 \text{ for each } R \in \mathcal{R}\}$$

and

$$\mathcal{Q} = \{N \in \text{mod } \Lambda \mid \text{Hom}_\Lambda(N, R) = 0 \text{ for each } R \in \mathcal{R}\},$$

then $\text{mod } \Lambda = \mathcal{P} \vee \mathcal{R} \vee \mathcal{Q}$, $\text{pd}_\Lambda M \leq 1$ for each $M \in \mathcal{P} \vee \mathcal{R}$, $\text{id}_\Lambda N \leq 1$ for each $N \in \mathcal{R} \vee \mathcal{Q}$, and $\text{Hom}_\Lambda(N, M) = 0$ and $\text{Ext}_\Lambda^1(M, N) = 0$ if either $M \in \mathcal{P} \vee \mathcal{R}$ and $N \in \mathcal{Q}$ or $M \in \mathcal{P}$ and $N \in \mathcal{R} \vee \mathcal{Q}$.

2. MODULE VARIETIES

2.1. Let Λ be the path algebra of a bound quiver (Δ, \mathfrak{R}) and $\mathbf{d} \in \mathbb{N}^{\Delta_0}$. By the variety $\text{mod}_\Lambda(\mathbf{d})$ of Λ -modules of dimension vector \mathbf{d} we mean the set consisting of all $M \in \text{rep}(\Delta, \mathfrak{R})$ such that $M_x = k^{d_x}$. By forgetting the spaces M_x , $x \in \Delta$, we identify $\text{mod}_\Lambda(\mathbf{d})$ with the Zariski-closed subset of $\prod_{\alpha \in \Delta_1} \mathbb{M}_{d_{t_\alpha} \times d_{s_\alpha}}(k)$. Observe that for each Λ -module N of dimension vector \mathbf{d} there exists $M \in \text{mod}_\Lambda(\mathbf{d})$ such that $M \simeq N$. We will usually assume that all considered Λ -modules are points of module varieties. Observe that the above construction can be performed for any pair (Δ', \mathfrak{R}') consisting of a quiver Δ' and a set of relations \mathfrak{R}' (not only for bound quivers). The varieties obtained in this way are called representation varieties.

2.2. Let Λ be an algebra, Δ its Gabriel quiver, and $\mathbf{d} \in \mathbb{N}^{\Delta_0}$. Then $\text{GL}(\mathbf{d}) = \prod_{x \in \Delta_0} \text{GL}_{d_x}(k)$ acts on $\text{mod}_\Lambda(\mathbf{d})$ by conjugation: $(gM)_\alpha = g_{t_\alpha} M_\alpha g_{s_\alpha}^{-1}$ for $g \in \text{GL}(\mathbf{d})$, $M \in \text{mod}_\Lambda(\mathbf{d})$, and $\alpha \in \Delta_1$. Observe that $M \simeq N$ for $M, N \in \text{mod}_\Lambda(\mathbf{d})$ if and only if $\mathcal{O}(M) = \mathcal{O}(N)$, where for $M \in \text{mod}_\Lambda(\mathbf{d})$ we denote by $\mathcal{O}(M)$ its $\text{GL}(\mathbf{d})$ -orbit. One easily calculates that $\dim \mathcal{O}(M) = \dim \text{GL}(\mathbf{d}) - \dim_k \text{End}_\Lambda(M)$ for each $M \in \text{mod}_\Lambda(\mathbf{d})$ [30, 2.2]. If \mathcal{U} is a $\text{GL}(\mathbf{d})$ -invariant subset of $\text{mod}_\Lambda(\mathbf{d})$ and

$M \in \mathcal{U}$, then we say that $\mathcal{O}(M)$ is maximal in \mathcal{U} if there is no $N \in \mathcal{U}$ such that $\mathcal{O}(M) \subseteq \overline{\mathcal{O}(N)}$ and $N \not\cong M$.

Fix $M, N \in \text{mod}_\Lambda(\mathbf{d})$. The formula for the dimension of an orbit implies in particular that $\dim_k \text{End}_\Lambda(M) < \dim_k \text{End}_\Lambda(N)$ provided $\mathcal{O}(N) \subseteq \overline{\mathcal{O}(M)}$ and $N \not\cong M$. Moreover, if there exists an exact sequence $0 \rightarrow N_1 \rightarrow M \rightarrow N_2 \rightarrow 0$ such that $N_1 \oplus N_2 \simeq N$, then $\mathcal{O}(N) \subseteq \overline{\mathcal{O}(M)}$ (see for example [17, Lemma 1.1]). On the other hand, it has been proved in [44, Corollary 6] that if Λ is tame and quasi-tilted, then $\mathcal{O}(N) \subseteq \overline{\mathcal{O}(M)}$ if and only if there exist exact sequences $0 \rightarrow U'_i \rightarrow V_i \rightarrow U''_i \rightarrow 0$, $i \in [0, n]$, for $n \in \mathbb{N}$, such that $M \simeq V_0$, $V_i \simeq U'_{i-1} \oplus U''_{i-1}$ for each $i \in [1, n]$, and $U'_n \oplus U''_n \simeq N$.

2.3. Let \mathcal{V} be an affine variety. Recall that $x \in \mathcal{V}$ is called a regular point of \mathcal{V} if the dimension of the tangent space $T_x \mathcal{V}$ to \mathcal{V} at x equals the maximum of the dimensions of the irreducible components of \mathcal{V} containing x (see for example [31, VI.1]). In particular, if \mathcal{V} is irreducible, then $x \in \mathcal{V}$ is a regular point of \mathcal{V} if and only if $\dim_k T_x \mathcal{V} = \dim \mathcal{V}$. In general, if \mathcal{V}' is an irreducible component of \mathcal{V} and $x \in \mathcal{V}'$ is a regular point of \mathcal{V} , then x does not belong to an irreducible component of \mathcal{V} different from \mathcal{V}' . In particular, in the above situation x is a regular point of \mathcal{V}' . We say that \mathcal{V} is regular in codimension one if the codimension of the complement of the set of regular points of \mathcal{V} is at least 2. For example, normal varieties are regular in codimension one [21, Section 11.2]. Recall, that \mathcal{V} is called normal if \mathcal{V} is irreducible and its coordinate ring is an integrally closed domain.

2.4. Let Λ be the path algebra of a bound quiver (Δ, \mathfrak{R}) and $\mathbf{d}', \mathbf{d}'' \in \mathbb{N}^{\Delta_0}$. Fix $U \in \text{mod}_\Lambda(\mathbf{d}')$ and $V \in \text{mod}_\Lambda(\mathbf{d}'')$. If $\rho = \lambda_1 \sigma_1 + \cdots + \lambda_n \sigma_n$, for $n \in \mathbb{N}_+$, $\lambda_1, \dots, \lambda_n \in k$, and paths $\sigma_1, \dots, \sigma_n$ in Δ , is a relation, then for $Z \in \prod_{\alpha \in \Delta_1} \mathbb{M}_{d'_\alpha \times d''_\alpha}(k)$ we put

$$Z_\rho = \sum_{i \in [1, n]} \sum_{j \in [1, m_i]} \lambda_i U_{\alpha_{i,1}} \cdots U_{\alpha_{i,j-1}} Z_{\alpha_{i,j}} V_{\alpha_{i,j+1}} \cdots V_{\alpha_{i,m_i}},$$

provided $\sigma_i = \alpha_{i,1} \cdots \alpha_{i,m_i}$ with $\alpha_{i,1}, \dots, \alpha_{i,m_i} \in \Delta_1$ and $m_i \in \mathbb{N}_+$ for each $i \in [1, n]$. Let $\mathbb{Z}(V, U)$ be the set of all $Z \in \prod_{\alpha \in \Delta_1} \mathbb{M}_{d'_\alpha \times d''_\alpha}(k)$ such that $Z_\rho = 0$ for each $\rho \in \mathfrak{R}$. With every $Z \in \mathbb{Z}(V, U)$ we associate the exact sequence $\xi^Z : 0 \rightarrow U \xrightarrow{f} W^Z \xrightarrow{g} V \rightarrow 0$ of Λ -modules, where $W_x^Z = U_x \oplus V_x$ for $x \in \Delta_0$, $W_\alpha = \begin{bmatrix} U_\alpha & Z_\alpha \\ 0 & V_\alpha \end{bmatrix}$ for $\alpha \in \Delta_1$, and f_x and g_x are the canonical injection and projection, respectively, for $x \in \Delta_0$. This assignment induces a surjective linear map $\mathbb{Z}(V, U) \rightarrow \text{Ext}_\Lambda^1(V, U)$. Let $\mathbb{B}(V, U)$ be the kernel of this map. Then

$$\dim_k \mathbb{B}(V, U) = -\dim_k \text{Hom}_\Lambda(V, U) + \sum_{x \in \Delta_0} d'_x d''_x$$

and, consequently,

$$\dim_k \mathbb{Z}(V, U) = \dim_k \operatorname{Ext}_\Lambda^1(V, U) - \dim_k \operatorname{Hom}_\Lambda(V, U) + \sum_{x \in \Delta_0} d'_x d''_x$$

(compare [16, 2.1]).

2.5. Let Λ be an algebra and \mathbf{d} a dimension vector. It can be easily verified that $T_M \operatorname{mod}_\Lambda(\mathbf{d})$ is a subspace of $\mathbb{Z}(M, M)$ for each $M \in \operatorname{mod}_\Lambda(\mathbf{d})$ (compare [43, 3.4]). Using this observation one may show for $M \in \operatorname{mod}_\Lambda(\mathbf{d})$ that if Λ is triangular, $\operatorname{gldim} \Lambda \leq 2$, and $\operatorname{Ext}_\Lambda^2(M, M) = 0$, then $\dim_k \mathbb{Z}(M, M) = a_\Lambda(\mathbf{d}) = \sum_{\alpha \in \Delta_1} d_{s_\alpha} d_{t_\alpha} - \sum_{\rho \in \mathfrak{A}} d_{s_\rho} d_{t_\rho}$ and M is a regular point of $\operatorname{mod}_\Lambda(\mathbf{d})$ (compare [8, Proposition 3.2]). Observe that if Λ is triangular and $\operatorname{gldim} \Lambda \leq 2$, then $a_\Lambda(\mathbf{d}) = \dim \operatorname{GL}(\mathbf{d}) - q_\Lambda(\mathbf{d})$.

2.6. Let Λ be an algebra, M a Λ -module, and \mathbf{d} a dimension vector. For $d \in \mathbb{N}$ we define the subset $\mathcal{H}_d^{\mathbf{d}}(M)$ of $\operatorname{mod}_\Lambda(\mathbf{d})$ consisting of all N such that $\dim_k \operatorname{Hom}_\Lambda(M, N) = d$. This is a (possibly empty) locally closed subset of $\operatorname{mod}_\Lambda(\mathbf{d})$. It follows from [46, Section 3.4] (see also [45, Section 2]) that if $N \in \mathcal{H}_d^{\mathbf{d}}(M)$, then $T_N \mathcal{H}_d^{\mathbf{d}}(M)$ is contained in the subspace $\mathbb{Z}_M(N, N)$ of $\mathbb{Z}(N, N)$ consisting of all Z with $\dim_k \operatorname{Hom}_\Lambda(M, W^Z) = 2 \cdot \dim_k \operatorname{Hom}_\Lambda(M, N)$.

2.7. Let Λ be an algebra, and let \mathbf{d}' and \mathbf{d}'' be dimension vectors. For $d \in \mathbb{N}$ we define the subset $\mathcal{E}_d^{\mathbf{d}', \mathbf{d}''}$ of $\operatorname{mod}_\Lambda(\mathbf{d}') \times \operatorname{mod}_\Lambda(\mathbf{d}'')$ consisting of all (U, V) such that $\operatorname{Hom}_\Lambda(V, U) = 0$ and $\dim_k \operatorname{Ext}_\Lambda^1(V, U) = d$. It follows from [19, Lemma 4.3] that $\mathcal{E}_d^{\mathbf{d}', \mathbf{d}''}$ is a (possibly empty) locally closed subset of $\operatorname{mod}_\Lambda(\mathbf{d}') \times \operatorname{mod}_\Lambda(\mathbf{d}'')$. Let $(U, V) \in \mathcal{E}_d^{\mathbf{d}', \mathbf{d}''}$. Then $T_{U, V} \mathcal{E}_d^{\mathbf{d}', \mathbf{d}''}$ is contained in the subspace of $\mathbb{Z}(U, U) \times \mathbb{Z}(V, V)$ consisting of all (Z', Z'') such that the class of $\xi^{Z'} \circ \xi + \xi \circ \xi^{Z''}$ is zero in $\operatorname{Ext}_\Lambda^2(V, U)$ for each $[\xi] \in \operatorname{Ext}_\Lambda^1(V, U)$ [7, Proposition 3.3]. In particular, if Λ is triangular, $\operatorname{gldim} \Lambda \leq 2$, $\operatorname{Ext}_\Lambda^2(U, U) = 0 = \operatorname{Ext}_\Lambda^2(V, V)$, and there exists an exact sequence $\xi : 0 \rightarrow U \rightarrow M \rightarrow V \rightarrow 0$ such that either $\operatorname{Ext}_\Lambda^2(M, U) = 0$ or $\operatorname{Ext}_\Lambda^2(V, M) = 0$, then

$$\dim_k T_{U, V} \mathcal{E}_d^{\mathbf{d}', \mathbf{d}''} \leq a(\mathbf{d}') + a(\mathbf{d}'') - \dim_k \operatorname{Ext}_\Lambda^2(V, U).$$

Indeed, our assumptions imply that the map $\mathbb{Z}(U, U) \times \mathbb{Z}(V, V) \rightarrow \operatorname{Ext}_\Lambda^2(V, U)$, $(Z', Z'') \mapsto [\xi^{Z'} \circ \xi + \xi \circ \xi^{Z''}]$, is surjective, hence the claim follows.

2.8. Let Λ an algebra and \mathbf{d} a dimension vector. For a full subcategory \mathcal{C} of $\operatorname{mod} \Lambda$ by $\mathcal{C}(\mathbf{d})$ we denote $\operatorname{mod}_\Lambda(\mathbf{d}) \cap \mathcal{C}$. Let \mathcal{P} be the full subcategory of $\operatorname{mod} \Lambda$ consisting of the modules of projective dimension at most 1. Then $\mathcal{P}(\mathbf{d})$ is an open subset of $\operatorname{mod}_\Lambda(\mathbf{d})$. Moreover, if Λ

is triangular and $\mathcal{P}(\mathbf{d}) \neq \emptyset$, then $\mathcal{P}(\mathbf{d})$ is irreducible. Finally, if additionally $\text{gldim } \Lambda \leq 2$, then $\dim \mathcal{P}(\mathbf{d}) = a_\Lambda(\mathbf{d})$ (see [3, Proposition 3.1] for all the above claims).

2.9. Let Λ be an algebra and \mathbf{d}' , \mathbf{d}'' dimension vectors. If $\mathcal{U}' \subseteq \text{mod}_\Lambda(\mathbf{d}')$ and $\mathcal{U}'' \subseteq \text{mod}_\Lambda(\mathbf{d}'')$, then $\mathcal{U}' \oplus \mathcal{U}''$ denotes the set of all $M \in \text{mod}_\Lambda(\mathbf{d}' + \mathbf{d}'')$ such that $M \simeq M' \oplus M''$ for some $M' \in \mathcal{U}'$ and $M'' \in \mathcal{U}''$. Obviously, if \mathcal{U}' and \mathcal{U}'' are constructible (irreducible), then $\mathcal{U}' \oplus \mathcal{U}''$ is also constructible (irreducible, respectively). Moreover,

$$\begin{aligned} \dim(\mathcal{U}' \oplus \mathcal{U}'') &= \dim \mathcal{U}' + \dim \mathcal{U}'' + \dim \text{GL}(\mathbf{d}' + \mathbf{d}'') \\ &\quad - \dim \text{GL}(\mathbf{d}') - \min\{\dim_k \text{Hom}_\Lambda(U', U'') \mid U' \in \mathcal{U}', U'' \in \mathcal{U}''\} \\ &\quad - \dim \text{GL}(\mathbf{d}'') - \min\{\dim_k \text{Hom}_\Lambda(U'', U') \mid U' \in \mathcal{U}', U'' \in \mathcal{U}''\} \end{aligned}$$

(see for example [6, Lemma 3.4]).

3. GEOMETRIC BISECTIONS

In order to make possible to apply the arguments used in the proof of Proposition 3.1 in other situations, we start with developing the following technical concept. If Λ is an algebra, then a pair $(\mathcal{X}, \mathcal{Y})$ of full subcategories of $\text{mod } \Lambda$, which are closed under direct sums, will be called a geometric bisection of $\text{mod } \Lambda$ if the following conditions are satisfied:

- (1) $\mathcal{X} \vee \mathcal{Y} = \text{mod } \Lambda$,
- (2) $\text{pd}_\Lambda X \leq 1$ for each $X \in \mathcal{X}$ and $\text{id}_\Lambda Y \leq 1$ for each $Y \in \mathcal{Y}$,
- (3) $\text{Hom}_\Lambda(Y, X) = 0$ and $\text{Ext}_\Lambda^1(X, Y) = 0$ for all $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$,
- (4) if \mathbf{d} is a dimension vector, then $\mathcal{X}(\mathbf{d})$ and $\mathcal{Y}(\mathbf{d})$ are open subsets of $\text{mod}_\Lambda(\mathbf{d})$.

An example of a geometric bisection is provided by $(\mathcal{F}(T), \mathcal{T}(T))$ for a directing tilting module T over an algebra Λ (which is necessarily tilted). Indeed, the first three conditions are just the properties of directing tilting modules presented in (1.8). The remaining one follows from the upper-semicontinuity of the functions

$$\text{mod}_\Lambda(\mathbf{d}) \ni M \mapsto \dim_k \text{Hom}_\Lambda(T, M), \dim_k \text{Ext}_\Lambda^1(T, M) \in \mathbb{Z}$$

[19, Lemma 4.3].

Another example of a geometric bisection can be constructed for a concealed-canonical algebra Λ . Indeed, let \mathcal{P} , \mathcal{R} and \mathcal{Q} be as in (1.9), $\mathcal{X} = \mathcal{P}$, and $\mathcal{Y} = \mathcal{R} \vee \mathcal{Q}$. Then it remains to show that the last condition of the definition is satisfied. However, the proofs of [6, Lemmas 3.7 and 3.8] given for canonical algebras generalize easily to the case of concealed canonical algebras.

Let Λ be an algebra of global dimension at most 2 with a geometric bisection $(\mathcal{X}, \mathcal{Y})$. Observe that this means in particular that Λ is quasi-tilted. If $\mathcal{X}(\mathbf{d}) \neq \emptyset$ for a dimension vector \mathbf{d} , then $\overline{\mathcal{X}(\mathbf{d})} = \overline{\mathcal{P}(\mathbf{d})}$,

hence in particular $\overline{\mathcal{X}(\mathbf{d})}$ is an irreducible component of $\text{mod}_\Lambda(\mathbf{d})$ of dimension $a_\Lambda(\mathbf{d})$ (see (2.8)).

Proposition 3.1. *Let Λ be a quasi-tilted algebra with a geometric bisection $(\mathcal{X}, \mathcal{Y})$. Let \mathbf{d} be a dimension vector such that $\mathcal{X}(\mathbf{d}) \neq \emptyset$ and for each $N \in \overline{\mathcal{X}(\mathbf{d})} \setminus \mathcal{X}(\mathbf{d})$ such that $\mathcal{O}(N)$ is maximal in $\overline{\mathcal{X}(\mathbf{d})} \setminus \mathcal{X}(\mathbf{d})$ there exists an exact sequence $0 \rightarrow N_1 \rightarrow M \rightarrow N_2 \rightarrow 0$ with $M \in \mathcal{X}(\mathbf{d})$ and $N_1 \oplus N_2 \simeq N$. Then $\overline{\mathcal{X}(\mathbf{d})}$ is regular in codimension one.*

The proof basically repeats the arguments from [7, Proof of the main result], however some changes are necessary, hence we include it here for completeness.

Proof. Since $\text{pd}_\Lambda M \leq 1$ for each $M \in \mathcal{X}(\mathbf{d})$, $\mathcal{X}(\mathbf{d})$ consists of points which are regular in $\overline{\mathcal{X}(\mathbf{d})}$. Thus it suffices to show that for each irreducible component \mathcal{V} of $\overline{\mathcal{X}(\mathbf{d})} \setminus \mathcal{X}(\mathbf{d})$ there exists an open subset \mathcal{U} of \mathcal{V} consisting of regular points of $\overline{\mathcal{X}(\mathbf{d})}$. Fix such a component \mathcal{V} . Our assumptions imply that there exist dimension vectors \mathbf{d}' and \mathbf{d}'' such that $\mathcal{X}(\mathbf{d}') \oplus \mathcal{Y}(\mathbf{d}'')$ contains an open subset \mathcal{U}' of \mathcal{V} . Put

$$p = \min\{\text{pd}_\Lambda N \mid N \in \mathcal{U}'\}, \quad d' = \min\{\dim_k \text{End}_\Lambda(N) \mid N \in \mathcal{U}'\},$$

and

$$d'' = \min\{\dim_k \text{Ext}_\Lambda^1(N, N) \mid N \in \mathcal{U}'\}.$$

Let \mathcal{U} be the set of all $N \in \mathcal{U}'$ such that N does not belong to an irreducible component of $\overline{\mathcal{X}(\mathbf{d})} \setminus \mathcal{X}(\mathbf{d})$ different from \mathcal{V} , $\text{pd}_\Lambda N = p$, $\dim_k \text{End}_\Lambda(N) = d'$, and $\dim_k \text{Ext}_\Lambda^1(N, N) = d''$. Then \mathcal{U} is an open subset of \mathcal{V} . We show that N is a regular point of $\overline{\mathcal{X}(\mathbf{d})}$ for each $N \in \mathcal{U}$. The claim is obvious if $p \leq 1$, thus we may assume that $\text{pd}_\Lambda N = 2$ for each $N \in \mathcal{U}$.

We first show that for each $N \in \mathcal{U}$ there exists an exact sequence $0 \rightarrow N' \rightarrow M \rightarrow N'' \rightarrow 0$ with $M \in \mathcal{X}(\mathbf{d})$, $N' \in \mathcal{X}(\mathbf{d}')$, $N'' \in \mathcal{Y}(\mathbf{d}'')$, and $N' \oplus N'' \simeq N$. Since $\mathcal{O}(N)$ is maximal in $\overline{\mathcal{X}(\mathbf{d})} \setminus \mathcal{X}(\mathbf{d})$, there exists an exact sequence $\xi : 0 \rightarrow N_1 \rightarrow M_0 \rightarrow N_2 \rightarrow 0$ such that $N_1 \oplus N_2 \simeq N$ and $M_0 \in \mathcal{X}(\mathbf{d})$. Choose indecomposable direct summands N'_1 and N'_2 of N_1 and N_2 , respectively, such that $p \circ \xi \circ i : 0 \rightarrow N'_1 \rightarrow M' \rightarrow N'_2 \rightarrow 0$ does not split, where $p : N_1 \rightarrow N'_1$ and $i : N'_2 \rightarrow N_2$ are the canonical projection and injection, respectively. Write $N = N_0 \oplus N'_1 \oplus N'_2$ and let $M = N_0 \oplus M'$. Then $N \not\cong M$, $\mathcal{O}(N) \subseteq \overline{\mathcal{O}(M)}$, and $\mathcal{O}(M) \subseteq \overline{\mathcal{O}(N_0)}$. Consequently, the maximality of $\mathcal{O}(N)$ in $\overline{\mathcal{X}(\mathbf{d})} \setminus \mathcal{X}(\mathbf{d})$ implies that $M \in \mathcal{X}$. This immediately implies that $N_0 \oplus N'_1 \in \mathcal{X}$. Moreover, $N'_2 \in \mathcal{Y}$, since $\text{pd}_\Lambda N = 2$ and N'_2 is indecomposable, hence by adding $0 \rightarrow N_0 \rightarrow N_0 \rightarrow 0 \rightarrow 0$ to $p \circ \xi \circ i$ we obtain the desired exact sequence.

Now we prove that N is regular in $\overline{\mathcal{X}(\mathbf{d})}$ for each $N \in \mathcal{U}$. Let $0 \rightarrow N' \rightarrow M \rightarrow N'' \rightarrow 0$ be an exact sequence with $M \in \mathcal{X}(\mathbf{d})$, $N' \in \mathcal{X}(\mathbf{d}')$, $N'' \in \mathcal{Y}(\mathbf{d}'')$, and $N' \oplus N'' \simeq N$. Obviously, we may

assume that $N = N' \oplus N''$, i.e. $N \in \text{mod}_\Lambda(\mathbf{d}') \times \text{mod}_\Lambda(\mathbf{d}'')$, where we identify $\text{mod}_\Lambda(\mathbf{d}') \times \text{mod}_\Lambda(\mathbf{d}'')$ with

$$\left\{ \left(\begin{bmatrix} U_\alpha & 0 \\ 0 & V_\alpha \end{bmatrix} \right)_{\alpha \in \Delta_1} \mid U \in \text{mod}_\Lambda(\mathbf{d}'), V \in \text{mod}_\Lambda(\mathbf{d}'') \right\} \subseteq \text{mod}_\Lambda(\mathbf{d}).$$

Let \mathcal{U}_0 be the intersection of \mathcal{U} with $\text{mod}_\Lambda(\mathbf{d}') \times \text{mod}_\Lambda(\mathbf{d}'')$. Then \mathcal{U}_0 is an open subset of $(\text{mod}_\Lambda(\mathbf{d}') \times \text{mod}_\Lambda(\mathbf{d}'')) \cap \overline{\mathcal{X}(\mathbf{d})}$. Moreover, direct calculations show that $\mathcal{U}_0 \subseteq \mathcal{E}_d^{\mathbf{d}', \mathbf{d}''}$ for $d = d'' - d' + q_\Lambda(\mathbf{d}') + q_\Lambda(\mathbf{d}'') + \langle \mathbf{d}', \mathbf{d}'' \rangle_\Lambda$. Consequently,

$$\begin{aligned} \dim_k T_N((\text{mod}_\Lambda(\mathbf{d}') \times \text{mod}_\Lambda(\mathbf{d}'')) \cap \overline{\mathcal{X}(\mathbf{d})}) &= \dim_k T_N \mathcal{U}_0 \\ &\leq \dim_k T_{N', N''} \mathcal{E}_d^{\mathbf{d}', \mathbf{d}''} \leq a_\Lambda(\mathbf{d}') + a_\Lambda(\mathbf{d}'') - \dim_k \text{Ext}_\Lambda^2(N'', N') \end{aligned}$$

according to (2.7).

Now we show that $T_N \overline{\mathcal{X}(\mathbf{d})} \leq a(\mathbf{d})$ and this will finish the proof. Indeed, the above inequality implies that

$$\begin{aligned} T_N \overline{\mathcal{X}(\mathbf{d})} &\leq a_\Lambda(\mathbf{d}') + a_\Lambda(\mathbf{d}'') - \dim_k \text{Ext}_\Lambda^2(N'', N') \\ &\quad + \dim_k \mathbb{Z}(N', N'') + \dim_k \mathbb{Z}(N'', N') \end{aligned}$$

and direct calculations prove the claim. \square

4. PROOF OF THE MAIN RESULT

We start with presenting the following consequence of [8, Theorems 1 and 2]. If Λ is a tame quasi-tilted algebra and \mathbf{d} is the dimension vector of an indecomposable Λ -module, then $\text{mod}_\Lambda(\mathbf{d})$ has at most two irreducible components and each irreducible component of $\text{mod}_\Lambda(\mathbf{d})$ has dimension $a_\Lambda(\mathbf{d})$. Moreover, if $\text{mod}_\Lambda(\mathbf{d})$ is irreducible, then $\text{mod}_\Lambda(\mathbf{d})$ is normal, hence in particular regular in codimension one. On the other hand, if $\text{mod}_\Lambda(\mathbf{d})$ is not irreducible, then we may assume that \mathbf{d} is sincere and one of the following conditions is satisfied:

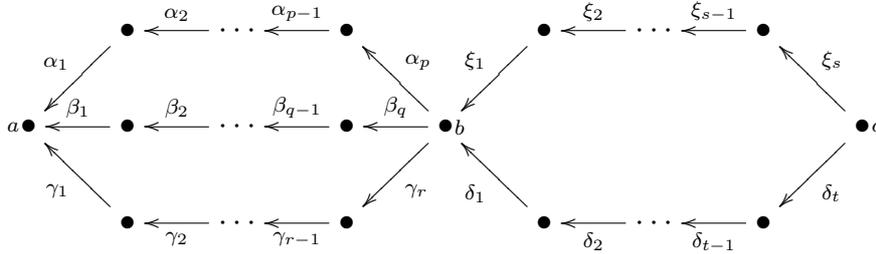
- (1) $\mathbf{d} = \mathbf{h} + \mathbf{d}'$ for nonzero connected dimension vectors \mathbf{h} and \mathbf{d}' with disjoint supports such that $q_\Lambda(\mathbf{h}) = 0$, $q_\Lambda(\mathbf{d}') = 1$, and $d'_x \leq 1$ each vertex x of the Gabriel quiver of Λ ,
- (2) $\mathbf{d} = \mathbf{h}' + \mathbf{h}''$ for connected dimension vectors \mathbf{h}' and \mathbf{h}'' such that $q_\Lambda(\mathbf{h}') = 0 = q_\Lambda(\mathbf{h}'')$, $\langle \mathbf{h}', \mathbf{h}'' \rangle_\Lambda = 1$, and $\langle \mathbf{h}'', \mathbf{h}' \rangle_\Lambda = 0$.

We remark that in both cases $q_\Lambda(\mathbf{d}) = 1$, hence there exists a uniquely determined (up to isomorphism) indecomposable Λ -module X of dimension vector \mathbf{d} . We study now the above described cases. In the below considerations we will use freely information about $\text{mod } \Lambda$ and $\text{mod}_\Lambda(\mathbf{d})$ for Λ and \mathbf{d} as above obtained in [8, 9].

First assume that the condition (1) is satisfied. In this case Λ is either tilted or concealed canonical, hence in particular it possesses a geometric bisection $(\mathcal{X}, \mathcal{Y})$. Up to duality, we may assume that $X \in \mathcal{X}$. A discussion of this case presented in [8, Section 5] shows that

under this assumption one of the irreducible components of $\text{mod}_\Lambda(\mathbf{d})$ is $\overline{\mathcal{X}(\mathbf{d})}$. Moreover, if $\mathcal{O}(M)$ is maximal in $\overline{\mathcal{X}(\mathbf{d})}$ for $M \in \text{mod}_\Lambda(\mathbf{d})$, then $M \in \mathcal{X}(\mathbf{d})$. Indeed, one shows that the maximal $\text{GL}(\mathbf{d})$ -orbits in $\overline{\mathcal{X}(\mathbf{d})}$ are precisely the maximal $\text{GL}(\mathbf{d})$ -orbits in $\text{mod}_\Lambda(\mathbf{d})$ which are contained in $\mathcal{X}(\mathbf{d})$. The proof of this fact requires a more detailed description of mod_Λ as given in [8, Section 5] and we leave it to the reader as an exercise. In particular, (2.2) implies that for each $N \in \overline{\mathcal{X}(\mathbf{d})} \setminus \mathcal{X}(\mathbf{d})$ such that $\mathcal{O}(N)$ is maximal in $\overline{\mathcal{X}(\mathbf{d})} \setminus \mathcal{X}(\mathbf{d})$ there exists an exact sequence $0 \rightarrow N_1 \rightarrow M \rightarrow N_2 \rightarrow 0$ with $M \in \mathcal{X}(\mathbf{d})$ and $N_1 \oplus N_2 \simeq N$. Consequently, $\overline{\mathcal{X}(\mathbf{d})}$ is regular in codimension one according to Proposition 3.1. The other irreducible component of $\text{mod}_\Lambda(\mathbf{d})$ in the case (1) is (isomorphic to) $\text{mod}_\Lambda(\mathbf{h}) \times \text{mod}_\Lambda(\mathbf{d}')$. The results of [8] imply that both factors are normal, hence regular in codimension one, and the claim follows for this component.

Now we study the case (2). The irreducible components of $\text{mod}_\Lambda(\mathbf{d})$ appearing in this case are described in [9]. This description implies that one of the irreducible components of $\text{mod}_\Lambda(\mathbf{d})$ is $\overline{\mathcal{O}(X)}$. Moreover, X is directing, hence the claim follows from [13, Theorem 1.1]. Thus it remains to study the other irreducible component \mathcal{V} of $\text{mod}_\Lambda(\mathbf{d})$. We describe this situation more precisely. For $p, q, r, s, t \in \mathbb{N}_+$ let $\Delta(p, q, r, s, t)$ be the quiver



and

$$\mathfrak{R}(p, q, r, s, t) = \{ \alpha_1 \cdots \alpha_p - \beta_1 \cdots \beta_q + \gamma_1 \cdots \gamma_r, \\ \alpha_p \xi_1, \beta_q \xi_1 \cdots \xi_s - \beta_q \delta_1 \cdots \delta_t, \gamma_r \delta_1 \}.$$

Then either Λ or Λ^{op} is isomorphic to $k\Delta(p, q, r, s, t)/\mathfrak{R}(p, q, r, s, t)$ for some $p, q, r, s, t \in \mathbb{N}_+$. Observe that if at least one of the numbers p, q, r equals 1, then $(\Delta(p, q, r, s, t), \mathfrak{R}(p, q, r, s, t))$ is not a bound quiver. In this case a corresponding bound quiver is obtained by removing an appropriate arrow (this arrow may be not uniquely determined if at least two of the numbers p, q, r equal 1) and modifying $\mathfrak{R}(p, q, r, s, t)$ accordingly (see [10, Section 2] for the list of bound quivers obtained in this way). This operation induces an isomorphism of the corresponding representation varieties, hence we may work with the above family of algebras in order to reduce the number of considered cases.

Thus assume that $\Lambda = k\Delta/\mathfrak{R}(p, q, r, s, t)$ for some $p, q, r, s, t \in \mathbb{N}_+$, where we put $\Delta = \Delta(p, q, r, s, t)$. Let Δ' (Δ'') be the minimal convex subquiver of Δ containing a and b (b and c , respectively). If \mathbf{h}' and \mathbf{h}'' are as in (2), then

$$h'_x = \begin{cases} 1 & x \in \Delta'_0 \\ 0 & x \notin \Delta'_0 \end{cases} \quad \text{and} \quad h''_x = \begin{cases} 1 & x \in \Delta''_0 \\ 0 & x \notin \Delta''_0 \end{cases}.$$

For $\lambda \in k \setminus \{0, 1\}$ we define the representation $H'(\lambda)$ of Δ of dimension vector \mathbf{h}' by

$$H'(\lambda)_\alpha = \begin{cases} \lambda & \alpha = \alpha_1, \\ \lambda + 1 & \alpha = \beta_1, \\ 1 & \alpha \in \Delta'_1, \alpha \neq \alpha_1, \beta_1, \\ 0 & \alpha \in \Delta''_1. \end{cases}$$

Similarly, we define the representation $H''(\lambda)$ of Δ of dimension vector \mathbf{h}'' for each $\lambda \in k \setminus \{0\}$. Moreover, for each $i \in [1, p]$ we define the representation $H'(\alpha_i)$ of Δ of dimension vector \mathbf{h}' by

$$H'(\alpha_i)_\alpha = \begin{cases} 0 & \alpha = \alpha_i, \\ 1 & \alpha \in \Delta'_1, \alpha \neq \alpha_i, \\ 0 & \alpha \in \Delta''_1. \end{cases}$$

Similarly, we define the representations $H'(\beta_i)$, $i \in [1, q]$, (in this case -1 has to appear in the definition) and $H'(\gamma_i)$, $i \in [1, r]$, of dimension vector \mathbf{h}' , and the representations $H''(\xi_i)$, $i \in [1, s]$, and $H''(\delta_i)$, $i \in [1, t]$, of dimension vector \mathbf{h}'' . Let \mathcal{U} be the union of $\mathcal{O}(H'(u) \oplus H''(v))$, $u \in (k \setminus \{0, 1\}) \cup \Delta'_1$, $v \in (k \setminus \{0\}) \cup \Delta''_1$. Then \mathcal{V} is the closure of \mathcal{U} .

Now we show that $\mathcal{V} = \text{mod}_\Lambda(\mathbf{h}') \oplus \text{mod}_\Lambda(\mathbf{h}'')$. Since $\mathcal{U} \subseteq \text{mod}_\Lambda(\mathbf{h}') \oplus \text{mod}_\Lambda(\mathbf{h}'')$ and $\text{mod}_\Lambda(\mathbf{h}') \oplus \text{mod}_\Lambda(\mathbf{h}'')$ is irreducible, it suffices to show that $\text{mod}_\Lambda(\mathbf{h}') \oplus \text{mod}_\Lambda(\mathbf{h}'')$ is closed. However, it can be easily checked that $M \in \text{mod}_\Lambda(\mathbf{h}') \oplus \text{mod}_\Lambda(\mathbf{h}'')$ if and only if $\text{rk}[(M_\alpha^{\text{tr}})_{\substack{\alpha \in \Delta_1 \\ s_\alpha = b}}] \leq 1$, $\text{rk}[(M_\alpha)_{\substack{\alpha \in \Delta_1 \\ t_\alpha = b}}] \leq 1$, and $M_{\alpha'} M_{\alpha''} = 0$ for all $\alpha', \alpha'' \in \Delta_1$ with $s_{\alpha'} = b = t_{\alpha''}$.

As the next step we show that $\dim(\mathcal{V} \setminus \mathcal{U}) \leq \dim \mathcal{V} - 2$. Let \mathcal{U}' be the union of $\mathcal{O}(H'(u))$, $u \in (k \setminus \{0, 1\}) \cup \Delta'_1$. Then $\text{mod}_\Lambda(\mathbf{h}') \setminus \mathcal{U}'$ consists of $M \in \text{mod}_\Lambda(\mathbf{h}')$ such that $M_{\alpha_i} = M_{\beta_j} = M_{\gamma_l} = 0$ for some $i \in [1, p]$, $j \in [1, q]$, and $l \in [1, r]$. Thus $\dim(\text{mod}_\Lambda(\mathbf{h}') \setminus \mathcal{U}') = \dim \text{mod}_\Lambda(\mathbf{h}') - 2$. Similarly, $\dim(\text{mod}_\Lambda(\mathbf{h}'') \setminus \mathcal{U}'') = \dim \text{mod}_\Lambda(\mathbf{h}'') - 2$, where \mathcal{U}'' is the union of $\mathcal{O}(H''(v))$, $v \in (k \setminus \{0\}) \cup \Delta''_1$. Since

$$\mathcal{V} \setminus \mathcal{U} = (\mathcal{U}' \oplus \text{mod}_\Lambda(\mathbf{h}'')) \cup (\text{mod}_\Lambda(\mathbf{h}') \oplus \mathcal{U}''),$$

the inequality $\dim(\mathcal{V} \setminus \mathcal{U}) \leq \dim \mathcal{V} - 2$ follows from (2.9).

As the last step of the proof we show that \mathcal{U} consists of regular points of \mathcal{V} . Let S be the simple Λ -module at b , i.e. $S_x = \delta_{x,b}k$ for

$x \in \Delta_0$, where $\delta_{x,y}$ is the Kronecker delta. Then $\mathcal{U} \subseteq \mathcal{H}_1^{\mathbf{d}}(S)$ and $\mathcal{H}_1^{\mathbf{d}}(S)$ contains an open subset of \mathcal{V} . Consequently, $T_M \mathcal{V} \subseteq \mathbb{Z}_S(M, M)$ for each $M \in \mathcal{U}$ (2.6). However, if $M = H'(u) \oplus H''(v)$ for $u \in (k \setminus \{0, 1\}) \cup \Delta'_1$ and $v \in (k \setminus \{0\}) \cup \Delta''_1$, then direct calculations show that

$$\begin{aligned} \mathbb{Z}_S(M, M) = \mathbb{Z}(H'(u), H'(u)) \oplus \mathbb{Z}(H''(v), H''(v)) \\ \oplus \mathbb{Z}(H'(u), H''(v)) \oplus \mathbb{B}(H''(v), H'(u)). \end{aligned}$$

By applying formulas from (2.4) we get $\dim_k \mathbb{Z}_S(M, M) \leq a_{\Lambda}(\mathbf{d})$, and this finishes the proof.

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