SIMULTANEOUS PING-PONG PARTNERS IN $PSL_n(\mathbb{Z})$.

G. SOIFER, S. VISHKAUTSAN

Abstract. We show that for any finite set F of nonidentity elements in $PSL_n(\mathbb{Z})$ for $n \geq 3$, consisting of hyperbolic, finite order or unipotent elements, there exists an element g of infinite order in $PSL_n(\mathbb{Z})$ such that for any $h \in F$, the subgroup $\langle g, h\rangle$ generated by g and h is canonically isomorphic to the free product $\langle g\rangle * \langle h\rangle$. We also show that the set of such elements in $PSL_n(\mathbb{Z})$ is Zariski dense in $PSL_n(\mathbb{R})$.

Contents

1. Introduction

This paper gives a partial positive answer to the following question due to P. de la Harpe (cf. [2], [5]):

Question 1.1. *Let F be any finite set of nonidentity elements in $PSL_n(\mathbb{Z})$ for $n \geq 3$. Does there exist an element g of infinite order in $PSL_n(\mathbb{Z})$ such that for any $h \in F$, the subgroup $\langle g, h\rangle$ generated by g and h is canonically isomorphic to the free product $\langle g\rangle * \langle h\rangle$?*

Date: November 2007.

This property in a group is called $P_{naï}$ by de la Harpe, and has already been demonstrated in [2] for $PSL_2(\mathbb{Z})$ and more generally for Zariski-dense subgroups of connected simple Lie groups with $\mathbb{R}$-rank 1 and trivial center.

It is a well known fact that for each element $g \in GL_n(k)$ (where k is any local field) there exists a unique decomposition $g = g_s g_u$ where g_s is semisimple (diagonalizable), g_u is unipotent and $g_s g_u = g_u g_s$. This is known as the multiplicative Jordan decomposition (see [3]). We call an element g in a linear group *hyperbolic* if both g and g^{-1} have unique eigenvalues with multiplicity 1 of maximal modulus. Such elements have unique fixed points in projective space attracting the whole space except for a hyperplane (we give precise definitions of hyperbolic elements and attracting points in Section 4). It is clear that g is hyperbolic if and only if g_s is hyperbolic.

In this article we prove the property described by de la Harpe for an important subset of $PSL_n(\mathbb{Z})$:

Theorem 1.2. *Let F be any finite set of nonidentity elements in $PSL_n(\mathbb{Z})$ for $n \geq 3$, such that for every $h \in F$, h_s is either hyperbolic, of finite order or trivial. Then there exists an element g of infinite order in $PSL_n(\mathbb{Z})$ such that for any $h \in F$, the subgroup $\langle g, h\rangle$ generated by g and h is canonically isomorphic to the free product $\langle g\rangle * \langle h\rangle$.*

We call elements g such as in theorem 1.2 *ping-pong partners* for the elements in the set F (This terminology will be explained in section 2). We will show that there are many such ping-pong partners g in $PSL_n(\mathbb{Z})$:

Theorem 1.3. *Let Ω_F be the set of all ping-pong partners for a set F as in theorem 1.2. Then Ω_F is Zariski dense in $PSL_n(\mathbb{R})$.*

Question 1.1 is a special case of a more general question of P. de la Harpe (cf. [2], [6]):

Question 1.4. *Let G be a connected, semisimple Lie group without compact factors and with finite center, and let Γ be a center-free subgroup whose image is Zariski-dense in the adjoint group $Ad(G)$. Does Γ enjoy the property that, for any finite set F of nonidentity elements of Γ, there exists an element g of infinite order such that for any $h \in F$, the subgroup $\langle g, h\rangle$ generated by g and h is canonically isomorphic to the free product $\langle g\rangle * \langle h\rangle$?*

This question was answered for "most" semisimple groups by T. Poznanski (cf. [8]):

Theorem 1.5. *Let G be a connected, semisimple Lie group without compact factors and with finite center. Let Γ be a center-free subgroup whose image is Zariski-dense in the adjoint group $Ad(G)$. Let G_b be the subgroup generated by the almost simple factors of G of type A_n or D_{2n+1} for $n > 1$, or E_6. If $\Gamma \bigcap G_b = 1$, then for any finite set F of nonidentity elements of Γ, there exists an element g of infinite order such that for any $h \in F$, the subgroup $\langle g, h\rangle$ generated by g and h is canonically isomorphic to the free product $\langle g\rangle * \langle h\rangle$.*

Notice that $PSL_n(\mathbb{R})$ is of type A_n making theorem 1.5 inapplicable for theorem 1.1.

In this article we employ techniques introduced by J. Tits[10] and extended in articles such as [7] and [1] among many others. The techniques are based on dynamic properties of transformations in projective space.

The authors would like to thank G.Noskov pointed out our attention on de la Harpe question (see Harpe-Open-Question1) and very useful discussions; several institutions and foundations for their support during the preparation of this paper: Bielefeld University, Yale University, SFB 701 "Spektrale Strukturen und Topologische Methoden in der Mathematik", USA- Israel Binational Science foundation under BSF grant 2004010, the Emmy Noether Research Institute for Mathematics, Bar-Ilan University.

2. The Ping-Pong Lemma

The following criterion is central in identifying free subgroups in groups [10].

Theorem 2.1 (Ping-Pong Lemma). *Let G be a group acting upon a set M. Let I be an index set, and $\{M_i|i \in I\}$ be subsets of M. Let $\{g_i|g_i \in G, i \in I\}$ be a set generating G. If there exists $m_0 \in M$ such that $m_0 \notin \bigcup_{i\in I} M_i$ and $\forall i \in I, j \in I, z \in \mathbb{Z}$ $g_i^z(m_0 \cup M_j) \subseteq M_i$ when $g_i^z \neq 1$ then the group $G = \langle g_i \rangle_{i\in I}$ is canonically isomorphic to the free product $*_{i\in I}\langle g_i\rangle$.*

Proof. Suppose $\langle g_i|i \in I\rangle$ is not free, i.e. there exist $n_{i_1}, \ldots, n_{i_k} \in \mathbb{Z}$ such that $g_{i_1}^{n_{i_1}} \cdots g_{i_k}^{n_{i_k}} = e$. In that case $g_{i_1}^{n_{i_1}} \cdots g_{i_k}^{n_{i_k}}(m_0) = e(m_0) = m_0$. However, $g_{i_1}^{n_{i_1}} \cdots g_{i_k}^{n_{i_k}}(m_0) = g_{i_1}^{n_{i_1}}(g_{i_2}^{n_{i_2}} \cdots g_{i_k}^{n_{i_k}}(m_0)) \in M_{i_1}$ (by inductive reasoning) and because $m_0 \notin \bigcup_1^n M_i$ we get a contradiction. □

Remark 2.2. *It is only necessary to require in lemma 2.1 that $g_i(m_0) \in M_i$ for some $i \in I$. This is because $g_i^{-1}hg_i = 1 \iff h = 1$ for $h \in G$ (stretching the metaphor, the players can still play ping-pong even if only one of them knows how to serve). Also notice that the subsets M_i can intersect non-trivially.*

Definition 2.3. Let G be a group, and $g, h \in G$. g is a *ping-pong partner* for h if the subgroup $\langle g, h\rangle$ generated by g and h is canonically isomorphic to the free product $\langle g\rangle * \langle h\rangle$.

Remark 2.4. *We also say that g, h in the definition above* play ping-pong *with each other.*

3. Standard Metric on Projective Space

Let V be an n-dimensional vector space over the local field k, and let P be the corresponding projective space of V.

For using the dynamic properties of projective transformations, we need a stronger topology than the Zariski topology on projective space. We define the *standard metric* on P(cf. Breuillard & Gelander [4]):

For any $[v],[w] \in P$ we set d to

$$d\big([v],[w]\big) = \frac{\|v \wedge w\|}{\|v\| \cdot \|w\|}$$

($v \wedge w$ is the standard vector product.)

Notice that this is the same as defining the distance between the 2 points to be the sine of the angle between any 2 representatives of $[v]$ and $[w]$.

Proposition 3.1. *d is a well defined distance function on P, and it induces the canonical topology inherited from the local field k.*

Proof. First we show that d is well defined. Since v and w are both nonzero by the definition P, we never get 0 in the denominator. Let v' (resp. w') be another representative of $[v]$ (resp. $[w]$). This means there exists $\lambda_1, \lambda_2 \in k$ such that $v' = \lambda_1 v$ and $w' = \lambda_2 w$.

$$d\big([v'],[w']\big) = \frac{\|v' \wedge w'\|}{\|v'\| \cdot \|w'\|} = \frac{\|\lambda_1 v \wedge \lambda_2 w\|}{\|\lambda_1 v\| \cdot \|\lambda_2 w\|} = \frac{|\lambda_1|\,|\lambda_2|\,\|v \wedge w\|}{|\lambda_1|\,|\lambda_2|\,\|v\| \cdot \|w\|} = d\big([v],[w]\big).$$

P is isomorphic to a quotient space of the sphere S^{n-1}, where the angle metric is equivalent to the Euclidean distance function, inducing on S^{n-1} the canonical topology inherited from k. d is equivalent to the quotient metric of the angle metric on P inducing the canonical topology inherited from the local field k. □

In the rest of this paper, when we refer to topological properties of P they are in the topology induced from the metric d defined above, unless stated otherwise.

4. Proximality

The contents in this section are based on Tits[10].

Let V be an n-dimensional vector space over a local field k, and let P be the corresponding projective space of V.

Definition 4.1. For any $g \in GL(V)$, let $f(t) = \prod_{i=1}^{n}(t-\lambda_i)$ be it's characteristic polynomial. We look at the set of eigenvalues of maximal modulus of g, $\Omega := \{\lambda_i, |\lambda_i| = max\{|\lambda_j|\}_{1\le j\le n}\}$. $f(t) = f_1(t)f_2(t)$ where $f_1(t) = \prod_{\lambda_i\in\Omega}(t-\lambda_i)$ and $f_2(t) = \prod_{\lambda_i\notin\Omega}(t-\lambda_i)$. We define $A(g)$ to be the subspace of P corresponding to $Ker(f_1(g))$, and call it the *attracting subspace* of g. We define $A'(g)$ to be the subspace of P corresponding to $Ker(f_2(g))$, and call it the *repulsing subspace* of g.

Definition 4.2. An element $g \in GL(V)$ is called *proximal* if g has a unique eigenvalue λ of maximal absolute value of algebraic multiplicity one. For a proximal element g we denote by $\lambda(g)$ the unique eigenvalue of maximal modulus of g.

Remark 4.3. *If $k = \mathbb{C}$ then $\lambda(g)$ has to be real (otherwise $\overline{\lambda(g)}$ the complex conjugate of $\lambda(g)$ will also be an eigenvalue of maximal modulus).*

Remark 4.4. *For a proximal element $g \in SL_n(k)$ we have $|\lambda(g)| > 1$, since the determinant of g is 1.*

Remark 4.5. *For a proximal element $g \in GL(V)$, $A(g)$ is a point in P and is called the* attracting point *of g. $A'(g)$ is a projective hyperplane.*

It is easy to see why the following two propositions are true.

Proposition 4.6. *If $g \in GL(V)$ is proximal, then so is g^n for $n \in \mathbb{Z}$ where $n > 0$. $\lambda(g^n) = \lambda(g)^n$. $A(g^n) = A(g)$ and $A'(g^n) = A'(g^n)$.*

Proposition 4.7. *Let $g \in GL_n(k)$. Then for any $h \in GL_n(k)$ we have $A(hgh^{-1}) = hA(g)$ and $A'(hgh^{-1}) = hA'(g)$*

Corollary 4.8. *Let $g \in GL_n(k)$ be proximal. Then for any $h \in GL_n(k)$, hgh^{-1} is proximal, and $\lambda(hgh^{-1}) = \lambda(g)$.*

The following proposition explains the name "attracting point" (For proof see [10]):

Proposition 4.9. *Let $g \in GL(V)$ be a proximal element, $K \subset P$ be a compact set such that $K \cap A'(g) = \emptyset$, and r be any positive number in $\mathbb{R}$. Then for every neighborhood U of $A(g)$ there exists $M \in \mathbb{Z}$, $M > 0$, such that for all $m \geq M$, $g^m K \subset U$ and $\|g^m|_K\| < r$.*

Conversely we have a sufficient condition for an element being proximal (For proof see [10]):

Proposition 4.10. *Let $g \in GL(V)$ and let $K \subset P$ be a compact set. Let $int(K)$ denote the interior of K in P. If for some $m \in \mathbb{Z}, m > 0$, $g^m K \subset int(K)$ and $\|g^m|_K\| < 1$ then g is proximal and $A(g) \in int(K)$.*

Proposition 4.11. *If $g \in GL(V)$ is proximal, then there exists a small enough neighborhood W of e (the unit element in $GL(V)$) such that $w_1 g w_2$ is proximal for any $w_1, w_2 \in W$.*

Proof. It suffices to prove the proposition for gw for any w in W, since we can conjugate $w_1 g w_2$ by w_1^{-1} and get $g w_2 w_1$ and use property 4.7.

Given a compact $K \subset P$ such that $A(g) \in int(K)$, we know by proposition 4.9 that there exists a large enough power m such that $\|g^m|_K\| < r$ for some $r < 1$. Moreover, we have $A(g) \in int(g^m K)$, and we can assume $g(g^{m-1}(K)) \subset int(g^{m-1}K)$. We can choose a small enough neighborhood W of e such that $wg^m K \subset int(g^{m-1}K)$, and such that $\|w\| < r^{-1}$. Therefore we have $gw(g^m K) \subset int(g^m K)$ and $\|gw|_{g^m K}\| < 1$. Using proposition 4.10 we get that gw is proximal, and $A(gw)$ is in $int(g^m K)$. □

Definition 4.12. We will call $g \in GL(V)$ *hyperbolic* if both g and g^{-1} are proximal.

(Hyperbolic elements are called *very proximal* by Poznansky [8].)

5. Transversality

Definition 5.1. For any two elements $g, h \in GL(V), g \neq h$, g and h are called *transversal* if $A(g) \cup A(g^{-1}) \subset P - A'(h) - A'(h^{-1})$, and $A(h) \cup A(h^{-1}) \subset P - A'(g) - A'(g^{-1})$

The following important theorem was proved by Tits [10]:

Theorem 5.2. *Let F be a finite set of hyperbolic elements in $GL_n(k)$. Suppose that the elements in F are transversal in pairs. Then there exists $M \in \mathbb{Z}, M > 0$ such that for any integer $m > M$, the set $F^m = \{g^m | g \in F\}$ is free in $GL_n(k)$.*

(A subset F of a group G is said to be *free* if the inclusion $F \to G$ extends to an injective homomorphism of the free group generated by F into G).

Proof. We start by choosing a point p not in $A(g), A'(g), A(g), A'(g)$ for all g in F. We choose for every g in F compact neighborhoods U_g of $A(g)$ and $U_{g^{-1}}$ of $A(g^{-1})$ such that for all h in F we have:

$$(U_g \cup U_{g^{-1}}) \cap (A'(h) \cup A'(h^{-1})) = \emptyset.$$

Such a system exists because g is transversal with any h in F. By proposition 4.9 there exists $M \in \mathbb{Z}, M > 0$ such that for any $m \geq M$, $z \in \mathbb{Z}$ and $g \in F$ we have $g^{mz}p \in U_g \cup U_{g^{-1}}$. We can then apply the ping-pong lemma to show that F^m is free for any $m \geq M$. □

Obviously any two transversal hyperbolic elements are ping-pong partners when taken in a large enough power. However this does not mean that these two elements are ping-pong partners themselves. The notion of transversality does not take into account the orbit of the starting point chosen. In section 8 we show a method of choosing a ping-pong partner that takes this into account, providing a ping-pong partner for the actual element and not for some power of it. The techniques used in proving it draw from the proof of theorem 5.2 and the notion of transversality.

6. Some Facts from Ergodic Theory

We recall some well known facts from Ergodic Theory (See [11]).

Definition 6.1. Let Let $(X, \mathscr{S}, \mu)$ be a measure space. We call a transformation T a *measure preserving transformation* if for all $A \in \mathscr{S}$, we have that $\mu(T^{-1}(A)) = \mu(A)$.

Theorem 6.2 (Poincaré Recurrence Theorem)**.** *Let $(X, \mathscr{S}, \mu)$ be a measure space where $\mu(X) < \infty$ and let $f\colon X \to X$ be a measure preserving transformation. For any $E \in \mathscr{S}$, the set of those points $x \in E$ such that $f^n(x) \notin E$ for all $n > 0$ has zero measure. That is, almost every point of E returns to E. In fact, almost every point returns* infinitely often*; i.e.*

$$\mu\left(\{x \in E : \text{ there exists } N \text{ such that } f^n(x) \notin E \text{ for all } n > N\}\right) = 0.$$

Proof. We define a descending chain of sets $A_i = \bigcup_{k=i}^{\infty} f^{-k}(E)$. We have $A_i = f^{j-i}(A_j)$, for all $0 \leq j \leq i$. Therefore $\mu(A_i) = \mu(A_j)$ for all $i, j \geq 0$. Since $E \subset A_0$, then for all $n > 0$ we have $E - A_n \subset A_0 - A_n$, and $\mu(E - A_n) \leq \mu(A_0 - A_n) = \mu(A_0) - \mu(A_n) = 0$. So $\mu(E - \bigcap_{n=1}^{\infty} A_n) = \mu(\bigcup_{n=1}^{\infty}(E - A_n)) = 0$. $E - \bigcap_{n=1}^{\infty} A_n$ is exactly the set of points in E not returning an infinite number of times to E. □

Definition 6.3. Let $(X, \mathscr{S}, \mu)$ be a finite measure space, and let $f\colon X \to X$ be a measure preserving transformation. Then f is called *ergodic* if for every set $E \in \mathscr{S}$, $f(E) = E$ means that $\mu(E) = 0$ or $\mu(E) = 1$.

Theorem 6.4. *Let $(X, \mathscr{S}, \mu)$ be a finite measure space. If a measure preserving map $f\colon X \to X$ is ergodic then for all sets $E, F \in \mathscr{S}$ such that $\mu(E) > 0$ and $\mu(F) > 0$, the set of points $x \in E$ such that $f^n(x) \notin F$ for all $n > 0$ has zero measure. That is, almost every point of E visits F. In fact, almost every point of E visits F* infinitely often*; i.e.*

$$\mu\left(\{x \in E : \text{ there exists } N \text{ such that } f^n(x) \notin F \text{ for all } n > N\}\right) = 0.$$

Proof. We define $A_0 = E$ and $A_i = f^i(E)$ for $i > 0$. Noticing that $f(\bigcup_{i=0}^{\infty} A_i) = \bigcup_{i=0}^{\infty} A_i$, we get that $\mu(\bigcup_{i=0}^{\infty} A_i)$ is either 0 or 1. Since $\bigcup_{i=0}^{\infty} A_i \supset E$, we get $\mu(\bigcup_{i=0}^{\infty} A_i) = 1$. Therefore $\bigcup_{i=0}^{\infty} A_i \cap F \neq \emptyset$, meaning there exists x in E and an integer $j > 0$ such that $f^j(x)$ is in F.

Let $\tilde{E} = \{x \in E : \text{ there exists } N \text{ such that } f^n(x) \notin F \text{ for all } n > N\}$. To prove $\mu(\tilde{E}) = 0$ we notice that $\tilde{E} = \bigcup_{i=1}^{\infty} B_i \setminus B_{i-1}$ where $B_0 = \emptyset$ and $B_i = \{x \in E : f^n(x) \notin F \text{ for all } n \geq i\}$. The sets $\tilde{B}_i = B_i \setminus B_{i-1}$ $(i \geq 1)$ are pairwise disjoint. Therefore $\mu(\tilde{E}) = \sum_{i=1}^{\infty} \mu(B_i \setminus B_{i-1})$. But $\mu(\tilde{B}_i) = 0$ for all $i \geq 1$ for otherwise we can define $h = f^{i+1}$ and using the same reasoning as for E and f get that there exists $x \in \tilde{B}_i$ such that $f^k \in F$ for some $k > i + 1$. Therefore $\mu(\tilde{E}) = 0$. □

Theorem 6.5 (Moore's Ergodicity Theorem [12])**.** *Let $G = \Pi G_i$ be a finite product of connected non-compact simple Lie groups with finite center. Let $\Gamma \subset G$ be an irreducible lattice. If $H \subset G$ is a closed subgroup and H is not compact, then H is ergodic on G/Γ.*

Corollary 6.6. *The action of the group generated by a hyperbolic element $g \in PSL_n(\mathbb{R})$ on $SL_n(\mathbb{R})/SL_n(\mathbb{Z})$ is ergodic.*

Proof. To use Moore's Ergodicity Theorem 6.5, we take the closure of the group generated by g in $PSL_n(\mathbb{R})$ and all that remains to be proved is that this group is compact. Since g is hyperbolic we have $|\lambda(g)| > 1$. Therefore the closure of the group generated by g is unbounded in $PSL_n(\mathbb{R})$ i.e. a non-compact subgroup of $PSL_n(\mathbb{R})$. □

7. Selberg's Lemma

Selberg's Lemma conveniently allows us to work with elements from $SL_n(\mathbb{R})$ and then replace them with elements from $SL_n(\mathbb{Z})$ having the same properties (See [9]).

Definition 7.1. Let G be a connected Lie group and H a subgroup of G. H is said to have *property (S)* in G if for any neighborhood Ω of e (unit element in G) and any element $g \in G$, there exists an integer $k \geq 0$ (depending on g and Ω), and $w_1, w_2 \in \Omega$ and $h \in H$ such that $g^k = w_1 h w_2$.

Lemma 7.2 (Selberg). *If H is a closed subgroup of G such that G/H has a finite invariant measure then H has property (S) in G.*

Proof. Take an element $g \in G$ (we can assume $g \neq e$), and any open neighborhood $W \subset G$ of e. Using Poincaré's Recurrence Theorem (6.2) (notice that $g^k H$ is measure preserving due to the left-invariance of the measure under all elements in G/H), we get that for some $k \in \mathbb{Z}$, $k > 0$, $g^k WH \cap WH \neq \emptyset$. From this we immediately get the result of the lemma. □

Since $SL_n(\mathbb{Z})$ is a lattice of $SL_n(\mathbb{R})$ we have:

Corollary 7.3. *$SL_n(\mathbb{Z})$ possesses property (S) in $SL_n(\mathbb{R})$.*

Remark 7.4. *Notice that $g^k = w_1 h w_2$ means that $h = w_1^{-1} g^k w_2^{-1}$, so that we found an element $h \in SL_n(\mathbb{Z})$ in a neighborhood as small as we like of some power of g, for every $g \in SL_n(\mathbb{R})$.*

8. Existence of Ping-pong partners in $PSL_n(\mathbb{Z})$

In this section we provide a sufficient condition for the existence of ping-pong partners in $PSL_n(\mathbb{Z})$. In the following sections we will use this lemma to prove the existence of ping-pong partners for the individual cases of finite order, unipotent and hyperbolic elements.

Lemma 8.1. *Let V be an n-dimensional vector space over a local field k. Let P be the projective space of V. Given a finite set of nonequal points $x_0, \ldots, x_m \in P$ there exists a hyperplane in P passing through x_0 but not through $x_1, \ldots, x_m$.*

Proof. We employ the duality of points and hyperplanes in P. Each hyperplane in P corresponds to a hyperplane in V. Each hyperplane in V can be represented by a functional $f \in V^*$ (V^* is the dual space of V - the space of all linear functions from V to k). Given $f \in V^*$, the set $V_f = \{v \in V | f(v) = 0\}$ is a hyperplane in V. On the other hand, for $v \in V$, $f(v) = 0$ iff $f(\lambda v) = \lambda f(v) = 0$ for all $\lambda \in k$. Therefore if $f(v) \neq 0$ then the corresponding point of v in P is not on the hyperplane in P corresponding to the hyperplane V_f.

We now use the Zariski-topology defined on the linear space V^* (having dimension n). The following variety defines the set of all functionals $f \in V^*$ such that the hyperplane in P corresponding to V_f satisfies the lemma:

$$W = \{f \in V^* | f(x_0) = 0, f(x_1) \neq 0, \ldots, f(x_m) \neq 0\} \tag{1}$$

It is obvious that for each $f \in W$, the hyperplane in P corresponding to V_f passes through x_0 but doesn't pass through $x_i, 1 \leq i \leq m$.

Since x_0 defines a functional on V^*, the set $U = \{f \in V^* | f(x_0) = 0\}$ is a hyperplane in V^* of dimension $n-1$, from which we remove components $\{f \in V^* | f(x_0) = 0, f(x_i) = 0\}$ for all $1 \leq i \leq m$. Each component is an intersection between two non-coinciding hyperplanes in V^* and is therefore of dimension $n-2$. The union of all components is of dimension $n-2$ and is therefore not equal to all of U, and therefore W is non-empty. For any element $f \in W$, the corresponding hyperplane of V_f in P satisfies the lemma. □

Lemma 8.2. *Given an element $g \in PSL_n(\mathbb{C})$, a point $x \in P$, a compact neighborhood $U \subset P$ of the set $\{g^i(x) | i \in \mathbb{Z}, i \neq 0, g^i(x) \neq x\}$, a compact neighborhood $W \subset P$ of x disjoint from U such that $g^i(W) \subset U$ when $g^i(x) \neq x$, $i \in \mathbb{Z}$, and a hyperplane $L \subset P$ passing through x and not intersecting U, there exists an element $h \in PSL_n(\mathbb{Z})$ such that g and h play ping-pong.*

Proof. First, we replace L by two different hyperplanes $L_1, L_2 \subset P$ that intersect W but not U and such that $x \notin L_1 \cup L_2$. We do this by choosing hyperplanes in a small enough neighborhood of L (in the sense of the metric defined in section 3) that do not intersect $U \cup x$. Since this is an open condition on V^* there must exist 2 different hyperplanes satisfying it.

We denote by $\bar{L}_1$ and $\bar{L}_2$ the corresponding linear hyperplanes in V of L_1 and L_2.

We choose points $[v_1] \in (L_1 \setminus L_2) \cap W$ and $[v_2] \in (L_2 \setminus L_1) \cap W$, and define a linear map in $SL_n(\mathbb{R})$:

$$h(v) := \begin{cases} 2v & \text{if } v \in [v_1], \\ v & \text{if } [v] \in L_1 \cap L_2, \\ \frac{1}{2}v & \text{if } v \in [v_2]. \end{cases} \tag{2}$$

We claim h is a hyperbolic element in $SL_n(\mathbb{R})$: The eigenvalues are $2, 1, \frac{1}{2}$ and their multiplication is 1; h is well defined because the subspace of V spanned by $v_1, \bar{L}_1 \cap \bar{L}_2, v_2$ is of dimension n; finally we have $A(h) = [v_1], A(h^-1) = [v_2] \in W$, $A'(h) = L_1$ and $A'(h^-1) = L_2$.

We choose a point $x_0 \in P$ such that $x_0 \notin A'(h) \cup A'(h^-1)$. Using proposition 4.9 we can choose resp. compact neighborhoods K_1, K_2 of $A(h), A(h^-1)$

such that $K_1 \cup K_2 \subset W$ and there exists $M \in \mathbb{Z}$ such that $h^i(x_0) \in K_1$ for all $i \geq M$ and $h^i(x_0) \in K_2$ for all $i \leq -M$. We replace h with h^M. Then if we define $K := K_1 \cup K_2$ we get for all $i \in \mathbb{Z}$ that $h^i(\{m_0\} \cup U) \subset K$ and $g^i(K) \subset U$ when $g^i \neq 1$.

Noticing remark 2.2 we get that g, h satisfy the Ping-Pong Lemma (2.1), and thus $\langle g, h\rangle = \langle g\rangle * \langle h\rangle$.

We now want to replace h with an element in $PSL_n(\mathbb{Z})$. We use Selberg's Lemma (7.2) for this. Using remark 7.4, we find an element $\tilde{h} \in PSL_n(\mathbb{Z})$ such that $\tilde{h} = w_1 h^j w_2$ for some $j \in \mathbb{Z}$, $j > 0$ and $w_1, w_2 \in W$ where W is a compact neighborhood of e. Using propositions 4.6 4.11 we can choose W small enough such that $\tilde{h} \in PSL_n(\mathbb{Z})$ is hyperbolic with the same properties as h. And so $\langle g, \tilde{h}\rangle = \langle g\rangle * \langle \tilde{h}\rangle$.

□

9. Ping-pong partners for elements of finite order in $PSL_n(\mathbb{Z})$

Proposition 9.1. *Let $g \in PSL_n(\mathbb{Z})$ be an element of finite order. This means that $g^m = e$ for a minimal $m > 1$. Then there exists a hyperbolic element $h \in PSL_n(\mathbb{Z})$ such that g and h are ping-pong partners.*

Proof. We choose any vector $v \in P$ to get a finite sequence v, $g(v)$, $g^2(v)$, …, $g^{m-1}(v)$. Using lemma 8.1, it is possible to choose a hyperplane L in P such that $v \in L$ and $g(v), g^2(v), ..., g^{m-1}(v) \notin L$. Since the chosen hyperplane is closed in P(therefore compact in P) and the finite point set $S = \{v, g(v), g^2(v), ...g^{m-1}(v)\}$ is compact in P, the distance between L and S is positive and we can choose a compact neighborhood W of v such that $g(W), g^2(W), \dots, g^{m-1}(W)$ intersect L trivially and intersect each other trivially in pairs. Define $U := g(W) \cup \dots \cup g^{m-1}(W)$. Now we have fulfilled all the requirements of lemma 8.2, so there exists $h \in PSL_n(\mathbb{Z})$ that is a ping-pong partner for g. □

10. Ping-pong partners for hyperbolic elements in $PSL_n(\mathbb{Z})$

We will now prove that every hyperbolic element in $PSL_n(\mathbb{Z})$ has a ping-pong partner in $PSL_n(\mathbb{Z})$:

Proposition 10.1. *If $g \in PSL_n(\mathbb{Z})$ is hyperbolic, then g has a ping-pong partner in $PSL_n(\mathbb{Z})$.*

Proof. Given any point $x \in P - A(g) - A'(g) - A(g^{-1}) - A'(g^{-1})$, we use lemma 8.1 to choose a hyperplane L passing through x and not passing through $A(g), A(g^{-1})$. We choose a compact neighborhood U of $A(g)$ and $A(g^{-1})$ not intersecting with L. Using Property 4.9, we know that only a finite number of points $g^i(x)$ are not in U, for $i \in \mathbb{Z}$. That is there exists $M \in \mathbb{Z}$, $M > 0$, such that for all $|i| \geq M$ we have $g^i(x) \in U$.

The hyperplane L can still intersect a finite number of points $g^i(x) \notin U$. The distance δ between L and U is positive (L and U are compact), and the set of hyperplanes passing through x and are at a distance less than δ

from L is an open nonempty set in V^*. This open nonempty set is locally of dimension n. Using the same reasoning as in the proof of lemma 8.1 (only locally), we know that there exists a hyperplane in this open set that passes through x but does not pass through any other $g^i(x)$, for $i \in \mathbb{Z}$, nor does it intersect U. We replace L with this hyperplane.

We choose a compact neighborhood W of x small enough so that for all $|i| < M$, $g^i(W)$ doesn't intersect L, and for $|i| \geq M$, $g^i(W) \subset U$. We redefine U to be $U \cup \bigcup_{0<|i|<N_0} g^i(W)$.

We use lemma 8.2 to find an element $h \in PSL_n(\mathbb{Z})$ that is a ping-pong partner for g. □

11. Ping-pong partners for unipotent elements in $PSL_n(\mathbb{Z})$

We begin by exploring the dynamic properties of unipotent elements in $PSL_n(\mathbb{Z})$ as actions on the projective space P corresponding to an n-dimensional vector space V over $\mathbb{C}$.

Let $g \in PSL_n(\mathbb{Z})$ be a unipotent element, and let $\bar{g} \in SL_n(\mathbb{Z})$ be a representative of g. $\bar{g}$ has a Jordan canonical form $\tilde{g}$ as a block matrix:

$$\tilde{g} = \begin{pmatrix} B_1 & & \\ & \ddots & \\ & & B_k \end{pmatrix}_{n\times n} \tag{3}$$

Each block B_i has 1s on the diagonal and 1s on the next upper diagonal and 0s elsewhere:

$$B_i = \begin{pmatrix} 1 & 1 & 0 & \dots & 0 \\ 0 & 1 & 1 & \dots & 0 \\ \vdots & & \ddots & \ddots & \\ \vdots & & & 1 & 1 \\ 0 & \dots\dots & & 0 & 1 \end{pmatrix}_{m\times m} \tag{4}$$

Notice that:

$$B_i^k = \begin{pmatrix} 1 & k & \binom{k}{2} & \dots & & \binom{k}{m-1} \\ 0 & 1 & k & \binom{k}{2} & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \\ \vdots & & & 1 & k & \binom{k}{2} \\ & & & & 1 & k \\ 0 & \dots\dots & & & 0 & 1 \end{pmatrix}_{m\times m} \tag{5}$$

Given a vector $x = (a_1, \dots, a_m)^T$ in W_i (the subspace of V corresponding to B_i), we apply B_i^k to x. Assuming $a_m \neq 0$ we get $\lim_{k\to\infty} \binom{k}{m-1}^{-1} B_i^k(x) =$

$(a_m, 0, \dots, 0)^T$. We see that B_i^k converges to a composition of a projection and rotation.

Given M as the size of the maximal blocks in $\tilde{g}$ and vectors $x_i = (a_1, \dots, a_{M_i})^T$ in W_i where we assume $a_{M_i} \neq 0$, we get $\lim_{k\to\infty} \binom{k}{M-1}^{-1} B_i^k(x_i) = 0$ when the size of the block B_i is smaller than M, and $\lim_{k\to\infty} \binom{k}{M-1}^{-1} B_i^k(x_i) = (a_{M_i}, 0, \dots, 0)^T$ when the size of B_i is M. Thus, given any point $x \in P$ such that it does not intersect any of the relevant hyperplanes ($a_{M_i} = 0$), we get a dynamic convergence towards a single point (dependant upon the choice of the original point). Also notice that this point of attraction is fixed by $\tilde{g}$.

Since the unipotent elements in $PSL_n(\mathbb{Z})$ are a subgroup of $PSL_n(\mathbb{Z})$, g^{-1} is also a unipotent element, and it is not hard to check that g^{-1} has the same attracting point as g. Therefore it is possible to choose a point in P not lying in any of the "bad" hyperplanes for g and g^{-1}, and we can build the same scenario for lemma 8.2 as for hyperbolic elements, proving the following property:

Proposition 11.1. *If $g \in PSL_n(\mathbb{Z})$ is unipotent, then g has a ping-pong partner in $PSL_n(\mathbb{Z})$.*

12. Multiplication of finite order elements with unipotent elements

In this section we prove that combinations of finite order elements with unipotent elements (i.e. elements g in $PSL_n(\mathbb{Z})$ with Jordan decomposition $g = g_s g_u$ where g_s is of finite order and g_u is unipotent) also have ping-pong partners in $PSL_n(\mathbb{Z})$.

Proposition 12.1. *Let $g \in PSL_n(Z)$ be an element such that it's Jordan decomposition is of the form where g_s is of finite order. Then g has a ping-pong partner in $PSL_n(\mathbb{Z})$.*

Proof. Since g_s and g_u commute, we have $g^m = (g_s g_u)^m = g_s^m g_u^m$. Also notice that $g^{-1} = g_s^{-1} g_u^{-1}$. We know that if we choose a point not intersecting any of g_u's "bad" hyperplanes we get that the orbit of this point under g_u converges to a point. Since g_s is of a finite order the orbit of the point chosen has a finite number of limit points under g. Thus we can contain the orbit in a compact set and use lemma 8.2, proving the existence a ping-pong partner for g. □

13. Simultaneous ping-pong partners for elements in $PSL_n(\mathbb{Z})$

In this chapter we start with $F = \{g_1, \dots, g_s\}$ - a finite set of elements in $PSL_n(\mathbb{Z})$ as in theorem 1.2.

We denote by Ω_F the set of all elements in $PSL_n(\mathbb{Z})$ that play ping-pong simultaneously with all elements in F.

Theorem 13.1. $\Omega_F \neq \emptyset$.

Proof. When we used lemma 8.1 in the proofs of propositions 9.1, 10.1 and 11.1 we chose a point not in a closed set and a hyperplane not intersecting a finite set of points in the vector space. Given the finite set of elements F, we can build a finite union of these closed sets and find a point and a hyperplane fulfilling the requirements of lemma 8.2 (under a finite union the sets will remain compact and closed). Thus the hyperbolic element that we built is in fact a ping-pong partner for all elements in F. □

This proves theorem 1.2.

14. Zariski-density of simultaneous ping-pong partners in $PSL_n(\mathbb{Z})$

In this section we prove theorem 1.3.

Definition 14.1. Let G be any group. A subset H of G is called *pro-finitely dense* if for every $\langle e\rangle \neq N \lhd_f G$ (normal subgroup of finite index of G), $HN = G$.

Let F be a set of elements as in the previous section.

Lemma 14.2. *Ω_F is pro-finitely dense in $PSL_n(\mathbb{Z})$.*

Proof. Given a finite index normal subgroup $N \lhd PSL_n(\mathbb{Z})$, we need to show that $\Omega_F N = PSL_n(\mathbb{Z})$. That is, we need to show that Ω_F contains a representative of each left coset of N. In other words, we need to show that for every $\gamma_0 \in PSL_n(\mathbb{Z})$, $\gamma_0 N \cap \Omega_F \neq \emptyset$.

If N is a finite index normal subgroup of $SL_n(\mathbb{Z})$ then N is also a lattice of $SL_n(\mathbb{R})$. And so, $SL_n(\mathbb{R})/N$ is a finite measure space.

Let $W \subset SL_n(\mathbb{R})$ be any neighborhood of the identity element e, and let g be any hyperbolic element from Ω_F (we have seen such an element exists in the previous section). We can now use theorem 6.6. We know that g^s is ergodic on $SL_n(\mathbb{R})/N$, for all $s \in \mathbb{Z}, s \neq 0$. We notice that $\mu(W\gamma_0 N) > 0$ and $\mu(WN) > 0$. Using theorem 6.4, we know that there exists an infinite set of integers $M \subset \mathbb{Z}$, $M > 0$, such that $g^k WN \cap W\gamma_0 N \neq \emptyset$ for $k \in M$. Therefore there exists an element $w_1 g^k w_2 \in \gamma_0 N$, where $k > 0$, $w_1, w_2 \in W$. Using propositions 4.6 and 4.11 we can find a small enough neighborhood W and a large enough power k such that $w_1 g^k w_2 \in \Omega_F$. □

We cite the following lemma from [7] (proposition 2.3):

Lemma 14.3. *Let G be a finitely generated subgroup of $GL_n(k)$, for an arbitrary field k. If H is a pro-finitely dense subgroup of G, then H is Zariski-dense in G.*

A careful examination of the proof of this lemma shows that it remains true for pro-finitely dense subsets, and so we get that Ω_F is Zariski-dense in $PSL_n(\mathbb{Z})$. Since $PSL_n(\mathbb{Z})$ is Zariski-dense in $PSL_n(\mathbb{R})$ we have proved theorem 1.3.

References

[1] H. Abels, G. A. Margulis and G. A. Soifer, *Semigroups containing proximal linear maps*, Israel J. Math 91 (1995), no. 1-3, 1-30.

[2] M.Bekka, M. Cowling, and P. de la Harpe, *Some groups whose reduced C^*-algebra is simple*, Inst. Hautes Études Sci. Publ. Math. (1994), no.80, 117-134.

[3] A. Borel, *Linear Algebraic Groups (2nd ed.)*, Springer-Verlag New York, Inc. (1991).

[4] E. Breuillard, T. Gelander, *On dense free subgroups of Lie groups*, J. Algebra **261** (2003), no. 2, p. 448–467.

[5] P. de la Harpe. *Topics in Geometric Group Theory*, University Of Chicago Press (2000).

[6] P. de la Harpe, *On simplicity of reduced C^*-algebras of groups*, arXiv:math.OA/0509450.

[7] G. A. Margulis and G. A. Soifer, *Maximal subgroups of infinite index in finitely generated linear groups*, J. Algebra 69 (1981), no.1, 1-23.

[8] T. Poznansky *Existence of simultaneous ping-pong partners in linear groups*, Ph.D. dissertation, Yale University, 45 pp (2006).

[9] M.S. Raghunathan, *Discrete subgroups of Lie groups*, Springer (1972)

[10] J. Tits. *Free subgroups in linear groups*, J. Algebra 20 (1972), 250-270.

[11] P. Walters, *An Introduction to Ergodic Theory*, Springer-Verlag New York, Inc. (1982).

[12] Robert J. Zimmer, *Ergodic Theory and Semisimple Groups*, Birkhäuser (1984).