Monopoles over 4–manifolds containing long necks, II

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Abstract

We establish a gluing theorem for monopoles over 4–manifolds containing long necks. The theorem is stated in terms of an ungluing map defined explicitly in terms of data that appear naturally in applications. Orientations of moduli spaces are handled using Benevieri–Furi's concept of orientations of Fredholm operators of index 0.

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1 Introduction

In this paper we prove a gluing theorem for monopoles suitable for the construction of Floer homology groups in the simplest cases and for establishing certain gluing formulae for Seiberg–Witten invariants of 4–manifolds (to be discussed in [12, 13]).

There is now a large literature on gluing theory for instantons and monopoles. The theory was introduced by Taubes [23, 24], who used it to obtain existence results for self-dual connections over closed 4–manifolds. It was further developed in seminal work of Donaldson [3], see also Freed–Uhlenbeck [9]. General gluing theorems for instantons over connected sums were proved by Donaldson [4] and Donaldson–Kronheimer [6]. In the setting of instanton Floer theory there is a highly readable account in Donaldson [5], see also Floer [8] and Fukaya [14]. Gluing with degenerate asymptotic limits was studied by Morgan–Mrowka [19]; part of their work was adapted to the context of monopoles by Safari [21]. Nicolaescu [20] established gluing theorems for monopoles in certain situations, including one involving gluing obstructions. Marcolli–Wang [17] discuss gluing theory in connection with monopole Floer homology. For monopoles over closed 3–manifolds split along certain tori, see Chen [2]. Finally, gluing theory is a key ingredient in a large programme of Feehan–Leness [7] (using ideas of Pidstrigach–Tyurin) for proving Witten's conjecture relating Donaldson and Seiberg–Witten invariants.

As should be evident from this brief survey, there are many different hypotheses under which one can consider the gluing problem. This paper does not aim at the utmost generality, but is an expositary account of gluing in what might be called the favourable cases. More precisely, we will glue precompact families of regular monopoles over 4–manifolds with tubular ends, under similar general assumptions as in [11]. Although obstructed gluing is not discussed explicitly, we will show in [12] how the parametrized version of our gluing theorem can be used to handle one kind of gluing obstructions.

One source of difficulty when formulating a gluing theorem is that gluing maps are in general not canonical, but rather depend on various choices hidden in their construction. We have therefore chosen to express our gluing theorem as a statement about an *ungluing map*, which is explicitly defined in terms of data that appear naturally in applications.

If X is a 4–manifold with tubular ends and $X^{(T)}$ the glued manifold as in [11], then the first component of the ungluing map involves restricting monopoles over $X^{(T)}$ to some fixed compact subset $K \subset X$ (which may also be regarded as a subset of $X^{(T)}$ when each T_j is large). In the case of gluing along a reducible critical point, the ungluing map has an additional component which reads off the U(1) gluing parameter by measuring the holonomy along a path running once through the corresponding neck in $X^{(T)}$.

Ungluing maps of a different kind were studied already in [3, 9], but later authors have mostly formulated gluing theorems in terms of gluing maps, usually without characterizing these maps uniquely.

The proof of the gluing theorem is divided into two parts: surjectivity and injectivity of the ungluing map. In the first part the (quantitative) inverse function theorem is used to construct a smooth local right inverse $\hat{\zeta}$ of an "extended monopole map" $\hat{\Xi}$. In the second part the inverse function theorem is applied a second time to show, essentially, that the image of $\hat{\zeta}$ is not too small. There are many similarities with the proof of the gluing theorem in [6], but also some differences. For instance, we do not use the method of continuity, and we handle gluing parameters differently.

It may be worth mentioning that the proof does not depend on unique continuation for monopoles (only for harmonic spinors), as we do not know whether solutions to our perturbed monopole equations satisfy any such property. (Unique continuation for genuine monopoles was used in [11, Proposition 4.3] in the discussion of perturbations, but this has little to do with gluing theory.) Therefore, in the injectivity part of the proof, we argue by contradiction, restricting monopoles to ever larger subsets $\tilde{K} \subset X$. This is also reflected in the statement of the theorem, which would have been somewhat simpler if unique continuation were available.

We also give a detailed account of orientations of moduli spaces, using Benevieri–Furi's concept of orientations of Fredholm operators of index 0 [1]. This seems simpler to us than the standard approach using determinant line bundles. Our main result here, Theorem 4.1, tells when the ungluing map preserves resp. reverses orientation. The length of this part of the paper is much due to the fact that we allow gluing along reducible critical points and that we work with (multi)framed moduli spaces (as a means of handling reducibles over the 4–manifolds). Although the orientability of monopole moduli spaces over closed 4–manifolds is well documented (see [18, 22]) and the behaviour of the determinant line bundle under gluing along irreducibles is described in [5], beyond this the existing literature seems short on details concerning the orientation issue in the type of gluing problem considered in this paper.

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2 The gluing theorem

2.1 Statement of theorem

Consider the situation of [11, Subsection 1.4], but without assuming any of the conditions (B1),(B2),(C). We now assume that every component of X contains an end $\mathbb{R}_+ \times Y_j$ or $\mathbb{R}_+ \times (-Y_j)$ (ie an end that is being glued). Fix non-degenerate monopoles α_j over Y_j and α'_j over Y'_j. (These should be smooth configurations rather than gauge equivalence classes of such.) Suppose α_j is reducible for $1 \le j \le r_0$ and irreducible for $r_0 < j \le r$, where $0 \le r_0 \le r$. We consider monopoles over X and $X^{(T)}$ that are asymptotic to α'_j over $\mathbb{R}_+ \times Y'_j$ and (in the case of X) asymptotic to α_j over $\mathbb{R}_+ \times (\pm Y_j)$. These monopoles build moduli spaces

$$M_{\mathfrak{b}} = M_{\mathfrak{b}}(X; \vec{\alpha}, \vec{\alpha}, \vec{\alpha}'), \qquad M_{\mathfrak{b}}^{(T)} = M_{\mathfrak{b}}(X^{(T)}; \vec{\alpha}').$$

Here $\mathfrak{b} \subset X$ is a finite subset to be specified in a moment, and the subscript indicates that we only divide out by those gauge transformations that restrict to the identity on $\mathfrak{b}$, see [11, Subsection 3.4]. The ungluing map $\mathbf{f}$ will be a diffeomorphism between certain open subsets of $M_{\mathfrak{b}}^{(T)}$ and $M_{\mathfrak{b}}$ when

$$\check{T} := \min(T_1, \ldots, T_r)$$

is large.

When gluing along the critical point α_j, the stabilizer of α_j in $\mathcal{G}_{Y_j}$ appears as a “gluing parameter”. This stabilizer is a copy of U(1) if α_j is reducible and trivial otherwise. When α_j is reducible we will read off the gluing parameter by means of the holonomy of the connection part of the glued monopole along a path γ_j in $X^{(T)}$ which runs once through the neck

$[-T_j, T_j] \times Y_j$. To make this precise, for $1 \le j \le r_0$ fix $y_j \in Y_j$ and smooth paths

$$\gamma_j^\pm : [-1, \infty) \to X$$

such that $\gamma_j^\pm(t) = \iota_j^\pm(t, y_j)$ for $t \ge 0$ and $\gamma_j^\pm([-1,0]) \subset X_{:0}$. Let $\mathfrak{b}$ denote the collection of all the start-points $o_j^\pm := \gamma_j^\pm(-1)$. (We do not assume that these are distinct.) Note in passing that we then have

$$M_{\mathfrak{b}}^* = M_{\mathfrak{b}}.$$

Define the smooth path

$$\gamma_j : I_j = [-T_j - 1, T_j + 1] \to X^{(T)}$$

by

$$\gamma_j(t) = \begin{cases} \pi_T \gamma_j^+(T_j + t), & -T_j - 1 \le t < T_j, \\ \pi_T \gamma_j^-(T_j - t), & -T_j < t \le T_j + 1, \end{cases}$$

where $\pi_T : X^{\{T\}} \to X^{(T)}$ is as in [11, Subsection 1.4].

Choose a reference configuration $S_o = (A_o, \Phi_o)$ over X with limit α_j over $\mathbb{R}_+ \times (\pm Y_j)$ and α_j' over $\mathbb{R} \times Y_j'$. Let $S_o' = (A_o', \Phi_o')$ denote the reference configuration over $X^{(T)}$ obtained from S_o in the obvious way when gluing the ends. Precisely speaking, S_o' is the unique smooth configuration over $X^{(T)}$ which agrees with S_o over $\mathrm{int}(X_{:T})$ (which can also be regarded as a subset of X).

If $P \to X^{(T)}$ temporarily denotes the principal $\mathrm{Spin}^c(4)$–bundle defining the spin^c structure, then the holonomy of a spin^c connection A in P along γ_j is a $\mathrm{Spin}^c(4)$–equivariant map

$$\mathrm{hol}_{\gamma_j}(A) : P_{o_j^+} \to P_{o_j^-}.$$

Because A and A_o' map to the same connection in the tangent bundle of $X^{(T)}$, there is a unique element $\mathrm{Hol}_j(A)$ in $\mathrm{U}(1)$ (identified with the kernel of $\mathrm{Spin}^c(4) \to \mathrm{SO}(4)$) such that

$$\mathrm{hol}_{\gamma_j}(A) = \mathrm{Hol}_j(A) \cdot \mathrm{hol}_{\gamma_j}(A_o').$$

Explicitly,

$$\mathrm{Hol}_j(A) = \exp\left(- \int_{I_j} \gamma_j^*(A - A_o') \right), \tag{1}$$

where as usual $A - A'_o$ is regarded as an imaginary valued 1–form on $X^{(T)}$. For gauge transformations $u : X^{(T)} \to \mathrm{U}(1)$ we have

$$\mathrm{Hol}_j(u(A)) = u(o_j^-) \cdot \mathrm{Hol}_j(A) \cdot u(o_j^+)^{-1}. \tag{2}$$

In particular, there is a natural smooth map

$$\mathrm{Hol} : M_{\mathfrak{b}}^{(T)} \to \mathrm{U}(1)^{r_0}, \quad [A, \Phi] \mapsto (\mathrm{Hol}_1(A), \dots, \mathrm{Hol}_{r_0}(A))$$

which is equivariant with respect to the appropriate action of

$$\mathbb{T} := \mathrm{Map}(\mathfrak{b}, \mathrm{U}(1)) \approx \mathrm{U}(1)^b,$$

where $b = |\mathfrak{b}|$.

Consider for the moment an arbitrary compact codimension 0 submanifold $K \subset X$ containing $\mathfrak{b}$. Let $D^{(T)}$ be the subgroup of $H^1(X^{(T)}; \mathbb{Z})$ consisting of those classes whose restriction to each Y'_j is zero. Let D_K be the cokernel of the restriction map $D^{(T)} \to H^1(K; \mathbb{Z})$. Here $\breve{T}$ should be so large that K may be regarded as a subset of $X^{(T)}$, and D_K is then obviously independent of T. In the following we use the L^p_1 configuration spaces etc introduced in [11, Subsection 2.5]. Let $\breve{\mathcal{G}}_{\mathfrak{b}}(K)$ be the kernel of the (surjective) group homomorphism

$$\mathcal{G}(K) \to \mathbb{T} \times D_K, \quad u \mapsto (u|_{\mathfrak{b}}, [u]),$$

where $[u]$ denotes the image in D_K of the homotopy class of u regarded as an element of $H^1(K; \mathbb{Z})$. Set

$$\breve{\mathcal{B}}_{\mathfrak{b}}(K) = \mathcal{C}(K)/\breve{\mathcal{G}}_{\mathfrak{b}}(K), \qquad \breve{\mathcal{B}}^*_{\mathfrak{b}}(K) = \mathcal{C}^*_{\mathfrak{b}}(K)/\breve{\mathcal{G}}_{\mathfrak{b}}(K).$$

On both these spaces there is a natural action of $\mathbb{T} \times D_K$. Note that D_K acts freely and properly discontinuously on the (Hausdorff) Banach manifold $\breve{\mathcal{B}}^*_{\mathfrak{b}}(K)$ with quotient $\mathcal{B}^*_{\mathfrak{b}}(K)$.

It is convenient here to agree once and for all that the Sobolev exponent $p > 4$ is to be an even integer. This ensures that our configuration spaces admit smooth partitions of unity, which is needed in Subsections 3.1 and 2.4 (but not in the proof of Theorem 2.1).

Fix a $\mathbb{T}$–invariant open subset $G \subset M_{\mathfrak{b}}$ whose closure $\overline{G}$ is compact and contains only regular points. (Of course, G is the pre-image of an open set G' in M, but G' may not be a smooth manifold due to reducibles and we therefore prefer to work with G.)

Definition 2.1 *By a* kv-pair *we mean a pair* (K, V) *where*

- *$K \subset X$ is a compact codimension 0 submanifold which contains $\mathfrak{b}$ and intersects every component of X,*

- *$V \subset \breve{\mathcal{B}}_{\mathfrak{b}}(K)$ is a $\mathbb{T}$–invariant open subset containing $R_K(\overline{G})$, where R_K denotes restriction to K.*

We define a partial ordering $\leq$ on the set of all kv-pairs, by decreeing that

$$(K', V') \leq (K, V)$$

if and only if $K \subset K'$ and $R_K(V') \subset V$.

Now fix a kv-pair (K, V) which satisfies the following two additional assumptions: firstly, that $V \subset \breve{\mathcal{B}}^*_{\mathfrak{b}}(K)$; secondly, that if X_e is any component of X which contains a point from $\mathfrak{b}$ then $X_e \cap K$ is connected. The second condition ensures that the image of $R_K : M_{\mathfrak{b}} = M^*_{\mathfrak{b}} \to \breve{\mathcal{B}}_{\mathfrak{b}}(K)$ lies in $\breve{\mathcal{B}}^*_{\mathfrak{b}}(K)$.

Suppose we are given a $\mathbb{T}$–equivariant smooth map

$$q : V \to M_{\mathfrak{b}} \tag{3}$$

such that $q(\omega|_K) = \omega$ for all $\omega \in \overline{G}$. (If $\mathbb{T}$ acts freely on $\overline{G}$ then such a map always exists when K is sufficiently large, see Subsection 2.4. In concrete applications there is often a natural choice of q, see Subsections 3.1, 3.2.)

Let $X^{\#}$ and the forms $\tilde{\eta}_j, \tilde{\eta}'_j$ be as in [11, Subsection 1.4], and choose $\lambda_j, \lambda'_j > 0$.

Theorem 2.1 *Suppose there is class in $H^2(X^{\#})$ whose restrictions to Y_j and Y'_j are $\lambda_j\tilde{\eta}_j$ and $\lambda'_j\tilde{\eta}'_j$, respectively, and suppose the perturbation parameters $\vec{\mathfrak{p}}, \vec{\mathfrak{p}}'$ are admissible for $\vec{\alpha}'$. Then there exists a kv-pair $(\tilde{K}, \tilde{V}) \leq (K, V)$ such that if (K', V') is any kv-pair $\leq (\tilde{K}, \tilde{V})$ then the following holds when $\check{T}$ is sufficiently large. Set*

$$H^{(T)} := \left\{\omega \in M^{(T)}_{\mathfrak{b}} \, : \, \omega|_{K'} \in V'\right\},$$

$$\mathbf{q} : H^{(T)} \to M_{\mathfrak{b}}, \quad \omega \mapsto q(\omega|_K).$$

Then $\mathbf{q}^{-1}G$ consists only of regular monopoles (hence is a smooth manifold), and the $\mathbb{T}$–equivariant map $\mathbf{f} := \mathbf{q} \times Hol$ restricts to a diffeomorphism

$$\mathbf{q}^{-1}G \to G \times U(1)^{r_0}.$$

Remarks. 1. When $\check{T}$ is large then $K' \subset X$ can also be regarded as a subset of $X^{(T)}$, in which case the expression $\omega|_{K'}$ in the definition of $H^{(T)}$ makes sense.

2. Except for the equivariance of $\mathbf{f}$, the theorem remains true if one leaves out all assumptions on $\mathbb{T}$–invariance resp. –equivariance on G and q, and on V in Definition 2.1, above. However, it is hard to imagine any application that would not require equivariance of $\mathbf{f}$.

3. The theorem remains true if one replaces $\breve{\mathcal{B}}_{\mathfrak{b}}(K)$ and $\breve{\mathcal{B}}^*_{\mathfrak{b}}(K)$ by $\mathcal{B}_{\mathfrak{b}}(K)$ and $\mathcal{B}^*_{\mathfrak{b}}(K)$ above. However, working with $\breve{\mathcal{B}}$ gives more flexibility in the construction of maps q, see Subsection 3.2.

4. Concerning admissibility of perturbation parameters, see the remarks after [11, Theorem 1.4]. Note that the assumption on $\lambda_j\tilde{\eta}_j$ and $\lambda'_j\tilde{\eta}'_j$ in the theorem above is weaker than either of the conditions (B1) and (B2) in [11]. However, in practice the gluing theorem is only useful in conjunction with a compactness theorem, so one may still have to assume (B1) or (B2).

The proof of Theorem 5.1 has two parts. The first part consists in showing that $\mathbf{f}$ has a smooth local right inverse around every point in $\overline{G}\times \mathrm{U}(1)^{r_0}$ (Proposition 2.1 below). In the second part we will prove that $\mathbf{f}$ is injective on $\mathbf{q}^{-1}\overline{G}$. (Proposition 2.2 below).

2.2 Surjectivity

The next two subsections are devoted to the proof of Theorem 2.1. Both parts of the proof make use of the same set-up, which we now introduce.

We first choose weight functions for our Sobolev spaces over X and $X^{(T)}$. Let $\sigma_j, \sigma'_j \geq 0$ be small constants and $w : X \to \mathbb{R}$ a smooth function which is equal to $\sigma_j t$ on $\mathbb{R}_+ \times (\pm Y_j)$ and equal to $\sigma'_j t$ on $\mathbb{R}_+ \times Y'_j$. As usual, we require $\sigma_j > 0$ if α_j is reducible (ie for $j = 1, \dots, r_0$), and similarly for σ'_j. For $j = 1, \dots, r$ choose a smooth function $w_j : \mathbb{R} \to \mathbb{R}$ such that $w_j(t) = -\sigma_j|t|$ for $|t| \geq 1$. We will always assume $\check{T} \geq 4$, in which case we can define a weight function $\kappa : X^{(T)} \to \mathbb{R}$ by

$$\kappa = w \quad \text{on } X^{(T)} \setminus \cup_j [-T_j, T_j] \times Y_j,$$
$$\kappa(t, y) = \sigma_j T_j + w_j(t) \quad \text{for } (t, y) \in [-T_j, T_j] \times Y_j.$$

Let $\mathcal{C}$ denote the $L^{p,w}_1$ configuration space over X defined by the reference configuration S_o, and let $\mathcal{C}'$ denote the $L^{p,\kappa}_1$ configuration space over $X^{(T)}$ defined by S'_o. Let $\mathcal{G}_{\mathfrak{b}}, \mathcal{G}'_{\mathfrak{b}}$ be the corresponding groups of gauge transformations and $\mathcal{B}_{\mathfrak{b}}, \mathcal{B}'_{\mathfrak{b}}$ the corresponding orbit spaces.

Now fix $(\omega_0, z) \in \overline{G} \times \mathrm{U}(1)^{r_0}$. Our immediate goal is to construct a smooth local right inverse of $\mathbf{f}$ around this point, but the following set-up will also be used in the injectivity part of the proof.

Choose a smooth representative $S_0 \in \mathcal{C}$ for ω_0 which is in temporal gauge over the ends of X. (This assumption is made in order to ensure exponential decay of S_0.) Set $d = \dim M_{\mathfrak{b}}$ and let $\pi : \mathcal{C} \to \mathcal{B}_{\mathfrak{b}}$ be the projection. By the local slice theorem we can find a smooth map

$$\mathbf{S} : \mathbb{R}^d \to \mathcal{C}$$

such that $\mathbf{S}(0) = S_0$ and such that $\varpi = \pi \circ \mathbf{S}$ is a diffeomorphism onto an open subset of $M_{\mathfrak{b}}$.

We will require one more property of $\mathbf{S}$, involving holonomy. If $a \in L^{p,w}_1(X; i\mathbb{R})$ then we define $\mathrm{Hol}^{\pm}_j(A_o + a) \in \mathrm{U}(1)$ by

$$\mathrm{Hol}^{\pm}_j(A_o + a) = \exp\left(-\int_{[-1,\infty)} (\gamma^{\pm}_j)^* a\right).$$

The integral exists because, by the Sobolev embedding $L^p_1 \subset C^0_B$ in $\mathbb{R}^4$ for $p > 4$, we have

$$\|e^w a\|_\infty \le C\|e^w a\|_{L^p_1} = C\|a\|_{L^{p,w}_1} \tag{4}$$

for some constant C. It is clear that $\mathrm{Hol}^{\pm}_j$ is a smooth function on $\mathcal{C}$. Because any smooth map $\mathbb{R}^d \to \mathrm{U}(1)$ factors through $\exp : \mathbb{R}i \to \mathrm{U}(1)$, we can arrange, after perhaps modifying $\mathbf{S}$ by a smooth family of gauge transformations that are all equal to 1 outside the ends $\mathbb{R}_+ \times Y_j$ and constant on $[1,\infty) \times Y_j$, that

$$\mathrm{Hol}^+_j(\mathbf{S}(v)) \cdot (\mathrm{Hol}^-_j(\mathbf{S}(v)))^{-1} = z_j \tag{5}$$

for $j = 1, \ldots, r_0$ and every $v \in \mathbb{R}^d$. Here $\mathrm{Hol}^{\pm}_j(\mathbf{S}(v))$ denotes the holonomy, as defined above, of the connection part of the configuration $\mathbf{S}(v)$, and the z_j are the coordinates of z.

Lemma 2.1 *Let E, F, G be Banach spaces, $S : E \to F$ a bounded operator and $T : E \to G$ a surjective bounded operator such that*

$$S + T : E \to F \oplus G, \quad x \mapsto (Sx, Tx)$$

is Fredholm. Then T has a bounded right inverse.

Proof. Because $S+T$ is Fredholm there is a bounded operator $A : F\oplus G \to E$ such that $(S+T)A-I$ is compact. Set $A(x, y) = A_1x+A_2y$ for $(x, y) \in F\oplus G$. Then

$$TA_2 - I : G \to G$$

is compact, hence TA_2 is Fredholm of index 0. Using the surjectivity of T and the fact that any closed subspace of finite dimension or codimension in a Banach space is complemented, it is easy to see that there is a bounded operator $K : G \to E$ (with finite-dimensional image) such that $T(A_2 + K)$ is an isomorphism. $\square$

Let

$$\Theta : \mathcal{C} \to L^{p,w}$$

we be Seiberg–Witten map over X. By assumption, every point in $\overline{G}$ is regular, so in particular ω_0 is regular, which means that $D\Theta(S_0) : L^{p,w}_1 \to L^{p,w}$ is surjective. Let Φ be the spinor part of S_0 and define $\mathcal{I}_\Phi$ as in [11, Subsection 2.3]. Then

$$\mathcal{I}^*_\Phi + D\Theta(S_0) : L^{p,w}_1 \to L^{p,w}$$

is Fredholm, so by Lemma 2.1 $D\Theta(S_0)$ has a bounded right inverse Q. (This can also be deduced from [11, Proposition 2.2 (ii)].)

Let $\mathsf{r} : \mathbb{R} \to \mathbb{R}$ be a smooth function such that $\mathsf{r}(t) = 1$ for $t \leq 0$ and $\mathsf{r}(t) = 0$ for $t \geq 1$. For $\tau \geq 1$ set $\mathsf{r}_\tau(t) = \mathsf{r}(t - \tau)$ and let $S_{v,\tau}$ be the configuration over X which agrees with $\mathbf{S}(v)$ away from the ends $\mathbb{R}_+ \times (\pm Y_j)$ and satisfies

$$S_{v,\tau} = (1 - \mathsf{r}_\tau)\underline{\alpha}_j + \mathsf{r}_\tau \mathbf{S}(v)$$

over $\mathbb{R}_+ \times (\pm Y_j)$. Here $\underline{\alpha}_j$ denotes, as before, the translationary invariant monopole over $\mathbb{R} \times Y_j$ determined by α_j. For each v we have

$$\|S_{v,\tau} - \mathbf{S}(v)\|_{L^{p,w}_1} \to 0 \qquad \text{as } \tau \to \infty.$$

Therefore, when τ is sufficiently large, the operator

$$D\Theta(S_{0,\tau}) \circ Q : L^{p,w} \to L^{p,w}$$

will be invertible, and we set

$$Q_\tau = Q(D\Theta(S_{0,\tau}) \circ Q)^{-1} : L^{p,w} \to L^{p,w}_1,$$

which is then a right inverse of $D\Theta(S_{0,\tau})$. It is clear that the operator norm $\|Q_\tau - Q\| \to 0$ as $\tau \to \infty$.

For the remainder of the proof of Theorem 2.1, the term 'constant' will always refer to a quantity that is independent of τ, T, unless otherwise indicated. The symbols $C_1, C_2, \ldots$ and $c_1, c_2, \ldots$ will each denote at most one constant, while other symbols may denote different constants in different contexts.

Consider the configuration space

$$\mathcal{C}_j = \underline{\alpha}_j + L^{p,-w_j}_1$$

over $\mathbb{R} \times Y_j$ and the Seiberg–Witten map

$$\Theta_j : \mathcal{C}_j \to L^{p,-w_j}.$$

As explained in [11, Subsection 3.4] there is an identification

$$\mathcal{I}^*_{\underline{\alpha}} + D\Theta_j(\underline{\alpha}_j) = \frac{d}{dt} + P_\alpha.$$

By the results of [5] the operator on the right hand side defines a Fredholm operator $L^{p,w_j}_1 \to L^{p,w_j}$, and this must be surjective because of the choice of weight function w_j. In particular,

$$D\Theta_j(\underline{\alpha}_j) : L^{p,w_j}_1 \to L^{p,w_j} \tag{6}$$

is surjective, hence has a bounded right inverse P_j by Lemma 2.1. (Here one cannot appeal to [11, Proposition 2.2 (ii)].)

Let $\Theta' : \mathcal{C}' \to L^{p,\kappa}$ be the Seiberg–Witten map over $X^{(T)}$. When $\check{T} > \tau + 1$ then by splicing $S_{v,\tau}$ in the natural way one obtains a smooth configuration $S_{v,\tau,T}$ over $X^{(T)}$. There is a constant $C_0 \gg 0$ such that if

$$\check{T} > \tau + C_0 \tag{7}$$

then we can splice the right inverses Q_τ and $P_1, \dots, P_r$ to obtain a right inverse $Q_{\tau,T}$ of

$$D\Theta'(S_{0,\tau,T}) : L^{p,\kappa}_1 \to L^{p,\kappa}$$

which satisfies

$$\|Q_{\tau,T}\| \le C \left(\|Q_\tau\| + \sum_j \|P_j\| \right)$$

for some constant C, see the appendix. Since $\|Q_\tau\|$ is bounded in τ (ie as a function of τ), we see that $\|Q_{\tau,T}\|$ is bounded in τ, T.

The inequality (7) will be assumed from now on.

We now introduce certain 1–forms that will be added to the configurations $S_{v,\tau,T}$ in order to make small changes to the holonomies Hol_j. For any $c = (c_1, \dots, c_{r_0}) \in \mathbb{R}^{r_0}$ define the 1–form $\theta_{c,\tau}$ over $X^{(T)}$ by

$$\theta_{c,\tau} = \begin{cases} 0 & \text{outside } \bigcup_{j=1}^{r_0} [-T_j, T_j] \times Y_j, \\ ic_j \mathsf{r}'_{\tau+1-T_j} dt & \text{on } [-T_j, T_j] \times Y_j, \quad j = 1, \dots, r_0. \end{cases}$$

where $\mathsf{r}'_s(t) = \frac{d}{dt}\mathsf{r}_s(t)$. Set

$$E = \mathbb{R}^d \times \mathbb{R}^{r_0} \times L^{p,\kappa}(X^{(T)}; i\Lambda^+ \oplus \mathbb{S}^-).$$

For $0 < \epsilon < 1$ let $B_\epsilon \subset E$ be the open ϵ–ball about 0. Define a smooth map $\zeta : E \to \mathcal{C}'$ by

$$\zeta(v, c, \xi) = S_{v,\tau,T} + \theta_{c,\tau} + Q_{\tau,T}\xi, \tag{8}$$

where $\theta_{c,\tau}$ is added to the connection part of $S_{v,\tau,T}$.

When deciding where to add the perturbation 1–form $\theta_{c,\tau}$ one has to balance two concerns. One the one hand, because the weight function κ increases exponentially as one approaches the middle of the necks $[-T_j, T_j] \times Y_j$, $j = 1, \dots, r_0$, it is desirable to add $\theta_{c,\tau}$ as close to the boundaries of these necks as possible. On the other hand, in order for Lemma 2.4 below to work, the spinor field of $S_{v,\tau,T}$ needs to be "small" in the perturbation region. We have chosen to add $\theta_{c,\tau}$ at the negative end of the cut-off region, where the spinor field is zero.

Although we will sometimes use the notation $\zeta(x)$, we shall think of ζ as a function of three variables v, c, ξ, and $D_j\zeta$ will denote the derivative of ζ with respect to the j'th variable. Similarly for other functions on (subsets of) E that we will define later. Set

$$\sigma = \max(\sigma_1, \dots, \sigma_r).$$

Notice that if $r_0 = 0$, ie if we are not gluing along any reducible critical point, then we may take $\sigma = 0$.

Lemma 2.2 *There exists a constant $C_1 > 0$ such that for $x \in E$ the following hold:*

(i) $\|D_1\zeta(x)\|, \|D^2\zeta(x)\| < C_1$ *if* $\|x\| < 1$,

(ii) $\|D_2\zeta(x)\| < C_1 e^{\sigma\tau}$,

(iii) $\|D_3\zeta(x)\| < C_1$.

Proof. To prove (ii), note that if $r_0 > 0$ and $c = (c_1, \dots, c_{r_0})$ then

$$\left\| \frac{\partial \zeta(v, c, \xi)}{\partial c_j} \right\|_{L_1^{p,\kappa}} = \text{const} \cdot e^{\sigma_j \tau}.$$

The other two statements are left to the reader. □

Let $\mathcal{C}'_1$ be the set of all $S \in \mathcal{C}'$ such that $[S|_K] \in V$, $q(S|_K) \in \varpi(\mathbb{R}^d)$, and $\mathrm{Hol}_j(S) \neq -z_j$ for $j = 1, \dots, r_0$. Then $\mathcal{C}'_1$ is an open subset of $\mathcal{C}'$, and there are unique smooth functions

$$\eta_j : \mathcal{C}'_1 \to (-\pi, \pi)$$

such that $\mathrm{Hol}_j(S) = z_j \exp(i\eta_j(S))$. Set $\eta = (\eta_1, \dots, \eta_{r_0})$ and define

$$\hat{\Xi} = (\varpi^{-1} \circ q \circ R_K, \eta, \Theta') : \mathcal{C}_1' \to E.$$

A crucial point in the proof of Theorem 2.1 will be the construction of a smooth local right inverse of $\hat{\Xi}$, defined in a neighbourhood of 0. The map ζ is a first approximation to such a local right inverse. The construction of a genuine local right inverse will involve an application of the quantitative inverse function theorem (see Lemma 2.7 below).

From now on we will take τ so large that $K \subset X_{:\tau}$ and

$$\mathrm{Hol}_j^+(S_{0,\tau}) \cdot (\mathrm{Hol}_j^-(S_{0,\tau}))^{-1} \neq -z_j$$

for $j = 1, \dots, r_0$. Note that the left hand side of this equation is equal to $\mathrm{Hol}_j(S_{0,\tau,T})$ whenever $T_j > \tau + 1$. There is then a constant $\epsilon > 0$ such that $\zeta(B_\epsilon) \subset \mathcal{C}_1'$, in which case we have a composite map

$$\Xi = \hat{\Xi} \circ \zeta : B_\epsilon \to E.$$

Choose $\lambda > 0$ so that none of the operators $\tilde{H}_{\alpha_j}$ $(j = 1, \dots, r)$ and $\tilde{H}_{\alpha_j'}$ $(j = 1, \dots, r')$ has any eigenvalue of absolute value $\leq \lambda$. (The notation $\tilde{H}_\alpha$ was introduced in [11, Subsection 6.1].) Recall that we assume the σ_j are small and non-negative, so in particular we may assume $6\sigma < \lambda$.

Lemma 2.3 *There is a constant $C_2 < \infty$ such that*

$$\|\Xi(0)\| \leq C_2 e^{(\sigma-\lambda)\tau}.$$

Proof. The first two components of $\Xi(0) = \hat{\Xi}(S_{0,\tau,T})$ are in fact zero: the first one because $S_{0,\tau,T} = S_0$ over K, the second one because the dt–component of

$$S_{0,\tau} - S_0 = (1 - \mathsf{r}_\tau)(\underline{\alpha}_j - S_0)$$

vanishes on $[1, \infty) \times (\pm Y_j)$ since S_0 and $\underline{\alpha}_j$ are both in temporal gauge there.

The third component of $\Xi(0)$ is $\Theta'(S_{0,\tau,T})$. It suffices to consider τ so large that the $\mathfrak{p}$–perturbations do not contribute to $\Theta'(S_{0,\tau,T})$, which then vanishes outside the two bands of length 1 in $[-T_j, T_j] \times Y_j$ centred at $t = \pm(T_j - \tau - \frac{1}{2})$, $j = 1, \dots, r$. Our exponential decay result from [11] says that for every $k \geq 0$ there is a constant C_k' such that for every $(t, y) \in \mathbb{R}_+ \times (\pm Y_j)$ we have

$$|\nabla^k (S_0 - \underline{\alpha}_j)|_{(t,y)} \leq C_k' e^{-\lambda t}.$$

Consequently,

$$\|\Theta'(S_{0,\tau,T})\|_\infty \le \text{const}\cdot(e^{-\lambda\tau}+e^{-2\lambda\tau}) \le \text{const}\cdot e^{-\lambda\tau}.$$

This yields

$$\|\Xi(0)\| = \|\Theta'(S_{0,\tau,T})\|_{L^{p,\kappa}} \le \text{const}\cdot e^{(\sigma-\lambda)\tau}. \qquad \square$$

Lemma 2.4 *There is a constant $C_3<\infty$ such that for sufficiently large τ the following hold:*

(i) $\|D\Xi(0)\|\le C_3$,

(ii) *$D\Xi(0)$ is invertible and $\|D\Xi(0)^{-1}\|\le C_3$.*

Proof. By construction, the derivative of Ξ at 0 has the form

$$D\Xi(0) = \begin{pmatrix} I & 0 & \beta_1 \\ \delta_2 & I & \beta_2 \\ \delta_3 & 0 & I \end{pmatrix},$$

where the k'th column is the k'th partial derivative and I the identity map.

The middle top entry in the above matrix is zero because $\theta_{c,\tau}$ vanishes on K. The middle bottom entry is zero because $S_{0,\tau,T}=\underline{\alpha}_j$ on the support of $\theta_{c,\tau}$ and the spinor field of $\underline{\alpha}_j$ is zero (for $j=1,\dots,r_0$). Adding $\theta_{c,\tau}$ to $S_{0,\tau,T}$ therefore has the effect of altering the latter by a gauge transformation over $[-T_j+\tau+1,-T_j+\tau+2]\times Y_j$, $j=1,\dots,r_0$.

We claim that β_k is bounded in τ,T for $k=1,2$. For $k=1$ this is obvious from the boundedness of $Q_{\tau,T}$. For $k=2$ note that the derivative of $\eta_j:\mathcal{C}'_1\to(-\pi,\pi)$ at any $S\in\mathcal{C}'_1$ is

$$D\eta_j(S)(a,\phi) = i\int_{I_j}\gamma_j^* a \tag{9}$$

where a is an imaginary valued 1–form and ϕ a positive spinor. Because of the weights used in the Sobolev norms it follows that $D\eta(S)$ is (independent of S and) bounded in τ,T (see (4)). This together with the bound on $Q_{\tau,T}$ gives the desired bound on β_2.

Note that, for $k=2,3$, $\|\delta_k\|$ is independent of T when $\tau\gg 0$, and routine calculations show that $\|\delta_k\|\to 0$ as $\tau\to\infty$. (In the case of δ_2 this depends on the normalization (5) of the holonomy of $\mathbf{S}(v)$.)

Write $D\Xi(0) = x - y$, where

$$x = \begin{pmatrix} I & 0 & \beta_1 \\ 0 & I & \beta_2 \\ 0 & 0 & I \end{pmatrix}, \qquad x^{-1} = \begin{pmatrix} I & 0 & -\beta_1 \\ 0 & I & -\beta_2 \\ 0 & 0 & I \end{pmatrix}.$$

When τ is so large that $\|y\|\,\|x^{-1}\| < 1$ then of course $\|yx^{-1}\| \le \|y\|\,\|x^{-1}\| < 1$, hence $x - y = (I - yx^{-1})x$ is invertible. Moreover,

$$(x-y)^{-1} - x^{-1} = x^{-1}[(I - yx^{-1})^{-1} - I] = x^{-1}\sum_{k=1}^{\infty}(yx^{-1})^k,$$

which gives

$$\|(x-y)^{-1} - x^{-1}\| \le \frac{\|x^{-1}\|^2\,\|y\|}{1 - \|x^{-1}\|\,\|y\|} \to 0 \quad \text{as } \tau \to \infty. \qquad \square$$

We now record some basic facts that will be used in the proof of Lemma 2.6 below.

Lemma 2.5 *If E_1, E_2, E_3 are Banach spaces, $U_j \subset E_j$ an open set for $j = 1, 2$, and $f : U_1 \to U_2$, $g : U_2 \to E_3$ smooth maps then the second derivate of the composite map $g \circ f : U_1 \to E_3$ is given by*

$$\begin{aligned} D^2(g \circ f)(x)(y,z) &= D^2g(f(x))(Df(x)y, Df(x)z) \\ &\quad + Dg(f(x))(D^2f(x)(y,z)) \end{aligned}$$

for $x \in U_1$ and $y, z \in E_1$.

Proof. Elementary. $\square$

It is also worth noting that embedding and multiplication theorems for L^q_k Sobolev spaces on $\mathbb{R}^4$ ($k \ge 0$, $1 \le q < \infty$) carry over to $X^{(T)}$, and that the embedding and multiplication constants are bounded functions of T.

Furthermore, a differential operator of degree d over $X^{(T)}$ which is translationary invariant over necks and ends induces a bounded operator $L^q_{k+d} \to L^q_k$ whose operator norm is a bounded function of T.

Lemma 2.6 *There is a constant $C_4 > 0$ such that $\|D^2\Xi(x)\| \le C_4$ whenever $\|x\| \le C_4^{-1}$ and $\tau \ge C_4$.*

Proof. We will say a quantity depending on x, τ is *s-bounded* if the lemma holds with this quantity in place of $D^2\Xi$.

Let Ξ_1, Ξ_2, Ξ_3 be the components of Ξ.

The assumption $K \subset X_{:\tau}$ ensures that $\Xi_1(v, c, \xi)$ is independent of c. It then follows from Lemma 2.5 and the bound on $Q_{\tau,T}$ that $D^2\Xi_1$ is s-bounded.

When v, c, ξ are small we have

$$\Xi_2(v, c, \xi) = \eta(S_{v,\tau,T} + \theta_{c,\tau} + Q_{\tau,T}\xi) = c + \eta(S_{v,\tau,T} + Q_{\tau,T}\xi).$$

Since $D\eta$ is constant, as noted above, we have $D^2\eta = 0$. From the bounds on $D\eta$ and $Q_{\tau,T}$ we then deduce that $D^2\Xi_2$ is s-bounded.

To estimate Ξ_3, we fix $h \gg 0$ and consider only $\tau \geq h$. It is easy to see that

$$\Xi_3(x)|_{X_{:h}}$$

is s-bounded. By restricting to small x and choosing h large we may arrange that the $\mathfrak{p}$–perturbations do not contribute to

$$\Xi_{3,j} := \Xi_3|_{[-T_j+h, T_j-h] \times Y_j}$$

for $j = 1, \ldots, r$. We need to show that each $D^2\Xi_{3,j}$ is s-bounded, but to simplify notation we will instead prove the same for $D^2\Xi_3$ under the assumption that the $\mathfrak{p}$ perturbations are zero.

First observe that for any configuration (A, Φ) over $X^{(T)}$ and any closed, imaginary valued 1–form a we have

$$\Theta'(A + a, \Phi) = \Theta'(A, \Phi) + (0, a \cdot \Phi).$$

Moreover,

$$\|a \cdot \Phi\|_{L^{p,\kappa}} = C\|a \cdot e^\kappa \Phi\|_p \leq C\|a\|_{2p}\,\|e^\kappa \Phi\|_{2p} \leq C'\|a\|_{2p}\,\|\Phi\|_{L^{p,\kappa}_1}$$

for some constants $C, C' < \infty$. Taking $(A, \Phi) = S_{v,\tau,T} + Q_{\tau,T}\xi$ and $a = \theta_{c,\tau}$ we see that $D_k D_2 \Xi_3(x)$ is s-bounded for $k = 1, 2, 3$.

Next note that the derivative of the Seiberg–Witten map $\Theta' : \mathcal{C}' \to L^{p,\kappa}$ at a point $S'_o + s_1$ has the form

$$D\Theta'(S'_o + s_1)s_2 = Ls_2 + B(s_1, s_2)$$

where B is a pointwise bilinear operator, and L a first order operator which is independent of s_1 and translationary invariant over necks and ends. This yields

$$\|D\Theta'(S'_o + s)\| \leq \text{const} \cdot (1 + \|s\|_{2p}). \tag{10}$$

Moreover, $D^2\Theta'(S) = B$ for all S, hence there is a constant $C'' < \infty$ such that

$$\|D^2\Theta'(S)\| \le C''$$

for all T.

Combining the above results on Θ' with Lemma 2.5 we see that $D_jD_k\Xi_3$ is s-bounded also when $j, k \neq 2$. □

Lemma 2.7 *There exist $c_5 > 0$ and $C_6 < \infty$ such that if $0 < \epsilon' < c_5\epsilon < c_5^2$ then for sufficiently large τ the following hold:*

(i) $\Xi : B_\epsilon \to E$ *is injective,*

(ii) *There is a (unique) smooth map $\Xi^{-1} : B_{\epsilon'} \to B_\epsilon$ such that $\Xi \circ \Xi^{-1} = I$,*

(iii) $\|D(\Xi^{-1})(x)\| \le C_6$ *for all $x \in B_{\epsilon'}$,*

(iv) $\|D^2(\Xi^{-1})(x)\| \le C_6$ *for all $x \in B_{\epsilon'}$,*

(v) $\|\Xi^{-1}(0)\| \le C_6 e^{(\sigma-\lambda)\tau}$.

Proof. For sufficiently large τ we have

$$\epsilon' + \|\Xi(0)\| < c_5\epsilon$$

by Lemma 2.3. Statements (i)–(iv) now follow from the inverse function theorem, [11, Proposition B.1], applied to the function $x \mapsto \Xi(x) - \Xi(0)$, together with Lemmas 2.4 and 2.6. To prove (v), set $h = \Xi^{-1}$, $x = \Xi(0)$ and take τ so large that $x \in B_{\epsilon'}$. Since Ξ is injective on B_ϵ we must have $h(x) = 0$, so

$$\|h(0)\| = \|h(x) - h(0)\| \le \|x\| \sup_{\|y\| \le \epsilon'} \|Dh(y)\|.$$

Now (v) follows from (iii) and Lemma 2.3. □

From now on we assume that $\epsilon, \epsilon', \tau$ are chosen so that the conclusions of the lemma are satisfied. Define

$$\hat{\zeta} = \zeta \circ \Xi^{-1} : B_{\epsilon'} \to \mathcal{C}'_1.$$

Then clearly

$$\hat{\Xi} \circ \hat{\zeta} = I.$$

Thus, $(v, c) \mapsto \hat{\zeta}(v, c, 0)$ is a "gluing map", ie for small v, c it solves the problem of gluing the monopole $\mathbf{S}(v)$ over X to get a monopole over $X^{(T)}$ with prescribed holonomy $z_j e^{ic_j}$ along the path γ_j for $j = 1, \dots, r_0$.

Lemma 2.8 *There is a constant $C_7 < \infty$ such that for $x \in B_{\epsilon'}$ one has*

$$\|D\hat{\zeta}(x)\|, \|D^2\hat{\zeta}(x)\| \le C_7 e^{\sigma\tau}.$$

Proof. This follows from Lemmas 2.2 and 2.7 and the chain rule. □

The following proposition refers to the situation of Subsection 2.1 and uses the notation of Theorem 2.1.

Proposition 2.1 *If (K', V') is any kv-pair $\le (K, V)$ then $\overline{G} \times U(1)^{r_0}$ can be covered by finitely many connected open sets W in $M_{\mathfrak{b}} \times U(1)^{r_0}$ such that if $\check{T}$ is sufficiently large then for each W there exists a smooth map $\mathbf{h} : W \to H^{(T)}$ whose image consists only of regular points and which satisfies $\mathbf{f} \circ \mathbf{h} = I$.*

Here we do not need any assumptions on $\tilde{\eta}_j, \tilde{\eta}'_j$ or on $\mathfrak{p}_j, \mathfrak{p}'_j$.

Proof. Let $(\omega_0, z) \in \overline{G} \times \mathrm{U}(1)^{r_0}$ and consider the set-up above, with τ so large that $K' \subset X_{:\tau}$ and ϵ so small that

$$\zeta(x)|_{K'} \in V' \qquad \text{for every } x \in B_\epsilon. \tag{11}$$

Note that taking ϵ small may require taking τ (and hence $\check{T}$) large, see Lemma 2.7. For any sufficiently small open neighbourhood $W \subset M_{\mathfrak{b}} \times \mathrm{U}(1)^{r_0}$ of (ω_0, z) we can define a smooth map $\nu : W \to \mathcal{C}'_1$ by the formula

$$\nu(\omega, a) = \hat{\zeta}(\varpi^{-1}(\omega), -i \log \frac{a}{z}, 0).$$

Here $\log e^u = u$ for any complex number u with $|\mathrm{Im}\, u| < \pi$, and $i \log \frac{a}{z} \in \mathbb{R}^{r_0}$ denotes the vector whose j'th component is $i \log \frac{a_j}{z_j}$. Because $\hat{\Xi} \circ \hat{\zeta} = I$ and the Seiberg–Witten map is the third component of $\hat{\Xi}$, the image of ν consists of regular monopoles. Let $\mathbf{h} : W \to \mathcal{B}'_{\mathfrak{b}}$ be the composition of ν with the projection $\mathcal{C}' \to \mathcal{B}'_{\mathfrak{b}}$. Unravelling the definitions involved and using (11) one finds that $\mathbf{h}$ has the required properties.

How large $\check{T}$ must be for this to work might depend on (ω_0, z). But $\overline{G} \times \mathrm{U}(1)^{r_0}$ is compact, hence it can be covered by finitely many such open sets W. If $\check{T}$ is sufficiently large then the above construction will work for each of these W. □

2.3 Injectivity

We now continue the discussion that was interrupted by Proposition 2.1. Set

$$\tilde{S} = S_{0,\tau,T}, \quad \hat{S} = \hat{\zeta}(0).$$

Lemma 2.9 *There is a constant $C_8 < \infty$ such that for sufficiently large τ one has*

$$\|\hat{S} - \tilde{S}\|_{L^{p,\kappa}_1} \le C_8 e^{(2\sigma-\lambda)\tau}, \quad \|\hat{S} - S'_o\|_{L^{p,\kappa}_1} \le C_8.$$

Proof. Set $\Xi^{-1}(0) = (v, c, \xi) \in B_\epsilon$. For sufficiently large τ we have

$$\begin{aligned}\|\hat{S} - \tilde{S}\|_{L^{p,\kappa}_1} &\le \|S_{v,\tau,T} - S_{0,\tau,T}\|_{L^{p,\kappa}_1} + \|\theta_{c,\tau} + Q_{\tau,T}\xi\|_{L^{p,\kappa}_1} \\ &\le \text{const} \cdot (\|v\| + e^{\sigma\tau}\|c\| + \|\xi\|) \\ &\le \text{const} \cdot e^{(2\sigma-\lambda)\tau},\end{aligned}$$

where we used Lemma 2.7 (v) to obtain the last inequality. Because $\|\tilde{S} - S'_o\|_{L^{p,\kappa}_1}$ is bounded in τ, T, and we assume $6\sigma < \lambda$, the second inequality of the lemma follows as well. □

For positive spinors Φ on $X^{(T)}$ it is convenient to extend the definition of $\mathcal{I}_\Phi$ to complex valued functions on $X^{(T)}$:

$$\mathcal{I}_\Phi f = (-df, f\Phi).$$

(However, $\mathcal{I}^*_\Phi$ will always refer to the formal adjoint of $\mathcal{I}_\Phi$ acting on imaginary valued functions.) When Φ is the spinor part of $S'_o, \tilde{S}, \hat{S}$ then the corresponding operators $\mathcal{I}_\Phi$ will be denoted $\mathcal{I}'_o, \tilde{\mathcal{I}}, \mathcal{I}$, respectively. (We omit the $\hat{}$ on $\mathcal{I}$ to simplify notation.) As in [11, Subsection 2.2] we define

$$\mathcal{E}' = \{f \in L^p_{2,\text{loc}}(X^{(T)}; \mathbb{C}) \,:\, \mathcal{I}'_o f \in L^{p,\kappa}_1\}.$$

We can take the norm to be

$$\|f\|_{\mathcal{E}'} = \|\mathcal{I}'_o f\|_{L^{p,\kappa}_1} + \sum_{x \in \mathfrak{b}} |f(x)|.$$

Lemma 2.10 *There is a constant $C_9 < \infty$ such that if $\mathbf{I}$ is any of the operators $\mathcal{I}'_o, \tilde{\mathcal{I}}, \mathcal{I}$ then for all $f \in \mathcal{E}'$ one has*

$$\|f\|_\infty \le C_9 \left(\|\mathbf{I}f\|_{L^{p,\kappa}} + \sum_{x \in \mathfrak{b}} |f(x)| \right).$$

Proof. We first prove the inequality for $\mathbf{I} = \tilde{\mathcal{I}}$ (the case of $\mathcal{I}'_o$ is similar, or easier). If X_e is any component of X and $0 \le \bar{\tau} \le \tau$ then for some constant $C_{\bar{\tau}} < \infty$ one has

$$\|f\|_\infty \le \text{const} \cdot \|f\|_{L^p_1} \le C_{\bar{\tau}} \left(\|\tilde{\mathcal{I}} f\|_p + \sum_{x \in \mathfrak{b} \cap X_e} |f(x)| \right)$$

for all L^p_1 functions $f:(X_e)_{:\bar\tau}\to\mathbb{C}$. Here the Sobolev inequality holds because $p>4$, whereas the second inequality follows from [11, Lemma 2.1]. We use part (i) of that lemma if the spinor field of S_0 is not identically zero on X_e, and part (ii) otherwise. (In the latter case $\mathfrak{b}\cap X_e$ is non-empty.)

When $\bar\tau,\tau$ are sufficiently large we can apply part (i) of the same lemma in a similar fashion to the band $[t,t+1]\times Y'_j$ provided $t\ge\bar\tau$ and α'_j is irreducible, and to the band $[t-1,t+1]\times Y_j$ provided $|t|\le T_j-\bar\tau-1$ and α_j is irreducible. To estimate $|f|$ over these bands when α'_j resp. α_j is reducible one can use [11, Lemma 2.2 (ii)]. This proves the lemma for $\mathbf{I}=\tilde{\mathcal{I}}$ (and for $\mathbf{I}=\mathcal{I}'_o$).

We now turn to the case $\mathbf{I}=\mathcal{I}$. Let ϕ denote the spinor part of $\hat S-\tilde S$. Then

$$\begin{aligned}\|\tilde{\mathcal{I}}f\|_{L^{p,\kappa}} &\le \|\mathcal{I}f\|_{L^{p,\kappa}}+\|f\phi\|_{L^{p,\kappa}}\\ &\le \|\mathcal{I}f\|_{L^{p,\kappa}}+\text{const}\cdot\left(\|\tilde{\mathcal{I}}f\|_{L^{p,\kappa}}+\sum_{x\in\mathfrak{b}}|f(x)|\right)\cdot\|\phi\|_{L^{p,\kappa}}.\end{aligned}$$

By Lemma 2.9 we have $\|\phi\|_{L^{p,\kappa}_1}\to 0$ as $\tau\to 0$, so for sufficiently large τ we get

$$\|\tilde{\mathcal{I}}f\|_{L^{p,\kappa}}\le\text{const}\cdot\left(\|\mathcal{I}f\|_{L^{p,\kappa}}+\sum_{x\in\mathfrak{b}}|f(x)|\right).$$

Therefore, the lemma holds with $\mathbf{I}=\mathcal{I}$ as well. □

Lemma 2.11 *There is a constant $C_{10}<\infty$ such that for all $f,g\in\mathcal{E}'$ and $\phi\in L^{p,\kappa}_1(X^{(T)};\mathbb{S}^+)$ one has*

(i) $\|fg\|\le C_{10}\|f\|\,\|g\|$,

(ii) $\|f\phi\|\le C_{10}\|f\|\,\|\phi\|$,

where we use the $L^{p,\kappa}_1$ norm on spinors and the $\mathcal{E}'$ norm on elements of $\mathcal{E}'$.

Proof. By routine calculation using Lemma 2.10 with $\mathbf{I}=\mathcal{I}'_o$ one easily proves (ii) and the inequality

$$\|d(fg)\|_{L^{p,\kappa}_1}\le\text{const}\cdot\|f\|_{\mathcal{E}'}\|g\|_{\mathcal{E}'}.$$

Now observe that by definition $g\Phi'_o\in L^{p,\kappa}_1$, where as before Φ'_o denotes the spinor field of the reference configuration S'_o. Applying (ii) we then obtain

$$\|fg\Phi'_o\|_{L^{p,\kappa}_1}\le\text{const}\cdot\|f\|_{\mathcal{E}'}\,\|g\Phi'_o\|_{L^{p,\kappa}_1}\le\text{const}\cdot\|f\|_{\mathcal{E}'}\,\|g\|_{\mathcal{E}'},$$

completing the proof of (i). □

Recall from [11, Subsection 2.4] that the Lie algebra $L\mathcal{G}'_{\mathfrak{b}}$ is the space of imaginary valued functions in $\mathcal{E}'$ that vanish on $\mathfrak{b}$.

Lemma 2.12 *There is a constant* $C_{11} > 0$ *such that for* $\tau > C_{11}$ *and all* $f \in L\mathcal{G}'_{\mathfrak{b}}$ *one has*

$$C_{11}^{-1}\|\mathcal{I}'_o f\|_{L^{p,\kappa}_1} \le \|\mathcal{I} f\|_{L^{p,\kappa}_1} \le C_{11}\|\mathcal{I}'_o f\|_{L^{p,\kappa}_1}.$$

Proof. Let ψ denote the spinor part of $\hat{S} - S'_o$. Then

$$\begin{aligned}\|f\psi\|_{L^{p,\kappa}_1} &\le \text{const}\cdot(\|f\|_\infty\|\psi\|_{L^{p,\kappa}_1} + \|df\|_{L^{2p,\kappa}}\|\psi\|_{2p})\\ &\le \text{const}\cdot\|\mathcal{I}f\|_{L^{p,\kappa}_1}\|\psi\|_{L^{p,\kappa}_1},\end{aligned}$$

and similarly with $\mathcal{I}'_o$ instead of $\mathcal{I}$. The lemma now follows from Lemma 2.9. □

We are going to use the inverse function theorem a second time, to show that the image of the smooth map

$$\begin{aligned}\Pi : L\mathcal{G}'_{\mathfrak{b}} \times B_{\epsilon'} &\to \mathcal{C}'_1,\\ (f,x) &\mapsto \exp(f)(\hat{\zeta}(x)).\end{aligned}$$

contains a "not too small" neighbourhood of $\hat{S}$. The derivative of Π at $(0,0)$ is

$$\begin{aligned}D\Pi(0,0) : L\mathcal{G}'_{\mathfrak{b}} \oplus E &\to L^{p,\kappa}_1,\\ (f,x) &\mapsto \mathcal{I}f + D\hat{\zeta}(0)x.\end{aligned}$$

To be concrete, let $L\mathcal{G}'_{\mathfrak{b}} \oplus E$ have the norm $\|(f,x)\| = \|f\|_{\mathcal{E}'} + \|x\|_E$.

Lemma 2.13 $D\Pi(0,0)$ *is a linear homeomorphism.*

Proof. By [11, Proposition 2.2], $\mathcal{I}^*\mathcal{I} : L\mathcal{G}'_{\mathfrak{b}} \to L^{p,\kappa}$ is a Fredholm operator with the same kernel as $\mathcal{I}$. Now, $\mathcal{I}$ is injective on $L\mathcal{G}'_{\mathfrak{b}}$, because $\hat{\zeta}$ maps into $\mathcal{C}'_1$ and therefore $[\hat{S}|_K] \in V \subset \mathcal{B}^*_{\mathfrak{b}}$. Since

$$W = \mathcal{I}^*\mathcal{I}(L\mathcal{G}'_{\mathfrak{b}})$$

is a closed subspace of $L^{p,\kappa}$ of finite codimension, we can choose a bounded operator

$$\pi : L^{p,\kappa} \to W$$

such that $\pi|_W = I$. Set

$$\mathcal{I}^{\#} = \pi \mathcal{I}^* : L_1^{p,\kappa} \to W.$$

Then

$$\mathcal{I}^{\#}\mathcal{I} : L\mathcal{G}'_{\mathfrak{b}} \to W$$

is an isomorphism. Furthermore,

$$\text{index}(\mathcal{I}^{\#} + D\Theta'(\hat{S})) = \dim\, M_{\mathfrak{b}}^{(T)} = \dim M_{\mathfrak{b}} + r_0,$$

where 'dim' refers to expected dimension (which in the case of $M_{\mathfrak{b}}$ is equal to the actual dimension of G), and the second equality follows from the addition formula for the index (see Corollary A.1). Consequently,

$$\text{index}(\mathcal{I}^{\#} + D\hat{\Xi}(\hat{S})) = 0.$$

We now compute

$$(\mathcal{I}^{\#} + D\hat{\Xi}(\hat{S})) \circ D\Pi(0,0) = \begin{pmatrix} \mathcal{I}^{\#}\mathcal{I} & B \\ 0 & I \end{pmatrix} : L\mathcal{G}'_{\mathfrak{b}} \oplus E \to W \oplus E, \qquad (12)$$

where $B : E \to W$. The zero in the matrix above is due to the fact that

$$D\hat{\Xi}(\hat{S})\mathcal{I}f = \left.\frac{d}{dt}\right|_0 \hat{\Xi}(e^{tf}(\hat{S})) = 0,$$

which holds because $\hat{\Xi}_1, \hat{\Xi}_2$ are $\mathcal{G}'_{\mathfrak{b}}$–invariant, $\hat{\Xi}_3$ is $\mathcal{G}'_{\mathfrak{b}}$–equivariant, and $\hat{\Xi}(\hat{S}) = 0$.

Since the right hand side of (12) is invertible, it follows that $\mathcal{I}^{\#} + D\hat{\Xi}(\hat{S})$ is a surjective Fredholm operator of index 0, hence invertible. Of course, this implies that $D\Pi(0,0)$ is also invertible. ☐

Lemma 2.14 *There is a constant $C_{12} < \infty$ such that for sufficiently large τ,*

$$\|D\Pi(0,0)^{-1}\| \le C_{12} e^{\sigma\tau}.$$

Proof. In this proof all unqualified norms are $L_1^{p,\kappa}$ norms. It follows from (9), (10) and Lemma 2.9 that $D\hat{\Xi}(\hat{S})$ is bounded in τ, T. Therefore there exists a constant $C < \infty$ such that

$$\|x\|_E = \|D\hat{\Xi}(\hat{S})(\mathcal{I}f + D\hat{\zeta}(0)x)\| \le C\|D\Pi(0,0)(f,x)\|$$

for all $f \in L\mathcal{G}'_{\mathfrak{b}}$ and $x \in E$. From Lemma 2.12 and Lemma 2.8 we get

$$\begin{aligned} C_{11}^{-1}\|\mathcal{I}'_o f\| &\le \|\mathcal{I} f\| \\ &\le \|D\Pi(0,0)(f,x)\| + \|D\hat{\zeta}(0)x\| \\ &\le \|D\Pi(0,0)(f,x)\| + C_7 e^{\sigma\tau}\|x\|_E \\ &\le (1 + CC_7 e^{\sigma\tau})\|D\Pi(0,0)(f,x)\|. \end{aligned}$$

This yields

$$\|f\|_{\mathcal{E}'} + \|x\|_E \le \text{const} \cdot e^{\sigma\tau}\|D\Pi(0,0)(f,x)\|. \qquad \square$$

Lemma 2.15 *There is a constant $C_{13} < \infty$ such that for sufficiently large τ one has*

$$\|D^2\Pi(f,x)\| \le C_{13}e^{\sigma\tau}$$

for all $f \in L\mathcal{G}'_{\mathfrak{b}}$, $x \in E$ such that $\|f\| < 1$ and $\|x\| < \epsilon'$.

Proof. For the purposes of this proof it is convenient to rescale the norm on $\mathcal{E}'$ so that we can take $C_{10} = 1$ in Lemma 2.11.

If $f, g \in \mathcal{E}'$ then $e^f g \in \mathcal{E}'$, and from Lemma 2.11 we obtain

$$\|e^f g\| \le \sum_{n=0}^{\infty} \frac{1}{n!}\|f^n g\| \le e^{\|f\|}\|g\|,$$

and similarly with a spinor $\phi \in L_1^{p,\kappa}$ instead of g.

The first two derivatives of $\exp : \mathcal{E}' \to 1 + \mathcal{E}'$ are

$$D\exp(f)g = g\exp(f), \qquad D^2\exp(f)(g,h) = gh\exp(f),$$

so

$$\|D\exp(f)\|, \|D^2\exp(f)\| \le \exp(\|f\|).$$

Let $\hat{\zeta}_1, \hat{\zeta}_2$ be the connection and spinor parts of ζ, respectively, and define Π_1, Π_2 similarly. Then

$$\Pi(f,x) = (\hat{\zeta}_1(x) - df, e^f \cdot \hat{\zeta}_2(x)).$$

We regard $\Pi(f,x)$ as a function of the the two variables f, x. Let $D_j\Pi$ denote the derivative of Π with respect to the j'th variable. Similarly for the second derivatives $D_jD_k\Pi$.

Applying Lemmas 2.8 and 2.11 we now find that

$$\begin{gathered} \|D_2^2\Pi_1(f,x)\| = \|D^2\hat{\zeta}_1(x)\| \le \text{const} \cdot e^{\sigma\tau}, \\ \|D_jD_k\Pi_2(f,x)\| \le \text{const} \cdot e^{\sigma\tau}, \quad j,k = 1,2 \end{gathered}$$

for $\|f\| < 1$ and $\|x\| < \epsilon'$. Since $D_jD_1\Pi_1 = 0$ for $j = 1, 2$, the lemma is proved. $\square$

In the following, $B(x; r)$ will denote the open r–ball about x (both in various Banach spaces and in $\mathcal{C}'$).

Lemma 2.16 *There exist constants $r_1, r_2 > 0$ such that for sufficiently large τ the image of Π contains the ball of radius $r_2e^{-3\sigma\tau}$ about $\hat{S}$ in $\mathcal{C}'$; more precisely one has*

$$B(\hat{S}; r_2e^{-3\sigma\tau}) \subset \Pi(B(0; r_1e^{-2\sigma\tau})).$$

Proof. We wish to apply the inverse function theorem [11, Proposition B.1] to the map Π restricted to a ball $B(0; R_1)$, where $R_1 \in (0, \epsilon']$ is to be chosen. For the time being let M, L, κ have the same meaning as in that proposition. By Lemma 2.15 we can take $M = C_{13}e^{\sigma\tau}$, and by Lemma 2.14 we have $\|L^{-1}\| \leq C_{12}e^{\sigma\tau}$. We need

$$0 \leq \kappa = \|L^{-1}\|^{-1} - R_1M.$$

This will hold if

$$R_1 \leq (C_{12}C_{13})^{-1}e^{-2\sigma\tau}.$$

When τ is large we can take R_1 to be the right hand side of this inequality. By [11, Proposition B.1], $\Pi(B(0; R_1))$ contains the ball $B(\hat{S}; R_2)$ where

$$R_2 = \frac{1}{2}R_1C_{12}^{-1}e^{-\sigma\tau} = \frac{1}{2}C_{12}^{-2}C_{13}^{-1}e^{-3\sigma\tau}. \qquad \square$$

Theorem 2.1 is a consequence of Proposition 2.1 and the following proposition:

Proposition 2.2 *Under the assumptions of Theorem 2.1, and using the same notation, there is a kv-pair $(K', V') \leq (K, V)$ such that $\mathbf{f}$ is injective on $\mathbf{q}^{-1}(\overline{G})$ for all sufficiently large $\check{T}$.*

The proof of Proposition 2.2 occupies the remainder of this subsection.

For any natural number m which is so large that $K \subset X_{:m}$, let V'_m be the set of all $\omega \in \breve{\mathcal{B}}_{\mathfrak{b}}(X_{:m})$ such that there exist a representative S of ω, and a configuration $\bar{S} = (\bar{A}, \bar{\Phi})$ over X representing an element of $\overline{G}$, such that

$$d_m(S, \bar{S}) := \int_{X_{:m}} |\bar{S} - S|^p + |\nabla_{\bar{A}}(\bar{S} - S)|^p < \frac{1}{m}. \tag{13}$$

Note that

$$d_m(u(S), u(\bar{S})) = d_m(S, \bar{S})$$

for any gauge transformation u over $X_{:m}$. In particular, V'_m is $\mathbb{T}$–invariant.

Lemma 2.17 *Let $\omega_n \in V'_{m_n}$ for $n = 1, 2, \dots,$ where $m_n \to \infty$. Then there exists for each n a representative S_n of ω_n such that a subsequence of S_n converges locally in L^p_1 over X to a smooth configuration representing an element of $\overline{G}$.*

Proof. By assumption there exist for each n a representative S_n of ω_n and a configuration $\bar{\mathsf{S}}_n$ over X representing an element of $\overline{G}$ such that

$$d_{m_n}(\mathsf{S}_n, \bar{\mathsf{S}}_n) < \frac{1}{m_n}. \tag{14}$$

After passing to a subsequence we may assume (since $\overline{G}$ is compact) that $[\bar{\mathsf{S}}_n]$ converges in $\overline{G}$ to some element $[\bar{S}]$, and we can choose $\bar{S}$ smooth. Since $M_{\mathfrak{b}} = M^*_{\mathfrak{b}}$, the local slice theorem guarantees that for large n we can find $u_n \in \mathcal{G}_{\mathfrak{b}}$ such that $\bar{S}_n = u_n(\bar{\mathsf{S}}_n)$ satisfies

$$\|\bar{S}_n - \bar{S}\|_{L^{p,w}_1} \to 0.$$

Set $S_n = u_n(\mathsf{S}_n)$, which is again a representative of ω_n. Let $\bar{A}, \bar{A}_n$ be the connection parts of $\bar{S}, \bar{S}_n$, respectively. Then (14) implies that $\bar{S}_n - S_n \to 0$ and $\nabla_{\bar{A}_n}(\bar{S}_n - S_n) \to 0$ locally in L^p over X, hence also $S_n \to \bar{S}$ locally in L^p over X. Now

$$\nabla_{\bar{A}}(S_n - \bar{S}) = \nabla_{\bar{A}_n}(S_n - \bar{S}_n) + \nabla_{\bar{A}}(\bar{S}_n - \bar{S}) + (\bar{A} - \bar{A}_n)(S_n - \bar{S}_n),$$

and each of the three terms on the right hand side converges to 0 locally in L^p over X (the third term because of the continuous multiplication $L^p_1 \times L^p \to L^p$ in $\mathbb{R}^4$ for $p > 4$). Hence $S_n \to \bar{S}$ locally in L^p_1 over X. ☐

Corollary 2.1 *For sufficiently large n one has that $R_K(V'_n) \subset V$.* ☐

Lemma 2.18 *Let $\omega_n \in V'_{m_n}$ for $n = 1, 2, \dots,$ where $m_n \to \infty$. Suppose $q(\omega_n|_K)$ converges in $M_{\mathfrak{b}}$ to an element g as $n \to \infty$. Then $g \in \overline{G}$, and there exists for each n a representative S_n of ω_n such that the sequence S_n converges locally in L^p_1 over X to a smooth configuration representing g.*

Proof. Let $\mathsf{S}_n, \bar{\mathsf{S}}_n$ be as in the proof of Lemma 2.17. First suppose that $[\bar{\mathsf{S}}_n]$ converges in $\overline{G}$ to some element $[\bar{S}]$, where $\bar{S}$ is smooth. Choosing $S_n, \bar{S}_n$ as in that proof we find again that $S_n \to \bar{S}$ locally in L^p_1 over X, hence

$$g = \lim_n q(S_n|_K) = q(\bar{S}|_K) = [\bar{S}].$$

We now turn to the general case when $[\bar{\mathsf{S}}_n]$ is not assumed to converge. Because $\overline{G}$ is compact, every subsequence of $[\bar{\mathsf{S}}_n]$ has a convergent subsequence whose limit must be g by the above argument. Hence $[\bar{\mathsf{S}}_n] \to g$. $\square$

Suppose we are given a sequence $\{m_n\}_{n=1,2,\dots}$ of natural numbers tending to infinity, and for each n an r–tuple $T(n)$ of real numbers such that

$$\check{T}(n) := \min_j T_j(n) > m_n.$$

Define $\mathbf{q}_n$ and $\mathbf{f}_n$ as in Theorem 2.1, with $K' = X_{:m_n}$ and $V' = V'_{m_n}$.

Lemma 2.19 *For $n = 1, 2, \dots$ suppose S_n is a smooth configuration over $X^{(T(n))}$ representing an element $\omega_n \in \mathbf{q}_n^{-1}(\overline{G})$, and such that*

$$\mathbf{f}_n(\omega_n) \to (\omega_0, z) \in \overline{G} \times U(1)^{r_0}$$

as $n \to \infty$. There exists a constant $C_{14} < \infty$ such that for sufficiently large τ the following holds for sufficiently large n. Let the map $\hat{\zeta} = \hat{\zeta}_n$ be defined as above and set $\hat{S}_n = \hat{\zeta}_n(0)$. Then there exists a smooth gauge transformation $u_n \in \mathcal{G}'_{\mathfrak{b}}$ such that

$$\|u_n(S_n) - \hat{S}_n\|_{L^{p,\kappa}_1} \le C_{14} e^{(3\sigma - \lambda)\tau}.$$

Note: This constant C_{14} depends on (ω_0, z) but not on the sequence S_n.

Before proving the lemma, we will use it to show that $\mathbf{f}_n$ is injective on $\mathbf{q}_n^{-1}(\overline{G})$ for some n. This will prove Proposition 2.2. Suppose $\omega_n, \omega'_n \in \mathbf{q}_n^{-1}(\overline{G})$ and $\mathbf{f}_n(\omega_n) = \mathbf{f}_n(\omega'_n)$, $n = 1, 2, \dots$. After passing to a subsequence we may assume that $\mathbf{f}_n(\omega_n)$ converges to some point $(\omega_0, z) \in \overline{G} \times \mathrm{U}(1)^{r_0}$. Combining Lemmas 2.16, 2.19 and the assumption $6\sigma < \lambda$ we conclude that if τ is sufficiently large then for sufficiently large n we can represent ω_n and ω'_n by configurations $\hat{\zeta}(x_n)$ and $\hat{\zeta}(x'_n)$, respectively, where $x_n, x'_n \in B_{\epsilon'}$. Now recall that $\hat{\Xi} \circ \hat{\zeta} = I$, and that the components $\hat{\Xi}_1, \hat{\Xi}_2$ are $\mathcal{G}'_{\mathfrak{b}}$–invariant whereas $\hat{\Xi}_3$ is the Seiberg–Witten map. Comparing the definitions of $\mathbf{f}_n$ and $\hat{\Xi}$ we conclude that

$$x_n = \hat{\Xi}(\hat{\zeta}(x_n) = \hat{\Xi}(\hat{\zeta}(x'_n) = x'_n,$$

hence $\omega_n = \omega'_n$ for large n. To complete the proof of Proposition 2.2 it therefore only remains to prove Lemma 2.19.

Proof of Lemma 2.19: In this proof, constants will be independent of the sequence S_n (as well as of τ as before).

By Lemma 2.18 we can find for each n an $L^p_{2,\mathrm{loc}}$ gauge transformation v_n over X with $v_n|_{\mathfrak{b}} = 1$ such that $S'_n = v_n(S_n)$ converges locally in L^p_1 over X

to a smooth configuration S' representing ω_0. A moment's thought shows that we can choose the v_n smooth, and we can clearly arrange that $S' = S_0$. Then for any $t \geq 0$ we have

$$\limsup_n \|S'_n - \hat{S}_n\|_{L_1^{p,\kappa}(X_{:t})} = \limsup_n \|S_0 - \hat{S}_n\|_{L_1^{p,\kappa}(X_{:t})} \leq \text{const} \cdot e^{(2\sigma-\lambda)\tau} \quad (15)$$

when τ is so large that Lemma 2.9 applies.

For $t \geq 0$ and any smooth configurations S over $X_{:t}$ consider the functional

$$\begin{aligned} E(S,t) = &\sum_{j=1}^{r} \lambda_j \left(\vartheta(S|_{\{t\}\times Y_j}) + \vartheta(S|_{\{t\}\times(-Y_j)}) \right) \\ &+ \sum_{j=1}^{r'} \lambda'_j \left(\vartheta(S|_{\{t\}\times Y'_j}) - \vartheta(\alpha'_j) \right), \end{aligned}$$

where in this formula $\{t\} \times (\pm Y_j)$ has the boundary orientation inherited from $X_{:t}$. (Recall that the Chern-Simons-Dirac functional ϑ changes sign when the orientation of the 3–manifold in question is reversed.) The assumption on λ_j, λ'_j and $\tilde{\eta}_j, \tilde{\eta}'_j$ in Theorem 2.1 implies that $E(S,t)$ depends only on the gauge equivalence class of S. Since ϑ is a smooth function on the $L^2_{1/2}$ configuration space by [11, Lemma 3.1], we obtain

$$E(S_n,t) = E(S'_n,t) \to E(S_0,t)$$

as $n \to \infty$. By our exponential decay results (see the proof of [11, Theorem 6.1]),

$$E(S_0,t) < \text{const} \cdot e^{-2\lambda t} \quad \text{for } t \geq 0.$$

It follows that

$$E(S_n,t) < \text{const} \cdot e^{-2\lambda t} \quad \text{for } n > N(t)$$

for some positive function N. By assumption the perturbation parameters $\vec{\mathfrak{p}}, \vec{\mathfrak{p}}'$ are admissible, hence there is a constant $C < \infty$ such that when $\check{T}(m_n) > C$, each of the $(r + r')$ summands appearing in the definition of $E(S_n,t)$ is non-negative. Explicitly, this yields

$$\begin{aligned} 0 \leq \vartheta(S_n|_{\{-T_j(n)+t\}\times Y_j}) - \vartheta(S_n|_{\{T_j(n)-t\}\times Y_j}) < \text{const} \cdot e^{-2\lambda t}, \\ 0 \leq \vartheta(S_n|_{\{t\}\times Y'_j}) - \vartheta(\alpha'_j) < \text{const} \cdot e^{-2\lambda t}, \end{aligned}$$

where the first line holds for $0 \leq t \leq T_j(n)$ and $j = 1, \dots, r$, the second line for $t \geq 0$ and $j = 1, \dots, r'$, and in both cases we assume $\check{T}(n) > C$ and $n > N(t)$.

In the following we will ignore the ends $\mathbb{R}_+ \times Y'_{j'}$ of X, ie we will pretend that $X^\#$ is compact. If $\alpha'_{j'}$ is irreducible then the argument for dealing with the end $\mathbb{R}_+ \times Y'_{j'}$ is completely analogous to the one given below for a neck $[-T_j, T_j] \times Y_j$, while if $\alpha'_{j'}$ is reducible it is simpler. (Compare the proof of [11, Proposition 6.3 (ii)].)

For the remainder of the proof of this lemma we will focus on one particular neck $[-T_j(n), T_j(n)] \times Y_j$ where $1 \leq j \leq r$. To simplify notation we will therefore mostly omit j from notation and write $T(n), Y, \alpha$ etc instead of $T_j(n), Y_j, \alpha_j$.

For $0 \leq t \leq T(n)$ set

$$B_t = [-T(n) + t, T(n) - t] \times Y,$$

regarded as a subset of $X^{(T(n))}$. By the above discussion there is a constant $t_1 > 0$ such that when n is sufficiently large, S_n will restrict to a genuine monopole over the band B_{t_1+3} by [11, Lemmas 4.1, 4.2, 4.3] and will have small enough energy over this band for [11, Theorem 6.2] to apply. That theorem then provides a smooth

$$\tilde{v}_n : B_{t_1} \to \mathrm{U}(1)$$

such that $S''_n = \tilde{v}_n(S_n|_{B_{t_1}})$ is in temporal gauge and

$$\|S''_n - \underline{\alpha}\|_{L^{p,\kappa}_1(B_t)} \leq \text{const} \cdot e^{(\sigma-\lambda)t}, \quad t \geq t_1.$$

Writing

$$S''_n - \hat{S}_n = (S''_n - \underline{\alpha}) + (\underline{\alpha} - \tilde{S}) + (\tilde{S} - \hat{S}_n)$$

we get

$$\limsup_{n\to\infty} \|S''_n - \hat{S}_n\|_{L^{p,\kappa}_1(B_t)} \leq \text{const} \cdot \left(e^{(2\sigma-\lambda)\tau} + e^{(\sigma-\lambda)t}\right) \qquad (16)$$

when $t \geq t_1$ and τ is so large that Lemma 2.9 applies.

To complete the proof of the lemma we interpolate between v_n and $\tilde{v}_n$ in the overlap region $\mathcal{O}_\tau = X_{:\tau} \cap B_{\tau-1}$. (This requires $\tau \geq t_1 + 1$.) The choice of this overlap region is somewhat arbitrary but simplifies the exposition. Define

$$w_n = \tilde{v}_n v_n^{-1} : \mathcal{O}_\tau \to \mathrm{U}(1).$$

Then

$$w_n(S'_n) = S''_n \quad \text{on } \mathcal{O}_\tau.$$

Set $x^\pm = \gamma(\pm(T-\tau))$, where $\gamma = \gamma_j$ is the path introduced in the beginning of this section. If α is reducible then by multiplying each $\tilde{v}_n$ by a constant and redefining w_n, S''_n accordingly we can arrange that $w_n(x^+) = 1$ for all n. These changes have no effect on the estimates above.

Lemma 2.19 is a consequence of the estimates (15), (16) together with the following sublemma (see the proof of [11, Proposition 6.3 (ii)].)

Sublemma 2.1 *There is a constant $C_{15} < \infty$ such that if $\tau \geq C_{15}$ then*

$$\limsup_{n\to\infty} \|w_n - 1\|_{L^p_2(\mathcal{O}_\tau)} \leq C_{15} e^{(2\sigma-\lambda)\tau}.$$

Proof of sublemma: If α is irreducible then the sublemma follows from inequalities (15), (16) and [11, Lemmas 6.9, 6.11]. (In this case the sublemma holds with $C_{15}e^{(\sigma-\lambda)\tau}$ as upper bound.)

Now suppose α is reducible. We will show that

$$\limsup_{n\to\infty} |w_n(x^-) - 1| \leq \text{const} \cdot e^{(2\sigma-\lambda)\tau} \tag{17}$$

for large τ. Granted this, we can prove the sublemma by applying [11, Lemma 6.9] and [11, Lemma 6.10 (ii)] to each component of $\mathcal{O}_\tau$.

In the remainder of the proof of the sublemma we will omit n from subscripts. To prove (17), define intervals

$$J_0 = [-T-1, -T+\tau], \quad J_1 = [-T+\tau, T-\tau], \quad J_2 = [T-\tau, T+1]$$

and for $k = 0, 1, 2$ set $\gamma^{(k)} = \gamma|_{J_k}$. Let $\mathrm{Hol}^{(k)}$ denote holonomy along $\gamma^{(k)}$ in the same sense as (1), ie $\mathrm{Hol}^{(k)}$ is the result of replacing the domain of integration I_j in that formula with J_k. Define $\delta^{(k)} \in \mathbb{C}$ by

$$\mathrm{Hol}^{(k)}(\hat{S}) = \mathrm{Hol}^{(k)}(S')(1 + \delta^{(k)}), \quad k = 0, 2,$$
$$\mathrm{Hol}^{(1)}(\hat{S}) = \mathrm{Hol}^{(1)}(S'')(1 + \delta^{(1)})$$

where as usual we mean holonomy with respect to the connection parts of the configurations. For large τ the estimates (15) and (16) give

$$|\delta^{(k)}| \leq \text{const} \cdot e^{(2\sigma-\lambda)\tau}$$

when n is sufficiently large.

Writing $h=\prod_{k=0}^2(1+\delta^{(k)})$ we have

$$z=\mathrm{Hol}(\hat{S})=\prod_{k=0}^{2}\mathrm{Hol}^{(k)}(\hat{S})=h\,\mathrm{Hol}^{(0)}(S')\,\mathrm{Hol}^{(1)}(S'')\,\mathrm{Hol}^{(2)}(S').$$

Now, by the definition of holonomy,

$$\mathrm{Hol}^{(1)}(S'')=\frac{\tilde{v}(x^+)}{\tilde{v}(x^-)}\,\mathrm{Hol}^{(1)}(S),$$

and there are similar formulas for $\mathrm{Hol}^{(k)}(S')$. Because $w(x^+)=1$ we obtain

$$z=h\,\mathrm{Hol}(S)\,w(x^-)^{-1}.$$

Setting $a=\mathrm{Hol}(S)z^{-1}$ we get

$$w(x^-)-1=ah-1=(a-1)h+h-1.$$

Since by assumption $a\to 1$ as $n\to\infty$, we have

$$|w(x^-)-1|\le \mathrm{const}\cdot\left(|a-1|+\sum_k|\delta^{(k)}|\right)\le \mathrm{const}\cdot e^{(2\sigma-\lambda)\tau}$$

for large n, proving the sublemma and hence also Lemma 2.19. □

This completes the proof of Proposition 2.2 and thereby also the proof of Theorem 2.1.

2.4 Existence of maps q

Let $G\subset M_{\mathfrak{b}}$ be as in Subsection 2.1. In this subsection we will show that there is always a map q as in (3) provided $\mathbb{T}$ acts freely on $\overline{G}$ and K is sufficiently large. It clearly suffices to prove the same with $\mathcal{B}^*_{\mathfrak{b}}(K)$ in place of $\breve{\mathcal{B}}^*_{\mathfrak{b}}(K)$.

Let $\mathcal{B},M$ denote the configuration and moduli spaces over X with the same asymptotic limits as $\mathcal{B}_{\mathfrak{b}},M_{\mathfrak{b}}$, but using the full group of gauge transformations $\mathcal{G}$ rather than $\mathcal{G}_{\mathfrak{b}}$.

Because $\mathcal{G}_{\mathfrak{b}}$ acts freely on $\mathcal{C}$, an element in $\mathcal{B}_{\mathfrak{b}}$ has trivial stabilizer in $\mathbb{T}$ if and only if its image in $\mathcal{B}$ is irreducible, ie when its spinor field does not vanish identically on any component of X.

Throughout this subsection, K will be a compact codimension 0 submanifold of X which contains $\mathfrak{b}$ and intersects every component of X.

Proposition 2.3 *If $\mathbb{T}$ acts freely on $\overline{G}$ then for sufficiently large K there exist a $\mathbb{T}$–invariant open neighbourhood $V \subset \mathcal{B}^*_{\mathfrak{b}}(K)$ of $R_K(\overline{G})$ and a $\mathbb{T}$–equivariant smooth map $q : V \to M_{\mathfrak{b}}$ such that $q(\omega|_K) = \omega$ for all $\omega \in G$.*

We first prove three lemmas. Let $H \subset M^*$ be the image of G. Because $\mathbb{T}$ is compact, the projection $\mathcal{B}_{\mathfrak{b}} \to \mathcal{B}$ is a closed map and therefore maps $\overline{G}$ to $\overline{H}$. Let $H_0 \subset M^*$ be any precompact open subset which contains $\overline{H}$ and whose closure consists only of regular points.

Lemma 2.20 *If K is sufficiently large then $R_K : M^* \to \mathcal{B}^*(K)$ restricts to an immersion on an open neighbourhood of $\overline{H}_0$.*

By 'immersion' we mean the same as in [16]. Since a finite-dimensional subspace of a Banach space is always complemented, the condition in our case is simply that the derivative of the map be injective at every point.

Proof. Fix $\omega = [S] \in \overline{H}_0$. We will show that R_K is an immersion at ω (hence in a neighbourhood of ω) when K is large enough. Since $\overline{H}_0$ is compact, this will prove the lemma.

Let $W \subset L^{p,w}_1$ be a linear subspace such that the derivative at S of the projection $S + W \to \mathcal{B}^*$ is a linear isomorphism onto the tangent space of M at ω. Let δ denote that derivative. For $t \geq 0$ so large that $\mathfrak{b} \subset X_{:t}$ let δ_t be the derivative at S of the natural map $S + W \to \mathcal{B}^*(X_{:t})$. We claim that δ_t is injective for $t \gg 0$. For suppose $\{w_n\}$ is a sequence in W such that $\|w_n\|_{L^{p,w}_1} = 1$ and $\delta_{t_n}(w_n) = 0$ for each n, where $t_n \to \infty$. Set $K_n = X_{:t_n}$. Then

$$w_n|_{K_n} = \mathcal{I}_\Phi f_n$$

for some $f_n \in L\mathcal{G}(K_n)$, where Φ is the spinor field of S. After passing to a subsequence we may assume that w_n converges in $L^{p,w}_1$ to some $w \in W$ (since W is finite-dimensional). By [11, Lemma 2.1] there exists for each n a constant $C_n < \infty$ such that for all $h \in L\mathcal{G}(K_n)$ one has

$$\|h\|_{L^p_2} \leq C_n \|\mathcal{I}_\Phi h\|_{L^p_1}.$$

It follows that f_n converges in L^p_2 over compact subsets of X to some function f. We obviously have $\mathcal{I}_\Phi f = w$, hence $f \in L\mathcal{G}$ and $\delta(w) = 0$. But this is impossible, since w has norm 1. This proves the lemma. □

Lemma 2.21 *If K is sufficiently large then the restriction map $H_0 \to \mathcal{B}^*(K)$ is a smooth embedding.* □

Proof. Because of Lemma 2.20 it suffices to show that R_K is injective on $\overline{H}_0$ when K is large.

Suppose $\omega_n, \omega'_n \in \overline{H}_0$ restrict to the same element in $\mathcal{B}(X_{:t_n})$ for $n = 1, 2, \dots$, where $t_n \to \infty$. Since $\overline{H}_0$ is compact we may assume, after passing to a subsequence, that ω_n, ω'_n converge in $\overline{H}_0$ to ω, ω' respectively. As in the proof of Lemma 3.3 one finds that $\omega = \omega'$. When n is large then $\overline{H}_0 \to \mathcal{B}(X_{:t_n})$ will be injective in a neighbourhood of ω by Lemma 2.20, hence $\omega_n = \omega'_n$ for n sufficiently large. □

For the time being, we will call a Banach space E *admissible*, if $x \mapsto \|x\|^r$ is a smooth function on E for some $r > 0$. (The examples we have in mind are L^p_k Sobolev spaces where p is an even integer.)

Lemma 2.22 *Let B be any second countable (smooth) Banach manifold modelled on an admissible Banach space. Then any submanifold Z of B possesses a tubular neighbourhood (in the sense of [16]).*

Proof. According to [16, p 96], if a Banach manifold admits partitions of unity then any *closed* submanifold possesses a tubular neigbourhood. Now observe that Z is by definition locally closed, hence $C = \overline{Z} \setminus Z$ is closed in B. But then Z is a closed submanifold of $B \setminus C$. In general, any second countable, regular T_1–space is metrizable, hence paracompact (see [15]). Because $B \setminus C$ is modelled on an admissible Banach space, the argument in [16] carries over to show that $B \setminus C$ admits partitions of unity. Therefore, Z possesses a tubular neighbourhood in $B \setminus C$, which also serves as a tubular neighbourhood of Z in B. □

Proof of Proposition 2.3: Choose K so large that $H_0 \to \mathcal{B}^*(K)$ is an embedding, with image Z, say. Let G_0 denote the preimage of H_0 in $M_{\mathfrak{b}}$.

Let $\mathcal{B}^{**}_{\mathfrak{b}}(K)$ be the open subset of $\mathcal{B}_{\mathfrak{b}}(K)$ consisting of those elements whose spinor does not vanish identically on any component of K. Then the projection $\pi : \mathcal{B}^{**}_{\mathfrak{b}}(K) \to \mathcal{B}^*(K)$ is a principal $\mathbb{T}$–bundle, and restriction to K defines a diffeomorphism

$$\iota : G_0 \to \pi^{-1} Z.$$

By Lemma 2.22 there is an open neighbourhood U of H_0 in $\mathcal{B}^*(K)$ and a smooth map

$$\rho : U \times [0, 1] \to \mathcal{B}^*(K)$$

such that $\rho(x, 1) \in Z$ for all x, and $\rho(x, t) = x$ if $x \in Z$ or $t = 0$. (In other words, ρ is a strong deformation retraction of U to Z.) After choosing a connection in the $\mathbb{T}$–bundle $\mathcal{B}^{**}_{\mathfrak{b}}(K)$ we can then construct a $\mathbb{T}$–invariant smooth retraction

$$\tilde{\rho} : \pi^{-1}(U) \to \pi^{-1}(Z)$$

by means of holonomy along the paths $t \mapsto \rho(t, x)$. Now set

$$q = \iota^{-1} \circ \tilde{\rho} : \pi^{-1}U \to G_0. \qquad \square$$

3 Applications

3.1 A model application

In this subsection we will show in a model case how the gluing theorem may be applied in combination with the compactness results of [11]. Here we only consider gluing along irreducible critical points. Examples of gluing along reducible critical points will appear in [12, 13]. The main result of this subsection, Theorem 3.1, encompasses both the simplest gluing formulae for Seiberg–Witten invariants (in situations where reducibles are not encountered) and, as we will see in the next subsection, the formula $d \circ d = 0$ for the standard Floer differential.

Recall that the Seiberg–Witten invariant of a closed spin^c 4–manifold (with $b_2^+ > 1$) can be defined as the number of points (counted with sign) in the zero-set of a generic section of a certain vector bundle over the moduli space. To obtain a gluing formula, this vector bundle and its section should be expressed as the pull-back of a vector bundle $E \to \breve{\mathcal{B}}^*(K)$ with section s, where $K \subset X$. In the proof of Theorem 3.1 below we will see how the section s gives rise in a natural way to a map q as in Theorem 2.1. Thus, the section s is being incorporated into the equations that the gluing map is required to solve. (We owe this idea to [5, p 99].)

We will now describe the set-up for our model application. Let X be as in [11, Subsection 1.4] with $r = 1$ and $r' \geq 0$, and set $Y = Y_1$. In other words, we will be gluing one single pair of ends $\mathbb{R}_+ \times (\pm Y)$ of X, but X may have other ends $\mathbb{R}_+ \times Y'_j$ not involved in the gluing. We assume $X^\#$ is connected, which means that X has one or two connected components. For $j = 1, \dots, r'$ fix a critical point $\alpha'_j \in \tilde{\mathcal{R}}_{Y'_j}$. Let μ be a 2–form and $\mathfrak{p}$ a perturbation parameter for Y, and let $\mu'_j, \mathfrak{p}'_j$ be similar data for Y'_j. Let each $\mathfrak{p}, \mathfrak{p}'_j$ have small C^1 norm. To simplify notation we write, for $\alpha, \beta \in \tilde{\mathcal{R}}_Y$,

$$M_{\alpha,\beta} = M(X; \alpha, \beta, \vec{\alpha}'), \qquad M^{(T)} = M(X^{(T)}; \vec{\alpha}').$$

We make the following assumptions:

- (Compactness) At least one of the conditions (B1), (B2) of [11] holds for some $\lambda_j, \lambda'_j > 0$,
- (Regularity) All moduli spaces over $\mathbb{R}\times Y$, $\mathbb{R}\times Y'_j$, and X contain only regular points, and
- (No reducibles) Given $\alpha_1, \alpha_2 \in \tilde{\mathcal{R}}_Y$ and $\alpha'_j \in \tilde{\mathcal{R}}_{Y'_j}$, if there exist a broken gradient line over $\mathbb{R}\times Y$ from α_1 to α_2 and for each j a broken gradient line over $\mathbb{R}\times Y'_j$ from α'_j to β'_j then $M(X;\alpha_1,\alpha_2,\vec{\alpha}')$ contains no reducible. (It then follows by compactness that $M^{(T)}$ contains no reducible when T is large.)

The regularity condition is stronger than necessary, because there are energy constraints on the moduli spaces that one may encounter in the situation to be considered, but we will not elaborate on this here.

Note that we have so far only developed a full transversality theory in the case when Y and each Y'_j are rational homology spheres; in the remaining cases the discussion here is therefore somewhat theoretical at this time.

Let $K \subset X$ be a compact codimension 0 submanifold which intersects every component of X. When $T \gg 0$ then K may also be regarded as a submanifold of $X^{(T)}$, and we have restriction maps

$$R_{\alpha,\beta} : M^*_{\alpha,\beta} \to \breve{\mathcal{B}}^*(K), \qquad R' : M^{(T)} \to \breve{\mathcal{B}}^*(K).$$

These take values in $\breve{\mathcal{B}}^*(K)$ rather than just in $\breve{\mathcal{B}}(K)$ because of the unique continuation property of harmonic spinors.

Suppose $E \to \breve{\mathcal{B}}^*(K)$ is an oriented smooth real vector bundle whose rank d is equal to the (expected) dimension of $M^{(T)}$. Choose a smooth section s of E such that the pull-back section $s_{\alpha,\beta} = R^*_{\alpha,\beta}s$ is transverse to the zero-section of the pull-back bundle $E_{\alpha,\beta} = R^*_{\alpha,\beta}E$ over $M^*_{\alpha,\beta}$ for each pair α,β. (Here the Sobolev exponent $p > 4$ should be an even integer to ensure the existence of smooth partitions of unity.) Set $s' = (R')^*s$, which is a section of $E' = (R')^*E$. We write $M_\alpha = M_{\alpha,\alpha} = M^*_{\alpha,\alpha}$ and $s_\alpha = s_{\alpha,\alpha}$ etc. Let $\hat{M}_\alpha$, $\hat{M}^{(T)}$ denote the zero-sets of s_α, s' respectively. By index theory we have

$$0 = \dim \hat{M}^{(T)} = \dim \hat{M}_\alpha + n_\alpha,$$

where $n_\alpha = 0$ if α is irreducible and $n_\alpha = 1$ otherwise. Thus, $\hat{M}_\alpha$ is empty if α is reducible.

Lemma 3.1 *If $\omega_n \in \hat{M}^{(T(n))}$ for $n = 1,2,\dots,$ where $T(n) \to \infty$, then a subsequence of ω_n chain-converges to an element of $\hat{M}_\alpha$ for some $\alpha \in \tilde{\mathcal{R}}^*_Y$.*

Moreover, if $\omega_n = [S_n]$ chain-converges to $[S] \in \hat{M}_\alpha$ then there exists for each n a smooth $u_n : X^{(T(n))} \to U(1)$ whose restriction to each end $\mathbb{R}_+ \times Y'_j$ is null-homotopic and such that the sequence $u_n(S_n)$ c-converges over X to S.

Proof. The statement of the first sentence follows from [11, Theorem 1.4] by dimension counting. Such maps u_n exist in general for chain-convergent sequences when the ω_n all have the same asymptotic limits over the ends $\mathbb{R}_+ \times Y'_j$, see [11]. □

Let $J \subset H^1(Y;\mathbb{Z})$ be the subgroup consisting of elements of the form $z|_Y$ where z is an element of $H^1(X^\#;\mathbb{Z})$ satisfying $z|_{Y'_j} = 0$ for $j = 1, \dots, r'$. This group J acts on the disjoint union

$$\hat{M}^u = \bigcup_{\alpha \in \tilde{\mathcal{R}}^*_Y} \hat{M}_\alpha,$$

permuting the summands.

Lemma 3.2 *The quotient $\hat{M} = \hat{M}^u/J$ is a finite set.*

Proof. By [11, Theorem 1.3] any sequence $\omega_n \in \hat{M}_{\alpha_n}$, $n = 1, 2, \dots$ has a chain-convergent subsequence, and for dimensional reasons the limit (well-defined up to gauge equivalence) must lie in some moduli space $\hat{M}_\beta$. Furthermore, if ω_n chain-converges to an element in $\hat{M}_\beta$ then $\hat{M}_{\alpha_n}$ is contained in the orbit $J \cdot \hat{M}_\beta$ for $n \gg 0$. Therefore, each $\hat{M}_\alpha$ is a finite set, and only finitely many orbits $J \cdot \hat{M}_\alpha$ are non-empty. This is equivalent to the statement of the lemma. □

Note that J is the largest subgroup of $H^1(Y;\mathbb{Z})$ which acts on $\hat{M}^u$ in a natural way. On the other hand, if $\hat{M}^u$ is non-emtpy then, since $H^1(Y;\mathbb{Z})$ acts freely on $\tilde{\mathcal{R}}_Y$, only subgroups $J' \subset J$ of finite index have the property that $\hat{M}^u/J'$ is finite.

Lemma 3.3 *There is a compact codimension 0 submanifold $K_0 \subset X$ such that the restriction map $\hat{M} \to \mathcal{B}(K_0)$ is injective.*

Proof. Let $[S_j] \in \hat{M}_{\beta_j}$, $j = 1, 2$, where each S_j is in temporal gauge over the ends of X (and therefore decays exponentially). Suppose there exists a sequence of smooth gauge transformations $u_n : X_{:t_n} \to \mathrm{U}(1)$ where $t_n \to \infty$, such that $u_n(S_1) = S_2$ over $X_{:t_n}$. After passing to a subsequence we can arrange that u_n c-converges over X to some gauge transformation u with $u(S_1) = S_2$. If $t \gg 0$ then $u|_{\{t\}\times(\pm Y)}$ will both be homotopic to a smooth

$v : Y \to \mathrm{U}(1)$ with $v(\alpha_1) = \alpha_2$. Hence $\hat{M}_{\alpha_1}, \hat{M}_{\alpha_2}$ lie in the same J–orbit, and S_1, S_2 represent the same element of $\hat{M}$ by [11, Proposition 2.5 (iii)].

Thus we can take $K_0 = X_{:t}$ for $t \gg 0$. $\square$

Now fix K_0 as in Lemma 3.3 and with $K \subset K_0$. Let $\{b_1, \dots, b_m\}$ be the image of the restriction map $R_{K_0} : \hat{M} \to \breve{\mathcal{B}}(K_0)$. Choose disjoint open neighbourhoods $W_j \subset \breve{\mathcal{B}}(K_0)$ of the points b_j. If $T \gg 0$ then

$$R_{K_0}(\hat{M}^{(T)}) \subset \bigcup_j W_j$$

by Lemma 3.1. For such T we get a natural map

$$g : \hat{M}^{(T)} \to \hat{M}.$$

It is clear that if g' is the map corresponding to a different choice of K_0 and neighbourhoods W_j then $g = g'$ for T sufficiently large.

Theorem 3.1 *For sufficiently large T the following hold:*

(i) *Every element of $\hat{M}^{(T)}$ is a regular point in $M^{(T)}$ and a regular zero of s'.*

(ii) *g is a bijection.*

Proof. If $\hat{M}$ is empty then, by Lemma 3.1, $\hat{M}^{(T)}$ is empty as well for $T \gg 0$, and there is nothing left to prove.

We now fix b_j and for the remainder of the proof omit j from notation. (Thus $b = b_j$, $W = W_j$ etc.) We will show that for $T \gg 0$ the set

$$\hat{B}^{(T)} = \{\omega \in \hat{M}^{(T)} : \omega|_{K_0} \in W\}$$

consists of precisely one element, and that this element is regular in the sense of (i). This will prove the theorem.

By definition, b is the restriction of some $\omega_0 \in \hat{M}_\alpha$. Choose an open neighbourhood $V \subset \breve{\mathcal{B}}^*(K)$ of $b|_K$ and a smooth map

$$\pi : E|_V \to \mathbb{R}^d$$

which restricts to a linear isomorphism on every fibre. Choose an open neighbourhood $V_0 \subset W$ of b such that $R_K(V_0) \subset V$. Let $G_+ \subset M_\alpha$ be a precompact open neighbourhood of ω_0 such that $R_{K_0}(\overline{G}_+) \subset V_0$. The assumption that ω_0 be a regular zero of s_α means that the composite map

$$G_+ \overset{R_K}{\to} V \overset{\pi \circ s}{\to} \mathbb{R}^d$$

is a local diffeomorphism at ω_0. We can then find an injective smooth map

$$p : \mathbb{R}^d \to M_\alpha$$

such that $p \circ \pi \circ s \circ R_K = \mathrm{id}$ in some open neighbourhood $G \subset G_+$ of ω_0. In particular, $p^{-1}(\omega_0) = \{0\}$ and p is a local diffeomorphism at 0. Set

$$q = p \circ \pi \circ s : V \to M_\alpha.$$

By Theorem 2.1 there is a kv-pair $(K', V') \le (K_0, V_0)$ such that if $T \gg 0$ then $\mathbf{q}^{-1}G$ consists only of regular monopoles and

$$\mathbf{f} = q \circ R_K : \mathbf{q}^{-1}G \to G$$

is a diffeomorphism. By Lemma 3.1 one has

$$\hat{B}^{(T)} = \mathbf{q}^{-1}G \cap (s')^{-1}(0) = \mathbf{f}^{-1}(\omega_0)$$

for $T \gg 0$. For such T the set $\hat{B}^{(T)}$ consists of precisely one point, and this point is regular in the sense of (i). $\square$

3.2 The Floer differential

Consider the situation of [11, Subsection 1.2]. Suppose a perturbation parameter $\mathfrak{p}$ of small C^1 norm has been chosen for which all moduli spaces $M(\alpha_1, \alpha_2)$ over $\mathbb{R} \times Y$ are regular. (This is possible at least when Y is a rational homology sphere, see [11].) Fix $\alpha_1, \alpha_2 \in \tilde{\mathcal{R}}^*_Y$ with

$$\dim M(\alpha_1, \alpha_2) = 2.$$

We will show that the disjoint union

$$\check{M} := \bigcup_{\beta \in \tilde{\mathcal{R}}^*_Y \setminus \{\alpha_1, \alpha_2\}} \check{M}(\alpha_1, \beta) \times \check{M}(\beta, \alpha_2)$$

is the boundary of a compact 1–manifold. (In other words, the standard Floer differential d satisfies $d \circ d = 0$ at least with $\mathbb{Z}/2$ coefficients.) To this end we will apply Theorem 3.1 to the case when X consists of two copies of $\mathbb{R} \times Y$, say

$$X = \mathbb{R} \times Y \times \{1, 2\},$$

and we glue $\mathbb{R}_+ \times Y \times \{1\}$ with $\mathbb{R}_- \times Y \times \{2\}$. Thus $r = 1$, $r' = 2$. We take $K = K_1 \cup K_2$, where $K_j = [0, 1] \times Y \times \{j\}$. In this case, $\breve{\mathcal{B}}^*(K)$ is the quotient of $\mathcal{C}^*(K)$ by the null-homotopic gauge transformations. The bundle E over $\breve{\mathcal{B}}^*(K)$ will be the product bundle with fibre $\mathbb{R}^2$. To define the section s of E, choose $\delta_1, \delta_2 > 0$ such that ϑ has no critical value in the set

$$(\vartheta(\alpha_2), \vartheta(\alpha_2) + \delta_2] \cup [\vartheta(\alpha_1) - \delta_1, \vartheta(\alpha_1)).$$

This is possible because we assume Condition (O1) of [11]. For any configuration S over $[0,1] \times Y$ set

$$s_j(S) = \int_0^1 \vartheta(S_t)\, dt - \vartheta(\alpha_j) - (-1)^j \delta_j.$$

Note that $s_j(S)$ does not change if we apply a null-homotopic gauge transformation to S.

A configuration over K consists of a pair (S_1, S_2) of configurations over $[0,1] \times Y$. Define a smooth function $s : \breve{\mathcal{B}}^*(K) \to \mathbb{R}^2$ (ie a section of E) by

$$s([S_1],[S_2]) = (s_1(S_1), s_2(S_2)).$$

If $[S]$ belongs to some moduli space $M(\beta_1, \beta_2)$ over $\mathbb{R} \times Y$ with $\beta_1 \neq \beta_2$ in $\tilde{\mathcal{R}}_Y$ then $\frac{d}{dt}\vartheta(S_t) < 0$ for all t by choice of $\mathfrak{p}$. Since $J = 0$, the natural map $\hat{M} \to \check{M}$ is therefore a bijection.

Let s'_j be the pull-back of s_j to $M^{(T)}$. Here $M^{(T)}$ is defined using [11, Equation 17] with $\mathfrak{q} = 0$, and so can be identified with $M(\alpha_1, \alpha_2)$. By [11, Theorem 1.3] the set

$$Z^{(T)} = \{\omega \in M^{(T)} \,:\, s'_1(\omega) = 0, \quad s'_2(\omega) \leq 0\}$$

is compact for all $T > 0$. If $T \gg 0$ then, by Theorem 3.1, $Z^{(T)}$ is a smooth submanifold of $M^{(T)}$, and the composition of the two bijections

$$\partial Z^{(T)} = \hat{M}^{(T)} \xrightarrow{g} \hat{M} \to \check{M}$$

yields the desired identification of $\check{M}$ with the boundary of a compact 1–manifold.

4 Orientations

In this section we discuss orientations of moduli spaces and determine when the ungluing map of Theorem 2.1 preserves resp. reverses orientation.

We will adopt the approach to orientations of Fredholm operators (and families of such) introduced by Benevieri–Furi [1], which was brought to our attention by Shuguang Wang [25]. This approach is more economical than the traditional one using determinant line bundles (see [6, 22, 10]) in the sense that it produces the orientation double cover directly. It also fits well in with gluing theory.

After reviewing Benevieri–Furi orientations in Subsection 4.1 we study orientations of unframed and (multi)framed moduli spaces, and the relationship between these, in Subsection 4.2. The framings require some extra care because of reducibles. The orientation cover $\lambda \to \mathcal{B} = \mathcal{B}(X;\vec{\alpha})$ is defined by the family of Fredholm operators $\mathcal{I}^*_S + D\Theta_S$ parametrized by $S \in \mathcal{C}$ (cf. [5, p 130]). Any section of λ (which is always trivial, see Proposition 4.2 below) defines an orientation of the regular part of the moduli space $M^*_{\mathfrak{b}}$ for any finite, oriented subset $\mathfrak{b} \subset X$. If all limits α_j are reducible then any homology orientation of X determines a section of λ, see Proposition 4.1 below. To relate ungluing maps to orientations we show that, in the notation of Subsection 2.2, any section of $\lambda \to \mathcal{B}$ determines a section of the orientation cover $\lambda' \to \mathcal{B}'$. (Here $\mathcal{B}, \mathcal{B}'$ are configuration spaces over $X, X^{(T)}$, respectively.) This is explained in Subsection 4.4 after some preparation in Subsection 4.3 concerning framings. With this background material in place, the result on ungluing maps, Theorem 4.1, is an easy consequence of earlier estimates. Subsection 4.5 addresses the question of whether gluing of orientations in the above sense is compatible with gluing of homology orientations in the case when all limits α_j, α'_j are reducible.

4.1 Benevieri–Furi orientations of Fredholm operators

We first review Benevieri–Furi's concept of orientability of a Fredholm operator $L : E \to F$ of index 0 between real Banach spaces. A *corrector* of L is a bounded operator $A : E \to F$ with finite dimensional image such that $L + A$ is an isomorphism. We introduce the following equivalence relation in the set $\mathcal{C}(L)$ of correctors of L. Given $A, B \in \mathcal{C}(L)$ set

$$P = L + A, \quad Q = L + B.$$

Let F_0 be any finite dimensional subspace of F containing the image of $A - B$. Then QP^{-1} is an automorphism of F which maps F_0 into itself. We call A and B *equivalent* if the map $F_0 \to F_0$ induced by QP^{-1} is orientation preserving (which holds by convention if $F_0 = 0$). This condition is independent of F_0. The set $\mathcal{C}(L)$ is now partitioned into two equivalence classes (unless $E = F = 0$), and we define an *orientation* of L to be a choice of an equivalence class, the elements of which are then called *positive* correctors. A corrector which is not positive is called *negative*.

Benevieri–Furi consider $Q^{-1}P$ instead of QP^{-1}, but it is easy to see that this yields the same equivalence relation.

Note that the equivalence classes are open and closed subsets of $\mathcal{C}(L)$ with respect to the operator norm. To see this, observe that $\mathcal{C}(L)$ is open among the bounded operators $E \to F$ of finite rank. Therefore, if B is a corrector sufficiently close to a given corrector A, then $A_t = (1-t)A + tB$ is a corrector for $0 \le t \le 1$. Since $\mathrm{im}(A_t - A) \subset \mathrm{im}(A) + \mathrm{im}(B)$, it follows by continuity that the A_t are all equivalent. In particular, A and B are equivalent.

If $L : E \to F$ is a Fredholm operator of arbitrary index then for any non-negative integers m, n we can form the operator

$$L_{m,n} : E \oplus \mathbb{R}^m \to F \oplus \mathbb{R}^n, \quad (x, 0) \mapsto (Lx, 0). \tag{18}$$

If L has index 0 then for any m there is a canonical correspondence between orientations of L and orientations of $L_{m,m}$ such that if A is a positive corrector of L then $A \oplus I_{\mathbb{R}^m}$ is a positive corrector of $L_{m,m}$. If L has index $k \neq 0$ then we define an orientation of L to be an orientation (in the above sense) of $L_{0,k}$ (if $k > 0$) or $L_{-k,0}$ (if $k < 0$).

Note that if A is a corrector of $L_{m,n}$ where $n - m = \mathrm{index}(L)$, and C an automorphism of $\mathbb{R}^m$ then A is equivalent to $A \circ (I_E \oplus C)$ if and only if $\det(C) > 0$, and similarly for automorphisms of $\mathbb{R}^n$.

It is clear that orientations of two Fredholm operators determine an orientation of their direct sum. Also, a complex linear Fredholm operator carries a canonical orientation (in this case we replace $\mathbb{R}$ by $\mathbb{C}$ in (18) and the orientation is then given by any complex linear corrector).

We now consider families of Fredholm operators. Let $\mathbf{E}, \mathbf{F}$ be Banach vector bundles over a topological space T, with fibres $\mathbf{E}_t, \mathbf{F}_t$ over $t \in T$. (We require that these satisfy the analogues of the vector bundle axioms VB 1-3 in [16, pp. 41-2] in the topological category.) Let $L(\mathbf{E}, \mathbf{F})$ denote the Banach vector bundle over T whose fibre over t is the Banach space of bounded operators $\mathbf{E}_t \to \mathbf{F}_t$. Suppose h is a (continuous) section of $L(\mathbf{E}, \mathbf{F})$ such that $h(t) : \mathbf{E}_t \to \mathbf{F}_t$ is a Fredholm operator of index 0 for every $t \in T$. If $\mathbf{E}_t \neq 0$ for every t then there is a natural double cover $\tilde{h} \to T$, the *orientation cover* of h, whose fibre over t consists of the two orientations of $h(t)$. If $U \subset T$ is an open subset and a a section of $L(\mathbf{E}, \mathbf{F})$ such that $a(t)$ has finite rank for all $t \in U$ then a defines a trivialization of $\tilde{h}$ over the open set of those $t \in U$ for which $h(t) + a(t)$ is an isomorphism. An *orientation* of h is by definition a section of $\tilde{h}$. If instead each $h(t)$ has index $k \neq 0$ then we define the orientation cover $\tilde{h}$ and orientations of h by first turning h into a family of index 0 operators as above and then applying the definitions just given for such families.

Wang [25] established a 1–1 correspondence between orientations of any family of index 0 Fredholm operators (between fixed Banach spaces) and orientations of its determinant line bundle. While we will make no use of determinant line bundles in this paper, we need to fix our convention for passing between orientations of a Fredholm operator $L : E \to F$ of arbitrary index and orientations of its determinant line,

$$\det(L) = \Lambda^{\max} \ker(L) \otimes \Lambda^{\max} \mathrm{coker}(L)^*.$$

Set $n = \dim \ker(L)$ and $m = \dim \mathrm{coker}(L)$. Choose bounded operators $A_1 : E \to \mathbb{R}^n$ and $A_2 : \mathbb{R}^m \to F$ which induce isomorphisms

$$\tilde{A}_1 : \ker(L) \to \mathbb{R}^n, \quad \tilde{A}_2 : \mathbb{R}^m \to \mathrm{coker}(L).$$

Then $A = \left(\begin{smallmatrix} 0 & A_2 \\ A_1 & 0 \end{smallmatrix}\right)$ is a corrector of $L_{m,n}$ which also defines an isomorphism $J_A : \det(L) \to \mathbb{R}$. Moreover, two such correctors A, B are equivalent if and only if $J_A J_B^{-1}$ preserves orientation. (To see this, note that after altering A_j, B_j by automorphisms of $\mathbb{R}^m$ or $\mathbb{R}^n$ as appropriate one can assume that $\tilde{A}_j = \tilde{B}_j$, in which case $(1-t)A + tB$ is a corrector of $L_{m,n}$ for every $t \in \mathbb{R}$.) This provides a 1–1 correspondence between orientations of L and orientations of $\det(L)$.

4.2 Orientations of moduli spaces

In the situation of [11, Subsection 3.4] set

$$\mathcal{S} = L^{p,w}_1(X; i\Lambda^1 \oplus \mathbb{S}^+), \quad \mathcal{F}_1 = L^{p,w}(X; i\mathbb{R}), \quad \mathcal{F}_2 = L^{p,w}(X; i\Lambda^+ \oplus \mathbb{S}^-) \tag{19}$$

and consider the family of Fredholm operators

$$\delta_S = \mathcal{I}^*_\Phi + D\Theta_S : \mathcal{S} \to \mathcal{F} := \mathcal{F}_1 \oplus \mathcal{F}_2$$

parametrized by $S = (A, \Phi) \in \mathcal{C}(X; \vec{\alpha})$. This family is gauge equivariant in the sense that

$$\delta_{u(S)}(us) = u\delta_S(s)$$

for any $s \in L^{p,w}_1$, $u \in \mathcal{G}$, where as usual u acts trivially on differential forms and by complex multiplication on spinors. Therefore, $\mathcal{G}$ acts continuously on the orientation cover $\tilde{\delta}$ such that the projection $\tilde{\delta} \to \mathcal{C}(X; \vec{\alpha})$ is

$\mathcal{G}$–equivariant. The local slice theorem and Lemma 4.1 below then show that $\tilde{\delta}$ descends to a double cover $\lambda \to \mathcal{B}(X;\vec{\alpha})$.

(Note that in the situation of [11, Subsection 2.4], the local slice theorem for the group $\mathcal{G}$ at a reducible point $(A,0) \in \mathcal{C}$ is easily deduced from the version of [11, Proposition 2.6] with $\mathfrak{b}$ consisting of one point from each component of X where Φ vanishes a.e.)

Lemma 4.1 *Let the topological group G act continuously on the spaces $Z, \tilde{Z}$, and let $\pi : \tilde{Z} \to Z$ be a G–equivariant covering map. Suppose any point in Z has arbitrarily small open neighbourhoods U such that for any $z \in U$ the set*

$$\{g \in G \,:\, gz \in U\}$$

is connected. Then the natural map $\tilde{\pi} : \tilde{Z}/G \to Z/G$ is a covering whose pull-back to Z is canonically isomorphic to π. Pull-back defines a 1-1 correspondence between (continuous) sections of $\tilde{\pi}$ and G–equivariant sections of π. If in addition G is connected then any section of π is G–equivariant.

Proof. Let $p : Z \to Z/G$ and $q : \tilde{Z} \to \tilde{Z}/G$. If U is as in the lemma and s is a section of π over U then for all $z \in U$, $g \in G$ with $gz \in U$ one has

$$s(gz) = gs(z).$$

Hence s descends to a section of $\tilde{\pi}$ over $p(U)$. If in addition $\pi^{-1}U$ is the disjoint union of open sets V_j each of which is mapped homeomorphically onto U by π then $q(V_j) \cap q(V_k) = \emptyset$ when $j \neq k$. Moreover, $\cup_j q(V_j) = \tilde{\pi}^{-1}p(U)$. $\square$

The following proposition extends a well known result in the case when X is closed (see [18, 22]). A related result was proved in [20, Prop. 4.4.18].

Proposition 4.1 *If each Y_j is a rational homology sphere and each α_j is reducible then any homology orientation of X canonically determines a section of $\lambda \to \mathcal{B}(X;\vec{\alpha})$.*

Proof. We may assume $\vec{\mathfrak{p}} = 0$, since rescaling $\vec{\mathfrak{p}}$ yields a homotopy of families δ. For any $(A,0) \in \mathcal{C}(X;\vec{\alpha})$ the operator $\delta_{(A,0)}$ is the connected sum of the operators $-d^* + d^+$ and D_A. While the homology orientation of X determines an orientation of $-d^* + d^+$ (whose cokernel we identify with $H^0 \oplus H^+$ rather than $H^+ \oplus H^0$, where H^+ now denotes the space of self-dual closed L^2 2–forms on X), the family of complex linear operators D_A carries a natural orientation which is preserved by the action of $\mathcal{G}$. This yields a section of λ over the reducible part $\mathcal{B}^{\rm red} \subset \mathcal{B} = \mathcal{B}(X;\vec{\alpha})$. Since the map $([A,\Phi],t) \mapsto [A,t\Phi]$, $0 \le t \le 1$ is a deformation retraction of $\mathcal{B}$ to $\mathcal{B}^{\rm red}$, we also obtain a section of λ over $\mathcal{B}$. $\square$

Returning to the situation discussed before Lemma 4.1, a section of λ determines an orientation of the regular part of the moduli space $M^*(X;\vec{\alpha})$. As we will now explain, it also determines an orientation of the regular part of $M^*_{\mathfrak{b}}(X;\vec{\alpha})$ for any finite oriented subset $\mathfrak{b} \subset X$. (By an orientation of $\mathfrak{b}$ we mean an equivalence class of orderings, two orderings being equivalent if they differ by an even permutation.)

Let $\mathcal{W}$ be the space of spinors that may occur in elements of $\mathcal{B}^*_{\mathfrak{b}}(X;\vec{\alpha})$; more precisely, $\mathcal{W}$ is the open subset of $\Phi_o + L^{p,w}_1$ consisting of those elements Φ such that $\mathfrak{b} \cup \mathrm{supp}(\Phi)$ intersects every component of X. For any such Φ the operator

$$\Delta_\Phi := \mathcal{I}^*_\Phi \mathcal{I}_\Phi = \Delta + |\Phi|^2 : L\mathcal{G} \to \mathcal{F}_1 \tag{20}$$

is injective on $L\mathcal{G}_{\mathfrak{b}}$, hence

$$\mathcal{V}_\Phi := \mathcal{F}_1 / \Delta_\Phi(L\mathcal{G}_{\mathfrak{b}})$$

has dimension $b := |\mathfrak{b}|$ by [11, Proposition 2.2 (i)]. Since $\Phi \mapsto \Delta_\Phi$ is a smooth map from $\Phi_o + L^{p,w}_1$ into the space of bounded operators $L\mathcal{G} \to \mathcal{F}_1$, the spaces $\mathcal{V}_\Phi$ form a smooth vector bundle $\mathcal{V}$ over $\mathcal{W}$. Because $\mathcal{W}$ is simply-connected, $\mathcal{V}$ is orientable. To specify an orientation it suffices to consider those Φ that do not vanish identically on any component of X. Given such a Φ, the operator (20) is an isomorphism, and we decree that a b–tuple $g_1, \dots, g_b \in \mathcal{F}_1$ spanning a linear complement of $\Delta_\Phi(L\mathcal{G}_{\mathfrak{b}})$ is *positive* if the determinant of the matrix

$$\left(-i\Delta_\Phi^{-1}(g_j)(x_k)\right)_{j,k=1,\dots,b}$$

is positive, where $(x_1, \dots, x_b)$ is any positive ordering of $\mathfrak{b}$.

It is natural to ask what it means for $\{g_j\}$ to be a positive basis for $\mathcal{V}_\Phi$ when $\Phi = 0$. Subsection 4.6 answers this question in the case $b = 1$.

For the purpose of understanding ungluing maps it is convenient to introduce local slices for the action of $\mathcal{G}_{\mathfrak{b}}$ that are defined by compactly supported functions on X. Given $S = (A,\Phi) \in \mathcal{C}^*_{\mathfrak{b}}(X;\vec{\alpha})$, choose compactly supported smooth functions $g_j, h_j : X \to i\mathbb{R}$, $j = 1, \dots, b$, such that

$$\int_X g_j h_k = \delta_{jk}, \tag{21}$$

where δ_{jk} is the Kronecker symbol, and such that $(g_1, \dots, g_b)$ represents a positive basis for $\mathcal{V}_\Phi$. (Note that there is a preferred choice of h_k, which lies in the linear span of the g_j's.) We define the operator $\mu : \mathcal{F}_1 \to \mathcal{F}_1$ by

$$\mu f = f - \sum_{j=1}^{b} g_j \int_X f h_j. \tag{22}$$

Clearly, this is a projection operator whose kernel is spanned by $g_1, \dots, g_b$. Furthermore, μ restricts to an isomorphism

$$\Delta_\Phi(L\mathcal{G}_{\mathfrak{b}}) \to \mathrm{im}(\mu). \tag{23}$$

Set $\mathcal{I}^{\#}_\Phi = \mu \circ \mathcal{I}^*_\Phi$ and

$$\delta_{\mu,S} := \mathcal{I}^{\#}_\Phi + D\Theta_S : \mathcal{S} \to \mathrm{im}(\mu) \oplus \mathcal{F}_2. \tag{24}$$

After composing with the inverse of (23), $\mathcal{I}^{\#}_\Phi$ becomes an operator of the same kind as considered in [11, Subsection 3.4]. Therefore, the local slice theorem [11, Proposition 2.6] applies, and if S represents a regular point of $M^*_{\mathfrak{b}}(X;\vec{\alpha})$ then an orientation of $\delta_{\mu,S}$ defines an orientation of the tangent space $T_{[S]}M^*_{\mathfrak{b}}(X;\vec{\alpha})$.

We will now relate orientations of δ_S to orientations of $\delta_{\mu,S}$. For any imaginary valued function f on X let $\mu' f \in \mathbb{R}^b$ have coordinates $\int_X f h_j$, $j = 1, \dots, b$. Choose non-negative integers ℓ, m with $m - \ell = \mathrm{index}(\delta_S)$ and set

$$\begin{gathered} \nu : \mathcal{F}_1 \oplus \mathcal{F}_2 \oplus \mathbb{R}^m \xrightarrow{\approx} \mathrm{im}(\mu) \oplus \mathcal{F}_2 \oplus \mathbb{R}^b \oplus \mathbb{R}^m, \\ (x_1, x_2, y) \mapsto (\mu x_1, x_2, \mu' x_1, y). \end{gathered}$$

To any corrector C of $(\delta_S)_{\ell,m}$ we associate a corrector $\mathsf{C}_{\mathfrak{b}}$ of $(\delta_{\mu,S})_{\ell,b+m}$ given by

$$(\delta_{\mu,S})_{\ell,b+m} + \mathsf{C}_{\mathfrak{b}} = \nu \circ ((\delta_S)_{\ell,m} + \mathsf{C}).$$

The map $\mathsf{C} \mapsto \mathsf{C}_{\mathfrak{b}}$ clearly respects the equivalence relation for correctors and therefore defines a 1–1 correspondence between orientations of δ_S and orientations of $\delta_{\mu,S}$. Moreover, for gauge transformations u one has

$$(u\mathsf{C}u^{-1})_{\mathfrak{b}} = u\mathsf{C}_{\mathfrak{b}}u^{-1},$$

where u acts by multiplication on spinors and trivially on the other components.

If $[S]$ is a regular point of $M^*_{\mathfrak{b}}(X;\vec{\alpha})$ then the above defined correspondence between orientations of δ_S and orientations of $M^*_{\mathfrak{b}}(X;\vec{\alpha})$ at $[S]$ does not depend on the choice of the $2b$–tuple $g_1, \dots, g_b, h_1, \dots, h_b$, because the space of such $2b$–tuples supported in a given compact subset of X is path-connected in the C^∞–topology.

The relationship between the orientations of $M^* = M^*(X;\vec{\alpha})$ and $M^*_{\mathfrak{b}} = M^*_{\mathfrak{b}}(X;\vec{\alpha})$ can be described explicitly as follows (assuming M^* is regular). Let $M^{**}_{\mathfrak{b}}$ be the part of $M^*_{\mathfrak{b}}$ that lies above M^*. Then $\pi : M^{**}_{\mathfrak{b}} \to M^*$ is a principal $\mathrm{U}(1)^b$–bundle whose fibres inherit orientations from $\mathrm{U}(1)^b$. If $(v_1,\dots,v_b)$ is a positive basis for the vertical tangent space of $M^{**}_{\mathfrak{b}}$ at ω and $(v_{b+1},\dots,v_d)$ a $(d-b)$–tuple of elements of $T_\omega M^{**}_{\mathfrak{b}}$ which maps to a positive basis for $T_{\pi(\omega)}M^*$ then $(v_1,\dots,v_d)$ is a positive basis for $T_\omega M^{**}_{\mathfrak{b}}$.

4.3 Gluing and the Laplacian

We continue the discussion of the previous subsection, but we now consider the situation of Subection 2.1, so that the ends of X are labelled as in [11, Subsection 1.4] and $\mathfrak{b} \subset X$ is the set of start-points of the paths $\gamma_j^\pm$, $j = 1,\dots,r_0$.

We define the function spaces $\mathcal{S}'$ and $\mathcal{F}' = \mathcal{F}'_1 \oplus \mathcal{F}'_2$ over $X^{(T)}$ just as the corresponding spaces $\mathcal{S},\mathcal{F}$ etc over X, replacing the weight function w by κ. We also define the space $\mathcal{W}'$ of spinors over $X^{(T)}$ and the oriented vector bundle $\mathcal{V}' \to \mathcal{W}'$ in the same way as $\mathcal{V} \to \mathcal{W}$, using the same set $\mathfrak{b}$.

Let $S = (A,\Phi) \in \mathcal{C}$ be a configuration over X such that $S - S_o$ is compactly supported, where S_o is the reference configuration over X. For large $\check{T}$ consider the glued configuration $S' = (A',\Phi')$ over $X^{(T)}$; this is the smooth configuration over $X^{(T)}$ which agrees with S over $\mathrm{int}(X_{:T})$. (This notation will also be used in later subsections. For the time being we are only interested in the spinors.) Let $g = (g_1,\dots,g_b)$ be as in the previous subsection.

Lemma 4.2 *If g represents a positive basis for $\mathcal{V}_\Phi$ then g also represents a positive basis for $\mathcal{V}'_{\Phi'}$ when $\check{T}$ is sufficiently large.*

(In this lemma it is not essential that X be a 4–manifold or that $\mathbb{S}^+$ be a spinor bundle, one could just as well use the more general set-up in [11, Subsection 2.1], at least if $p > \dim X$.)

Proof. There is one case where the lemma is obvious, namely when Φ does not vanish on any component of X and $g_j = \Delta_\Phi f_j$, where f_j is compactly supported. We will prove the general case by deforming a given set of data Φ, g to one of this special form. We begin by establishing a version of the lemma where 'positive basis' is replaced by 'basis' and one considers compact families of such data Φ, g. To make this precise, choose $\rho > 0$

such that $\text{supp}(g_j) \subset X_{:\rho}$ for each j, and let $\check{T} > \rho$. Let $\Gamma' \subset L^p_1(X;\mathbb{S}^+)$ and $\Gamma'' \subset C^\infty(X;(i\mathbb{R})^b)$ be the subspaces consisting of those elements that vanish outside $X_{:\rho}$. Let Γ (resp. Γ_T) be the set of pairs $(\phi, g) \in \Gamma' \times \Gamma''$ such that $\Phi_o + \phi \in \mathcal{W}$ (whence $\Phi'_o + \phi \in \mathcal{W}'$) and such that g represents a basis for $\mathcal{V}_{\Phi_o+\phi}$ (resp. $\mathcal{V}'_{\Phi'_o+\phi}$).

Sublemma 4.1 *If K is any compact subset of Γ then $K \subset \Gamma_T$ for $\check{T} \gg 0$.*

Assuming the sublemma for the moment, choose $\phi \in \Gamma'$ such that on each component of X exactly one of Φ, ϕ is zero. Choose smooth functions $f_j : X \to i\mathbb{R}$, $j = 1, \dots, b$, which are supported in $X_{:\rho}$ and satisfy $f_j(x_k) = i\delta_{jk}$, where $(x_1, \dots, x_b)$ is a positive ordering of $\mathfrak{b}$. Choose a small $\epsilon > 0$ and set $\tilde{g}_j = \Delta_{\Phi+\epsilon\phi} f_j$. Choose a path $(\mathbf{\Phi}(t), \mathbf{g}(t))$, $0 \le t \le 1$ in Γ from (Φ, g) to $(\Phi + \epsilon\phi, \tilde{g})$ such that $\mathbf{g}(t) = g$ for $0 \le t \le \epsilon$ and

$$\mathbf{\Phi}(t) = \begin{cases} \Phi + t\phi, & 0 \le t \le \epsilon, \\ \Phi + \epsilon\phi, & \epsilon \le t \le 1. \end{cases}$$

Let $\mathbf{\Phi}'(t)$ be the glued spinor over $X^{(T)}$ obtained from $\mathbf{\Phi}(t)$. By the sublemma, if $\check{T} \gg 0$ then for $0 \le t \le 1$ one has $(\mathbf{\Phi}'(t), \mathbf{g}(t)) \in \Gamma_T$. Since $\tilde{g}$ represents a positive basis for $\mathcal{V}'_{\Phi'+\epsilon\phi}$, it follows by continuity that g must represent a positive basis for $\mathcal{V}'_{\Phi'}$. This proves the lemma assuming the sublemma.

Proof of Sublemma 4.1: Suppose to the contrary that for $n = 1, 2, \dots$ there are $(\phi(n), g(n)) \in K \setminus \Gamma_{T(n)}$, where $\check{T}(n) \to \infty$. We may assume $(\phi(n), g(n)) \to (\phi, g)$ in K. Let V be the linear span of $g_1, \dots, g_b$ and V_n the linear span of $g_1(n), \dots, g_b(n)$. Set

$$\Phi'_n = \Phi'_o + \phi(n).$$

By assumption there exists a non-zero $f_n \in L\mathcal{G}'_{\mathfrak{b}}$ with $\Delta_{\Phi'_n} f_n \in V_n$. Choose real numbers σ, τ with $\rho < \sigma < \tau$. Since $\Delta_{\Phi'_o} f_n = 0$ outside $X_{:\rho}$, unique continuation implies that f_n cannot vanish identically on $X_{:\tau}$, so we may assume that

$$\|f_n\|_{L^p_2(X_{:\tau})} = 1.$$

We digress briefly to consider an injective bounded operator $J : E \to F$ between normed vector spaces and for fixed m a sequence of linear maps $P_n : \mathbb{R}^m \to E$ which converges in the operator norm to an injective linear map P. Then there is a constant $C < \infty$ such that $\|e\| < C\|Je\|$ for all e in a neighbourhood U of $P(S^{m-1})$. For large n one must have $P_n(S^{m-1}) \subset U$, hence

$$\|P_n x\| \le C\|JP_n x\|$$

for all $x \in \mathbb{R}^m$.

We apply this result with $m = b$, $E = L^p_1(X)$, $F = L^p(X)$, J the inclusion map, $P_n x = \sum_j x_j g_j(n)$, and $Px = \sum_j x_j g_j$. We conclude that there is a constant $C < \infty$ such that for sufficiently large n one has

$$\|v\|_{L^p_1} \le C\|v\|_{L^p}$$

for all $v \in V_n$. For such n,

$$\begin{aligned}
\|f_n\|_{L^p_3(X:\sigma)} &\le \text{const}\cdot\left(\|\Delta f_n\|_{L^p_1} + \|f_n\|_{L^p_2}\right)\\
&\le \text{const}\cdot\left(\|\Delta_{\Phi'_n} f_n\|_{L^p_1} + \||\Phi'_n|^2 f_n\|_{L^p_1} + 1\right)\\
&\le \text{const}\cdot\left(\|\Delta_{\Phi'_n} f_n\|_{L^p} + \|\Phi'_n\|^2_{L^p_1}\|f_n\|_{L^p_1} + 1\right)\\
&\le \text{const}\cdot(\|\Delta f_n\|_{L^p} + 1)\\
&\le \text{const},
\end{aligned}$$

where except in the first term all norms are taken over $X_{:\tau}$.

Let ψ_j, ψ'_j be the spinor parts of α_j, α'_j, respectively. Fix n for the moment and write $\bar\rho = T_j(n) - \rho$. Define $\bar\sigma$ similarly. Over $[-\bar\rho, \bar\rho] \times Y_j$ we then have

$$(-\partial_1^2 + \Delta_{\psi_j})f_n = 0,$$

where $\partial_1 = \frac{\partial}{\partial t}$ and $\Delta_{\psi_j} = \Delta_{Y_j} + |\psi_j|^2$. If h is any continuous real function on $[-\bar\rho, \bar\rho] \times Y_j$ satisfying $(-\partial_1^2 + \Delta_{\psi_j})h = 0$ on $(-\bar\rho, \bar\rho) \times Y_j$ then for any non-negative integer k and $t \in [-\bar\sigma + 1, \bar\sigma - 1]$ one has

$$\|h\|_{C^k([t-1,t+1]\times Y_j)} \le \text{const}\cdot\left(\|h\|_{L^2(\{-\bar\rho\}\times Y_j)} + \|h\|_{L^2(\{\bar\rho\}\times Y_j)}\right),$$

where the constant is independent of n, t. (To see this, expand h in terms of eigenvectors of Δ_{ψ_j} and note that each coefficient function c satisfies an equation $c'' = \lambda^2 c$, $\lambda \in \mathbb{R}$, which yields $(c^2)'' = 2(c')^2 + 2(\lambda c)^2 \ge 0$. Combine this with the usual elliptic estimates.) Similarly, if h is any bounded continuous function on $[\rho, \infty) \times Y'_j$ satisfying $(-\partial_1^2 + \Delta_{\psi'_j})h = 0$ on $(\rho, \infty) \times Y'_j$ then for any non-negative integer k and $t \ge \sigma$ one has

$$\|h\|_{C^k([t,t+1]\times Y'_j)} \le \text{const}\cdot\|h\|_{L^2(\{\rho\}\times Y'_j)},$$

for some constant independent of t.

After passing to a subsequence, we may therefore assume that f_n converges in L^p_2 over compact subsets of X to some function f, whose restriction to each end of X must be the sum of a constant function and an exponentially decaying one, the constant function being zero if the limiting spinor over that end (ψ_j or ψ'_j) is non-zero. In particular, $f \in L\mathcal{G}_{\mathfrak{b}}$. Furthermore,

$$\Delta_{\Phi_o+\phi} f \in V, \qquad \|f\|_{L^p_2(X_{:\tau})} = 1.$$

Since $\Delta_{\Phi_o+\phi}$ is injective on $L\mathcal{G}_{\mathfrak{b}}$ this contradicts the assumption that V is a linear complement of $\Delta_{\Phi_o+\phi}(L\mathcal{G}_{\mathfrak{b}})$ in $\mathcal{F}_1$.

This completes the proof of Sublemma 4.1 and thereby also the proof of Lemma 4.2. $\square$

4.4 Orientations and gluing

Let S, S' be as in the beginning of Subsection 4.3. For the time being we will consider a map μ defined by fixed but arbitrary b–tuples $\{g_j\}$, $\{h_j\}$ of imaginary valued, compactly supported, smooth functions on X satisfying the duality relation (21), where b is any non-negative integer. We will show that an orientation of $\delta_{\mu,S}$ canonically determines an orientation of $\delta_{\mu,S'}$ for large $\check{T}$. Set

$$\mathcal{F}_\mu = \mathrm{im}(\mu) \oplus \mathcal{F}_2, \quad \mathcal{F}'_\mu = \mathrm{im}(\mu) \oplus \mathcal{F}'_2.$$

Choose $\tau > 1$ so large that the functions g_j, h_j are all supported in $X_{:\tau}$, and define

$$\mathcal{S}_{:\tau} = L^p_1(X_{:\tau}; i\Lambda^1 \oplus \mathbb{S}^+).$$

Let $\mathcal{S}^{:\tau}$ be the subspace of $\mathcal{S}$ consisting of those elements that are supported in $X_{:\tau}$, and define $\mathcal{F}^{:\tau}_\mu \subset \mathcal{F}_\mu$ similarly. Set

$$\mathcal{C}^{:\tau} = S_o + \mathcal{S}^{:(\tau-1)}.$$

In other words, $\mathcal{C}^{:\tau}$ is the set of all $L^p_{1,\mathrm{loc}}$ configurations S over X such that $S - S_o$ is supported in $X_{:(\tau-1)}$. (The $\tau - 1$ is chosen here because of the non-local nature of our perturbations.) Suppose we are given a bounded operator

$$\mathsf{C} : \mathcal{S}_{:\tau} \oplus \mathbb{R}^\ell \to \mathcal{F}^{:\tau}_\mu \oplus \mathbb{R}^m \tag{25}$$

with finite dimensional image, where $m - \ell = \mathrm{index}(\delta_{\mu,S})$. Clearly, C induces linear maps

$$\mathcal{S} \oplus \mathbb{R}^\ell \to \mathcal{F}_\mu \oplus \mathbb{R}^m, \quad \mathcal{S}' \oplus \mathbb{R}^\ell \to \mathcal{F}'_\mu \oplus \mathbb{R}^m$$

(the latter when $\check{T} \geq \tau$); these will also be denoted by C. Fix an r_0–tuple of paths $\gamma = (\gamma_1, \dots, \gamma_{r_0})$ as in Subsection 2.1, and for any imaginary valued 1–form a on $X^{(T)}$ let $H_\gamma(a) \in \mathbb{R}^{r_0}$ have coordinates

$$H_{\gamma_j}(a) := \int_{\gamma_j} ia, \quad j = 1, \dots, r_0.$$

Lemma 4.3 *There exists a constant $C < \infty$ with the property that if C is any map as above and S any element of $\mathcal{C}^{:\tau}$ such that*

$$D := \delta_{\mu,S} + \mathsf{C} : \mathcal{S} \oplus \mathbb{R}^\ell \to \mathcal{F}_\mu \oplus \mathbb{R}^m$$

is invertible, then

$$E := \delta_{\mu,S'} + H_\gamma + \mathsf{C} : \mathcal{S}' \oplus \mathbb{R}^\ell \to \mathcal{F}'_\mu \oplus \mathbb{R}^{r_0} \oplus \mathbb{R}^m$$

is invertible when $\check{T} > C(\|D^{-1}\| + 1)$.

Proof. Let P_j be a bounded right inverse of the operator (6). As in Appendix A, if $\check{T} > \text{const}\cdot(\|D^{-1}\| + \sum_j \|P_j\|)$ then we can splice $D^{-1}, P_1, \dots, P_r$ to obtain a right inverse R of

$$\delta_{\mu,S'} + \mathsf{C} : \mathcal{S}' \oplus \mathbb{R}^\ell \to \mathcal{F}'_\mu \oplus \mathbb{R}^m.$$

(The present situation is slightly different from that in the appendix, but the construction there carries over.) Furthermore,

$$\|R\| \leq \text{const} \cdot (\|D^{-1}\| + \sum_j \|P_j\|).$$

Let $\mathsf{f} : \mathbb{R} \to \mathbb{R}$ be a smooth function such that $\mathsf{f}(t) = 0$ for $t \leq \frac{1}{2}$ and $\mathsf{f}(t) = 1$ for $t \geq 1$. Set

$$q_j(t) = \mathsf{f}(T_j - \tau + t)\mathsf{f}(T_j - \tau - t).$$

Thus, q_j approximates the characteristic function of the interval $[-T_j + \tau, T_j - \tau]$. For any $c = (c_1, \dots, c_{r_0}) \in \mathbb{R}^{r_0}$ let $\eta(c)$ be the imaginary valued 1–form on $X^{(T)}$ given by

$$\eta(c) = \begin{cases} 0 & \text{outside } \bigcup_{j=1}^{r_0} [-T_j, T_j] \times Y_j, \\ -(2T_j)^{-1} c_j q_j i \, dt & \text{on } [-T_j, T_j] \times Y_j, \quad j = 1, \dots, r_0. \end{cases}$$

For the present purposes it is convenient to rearrange summands and regard E as mapping into $(\mathcal{F}'_\mu \oplus \mathbb{R}^m) \oplus \mathbb{R}^{r_0}$. Set

$$L = R + \eta : (\mathcal{F}'_\mu \oplus \mathbb{R}^m) \oplus \mathbb{R}^{r_0} \to \mathcal{S}' \oplus \mathbb{R}^\ell.$$

Then EL takes the matrix form

$$\begin{pmatrix} I & 0 \\ \beta & I \end{pmatrix} + o, \tag{26}$$

where for large $\check{T}$ one has $\|\beta\| \leq \text{const} \cdot \|R\|$ and $\|o\| \leq \text{const} \cdot \check{T}^{-1}$, the constants being independent of S, T. As in the proof of Lemma 2.4 we conclude that EL is invertible when $\check{T} > \text{const} \cdot (\|\beta\| + 1)$, which holds if $\check{T} > \text{const} \cdot (\|D^{-1}\| + 1)$. Since E has index 0, it is invertible whenever EL is surjective. $\square$

Lemma 4.4 *Suppose $\mathsf{C}, \tilde{\mathsf{C}}$ are two maps as in (25) which define correctors of $(\delta_{\mu,S})_{\ell,m}$, and let $\gamma, \tilde{\gamma}$ be two r_0–tuples of paths as in Subsection 2.1. Then for sufficiently large $\check{T}$ the following holds: C and $\tilde{\mathsf{C}}$ define equivalent correctors of $(\delta_{\mu,S})_{\ell,m}$ if and only if $H_\gamma + \mathsf{C}$ and $H_{\tilde{\gamma}} + \tilde{\mathsf{C}}$ define equivalent correctors of $(\delta_{\mu,S'})_{\ell,r_0+m}$.*

Proof. We will use the same notation as in Lemma 4.3 and its proof. Let $\tilde{D}, \tilde{E}$ be defined as D, E, replacing C, γ by $\tilde{\mathsf{C}}, \tilde{\gamma}$. Observe that the image of $D - \tilde{D}$ is contained in $N \oplus \mathbb{R}^m$ for some finite dimensional subspace $N \subset \mathcal{F}^{:\tau}_\mu$, and the image of $E - \tilde{E}$ is then contained in $(N \oplus \mathbb{R}^m) \oplus \mathbb{R}^{r_0}$ (again rearranging summands). Moreover,

$$\tilde{E}E^{-1} = \tilde{E}L(EL)^{-1},$$

and $\tilde{E}L$ has the form

$$\begin{pmatrix} \tilde{D}'R & 0 \\ \beta_1 & I \end{pmatrix} + o_1, \tag{27}$$

where $\|\beta_1\|$ is bounded and $\|o_1\| \to 0$ as $\check{T} \to \infty$, and $\tilde{D}' = \delta_{\mu,S'} + \tilde{\mathsf{C}}$. From the description (26) of EL we see that $\tilde{E}E^{-1}$ also has the shape (27).

If $s \in \mathcal{F}^{:\tau}_\mu$ and ρ denotes restriction to $X_{:\tau}$, then

$$\begin{aligned} \|\rho\tilde{D}'Rs - \rho\tilde{D}D^{-1}s\|_{L^{p,w}} &\leq \text{const} \cdot \|\tilde{D}\| \cdot \|\rho Rs - \rho D^{-1}s\|_{L^{p,w}} \\ &\leq \text{const} \cdot \|\check{T}^{-1}\| \cdot \|\tilde{D}\| \cdot \Big(\|D^{-1}\| + \sum \|P_j\|\Big) \cdot \|s\|. \end{aligned}$$

It follows that as $\check{T} \to \infty$, the determinant of the endomorphism of $(N \oplus \mathbb{R}^m) \oplus \mathbb{R}^{r_0}$ induced by $\tilde{E}E^{-1}$ approaches the determinant of the endomorphism of $N \oplus \mathbb{R}^m$ induced by $\tilde{D}D^{-1}$. $\square$

Consider again the situation before Lemma 4.3. Choosing a corrector C of $(\delta_{\mu,S})_{\ell,m}$ of the kind (25) we obtain, by Lemma 4.3, an orientation of $(\delta_{\mu,S'})_{\ell,r_0+m}$ for $\check{T} \gg 0$, which we can extend by continuity to $\check{T} > 2\tau$. Now fix T with $\check{T} > 2\tau$, and let $\lambda_{\mu,S}$ (resp. $\lambda_{\mu,S'}$) denote the set consisting of the two orientations of $\delta_{\mu,S}$ (resp. $\delta_{\mu,S'}$). From Lemmas 4.3 and 4.4 we obtain a natural map

$$\lambda_{\mu,S} \to \lambda_{\mu,S'}.$$

There are two cases that we are interested in: One is when $b = 0$ (so that $\lambda_{\mu,S} = \lambda_S$). The other is when $b = |\mathfrak{b}|$ and $(g_0, \dots, g_b)$ defines a positive basis for $\mathcal{V}_\Phi$. Now letting μ refer to the second case, the preceeding discussion yields the following diagramme of bijections:

$$\begin{array}{ccc} \lambda_S & \longrightarrow & \lambda_{S'} \\ \downarrow & & \downarrow \\ \lambda_{\mu,S} & \longrightarrow & \lambda_{\mu,S'}, \end{array} \tag{28}$$

which commutes if and only if br_0 is even. (The reason for the sign is that one has to permute the summands $\mathbb{R}^b$ and $\mathbb{R}^{r_0}$ in the target spaces of the correctors.)

Turning now to the global picture, and taking $b = 0$, let $\mathcal{C}'_{[\tau]}$ denote the set of all $L^p_{1,\text{loc}}$ configurations $\tilde{S}$ over $X^{(T)}$ such that $\tilde{S} - S'_o$ is supported in $X_{:\tau}$. Then the gluing operation $S \mapsto S'$ defines a homeomorphism $u : \mathcal{C}^{:\tau} \to \mathcal{C}'_{[\tau]}$, and Lemmas 4.3 and 4.4 establish an isomorphism between the orientation cover of $\mathcal{C}^{:\tau}$ and the pull-back by u of the orientation cover of $\mathcal{C}'_{[\tau]}$. Combining this with Proposition 4.2 below we see that any section of $\lambda \to \mathcal{B}$ determines a section of the orientation cover $\lambda' \to \mathcal{B}'$. (Here $\mathcal{B}, \mathcal{B}'$ mean the same as in the beginning of Subsection 2.2).

Proposition 4.2 *If $X, \vec{\alpha}$ are as in [11, Subsection 3.4] then the orientation cover $\lambda \to \mathcal{B}(X; \vec{\alpha})$ is trivial.*

Proof. We may assume X is connected. Let $\mathfrak{b} \subset X$ consist of a single point. Let $\pi : \mathcal{B}_\mathfrak{b} \to \mathcal{B}$ be the projection, where $\mathcal{B} = \mathcal{B}(X; \vec{\alpha})$ etc. Since $\mathcal{B}$ is the quotient of $\mathcal{B}_\mathfrak{b}$ by the natural U(1) action, the local slice theorem and Lemma 4.1 imply that any section of $\pi^*\lambda$ descends to a section of λ. It therefore suffices to show that $\pi^*\lambda$ is trivial, or equivalently, that for any loop ℓ in $\mathcal{B}$ that lifts to $\mathcal{B}_\mathfrak{b}$ the pull-back $\ell^*\lambda$ is trivial. Since $\mathcal{C} \to \mathcal{B}_\mathfrak{b}$ is a (principal) fibre bundle, such a loop is the image of a path $z : [0,1] \to \mathcal{C}$ such that $z(1) = u(z(0))$ for some $u \in \mathcal{G}_{\mathfrak{b}}$. After altering the loop ℓ by a homotopy one can arrange that $u = 1$ and $z(t) = S_o$ (for all t) outside a compact subset of X.

(Here is one way to construct such a homotopy. For $0 \le s \le 1$ let $\xi_s = \zeta_s * \tilde{\zeta}_s$ be the composite of the two paths (both defined for $0 \le t \le 1$)

$$\zeta_s(t) = (1-s)z(t) + sS_o,$$
$$\tilde{\zeta}_s(t) = (1-t)\zeta_s(1) + tv_s(\zeta_s(0)),$$

where v_s is a path in $\mathcal{G}$ such that $v_0 = u$, and $v_1 = 1$ outside a compact subset of X. Clearly, $\xi_0 = \zeta_0$ is homotopic to z relative to $\{0,1\}$. Moreover, $v_s(\xi_s(0)) = \xi_s(1)$, and $\xi_1(t) = S_o$ where $v_1 = 1$.)

Now let $-X$ be the Riemannian manifold X with the opposite orientation and corresponding spin^c structure. Starting with $X \cup (-X)$ we form, for any $T > 0$, a compact manifold $W^{(T)}$ by gluing the j'th end of X with the j'th end of $-X$ to obtain a neck $[-T_j, T_j] \times Y_j$. Let S be any configuration over $-X$ which agrees with $\underline{\alpha}_j$ over the j'th end, and $z_1(t)$ the configuration over $W^{(T)}$ obtained by gluing S and $z(t)$. Then z_1 maps to a loop ℓ_1 in $\mathcal{B}(W^{(T)})$. By Proposition 4.2 the orientation cover $\lambda_1 \to \mathcal{B}(W^{(T)})$ is trivial. Now Lemmas 4.3 and 4.4 yield an isomorphism of $\mathbb{Z}/2$–bundles $\ell^*\lambda \to \ell_1^*\lambda_1$ when $\check{T}$ is large, hence $\ell^*\lambda$ is trivial. □

We now consider the situation of Theorem 2.1. Let $b = |\mathfrak{b}|$. Choose an orientation of $\lambda \to \mathcal{B}$, and let $\lambda' \to \mathcal{B}'$ have the associated orientation. Given an orientation of $\mathfrak{b}$, this orients the regular parts of $M_{\mathfrak{b}}$ and $M_{\mathfrak{b}}^{(T)}$. In terms of Diagramme 28 we are here using the top horizontal and vertical maps. Because the proof of the next theorem will use the bottom horizontal map, and the diagramme commutes if and only if br_0 is even, this causes a sign in the theorem. Note that b is the cardinality of the set $\mathfrak{b}$ of starting-points of the paths $\gamma_j^{\pm}$, which of course is even if these points are distinct.

Theorem 4.1 *In the situation of Theorem 2.1, if $\check{T}$ is sufficiently large then the diffeomorphism*

$$\mathbf{F} : \mathbf{q}^{-1}G \to U(1)^{r_0} \times G, \quad \omega \mapsto (Hol(\omega), \mathbf{q}(\omega))$$

is orientation preserving if br_0 is even and orientation reversing if br_0 is odd.

Proof. Given $\epsilon = \pm 1$, we will say that a map is ϵ*–preserving* if it changes orientations by the factor ϵ. A corrector of an oriented Fredholm operator of index 0 is called an ϵ–*corrector* if it is positive or negative according to the sign of ϵ.

Now set $\epsilon = (-1)^{br_0}$. In view of Proposition 2.1 it suffices to show that, for any given point $(z,\omega) \in \mathrm{U}(1)^{r_0} \times G$, the inverse $\mathbf{F}^{-1}$ is ϵ–preserving at (z,ω) when $\check{T}$ is sufficiently large.

Consider the set-up in Subsection 2.2, with $\varpi : \mathbb{R}^d \to M_{\mathfrak{b}}$ orientation preserving. Let $\pi : \mathcal{C}(K) \to \breve{\mathcal{B}}(K)$ be the projection. Then $f := \varpi^{-1} \circ q \circ \pi$ maps a small neighbourhood of $S_0|_K$ in $\mathcal{C}(K)$ to $\mathbb{R}^d$. Let

$$\mathsf{C} : L^p_1(K; i\Lambda^1 \oplus \mathbb{S}^+) \to \mathbb{R}^d$$

be the derivative of f at S_0. Let μ be as in (22), with Φ the spinor part of S_0. For $0 \le t \le 1$ set

$$S(t) = (1-t)S_0 + tS_{0,\tau-2}, \quad \delta(t) = \delta_{\mu,S(t)},$$
$$\hat{S}(t) = (1-t)\hat{S} + tS_{0,\tau-2,T}, \quad \delta'(t) = \delta_{\mu,\hat{S}(t)}.$$

(Thus, the τ in the proof of Theorem 2.1 corresponds to the present $\tau - 2$.) In the following, constants will be independent of τ, T. Because $q(\omega'|_K) = \omega'$ for all $\omega' \in G$, we see that C defines a positive corrector of

$$\delta(t)_{0,d} : \mathcal{S} \to \mathcal{F}_\mu \oplus \mathbb{R}^d$$

for $t = 0$. Hence, if $\tau >$ const (for a suitable constant) then C will define a positive corrector of $\delta(t)_{0,d}$ for $0 \le t \le 1$. We want to show that if $\tau >$ const then

$$E_t := \delta'(t) + H_\gamma + \mathsf{C} : \mathcal{S}' \to \mathcal{F}'_\mu \oplus \mathbb{R}^{r_0} \oplus \mathbb{R}^d$$

is an isomorphism for $0 \le t \le 1$ when $\check{T} \gg 0$. This is a Fredholm operator of index 0, so it suffices to show that it is surjective. As in the proof of Lemma 4.3 we can, for $\tau >$ const and $\check{T} > \tau +$ const, construct a right inverse R of

$$\delta'(1) + \mathsf{C} : \mathcal{S}' \to \mathcal{F}'_\mu \oplus \mathbb{R}^d$$

such that $\|R\|$ is bounded independently of τ, T. Set $L = R + \eta$ as in the said proof. For notational convenience we will here regard $E_t L$ as acting on

$$\left(\mathcal{F}'_\mu \oplus \mathbb{R}^d\right) \oplus \mathbb{R}^{r_0}.$$

Then there is the matrix representation

$$E_t L = \begin{pmatrix} (\delta'(t)) + \mathsf{C})R & \delta'(t)\eta \\ H_\gamma R & H_\gamma \eta \end{pmatrix}.$$

By Lemma 2.9 one has, for $\tau > \text{const}$,

$$\begin{aligned}\|(\delta'(t)+\mathsf{C})R - I\| &= \|(\delta'(t)-\delta'(1))R\| \\ &\le \text{const}\cdot\|\hat{S}(t)-\hat{S}(1)\|_{L^{p,\kappa}_1} \le \text{const}\cdot e^{(2\sigma-\lambda)\tau}.\end{aligned}$$

Furthermore, for $\tau > \text{const}$,

$$\begin{aligned}\|\delta'(t)\eta\| &\le \text{const}\cdot e^{\sigma\tau}\check{T}^{-1}, \\ \|H_\gamma R\| &\le \text{const}, \\ \|H_\gamma\eta - I\| &\le (\tau - \text{const})\cdot\check{T}^{-1}.\end{aligned}$$

Recalling the assumption $0 \le 6\sigma < \lambda$, we see that if $\tau > \text{const}$ then $E_t L$ (and hence E_t) will be invertible for $0 \le t \le 1$ when $\check{T} \gg 0$. Since $H_\gamma + \mathsf{C}$ is an ϵ–corrector of $\delta'(1)_{0,r_0+d}$, it must also be an ϵ–corrector of $\delta'(0)_{0,r_0+d}$, which in turn is equivalent to $\mathbf{F}$ being ϵ–preserving at $\mathbf{F}^{-1}(z,\omega)$. □

4.5 Homology orientations and gluing

In this subsection we will describe the "gluing of orientations" of Subsection 4.4 in terms of homology orientations in the simplest cases. This result will be needed in [12].

Let X be as in [11, Subsection 1.4] with $r = 1$, ie. only one pair of ends $\mathbb{R}_+ \times (\pm Y)$ is being glued. Suppose Y and each Y'_j are rational homology spheres. We assume the glued manifold $X^\#$ is connected, so that X has at most two components. As in Subsection 2.1 let γ be a path in $X^{(T)}$ running once through the neck $[-T,T]\times Y$, with starting-point x_0 and end-point x_1. If X is connected then we assume $x_0 = x_1$.

As before in this section, we will denote by $H^+(X)$ the space of self-dual closed L^2 2–forms on X. It is useful to observe here that orientations of $H^+(X)$ can be specified solely in terms of the intersection form on X. (We made implicit use of this already in the definition of homology orientation in [11, Subsection 1.1].) To see this, let V be any real vector space with a non-degenerate symmetric bilinear form $B : V\times V \to \mathbb{R}$ of signature (m,n), where $m > 0$ (the case $m = 0$ being trivial). Let $\mathcal{V}^+$ denote the space of all linearly independent m–tuples $(v_1,\dots,v_m)$ of elements of V such that B is positive definite on the linear span of $v_1,\dots,v_m$. Then $\mathcal{V}^+$ has exactly two path-components, and two such m–tuples $(v_1,\dots,v_m)$ and $(w_1,\dots,w_m)$ lie in the same component if and only if the matrix $(B(v_j,w_k))_{j,k=1,\dots,m}$ has positive determinant. In the case when B is the intersection form of X, a choice of a component of $\mathcal{V}^+$ determines orientations of both $H^+(X)$

and $H^+(X^{(T)})$ (since the intersection forms of X and $X^{(T)}$ are canonically isomorphic).

Given the ordering of the ends $\mathbb{R}_+ \times (\pm Y)$ of X there is a natural 1–1 correspondence between homology orientations of X and of $X^\#$. In general one can specify a homology orientation by choosing ordered bases for H^0, H^1, and H^+ (or equivalently, for the dual groups). If X has two components then the correspondence is given by replacing the basis (x_0) for $H_0(X^\#)$ with the ordered basis (x_0, x_1) for $H_0(X)$. If X is connected then we replace a given ordered basis $(e_1, \dots, e_\ell)$ for $H_1(X)$ (where $\ell = b_1(X)$) with the ordered basis $(-[\gamma], e_1, \dots, e_\ell)$ for $H_1(X^\#)$. (The sign in front of $[\gamma]$ is related to a sign appearing in the formula for Hol_j in (1).)

Now fix homology orientations of $X, X^\#$ which are compatible in the above sense. Let $\mathcal{B}, \mathcal{B}'$ be the configuration spaces over $X, X^{(T)}$ with reducible limits. According to Proposition 4.1 the chosen homology orientations determine an orientation o of $\lambda \to \mathcal{B}$ and an orientation o' of $\lambda' \to \mathcal{B}'$. On the other hand, λ' inherits a glued orientation $\tilde{o}$ from (λ, o) as specified in Subsection 4.4.

Proposition 4.3 **(i)** *If X is connected then $o' = \tilde{o}$.*

(ii) *If X has two components, then $o' = \tilde{o}$ if and only if $b_1(X) + b_2^+(X)$ is odd.*

Proof. Let $S_o = (A_o, 0)$ be a reference configuration over X as in Subsection 2.1 with reducible limit over each end. To simplify notation we will now write S, A instead of S_o, A_o. Let $S' = (A', 0)$ be the glued reference configuration over $X^{(T)}$. Set $L_A = d^+ \oplus D_A$, so that

$$\delta_S = -d^* + L_A : \mathcal{S} \to \mathcal{F}.$$

Set $b_1 = b_1(X)$, $b^+ = b_2^+(X)$, $m = \dim \ker(\delta_S)$, and $\ell = \dim \operatorname{coker}(L_A)$.

Choose smooth loops $\ell_1, \dots, \ell_{b^1}$ in $X_{:0}$ representing a positive basis for $H_1(X; \mathbb{R})$ and define

$$\mathsf{B}_1 : L^p_1(X_{:0}, i\Lambda^1) \to \mathbb{R}^{b_1}, \qquad a \mapsto \left(-\int_{\ell_j} ia \right)_{j=1,\dots,b_1}.$$

Choose a bounded complex-linear map

$$\mathsf{B}_2 : L^p_1(X_{:0}, \mathbb{S}^+) \to \mathbb{C}^{m_2}$$

whose composition with the restriction to $X_{:0}$ defines an isomorphism $\ker(D_A) \to \mathbb{C}^{m_2}$. Set

$$\mathsf{B} = \mathsf{B}_1 + \mathsf{B}_2 : \mathcal{S}_{:0} \to \mathbb{R}^{b_1} \oplus \mathbb{C}^{m_2} = \mathbb{R}^m.$$

Choose smooth imaginary-valued closed 2–forms $\omega_1, \dots, \omega_{b^+}$ on X which are supported in $X_{:0}$ and such that the cohomology classes $[-i\omega_1], \dots, [-i\omega_{b^+}]$ form a positive basis of a positive subspace for the intersection form of X. Then the self-dual parts $\omega_1^+, \dots, \omega_{b^+}^+$ map to a basis for $\operatorname{coker}(d^+)$ on both X and $X^{(T)}$, which in both cases is compatible with the chosen orientation of $H^+ = \operatorname{coker}(d^+)^*$. Choose smooth sections $\omega_{b^++1}, \dots, \omega_\ell$ of $\mathbb{S}_X^-$ which are supported in $X_{:0}$ and map to a positive basis for the real vector space $\operatorname{coker}(D_A)$ (with its complex orientation).

The remainder of the proof deals separately with the two cases.

Case (i): X is connected.

Let $g : X \to i\mathbb{R}$ be a smooth function supported in $X_{:0}$ and with $\int g = i$. Then for large T the orientations $o', \tilde{o}$ of $\delta_{S'}$ are both represented by the following corrector of $(\delta_{S'})_{\ell+1,m+1}$:

$$\mathcal{S}' \oplus \mathbb{R} \oplus \mathbb{R}^\ell \to \mathcal{F}' \oplus \mathbb{R} \oplus \mathbb{R}^m,$$
$$(\xi, t, z) \mapsto (tg + \sum_{j=1}^{\ell} z_j\omega_j, H_\gamma\xi, \mathsf{B}\xi),$$

where $H_\gamma\xi$ means H_γ applied to the 1–form part of ξ.

Case (ii): X has two components X_0, X_1, where $x_j \in X_j$.

Thus, $\mathbb{R}_+ \times Y \subset X_0$ and $\mathbb{R}_+ \times (-Y) \subset X_1$. For $j = 0, 1$ choose a smooth function $g_j : X_j \to i\mathbb{R}$ supported in $(X_j)_{:0}$ and with $\int g_j = i$. Set

$$\mathsf{C}' : \mathcal{S}' \oplus \mathbb{R} \oplus \mathbb{R}^\ell \oplus \mathbb{R} \to \mathcal{F}' \oplus \mathbb{R}^m \oplus \mathbb{R},$$
$$(\xi, t, z, t') \mapsto (tg_0 + \sum_{j=1}^{\ell} z_j\omega_j, \mathsf{B}\xi, t'),$$
$$\mathsf{C}_\gamma : \mathcal{S}' \oplus \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R}^\ell \to \mathcal{F}' \oplus \mathbb{R} \oplus \mathbb{R}^m,$$
$$(\xi, t, t', z) \mapsto (tg_0 + t'g_1 + \sum_{j=1}^{\ell} z_j\omega_j, H_\gamma\xi, \mathsf{B}\xi).$$

When T is large, C' and C_γ are both correctors of $(\delta_{S'})_{\ell+2,m+1}$ which represent the orientations $o', \tilde{o}$ of $\delta_{S'}$, respectively. Let C be the corrector of $(\delta_{S'})_{\ell+2,m+1}$ which has the same domain and target spaces as C_γ, and which is obtained from C' by interchanging summands as follows. If

$$(x, y, z) \in (\mathcal{S}' \oplus \mathbb{R}) \oplus \mathbb{R}^\ell \oplus \mathbb{R},$$
$$\mathsf{C}'(x, y, z) = (u, v, w) \in \mathcal{F}' \oplus \mathbb{R}^m \oplus \mathbb{R}$$

then $\mathsf{C}(x, z, y) = (u, w, v)$. As explained in Subsection 4.1, the correctors C, C' are equivalent if and only if $\ell + m$ is even. Set

$$E = \delta_{S'} + \mathsf{C}, \quad E_\gamma = \delta_{S'} + \mathsf{C}_\gamma.$$

We have

$$(\mathsf{C}_\gamma - \mathsf{C})(\xi, t, t', z) = (t' g_1, H_\gamma \xi - t', 0),$$

so the image N of $\mathsf{C}_\gamma - \mathsf{C}$ has dimension 2. We need to compute the determinant of the automorphism of N induced by $E_\gamma E^{-1}$. Let $s, s' \in \mathbb{R}$ and set

$$(\xi, t, t', z) = E^{-1}(s g_1, s', 0).$$

Write $\xi = (a, \phi) \in \Gamma(i\Lambda^1 \oplus \mathbb{S}^+)$. Then

$$-d^* a + t g_0 = s g_1, \quad t' = s'. \tag{29}$$

Integrating the first equation gives $t = s$. The equation

$$E_\gamma E^{-1} = (\mathsf{C}_\gamma - \mathsf{C}) E^{-1} + I$$

now yields

$$E_\gamma E^{-1}(s g_1, s', 0) = ((s + s') g_1, H_\gamma \xi, 0).$$

Note that ξ depends on s alone, so we can write $a = a(s)$. Set $\eta = H_\gamma(a(1))$. Then $E_\gamma E^{-1}|_N$ is represented by the matrix

$$\begin{pmatrix} 1 & 1 \\ \eta & 0 \end{pmatrix},$$

so $\mathsf{C}, \mathsf{C}_\gamma$ are equivalent correctors if and only if $\eta < 0$. We will show that $\eta > 0$ when T is large. This implies that $\mathsf{C}', \mathsf{C}_\gamma$ are equivalent correctors if and only if $\ell + m$ is odd. Since D_A is complex linear, this will prove part (ii) of the proposition.

Let $\{T_n\}$ be a sequence tending to ∞ and, working over $X^{(T_n)}$, set

$$(\xi_n, s_n, 0, z_n) = E^{-1}(s_n g_1, 0, 0),$$

or more explicitly,

$$\delta_{S'}(\xi_n) + \sum_j z_{n,j}\omega_j = s_n(g_1 - g_0), \quad \mathsf{B}\xi_n = 0,$$

where $s_n > 0$ is chosen such that

$$\|\xi_n\|_{L^2(X_{:2})} = 1.$$

(Because the supports of g_0 and g_1 are disjoint, Equation (29) shows that $a \neq 0$ over $X_{:0}$ when $s \neq 0$.) Write $\xi_n = (a_n, \phi_n)$. Equation (29) yields

$$s_n\|g_1\|_2^2 = -\int \langle a_n, dg_1\rangle \le \|dg_1\|_2,$$

hence the sequence s_n is bounded. An analogous argument applied to the equation

$$L_A(\xi_n) + \sum_j z_{n,j}\omega_j = 0$$

shows that the sequence z_n is bounded as well. Thus, $\delta_{S'}(\xi_n)$ is supported in $X_{:0}$, and for each $k \ge 0$ the C^k–norm of $\delta_{S'}(\xi_n)$ is bounded independently of n. Now recall from [11, Subsection 3.4] that over the neck $[-T_n, T_n] \times Y$ the operator $\delta_{S'}$ can be expressed in the form $\frac{\partial}{\partial t} + P$, where

$$P = \begin{pmatrix} 0 & -d^* & 0 \\ -d & *d & 0 \\ 0 & 0 & -\partial_B \end{pmatrix}$$

for some spin^c connection B over Y. Because of our non-degeneracy assumption on the critical points, the kernel of P consists of the constant functions in $i\Omega^0(Y)$. There is also a similar description of $\delta_{S'}$ over the ends $\mathbb{R}_+ \times Y'_j$. In general, if $(\frac{\partial}{\partial t} + P)\zeta = 0$ over a band $[0, \tau] \times Y$ and ζ involves only eigenvectors of P corresponding to positive eigenvalues then for any non-negative integer k and $1 \le t \le \tau - 2$, say, there is an estimate

$$\|\zeta\|_{C^k([t,t+1]\times Y)} \le \text{const} \cdot e^{-\rho t}\|\zeta\|_{L^2(\{0\}\times Y)},$$

where ρ is the smallest positive eigenvalue of P. This result immediately applies to ξ_n over the ends $\mathbb{R}_+ \times Y'_j$. Over the neck $[-T_n, T_n] \times Y$ one can write $\xi_n = \text{const} \cdot i\, dt + \xi_n^+ + \xi_n^-$ where $\xi_n^\pm$ involves only eigenvectors corresponding to positive/negative eigenvalues of P. One then obtains C^k–estimates on $\xi_n^\pm$ in terms of its L^2–norm over $\{\mp T_n\} \times Y$. It follows that after passing to a subsequence we may assume that ξ_n c-converges over X to some pair $\xi = (a, \phi)$ satisfying $\|\xi\|_{L^2(X_{:2})} = 1$. Of course, we may also assume that the sequences s_n, z_n converge, with limits s, z, say. Then

$$\delta_S \xi + \sum_j z_j \omega_j = s(g_1 - g_0), \quad \mathsf{B}\xi = 0.$$

Moreover,

$$\xi = \pm ci\, dt + \zeta_\pm \quad \text{on } \mathbb{R}_+ \times (\pm Y),$$

where $\zeta_\pm$ decays exponentially and

$$c = -\lim_{n\to\infty} \frac{H_\gamma(a_n)}{2T_n}.$$

On the other hand, Stokes' theorem yields

$$\int_{\{-T_n\}\times Y} *a_n = -\int_{(X_0)_{:0}} d^* a_n = -\int_{(X_0)_{:0}} s_n g_0 = -s_n i,$$

hence

$$ci \cdot \mathrm{Vol}(Y) = \int_{\{0\}\times Y} *a = \lim_n \int_{\{-T_n\}\times Y} *a_n = -si.$$

Thus, $c \cdot \mathrm{Vol}(Y) = -s \le 0$. If $c = 0$ then $\xi \neq 0$ would decay exponentially on all ends of X and satisfy $\delta_S \xi = 0$, contradicting $\mathsf{B}\xi = 0$. Therefore, $c < 0$, and $H_\gamma(a_n) > 0$ for large n.

This shows that $\eta > 0$ when T is large. □

4.6 Orientation of $\mathcal{V}_0$

In Subsection 4.2 the question arose what it means for $\{g_j\}$ to be a positive basis for $\mathcal{V}_\Phi$ when $\Phi = 0$. The following proposition answers this question when $b = 1$. This result is not needed elsewhere in this paper, but will be used in [12].

Proposition 4.4 *Let X be as in [11, Subsection 1.3]. Suppose X is connected, $b = 1$, and consider the bundle $\mathcal{V} \to \mathcal{W} = L^{p,w}_1(X; \mathbb{S}^+)$ with all weights σ_j positive. Then $g \in C^\infty_c(X; i\mathbb{R})$, represents a positive basis for $\mathcal{V}_0$ if and only if $\int_X g/i > 0$.*

Proof. It suffices to prove that $g = ih$ represents a positive basis when $\int_X h > 0$. Let $\mathfrak{b} = \{x\}$. By [11, Proposition 2.3] we may assume $h \ge 0$, $h(x) > 0$, and $\mathrm{supp}(h) \subset X_{:0}$. The proposition is then a consequence of the following lemma.

Lemma 4.5 *Let X, h be as above and v a smooth positive function on X whose restriction to each end $\mathbb{R}_+ \times Y_j$ is the pull-back of a function v_j on $\mathbb{R}_+$. Suppose f is a real function on X satisfying*

$$(\Delta + v)f = h, \quad df \in L_1^{p,w}.$$

Then $f \geq 0$, and $f > 0$ where $h > 0$.

The proof will make use of the following elementary result, whose proof is left to the reader.

Sublemma 4.2 *Suppose a, u are smooth real functions on $[0, \infty)$ such that $u'' = au$, $a > 0$, $u(0) > 0$, and u is bounded. Then $u > 0$ and $u' < 0$.* □

Proof of Lemma 4.5: We first study the behaviour of f on an end $\mathbb{R}_+ \times Y_j$. We omit j from notation and write $Y = Y_j$ etc. Set $\mathsf{f} = f|_{\mathbb{R}_+ \times Y}$. By [11, Proposition 2.1] the assumption $df \in L_1^{p,w}$ implies that $\mathsf{f}(t, \cdot)$ converges uniformly towards a constant function c as $t \to \infty$. Let $\{e_\nu\}$ be a maximal orthonormal set of eigenvectors of Δ_Y with corresponding eigenvalues λ_ν^2. Write

$$\mathsf{f}(t, y) = \sum_\nu u_\nu(t) e_\nu(y).$$

Then

$$u_\nu'' = (\lambda_\nu^2 + \mathsf{v}) u_\nu.$$

By the sublemma, either $u_\nu = 0$ or $u_\nu u_\nu' < 0$. Consequently,

$$\int_Y \mathsf{f}^2(t, y)\, dy = \sum_\nu u_\nu^2(t)$$

is a decreasing function of t, and

$$\max_{y \in Y} |\mathsf{f}(t, y)| \geq c$$

for all $t \geq 0$. In particular, if $c < 0$ then there exists a $(t, y) \in \mathbb{R}_+ \times Y$ with $\mathsf{f}(t, y) \leq c$. Hence, if $\inf f < 0$ then the infimum is attained.

Now, at any local minimum of f one has

$$vf = h - \Delta f \geq h,$$

so $f \geq 0$ everywhere. But then every zero of f is an absolute minimum, so $f > 0$ where $h > 0$. This proves the lemma and thereby also Proposition 4.4. □

5 Parametrized moduli spaces

Parametrized moduli spaces appear in many different situations in gauge theory, e.g. in the construction of 4–manifold invariants [6, 22] and Floer homology [5], and in connection with gluing obstructions [12]. A natural setting here would involve certain fibre bundles whose fibres are 4–manifolds. We feel, however, that gauge theory for such bundles in general deserves a separate treatment, and will therefore limit ourselves, at this time, to the case of a product bundle over a vector space. However, we take care to set up the theory in such a way that it would easily carry over to more general situations.

The main goal of this section is to extend the gluing theorem and the discussion of orientations to the parametrized case.

5.1 Moduli spaces

As in [11, Subsection 1.3] let X be a spin^c 4–manifold with Riemannian metric $\bar{g}$ and tubular ends $\overline{\mathbb{R}}_+ \times Y_j$, $j = 1, \dots, r$. Let W be a finite-dimensional Euclidean vector space and $\mathbf{g} = \{g_\mathsf{w}\}_{\mathsf{w}\in\mathsf{W}}$ a smooth family of Riemannian metrics on X all of which agree with $\bar{g}$ outside $X_{:0}$. We then have a principal SO(4)–bundle $P_{\mathrm{SO}}(\mathbf{g}) \to X \times \mathsf{W}$ whose fibre over (x, w) consists of all positive g_w–orthonormal frames in T_xX.

In the notation of [11, Subsection 3.1] let $P_{\mathrm{GL}^c} \to P_{\mathrm{GL}^+}$ be the spin^c structure on X. Denote by $P_{\mathrm{Spin}^c}(\mathbf{g})$ the pull-back of $P_{\mathrm{SO}}(\mathbf{g})$ under the projection $P_{\mathrm{GL}^c} \times \mathsf{W} \to P_{\mathrm{GL}^+} \times \mathsf{W}$. Then $P_{\mathrm{Spin}^c}(\mathbf{g})$ is a principal Spin^c(4)–bundle over $X \times \mathsf{W}$.

For $j = 1, \dots, r$ let $\alpha_j \in \mathcal{C}(Y_j)$ be a non-degenerate smooth monopole. Let $\mathcal{C}(g_\mathsf{w})$ denote the $L^{p,w}_1$ configuration space over X for the metric g_w and limits α_j, where p, w are as in [11, Subsection 3.4]. We will provide the disjoint union

$$\mathcal{C}(\mathbf{g}) = \bigcup_{\mathsf{w}\in\mathsf{W}} \mathcal{C}(g_\mathsf{w}) \times \{\mathsf{w}\}$$

with a natural structure of a (trivial) smooth fibre bundle over W. Let

$$\mathsf{v} : P_{\mathrm{Spin}^c}(\mathbf{g}) \to P_{\mathrm{Spin}^c}(g_0) \times \mathsf{W} \tag{30}$$

be any isomorphism of Spin^c(4)–bundles which covers the identity on $X \times \mathsf{W}$ and which outside $X_{:1} \times \mathsf{W}$ is given by the identification $P_{\mathrm{Spin}^c}(g_\mathsf{w}) = P_{\mathrm{Spin}^c}(g_0)$. There is then an induced isomorphism of SO(4)–bundles

$$P_{\mathrm{SO}}(\mathbf{g}) \to P_{\mathrm{SO}}(g_0) \times \mathsf{W},$$

since these are quotients of the corresponding $\mathrm{Spin}^c(4)$–bundles by the $\mathrm{U}(1)$–action. Such an isomorphism v can be constructed by means of the holonomy along rays of the form $\{x\}\times\overline{\mathbb{R}}_+\mathsf{w}$ where $(x,\mathsf{w})\in X\times\mathsf{W}$, with respect to any connection in $P_{\mathrm{Spin}^c}(\mathbf{g})$ which outside $X_{:1}\times\mathsf{W}$ is the pull-back of a connection in $P_{\mathrm{Spin}^c}(g_0)$. Then v induces a $\mathcal{G}=\mathcal{G}(X;\vec{\alpha})$–equivariant diffeomorphism

$$\mathcal{C}(g_{\mathsf{w}})\to\mathcal{C}(g_0) \tag{31}$$

for each w, where the map on the spin connections is obtained by identifying these with connections in the respective determinant line bundles and applying the isomorphism between these bundles induced by v. Putting together the maps (31) for all w yields a bijection

$$\mathsf{v}_*:\mathcal{C}(\mathbf{g})\to\mathcal{C}(g_0)\times\mathsf{W}.$$

If $\tilde{\mathsf{v}}$ is another isomorphism as in (30) then $\mathsf{v}_*(\tilde{\mathsf{v}}_*)^{-1}$ is smooth, hence we have obtained the desired structure on $\mathcal{C}(\mathbf{g})$. Furthermore, because of the gauge equivariance of v_* we also get a similar smooth fibre bundle structure on

$$\mathcal{B}^*_{\mathfrak{b}}(\mathbf{g})=\bigcup_{\mathsf{w}\in\mathsf{W}}\mathcal{B}^*_{\mathfrak{b}}(g_{\mathsf{w}})\times\{\mathsf{w}\} \tag{32}$$

for any finite subset $\mathfrak{b}\subset X$. The image of $(S,\mathsf{w})\in\mathcal{C}(\mathbf{g})$ in $\mathcal{B}(\mathbf{g})$ will be denoted $[S,\mathsf{w}]$.

We consider the natural smooth action of $\mathbb{T}$ on $\mathcal{B}^*_{\mathfrak{b}}(\mathbf{g})$ where an element of $\mathbb{T}$ maps each fibre $\mathcal{B}_{\mathfrak{b}}(g_{\mathsf{w}})$ into itself in the standard way. (There is another version of the gluing theorem where $\mathbb{T}$ acts non-trivially on W, see below.)

The principal bundle $P_{\mathrm{Spin}^c}(\mathbf{g})$ also gives rise to Banach vector bundles $\mathcal{S}(\mathbf{g}),\mathcal{F}(\mathbf{g}),\mathcal{F}_2(\mathbf{g})$ over W whose fibres over $\mathsf{w}\in\mathsf{W}$ are the spaces $\mathcal{S}(g_{\mathsf{w}}),\mathcal{F}(g_{\mathsf{w}}),\mathcal{F}_2(g_{\mathsf{w}})$ resp. defined as in (19) using the metric g_{w} on X.

Let $\dot{\Theta}:\mathcal{C}(\mathbf{g})\to\mathcal{F}_2(\mathbf{g})$ be the fibre-preserving monopole map whose effect on the fibre over $\mathsf{w}\in\mathsf{W}$ is the left hand side of [11, Equation (20)], interpreted in terms of the metric g_{w}. If we conjugate $\dot{\Theta}$ by the appropriate diffeomorphisms induced by v then we obtain the smooth $\mathcal{G}$–equivariant map

$$\dot{\Theta}_{\mathsf{v}}:\mathcal{C}(g_0)\times\mathsf{W}\to\mathcal{F}_2(g_0),$$
$$(A,\Phi,\mathsf{w})\mapsto\left((\mathsf{v}_{\mathsf{w}}(\hat{F}_A+\mathfrak{m}(A,\Phi)))^+-Q(\Phi)\,,\,\sum_j\mathsf{v}_{\mathsf{w}}(e_j)\cdot\nabla^{A+\mathsf{a}_{\mathsf{w}}}_{e_j}(\Phi)\right)$$

where the perturbation $\mathfrak{m}$ is smooth, hence $\dot{\Theta}$ is smooth. Here v_{w} denotes the isomorphism that v induces from the Clifford bundle of (X,g_{w}) to the

Clifford bundle of (X, g_0), and $\{e_j\}$ is a local g_{w}–orthonormal frame on X. Finally, if we temporarily let $\nabla^{(\mathsf{w})}$ denote the g_{w}–Riemannian connection in the tangent bundle of X then

$$\mathsf{a}_{\mathsf{w}} = \mathsf{v}_{\mathsf{w}}(\nabla^{(\mathsf{w})}) - \nabla^{(0)}.$$

Note that a_{w} is supported in $X_{:1}$.

In situations involving parametrized moduli spaces there will often be an additional perturbation which affects the equations only over some compact part of X. For the gluing theory one can consider quite generally perturbations given by an isomorphism v and a smooth $\mathcal{G}$–equivariant map

$$\mathfrak{o} : \mathcal{C}(X_{:\mathfrak{t}}, g_0) \times \mathsf{W} \to (\mathcal{F}_2)^{:\mathfrak{t}}(g_0)$$

for some $\mathfrak{t} \geq 0$, using notation introduced in Subsection 4.4. We require that the derivative of $\mathfrak{o}$ at any point be a compact operator. Let

$$\Theta : \mathcal{C}(\mathbf{g}) \to \mathcal{F}_2(\mathbf{g}) \tag{33}$$

be the map corresponding to $\Theta_{\mathsf{v}} := \dot{\Theta}_{\mathsf{v}} + \mathfrak{o}$. We define the parametrized moduli space $M_{\mathfrak{b}}(\mathbf{g})$ to be the image of $\Theta^{-1}(0)$ in $\mathcal{B}_{\mathfrak{b}}(\mathbf{g})$. By construction, v_* induces a homeomorphism

$$M_{\mathfrak{b}}(\mathbf{g}) \overset{\approx}{\to} \Theta_{\mathsf{v}}^{-1}(0)/\mathcal{G}_{\mathfrak{b}}.$$

A point in $M_{\mathfrak{b}}(\mathbf{g})$ is called *regular* if the corresponding zeros of Θ_{v} are regular (a regular zero being one where the derivative of Θ_{v} is surjective). This notion is independent of v. By the local slice theorem, the set of regular points in $M^*_{\mathfrak{b}}(\mathbf{g})$ is a smooth submanifold of $\mathcal{B}^*_{\mathfrak{b}}(\mathbf{g})$.

5.2 Orientations

Fix orientations of the vector space W and of the set $\mathfrak{b}$. For any $S \in \mathcal{C}(g_{\mathsf{w}})$ let

$$\delta_{S,\mathsf{w}} : \mathcal{S}(g_{\mathsf{w}}) \to \mathcal{F}(g_{\mathsf{w}})$$

be the Fredholm operator δ_S defined in terms of the metric g_{w}, now using the perturbed monopole map (33). The orientation cover of this family descends to a double cover $\lambda(\mathbf{g}) \to \mathcal{B}(\mathbf{g})$. (Note that the perturbation $\mathfrak{o}$ can be scaled down, so that an orientation of $\lambda(\mathbf{g})$ for $\mathfrak{o} = 0$ determines an orientation for any other $\mathfrak{o}$.) Clearly, any section of $\lambda(\mathbf{g})$ over $\mathcal{B}(g_0)$ extends uniquely to all of $\mathcal{B}(\mathbf{g})$. On the other hand, a section of $\lambda(\mathbf{g})$ determines an orientation of the regular part of $M^*_{\mathfrak{b}}(\mathbf{g})$, as we will now explain.

Let $T^v\mathcal{C}(\mathbf{g}) \subset T\mathcal{C}(\mathbf{g})$ be the subbundle of vertical tangent vectors. We can identify $T^v_{(S,\mathsf{w})}\mathcal{C}(\mathbf{g}) = \mathcal{S}(g_\mathsf{w})$. A choice of an isomorphism v_1 as in (30) determines a bundle homomorphism

$$P_1 : T\mathcal{C}(\mathbf{g}) \to T^v\mathcal{C}(\mathbf{g})$$

which is the identity on vertical tangent vectors. This yields a splitting

$$T_{(S,\mathsf{w})}\mathcal{C}(\mathbf{g}) = \mathcal{S}(g_\mathsf{w}) \oplus \mathsf{W}$$

into vertical and horizontal vectors (the latter making up the kernel of P_1 and being identified with W through the projection).

In general, a connection in a vector bundle $E \to \mathsf{W}$ determines for every element u of a fibre E_w a linear map $T_uE \to E_\mathsf{w}$, namely the projection onto the vertical part of the tangent space. Moreover, if $u = 0$ then this projection is independent of the connection. Together these projections form a smooth map $TE \to E$. Let

$$P_2 : T\mathcal{F}_2(\mathbf{g}) \to \mathcal{F}_2(\mathbf{g})$$

be such a map for $E = \mathcal{F}_2(\mathbf{g})$ determined by some isomorphism v_2.

Now let

$$\mathcal{I}^* : T^v\mathcal{C}(\mathbf{g}) \to \mathcal{F}_1$$

be the map which sends $s \in T^v_{(S,\mathsf{w})}\mathcal{C}(\mathbf{g})$ to $\mathcal{I}^*_S(s)$, where the * refers to the metric g_w. Set

$$\underline{\delta} := \mathcal{I}^* \circ P_1 + P_2 \circ D\Theta : T\mathcal{C}(\mathbf{g}) \to \mathcal{F}(\mathbf{g}),$$

where $D\Theta$ is the derivative of the map (33). By restriction of $\underline{\delta}$ we obtain bounded operators

$$\underline{\delta}_{S,\mathsf{w}} : T_{(S,\mathsf{w})}\mathcal{C}(\mathbf{g}) \to \mathcal{F}(g_\mathsf{w}).$$

Since the restriction of $\underline{\delta}_{S,\mathsf{w}}$ to the vertical tangent space $\mathcal{S}(g_\mathsf{w})$ is equal to the Fredholm operator $\delta_{S,\mathsf{w}}$, we conclude that $\underline{\delta}_{S,\mathsf{w}}$ is also Fredholm, and

$$\mathrm{ind}(\underline{\delta}_{S,\mathsf{w}}) = \mathrm{ind}(\delta_{S,\mathsf{w}}) + \mathsf{d},$$

where $\mathsf{d} = \dim \mathsf{W}$.

Choose non-negative integers ℓ, m with $\mathrm{ind}(\delta_{S,\mathsf{w}}) = m - \ell$ and an orientation preserving linear isomorphism $h : \mathsf{W} \to \mathbb{R}^\mathsf{d}$. If C is any corrector of $(\delta_{S,\mathsf{w}})_{\ell,m}$ then

$$\underline{\delta}_{S,\mathsf{w}} + h + \mathsf{C} : \mathcal{S}(g_\mathsf{w}) \oplus \mathsf{W} \oplus \mathbb{R}^\ell \to \mathcal{F}(g_\mathsf{w}) \oplus \mathbb{R}^\mathsf{d} \oplus \mathbb{R}^m$$

is an injective Fredholm operator of index 0, hence an isomorphism. The map $\mathsf{C} \mapsto h + \mathsf{C}$ respects the equivalence relation for correctors and therefore defines a 1-1 correspondence between orientations of $\delta_{S,\mathsf{w}}$ and orientations of $\underline{\delta}_{S,\mathsf{w}}$.

Fix $S \in \mathcal{C}^*_{\mathfrak{b}}(g_{\mathsf{w}})$, choose a map μ as in (22), and let

$$\underline{\delta}_{\mu,S,\mathsf{w}} := \mathcal{S}(g_{\mathsf{w}}) \oplus \mathsf{W} \to \mathcal{F}_{\mu}(g_{\mathsf{w}})$$

be the operator obtained from $\underline{\delta}_{S,\mathsf{w}}$ by replacing $\mathcal{I}^*_S$ by $\mu \circ \mathcal{I}^*_S$ (cf. (24)). Just as in the unparametrized case one establishes a 1-1 correspondence between orientations of $\underline{\delta}_{S,\mathsf{w}}$ and orientations of $\underline{\delta}_{\mu,S,\mathsf{w}}$.

Now suppose $[S,\mathsf{w}] \in M^*_{\mathfrak{b}}(\mathbf{g})$. Working in the trivialization v_1 and using the local slice theorem for the metric g_{w} one finds that $[S,\mathsf{w}]$ is a regular point of $M^*_{\mathfrak{b}}(\mathbf{g})$ if and only if $\underline{\delta}_{\mu,S,\mathsf{w}}$ is surjective, and in that case the projection $\mathcal{C}^*_{\mathfrak{b}}(\mathbf{g}) \to \mathcal{B}^*_{\mathfrak{b}}(\mathbf{g})$ induces an isomorphism

$$\ker(\underline{\delta}_{\mu,S,\mathsf{w}}) \xrightarrow{\approx} T_{[S,\mathsf{w}]} M^*_{\mathfrak{b}}(\mathbf{g}).$$

This establishes a 1-1 correspondence between orientations of $\delta_{S,\mathsf{w}}$ and orientations of $T_{[S,\mathsf{w}]} M^*_{\mathfrak{b}}(\mathbf{g})$. This correspondence is obviously independent of P_2, and it is independent of P_1 because the set of such operators form an affine space. It is also independent of μ for reasons explained earlier.

This associates to any orientation of $\lambda(\mathbf{g})$ an orientation of the regular part of $M^*_{\mathfrak{b}}(\mathbf{g})$.

5.3 The gluing theorem

We continue the discussion of the previous subsection, but we now specialize to the case when the ends of X are $\mathbb{R}_+ \times (\pm Y_j)$, $j = 1, \dots, r$ and $\mathbb{R}_+ \times Y'_j$, $j = 1, \dots, r'$, with non-degenerate limits α_j over $\mathbb{R}_+ \times (\pm Y_j)$ and α'_j over $\mathbb{R}_+ \times Y'_j$, as in Subsection 2.1. Let the paths $\gamma^{\pm}_j, \gamma_j$ and $\mathfrak{b} \subset X$ be as in that subsection. The family of metrics $\mathbf{g}$ on X defines, in a natural way, a smooth family of metrics $\{g(T,\mathsf{w})\}_{\mathsf{w} \in \mathsf{W}}$ on $X^{(T)}$ for any T. We retain our previous notation for configuration and moduli spaces over X, whereas those over $X^{(T)}$ will be denoted $\mathcal{C}'(\mathbf{g}), \mathcal{B}'_{\mathfrak{b}}(\mathbf{g}), M^{(T)}_{\mathfrak{b}}(\mathbf{g})$ etc. Fix an isomorphism v as in (30).

We first discuss gluing of orientations. The isomorphism v defines a corresponding isomorphism over $X^{(T)}$ and operators P_1, P_2 over both X and $X^{(T)}$. We then get families of Fredholm operators $\underline{\delta}, \underline{\delta}'$ parametrized by $\mathcal{C}(\mathbf{g}), \mathcal{C}'(\mathbf{g})$ resp. The procedure in Subsection 4.4 for gluing orientations carries over to this situation and yields a 1-1 correspondence between orientations of $\underline{\delta}$ and orientations of $\underline{\delta}'$. Given $S \in \mathcal{C}^{:0}(g_{\mathsf{w}})$, if $\lambda_{S,\mathsf{w}}$ resp. $\underline{\lambda}_{S,\mathsf{w}}$ denote the set of orientations of $\delta_{S,\mathsf{w}}$ resp. $\underline{\delta}_{S,\mathsf{w}}$, and similarly for the glued configuration $S' \in \mathcal{C}'(g_{\mathsf{w}})$, then the diagramme of bijections

$$\begin{array}{ccc} \lambda_{S,\mathsf{w}} & \longrightarrow & \lambda_{S',\mathsf{w}} \\ \downarrow & & \downarrow \\ \underline{\lambda}_{S,\mathsf{w}} & \longrightarrow & \underline{\lambda}_{S',\mathsf{w}} \end{array}$$

commutes if and only if $\mathsf{d}r_0$ is even.

Now fix an orientation of $\lambda(\mathbf{g}) \to \mathcal{B}_{\mathfrak{b}}(\mathbf{g})$ and let $\lambda'(\mathbf{g}) \to \mathcal{B}'_{\mathfrak{b}}(\mathbf{g})$ have the glued orientation. These orientations determine orientations of the regular parts of the moduli spaces $M_{\mathfrak{b}}(\mathbf{g})$ and $M^{(T)}_{\mathfrak{b}}(\mathbf{g})$, respectively, as specified in the previous subsection.

As before, a choice of reference configuration in $\mathcal{C}(g_0)$ gives rise to a glued reference configuration in $\mathcal{C}'(g_0)$ and a holonomy map

$$\mathcal{B}'_{\mathfrak{b}}(g_0) \to \mathrm{U}(1)^{r_0}.$$

Composing this with the map $M^{(T)}_{\mathfrak{b}}(\mathbf{g}) \to \mathcal{B}'_{\mathfrak{b}}(g_0)$ defined by the chosen isomorphism v yields a holonomy map

$$\mathrm{Hol} : M^{(T)}_{\mathfrak{b}}(\mathbf{g}) \to \mathrm{U}(1)^{r_0}.$$

Fix an open $\mathbb{T}$–invariant subset $G \subset M_{\mathfrak{b}}(\mathbf{g})$ whose closure is compact and contains only regular points.

By a kv-pair we mean as before a pair (K, V), where $K \subset X$ is a compact codimension 0 submanifold which contains $\mathfrak{b}$ and intersects every component of X, and V is an open $\mathbb{T}$–invariant neighbourhood of $R_K(\overline{G})$ in

$$\breve{\mathcal{B}}_{\mathfrak{b}}(K, \mathbf{g}) = \bigcup_{\mathsf{w} \in \mathsf{W}} \breve{\mathcal{B}}_{\mathfrak{b}}(K, g_{\mathsf{w}}) \times \{\mathsf{w}\}.$$

Now fix a kv-pair (K, V) satisfying similar additional assumptions as before: firstly, that $V \subset \breve{\mathcal{B}}^*_{\mathfrak{b}}(K, \mathbf{g})$; secondly, that if X_e is any component of X which contains a point from $\mathfrak{b}$ then $X_e \cap K$ is connected.

Suppose

$$q : V \to M_{\mathfrak{b}}(\mathbf{g})$$

is a smooth $\mathbb{T}$–equivariant map such that $q(\omega|_K) = \omega$ for all $\omega \in G$. (We do not require that q commute with the projections to W.) Choose $\lambda_j, \lambda'_j > 0$. Let 'admissibility of $\vec{\alpha}'$' be defined in terms of the parametrized moduli spaces $M^{(T)}_{\mathfrak{b}}(\mathbf{g})$ (see [11, Definition 7.3]).

Theorem 5.1 *Theorem 2.1 holds in the present situation if one replaces* $M_{\mathfrak{b}}$ *and* $M_{\mathfrak{b}}^{(T)}$ *by* $M_{\mathfrak{b}}(\mathbf{g})$ *and* $M_{\mathfrak{b}}^{(T)}(\mathbf{g})$*, respectively. Moreover, the diffeomorphism* $\mathbf{F}$ *defined as in Theorem 4.1 preserves or reverses orientations according as to whether* $(b+\mathsf{d})r_0$ *is even or odd.*

Proof. The proofs carry over without any substantial changes. □

There is another version of the theorem (which will be used in [12]) where the family of metrics $\mathbf{g}$ is constant (ie $g_{\mathsf{w}} = g_0$ for every w) and $\mathbb{T}$ acts smoothly on the manifold W. One then has a product action of $\mathbb{T}$ on

$$M_{\mathfrak{b}}(\mathbf{g}) = M_{\mathfrak{b}} \times \mathsf{W},$$

and the theorem holds in this setting as well. In fact, the action of $\mathbb{T}$ affects the proof in only one way, namely the requirement that $\tilde{K}$ be $\mathbb{T}$–invariant. To obtain this, let dist be a $\mathbb{T}$–invariant metric on the set W (arising for instance from a $\mathbb{T}$–invariant Riemannian metric) and replace the definition of d_m in (13) by

$$d_m((S,\mathsf{w}),(\bar{S},\bar{\mathsf{w}})) = \int_{X_{:m}} |\bar{S}-S|^p + |\nabla_{\bar{A}}(\bar{S}-S)|^p + \operatorname{dist}(\mathsf{w},\bar{\mathsf{w}}).$$

Then V'_m will be $\mathbb{T}$–invariant.

5.4 Compactness

In contrast to gluing theory, compactness requires more specific knowledge of the perturbation $\mathfrak{o}$, so we will here take $\mathfrak{o} = 0$. We observe that the notion of chain-convergence has a natural generalization to the parametrized situation, and that the compactness theorem [11, Theorem 1.4] carries over to sequences

$$[A_n, \Phi_n, \mathsf{w}_n] \in M_{\mathfrak{b}}(X^{(T(n))}, g(T(n),\mathsf{w}_n); \vec{\alpha}'_n)$$

provided the sequence w_n is bounded (and similarly for [11, Theorem 1.3]). The only new ingredient in the proof is the following simple fact: Suppose B is a Banach space, E,F vector bundles over a compact manifold, L,L' : $\Gamma(E) \to \Gamma(F)$ differential operators of order d, and $K : \Gamma(E) \to B$ a linear operator. If L satisfies an inequality

$$\|f\|_{L^p_k} \le C\left(\|Lf\|_{L^p_{k-d}} + \|Kf\|_B\right)$$

and L,L' are sufficiently close in the sense that

$$\|(L-L')f\|_{L^p_{k-d}} \le \epsilon\|f\|_{L^p_k}$$

for some constant $\epsilon > 0$ with $\epsilon C < 1$, then L' obeys the inequality

$$\|f\|_{L^p_k} \leq (1-\epsilon C)^{-1} C \left(\|L' f\|_{L^p_{k-d}} + \|Kf\|_B \right).$$

A Splicing left or right inverses

Let X be a Riemannian manifold with tubular ends as in [11, Subsection 1.4] but of arbitrary dimension. Let $E \to X$ be a vector bundle which over each end $\mathbb{R}_+ \times (\pm Y_j)$ (resp. $\mathbb{R}_+ \times Y'_j$) is isomorphic (by a fixed isomorphism) to the pull-back of a bundle $E_j \to Y_j$ (resp. $E'_j \to Y'_j$). Let $F \to X$ be another bundle of the same kind. Let $D : \Gamma(E) \to \Gamma(F)$ be a differential operator of order $d \geq 1$ which is translationary invariant over each end and such that for each j the restrictions of D to $\mathbb{R}_+ \times Y_j$ and $\mathbb{R}_+ \times (-Y_j)$ agree in the obvious sense. The operator D gives rise to a glued differential operator $D' : \Gamma(E') \to \Gamma(F')$ over $X^{(T)}$, where E', F' are the bundles over $X^{(T)}$ formed from E, F resp. Let k, ℓ, m be non-negative integers and $1 \leq p < \infty$. Let $L^p_k(X;F)_{:0}$ denote the subspace of $L^p_k(X;F)$ consisting of those elements that vanish a.e. outside $X_{:0}$. We can clearly also identify $L^p_k(X;F)_{:0}$ with a subspace of $L^p_k(X^{(T)};F')$. Let

$$V : L^p_{k+d}(X_{:0};E) \oplus \mathbb{R}^\ell \to L^p_k(X;F)_{:0} \oplus \mathbb{R}^m$$

be a bounded operator and set

$$\begin{aligned} P = D + V : L^p_{k+d}(X;E) \oplus \mathbb{R}^\ell &\to L^p_k(X;F) \oplus \mathbb{R}^m \\ (s,x) &\mapsto (Ds, 0) + V(s|_{X_{:0}}, x). \end{aligned}$$

Define the operator $P' = D' + V$ over $X^{(T)}$ similarly.

Proposition A.1 *If P has a bounded left (resp. right) inverse Q then for $\check{T} > C_1\|Q\|$ the operator P' has a bounded left (resp. right) inverse Q' with $\|Q'\| < C_2\|Q\|$. Here the constants $C_1, C_2 < \infty$ depend on the restriction of D to the ends $\mathbb{R}_+ \times (\pm Y_j)$ but are otherwise independent of P.*

For left inverses this was proved in a special case in [11, Lemma 5.4], and the general case is not very different. However, we would like to have the explicit expression for the right inverse on record, since this is used both in Subsection 2.2 and in Subsection 4.4.

Proof. Choose smooth functions $f_1, f_2 : \mathbb{R} \to \mathbb{R}$ such that $(f_1(t))^2 + (f_2(1-t))^2 = 1$ for all t, and $f_k(t) = 1$ for $t \leq \frac{1}{3}$, $k = 1, 2$. Define $\beta : X \to \mathbb{R}$

by

$$\beta = \begin{cases} f_1(t/(2T_j)) & \text{on each end } \mathbb{R}_+ \times Y_j, \\ f_2(t/(2T_j)) & \text{on each end } \mathbb{R}_+ \times (-Y_j), \\ 1 & \text{elsewhere,} \end{cases}$$

where t is the first coordinate on $\mathbb{R}_+ \times (\pm Y_j)$. If s' is a section of F' let the section $\overline{\beta}(s')$ of F be the result of pulling s' back to $X^{\{T\}}$ by means of $\pi^{\{T\}}$, multiplying by β, and then extending trivially to all of X. (The notation $\pi^{\{T\}}$ was introduced in [11, Subsection 1.4].) If $x \in \mathbb{R}^m$ set $\overline{\beta}(s', x) = (\overline{\beta}(s'), x)$. For any section s of E we define a section $\underline{\beta}(s)$ of E' as follows when $\check{T} \geq 3/2$. Outside $[-T_j+1, T_j-1] \times Y_j$ we set $\underline{\beta}(s) = s$. Over $[-T_j, T_j] \times Y_j$ let $\underline{\beta}(s)$ be the sum of the restrictions of the product βs to $[0, 2T_j] \times Y_j$ and $[0, 2T_j] \times (-Y_j)$, identifying both these bands with $[-T_j, T_j] \times Y_j$ by means of the projection $\pi^{\{T\}} : X^{\{T\}} \to X^{(T)}$. If $x \in \mathbb{R}^\ell$ set $\underline{\beta}(s, x) = (\underline{\beta}(s), x)$. Note that

$$\underline{\beta}\overline{\beta} = I.$$

Now suppose Q is a left or right inverse of P. Define

$$R' = \underline{\beta} Q \overline{\beta} : L^p_k(X^{(T)}; F') \oplus \mathbb{R}^m \to L^p_{k+d}(X^{(T)}; E') \oplus \mathbb{R}^\ell.$$

If $QP = I$ then a simple calculation yields

$$\|R'P' - I\| \leq C\check{T}^{-1}\|Q\|.$$

Therefore, if $\check{T} > C\|Q\|$ then $R'P'$ is invertible and $Q' = (R'P')^{-1}R'$ is a left inverse of P'. Similarly, if $PQ = I$ then

$$\|P'R' - I\| \leq C\check{T}^{-1}\|Q\|,$$

hence $Q' = R'(P'R')^{-1}$ is a right inverse of P' when $\check{T} > C\|Q\|$. In both cases the constant C depends on the restriction of D to the ends $\mathbb{R}_+ \times (\pm Y_j)$ but is otherwise independent of P. As for the bound on $\|Q'\|$, see the proof of Lemma 2.4. □

From the proposition one easily deduces the following version of the addition formula for the index, which was proved for first order operators in [5].

Corollary A.1 *If*

$$D : L^p_{k+d}(X; E) \to L^p_k(X; F)$$

is Fredholm, then for sufficiently large $\check{T}$,

$$D' : L^p_{k+d}(X^{(T)}; E') \to L^p_k(X^{(T)}; F')$$

is Fredholm with ind(D') = *ind*(D).

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