A generalized blow-up formula for Seiberg–Witten invariants

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Abstract

We prove a gluing formula for Seiberg–Witten invariants which describes in particular the behaviour of the invariant under blow-up and rational blow-down.

1 Introduction

In this paper we use the results of [5, 6] to establish a gluing formula for Seiberg–Witten invariants of certain 4–manifolds containing a negative definite piece. The formula describes in particular the behaviour of the Seiberg–Witten invariant under blow-up and under the rational blow-down procedure introduced by Fintushel–Stern [3]. While the formula will be known in principle to experts, to our knowledge no complete proof has previously been published. As for the classical blow-up formula, this was proved by Bauer [1, Corollary 4.2]. (Earlier, a proof had been announced by Salamon [9], and there is a sketch of a proof in Nicolaescu [8].) In the case of rational blow-down the formula was stated by Fintushel–Stern [3, Theorem 8.5] with a brief outline of a proof. Apart from providing a proof in the general case, the main motivation for writing this paper was to show how the parametrized version of the gluing theorem in [6] can be used to handle at least the simplest cases of obstructed gluing, thereby providing a unified approach to a wide range of gluing problems.

Before stating our results we explain how the Seiberg–Witten invariant, usually defined for closed 4–manifolds, can easily be generalized to compact spin^c 4–manifolds Z whose boundary $Y' = \partial Z$ satisfies $b_1(Y') = 0$ and admits a metric g of positive scalar curvature. (By a spin^c–manifold we mean as in [5] an oriented smooth manifold with a spin^c–structure.) As usual we also assume that $b_2^+(Z) > 1$. Let $\{Y'_j\}$ be the components of Y', which are rational homology spheres. Let $\tilde{Z}$ be the manifold with tubular ends obtained from Z by adding a half-infinite tube $\mathbb{R}_+ \times Y'$. Choose a Riemannian metric on $\tilde{Z}$ which agrees with $1 \times g$ on the ends. We consider the monopole equations on $\tilde{Z}$ perturbed solely by means of a smooth 2–form μ on $\tilde{Z}$ supported in Z as in [5, Equation 13]. Let $M = M(\tilde{Z})$ denote the moduli space of monopoles over $\tilde{Z}$ that are asymptotic over $\mathbb{R}_+ \times Y'_j$ to the unique (reducible) monopole over Y'_j. For generic μ the moduli space M will be free of reducibles and a smooth compact manifold of dimension

$$\dim M = 2h(Y') + \frac{1}{4}(c_1(\mathcal{L}_Z)^2 - \sigma(Z)) - b_0(Z) + b_1(Z) - b_2^+(Z),$$

see [5, Section 9]. Choose a base-point $x \in \tilde{Z}$ and let M_x be the framed moduli space defined just as M except that we now only divide out by those gauge transformations u for which $u(x) = 1$. Let $\mathbb{L} \to M$ be the complex line bundle whose sections are given by maps $s : M_x \to \mathbb{C}$ satisfying

$$s(u(\omega)) = u(x) \cdot s(\omega) \tag{1}$$

for all $\omega \in M_x$ and gauge transformations u. A choice of homology orientation of Z determines an orientation of M, and we can then define the Seiberg–Witten invariant of Z just as for closed 4–manifolds:

$$\mathrm{SW}(Z) = \begin{cases} \langle c_1(\mathbb{L})^k, [M] \rangle & \text{if} \quad \dim M = 2k \geq 0, \\ 0 & \text{if} \quad \dim M \text{ is negative or odd.} \end{cases}$$

The use of $\mathbb{L}$ rather than $\mathbb{L}^{-1}$ prevents a sign in Theorem 1 below. (Another justification is that, although $M_x \to M$ is a principal bundle with respect to the canonical U(1)–action, it seems more natural to regard that action as a *left* action.) This invariant $\mathrm{SW}(Z)$ depends only on the homology oriented spin^c–manifold Z, not on the choice of positive scalar curvature metric g on Y'; the proof of this is a special case of the proof of the generalized blow-up formula, which we are now ready to state.

Theorem 1 *Let Z be a connected, compact, homology oriented spin^c 4–manifold whose boundary $Y' = \partial Z$ satisfies $b_1(Y') = 0$ and admits a metric of positive scalar curvature, and such that $b_2^+(Z) > 1$. Suppose Z is separated by an embedded rational homology sphere Y admitting a metric of positive scalar curvature,*

$$Z = Z_0 \cup_Y Z_1,$$

where $b_1(Z_0) = b_2^+(Z_0) = 0$. Let Z_1 have the orientation, homology orientation, and spin^c structure inherited from Z. Then

$$SW(Z) = SW(Z_1) \quad \text{if} \quad \dim M(\tilde{Z}) \geq 0.$$

We will show in Section 2 that $\dim M(\tilde{Z}_0) \leq -1$. (A particular case of this was proved by different methods in [3, Lemma 8.3].) The addition formula for the index then yields

$$\dim M(\tilde{Z}) = \dim M(\tilde{Z}_0) + 1 + \dim M(\tilde{Z}_1) \leq \dim M(\tilde{Z}_1).$$

The following corollary describes the effect on the Seiberg–Witten invariant of both ordinary blow-up and rational blow-down:

Corollary 1 *Let Z_0, Z_0', Z_1 be compact, homology oriented spin^c 4–manifolds with $-\partial Z_1 = \partial Z_0 = \partial Z_0' = Y$ as spin^c manifolds, where Y is a spin^c rational homology sphere admitting a metric of positive scalar curvature. Suppose $b_2^+(Z_1) > 1$, $b_1(Z_0) = b_1(Z_0') = 0$, and $b_2(Z_0) = b_2^+(Z_0') = 0$. Let*

$$Z = Z_0 \cup_Y Z_1, \quad Z' = Z_0' \cup_Y Z_1$$

have the orientation, homology orientation and spin^c structure induced from Z_0, Z_0', Z_1. Then

$$SW(Z) = SW(Z') \quad \textit{if} \quad \dim M(Z') \geq 0.$$

Proof. Set $n_\pm = \dim M(\pm\tilde{Z}_0)$ and $W = Z_0 \cup_Y (-Z_0)$. Then

$$-1 = \dim M(W) = n_+ + 1 + n_-,$$

hence $n_\pm = -1$. Thus

$$\dim M(Z) = \dim M(\tilde{Z}_1) \geq \dim M(Z') \geq 0.$$

The theorem now yields

$$\mathrm{SW}(Z) = \mathrm{SW}(Z_1) = \mathrm{SW}(Z'). \qquad \square$$

2 Preliminaries on negative definite 4–manifolds

Let X be a connected spin^c Riemannian 4–manifold with tubular ends $\mathbb{R}_+ \times Y_j$, $j = 1, \ldots, r$, as in [5, Subsection 1.3]. Suppose each Y_j is a rational homology sphere and $b_1(X) = 0 = b_2^+(X)$. We consider the monopole equations on X perturbed only by means of a 2–form μ as in [5, Equation 13], where now μ is supported in a given non-empty, compact, codimension 0 submanifold $K \subset X$. Let $\alpha_j \in \mathcal{R}_{Y_j}$ be the reducible monopole over Y_j and $M_\mu = M(X; \vec{\alpha}; \mu; 0)$ the moduli space of monopoles over X with asymptotic limits $\vec{\alpha} = (\alpha_1, \ldots, \alpha_r)$. This moduli space contains a unique reducible point $\omega(\mu) = [A(\mu), 0]$. Let $\Omega^+_{X,K}$ denote the space of (smooth) self-dual 2–forms on X supported in K, with the C^∞ topology. Let p and w be the exponent and weight function used in the definition of the moduli space M_μ, as in [5, Subsection 3.4].

Lemma 1 *Let R be the set of all $\mu \in \Omega^+_{X,K}$ such that the operator*

$$D_{A(\mu)} : L^{p,w}_1(\mathbb{S}^+_X) \to L^{p,w}(\mathbb{S}^-_X) \tag{2}$$

is either injective or surjective. Then R is open and dense in $\Omega^+_{X,K}$.

Of course, whether the operator is injective or surjective for a given $\mu \in R$ is determined by its index, which is independent of μ.

Proof. By [5, Proposition 2.2 (ii)] and the proof of [5, Proposition 5.2], the operator

$$d^+ : \ker(d^*) \cap L^{p,w}_1 \to L^{p,w}$$

is an isomorphism. Therefore, if A_o is a reference connection over X with limits α_j as in [5, Subsection 3.4] then there is a unique (smooth) $a = a(\mu) \in L^{p,w}_1$ with

$$d^* a = 0, \qquad d^+ a = -\hat{F}^+(A_o) - i\mu.$$

Hence we can take $A(\mu) = A_o + a(\mu)$. Since the operator (2) has closed image, it follows by continuity of the map $\mu \mapsto A(\mu)$ that R is open in $\Omega^+_{X,K}$.

To see that R is dense, fix $\mu \in \Omega^+_{X,K}$ and write $A = A(\mu)$. Let W be a Banach space of smooth 1–forms on X supported in K as provided by [5, Lemma 8.2]. Using the unique continuation property of the Dirac operator it is easy to see that 0 is a regular value of the smooth map

$$\begin{aligned} h : W \times (L^{p,w}_1(\mathbb{S}^+_X) \setminus \{0\}) &\to L^{p,w}(\mathbb{S}^-_X), \\ (\eta, \Phi) &\mapsto D_{A+i\eta}\Phi. \end{aligned}$$

In general, if $f_1 : E \to F_1$ and $f_2 : E \to F_2$ are surjective homomorphisms between vector spaces then $f_1|_{\ker f_2}$ and $f_2|_{\ker f_1}$ have identical kernels and isomorphic cokernels. In particular, the projection $\pi : h^{-1}(0) \to W$ is a Fredholm map whose index at every point agrees with the index m of D_A. By the Sard-Smale theorem the regular values of π form a residual (hense dense) subset of W. If $\eta \in W$ is a regular value then we see that $D_{A+i\eta}$ is injective when $m \le 0$ and surjective when $m > 0$. Since the topology on W is stronger than the C^∞ topology it follows that R contains points of the form $\mu + d^+\eta$ arbitrarily close to μ. $\square$

Lemma 2 *Suppose the metric on each Y_j has positive scalar curvature. Let R' be the set of all $\mu \in \Omega^+_{X,K}$ such that the irreducible part M^*_μ is empty and the operator $D_{A(\mu)}$ in (2) is injective. Then R' is open and dense in $\Omega^+_{X,K}$.*

Proof. Recall that M^*_μ has expected dimension $2m-1$, where $m = \mathrm{ind}_{\mathbb{C}} D_{A(\mu)}$.

Suppose $m > 0$. We will show that this leads to a contradiction. Let R'' be the set of all $\mu \in \Omega^+_{X,K}$ for which M_μ is regular. (Note that the reducible point is regular precisely when $D_{A(\mu)}$ is surjective.) From Lemma 1 and [5, Proposition 8.2] one finds that R'' is dense in $\Omega^+_{X,K}$. (Starting with a given μ, first perturb it a little to make the reducible point regular, then a little more to make also the irreducible part regular.) But for any $\mu \in R''$ the moduli space M_μ would be compact with one reducible point, which yields a contradiction as in [4]. Therefore, $m \le 0$.

We now see, exactly as for R'', that R' is dense in $\Omega^+_{X,K}$. To prove that R' is open we use a compactness argument together with the following fact: For any given $\mu_0 \in R'$ there is a neighbourhood U of $\omega(\mu_0)$ in $\mathcal{B}(X;\vec{\alpha})$ such that

$$M^*_\mu \cap U = \emptyset$$

for any $\mu \in \Omega^+_{X,K}$ with $\|\mu-\mu_0\|_p$ sufficiently small. To prove this we work in a slice at $(A(\mu_0),0)$, ie we represent $\omega(\mu)$ (uniquely) by $(A,0)$ where $d^*(A-A(\mu_0))=0$, and we consider a point in M^*_μ represented by $(A+a,\phi)$ where $d^*a=0$. Note that since $b_1(X)=0$, the latter representative is unique up to multiplication of ϕ by unimodular constants.

Observe that there is a constant $C_1 < \infty$ such that if $\|\mu-\mu_0\|_p$ is sufficiently small then

$$\|\psi\|_{L^{p,w}_1} \le C_1 \|D_A\psi\|_{L^{p,w}}$$

for all $\psi \in L^{p,w}_1$. Hence if $L = (d^*+d^+, D_A)$ then for such μ one has

$$\|s\|_{L^{p,w}_1} \le C_2 \|Ls\|_{L^{p,w}}$$

for all $s \in L^{p,w}_1$. Denoting by SW_μ the Seiberg–Witten map over X for the perturbation form μ we have

$$0 = \mathrm{SW}_\mu(A+a,\phi) - \mathrm{SW}_\mu(A,0) = (d^+a - Q(\phi), D_A\phi + a\phi),$$

where Q is as in [5]. Taking $s=(a,\phi)$ we obtain

$$\|s\|_{L^{p,w}_1} \le C_2\|Ls\|_{L^{p,w}} \le C_3\|s\|^2_{L^{2p,w}} \le C_4\|s\|^2_{L^{p,w}_1}.$$

Since $s \neq 0$ we conclude that

$$\|s\|_{L^{p,w}_1} \geq C_4^{-1}.$$

Choose $\delta \in (0, C_4^{-1})$ and define

$$U = \{[A(\mu_0) + b, \psi] \,:\, \|(b,\psi)\|_{L^{p,w}_1} < \delta,\ d^*b = 0\}.$$

If $\|\mu - \mu_0\|_p$ is so small that $\|A - A(\mu_0)\|_{L^{p,w}_1} \leq C_4^{-1} - \delta$ then

$$\|(A + a - A(\mu_0), \phi)\|_{L^{p,w}_1} \geq \|s\|_{L^{p,w}_1} - \|A - A(\mu_0)\|_{L^{p,w}_1} \geq \delta,$$

hence $[A + a, \phi] \notin U$. $\square$

3 The extended monopole equations

We now return to the situation in Theorem 1. Set $X_j = \tilde{Z}_j$ for $j = 0, 1$. Choose metrics of positive scalar curvature on Y and Y' and a metric on the disjoint union $X = X_0 \cup X_1$ which agrees with the corresponding product metrics on the ends. Let Y be oriented as the boundary of Z_0, so that X_0 has an end $\mathbb{R}_+ \times Y$ and X_1 an end $\mathbb{R}_+ \times (-Y)$. Gluing these two ends of X we obtain as in [5, Subsection 1.4] a manifold $X^{(T)}$ for each $T > 0$.

Choose smooth monopoles α over Y and α'_j over Y'_j (these are reducible, and unique up to gauge equivalence). Let $S_o = (A_o, \Phi_o)$ be a reference configuration over X with these limits over the ends, and S'_o the associated reference configuration over $X^{(T)}$. Adopting the notation introduced in the beginning of [6, Subsection 2.2], let $\mathcal{C}$ be the corresponding $L^{p,w}_1$ configuration space over X and $\mathcal{C}'$ the corresponding $L^{p,\kappa}_1$ configuration space over $X^{(T)}$. For any finite subset $\mathfrak{b} \subset X_{:0} = Z_0 \cup Z_1$ let $\mathcal{G}_\mathfrak{b}, \mathcal{G}'_\mathfrak{b}$ be the corresponding groups of gauge transformations that restrict to 1 on $\mathfrak{b}$.

As in Section 2 we first consider the monopole equations over X and $X^{(T)}$ perturbed only by means of a self-dual 2–form $\mu = \mu_0 + \mu_1$, where μ_j is supported in Z_j. The corresponding moduli spaces will be denoted $M(X)$ and $M^{(T)} = M(X^{(T)})$. Of course, $M(X)$ is a product of moduli spaces over X_0 and X_1:

$$M(X) = M(X_0) \times M(X_1).$$

By Lemma 2 we can choose μ_0 such that $M(X_0)$ consists only of the reducible point (which we denote by $\omega_{\mathrm{red}} = [A_{\mathrm{red}}, 0]$), and such that the operator

$$D_{A_{\mathrm{red}}} : L^{p,w}_1(\mathbb{S}^+_{X_0}) \to L^{p,w}(\mathbb{S}^-_{X_0}) \tag{3}$$

is injective. By [5, Proposition 8.2] and unique continuation for self-dual closed 2–forms we can then choose μ_1 such that

- $M(X_1)$ is regular and contains no reducibles, and
- the irreducible part of $M^{(T)}$ is regular for all natural numbers T.

Set

$$k = -\mathrm{ind}_{\mathbb{C}}(D_{A_{\mathrm{red}}}) \geq 0.$$

If $k > 0$ then ω_{red} is not a regular point of $M(X_0)$ and we cannot appeal to the gluing theorem [6, Theorem 2.1] for describing $M^{(T)}$ when T is large. We will therefore introduce an extra parameter $z \in \mathbb{C}^k$ into the Dirac equation on Z_0, to obtain what we will call the "extended monopole equations", such that ω_{red} becomes a regular point of the resulting parametrized moduli space over X_0. This will allow us to apply the gluing theorem for parametrized moduli spaces, [6, Theorem 5.1].

We are going to add to the Dirac equation an extra term $\beta(A, \Phi, z)$ which will be a product of three factors:

(i) a holonomy term h_A (to achieve gauge equivariance)

(ii) a cut-off function $g(A, \Phi)$ (to retain an apriori pointwise bound on Φ)

(iii) a linear combination $\sum z_j \psi_j$ of certain negative spinors (to make ω_{red} regular).

We will now describe these terms more precisely.

(i) Choose an embedding $f : \mathbb{R}^4 \to \mathrm{int}(Z_0)$, and set $x_0 = f(0)$ and $U_0 = f(\mathbb{R}^4)$. For each $x \in U_0$ let $\gamma_x : [0,1] \to U_0$ be the path from x_0 to x given by

$$\gamma_x(t) = f(tf^{-1}(x)).$$

For any spin^c connection A over U_0 define the function $h_A : U_0 \to \mathrm{U}(1)$ by

$$h_A(x) = \exp\left(-\int_{[0,1]} \gamma_x^*(A - A_{\mathrm{red}}) \right),$$

cf. [6, Equation 1]. Note that h_A depends on the choice of A_{red}, which is only determined up to modification by elements of $\mathcal{G}$.

(ii) Set $K_0 = f(D^4)$, where $D^4 \subset \mathbb{R}^4$ is the closed unit disk. Choose a smooth function $g : \mathcal{B}^*(K_0) \to [0,1]$ such that $g(A, \Phi) = 0$ when $\|\Phi\|_{L^\infty(K_0)} \geq 2$ and $g(A, \Phi) = 1$ when $\|\Phi\|_{L^\infty(K_0)} \leq 1$. Extend g to $\mathcal{B}(K_0)$ by setting $g(A, 0) = 1$ for all A.

(iii) By unique continuation for the formal adjoint $D^*_{A_{\mathrm{red}}}$ there are smooth sections $\psi_1, \ldots, \psi_k$ of $\mathbb{S}^-_{X_0}$ supported in K_0 and spanning a linear complement of the image of the operator $D_{A_{\mathrm{red}}}$ in (3).

For any configuration (A,Φ) over X and $z=(z_1,\dots,z_k)\in\mathbb{C}^k$ define

$$\beta(A,\Phi,z)=g(A,\Phi)\,h_A\sum_{j=1}^k z_j\psi_j.$$

Note that for gauge transformations u over X one has

$$u(x_0)\,h_{u(A)}=u\,h_A.$$

Since g is gauge invariant, this yields

$$\beta(u(A),u\,\Phi,u(x_0)z)=u\cdot\beta(A,\Phi,z).$$

The following lemma is useful for estimating the holonomy term h_A:

Lemma 3 *Let $a=\sum a_j dx_j$ be a 1–form on the closed unit disk D^n in $\mathbb{R}^n$, $n>1$. For each $x\in D^n$ let $J(x)$ denote the integral of a along the line segment from 0 to x, ie*

$$J(x)=\sum_{j=1}^n x_j\int_0^1 a_j(tx)\,dt.$$

Then for any $q\geq 1$ and $r>qn$ and non-negative integer k there is a constant $C<\infty$ independent of a such that

$$\|J\|_{L^q_k(D^n)}\leq C\|a\|_{L^r_k(D^n)}.$$

Proof. If b is a function on D^n and χ the characteristic function of the interval $[0,1]$ then

$$\begin{aligned}\int_{D^n}\int_0^1 b(tx)\,dt\,dx &= \int_{D^n} b(x)\int_0^1 t^{-n}\chi(t^{-1}|x|)\,dt\,dx\\ &= \frac{1}{n-1}\int_{D^n}(|x|^{1-n}-1)\,b(x)\,dx.\end{aligned}$$

From this basic calculation the lemma is easily deduced. □

It follows from the lemma that $a\mapsto h_{A_{\mathrm{red}}+a}$ defines a smooth map $L^p_1(K_0;i\Lambda^1)\to L^q_1(K_0)$ provided $p>4q>16$. Hence, if $p>16$ (which we henceforth assume) then

$$\mathcal{C}(K_0)\times\mathbb{C}^k\to L^p(K_0;\mathbb{S}^-),\quad ((A,\Phi),z)\mapsto\beta(A,\Phi,z)$$

is a smooth map whose derivative at every point is a compact operator. Here $\mathcal{C}(K_0)$ is the L^p_1 configuration space over K_0.

The extended monopole equations for $((A,\Phi),z)\in\mathcal{C}\times\mathbb{C}^k$ are

$$\begin{aligned} \hat{F}^+_A + i\mu - Q(\Phi) &= 0,\\ D_A\Phi + \beta(A,\Phi,z) &= 0. \end{aligned} \tag{4}$$

(Cf. the holonomy perturbations of the instanton equations constructed in [2, 2 (b)].) We define actions of $\mathcal{G}$ and $\mathcal{G}'$ on $\mathcal{C}\times\mathbb{C}^k$ and $\mathcal{C}'\times\mathbb{C}^k$ respectively by

$$u(S,z) = (u(S), u(x_0)z).$$

Then the left hand side of (4) describes a $\mathcal{G}$–equivariant smooth map $\mathcal{C}\times\mathbb{C}^k\to L^{p,w}$.

For $\epsilon>0$ let $B^{2k}_\epsilon\subset\mathbb{C}^k$ denote the open ball of radius ϵ about the origin, and D^{2k}_ϵ the corresponding closed ball. For $0<\epsilon\le 1$ set

$${}_\epsilon M_{\mathfrak{b}}(X) = \{\text{solutions } ((A,\Phi),z)\in\mathcal{C}\times B^{2k}_\epsilon \text{ to } (4)\}\Big/\mathcal{G}_{\mathfrak{b}},$$

This moduli space is clearly a product of moduli spaces over X_0 and X_1:

$${}_\epsilon M_{\mathfrak{b}}(X) = {}_\epsilon M_{\mathfrak{b}_0}(X_0)\times M_{\mathfrak{b}_1}(X_1),$$

where $\mathfrak{b}_j = \mathfrak{b}\cap X_j$.

Noting that the equations (4) also make sense over $X^{(T)}$ we define

$${}_\epsilon M^{(T)}_{\mathfrak{b}} = \{\text{solutions } ((A,\Phi),z)\in\mathcal{C}'\times B^{2k}_\epsilon \text{ to } (4)\}\Big/\mathcal{G}'_{\mathfrak{b}}.$$

We define ${}^\epsilon M_{\mathfrak{b}}(X)$ and ${}^\epsilon M^{(T)}_{\mathfrak{b}}$ in a similar way as ${}_\epsilon M_{\mathfrak{b}}(X)$ and ${}_\epsilon M^{(T)}_{\mathfrak{b}}$, but with D^{2k}_ϵ in place of B^{2k}_ϵ.

Choose a base-point $x_1\in Z_1$. We will only consider the cases when $\mathfrak{b}$ is a subset of $\{x_0,x_1\}$, and we indicate $\mathfrak{b}$ by listing its elements (writing ${}_\epsilon M_{x_0,x_1}$ and ${}_\epsilon M$ etc).

Lemma 4 *Any element of ${}^1M(X_0)$ or ${}^1M^{(T)}$ has a smooth representative.*

Proof. Given Lemma 3 this is proved in the usual way. □

Lemma 5 *There is a $C<\infty$ independent of T such that $\|\Phi\|_\infty < C$ for all elements $[A,\Phi,z]$ of ${}^1M(X)$ or ${}^1M^{(T)}$.*

Proof. Suppose $|\Phi|$ achieves a local maximum ≥ 2 at some point x. If $x \notin K_0$ then one obtains a bound on $|\Phi(x)|$ using the maximum principle as in [7, Lemma 2]. If $x \in K_0$ then the same works because then $g(A, \Phi) = 0$. $\square$

Lemma 6 *${}^1M(X)$ and ${}^1M^{(T)}$ are compact for all $T > 0$.*

Proof. Given Lemmas 3 and 5, the second approach to compactness in [5] carries over. $\square$

We identify $M_{\mathfrak{b}_0}(X_0)$ with the set of elements of ${}_1M_{\mathfrak{b}_0}(X_0)$ with $z = 0$, and similarly for moduli spaces over $X, X^{(T)}$. It is clear from the definition of $\beta(A, \Phi, z)$ that ω_{red} is a regular point of ${}_1M(X_0)$. Since ${}_1M_{x_0}(X_0)$ has expected dimension 0, it follows that ω_{red} is an isolated point of ${}_1M_{x_0}(X_0)$. Because ${}^1M_{x_0}(X_0)$ is compact, there is an ϵ such that ${}_\epsilon M_{x_0}(X_0)$ consists only of the point ω_{red}. Fix such an ϵ for the remainder of the paper.

Lemma 7 *If $\omega_n \in {}_\epsilon M^{(T_n)}$ with $T_n \to \infty$ then a subsequence of $\{\omega_n\}$ chain-converges to (ω_{red}, ω) for some $\omega \in M(X_1)$.*

Proof. Again, this is proved as in [5] using the second approach to compactness. $\square$

Corollary 2 *If $T \gg 0$ then ${}_\epsilon M^{(T)}$ contains no element which is reducible over Z_1.* $\square$

4 Applying the gluing theorem

Let $\text{Hol} = \text{Hol}_1$ be defined as in [6, Equation 1] in terms of a path in $X^{(T)}$ from x_0 to x_1 running once through the neck.

By [6, Proposition 2.3], if $K_1 = (X_1)_{:\mathfrak{t}}$ with $\mathfrak{t} \gg 0$ then there is a U(1)–invariant open subset $V_1 \subset \mathcal{B}^*_{x_1}(K_1) = \mathcal{B}_{x_1}(K_1)$ containing $R_{K_1}(M_{x_1}(X_1))$, and a U(1)–equivariant smooth map

$$q_1 : V_1 \to M_{x_1}(X_1)$$

such that $q_1(\omega|_{K_1}) = \omega$ for all $\omega \in M_{x_1}(X_1)$. Here R_{K_1} denotes restriction to K_1. It follows from Lemma 7 that if T is sufficiently large then $\omega|_{K_1} \in V_1$ for all $\omega \in {}_\epsilon M^{(T)}_{x_1}$.

Proposition 1 *For all sufficently large T the moduli space ${}_\epsilon M^{(T)}_{x_1}$ is regular and the map*

$$ {}_\epsilon M^{(T)}_{x_1} \to M_{x_1}(X_1), \quad \omega \mapsto q_1(\omega|_{K_1}) \tag{5} $$

is an orientation preserving $U(1)$–equivariant diffeomorphism.

Proof. We will apply the version of [6, Theorem 5.1] with (in the notation of [6]) $\mathbb{T}$ acting non-trivially on W. Set

$$ G = {}_\epsilon M_{x_0,x_1}(X) = \{\omega_{\mathrm{red}}\} \times M_{x_1}(X_1), $$
$$ K = K_0 \cup K_1, $$
$$ V = \mathcal{B}_{x_0}(K_0) \times V_1 \times B^{2k}_\epsilon. $$

Note that G is compact and $\breve{\mathcal{G}}_{\mathfrak{b}}(K) = \mathcal{G}_{\mathfrak{b}}(K)$. Define

$$ q : V \to G, \quad (\omega_0, \omega_1, z) \mapsto (\omega_{\mathrm{red}}, q_1(\omega_1)). $$

In general, an element $(u_0, u_1) \in \mathrm{U}(1)^2$ acts on appropriate configuration and moduli spaces like any gauge transformation u with $u(x_j) = u_j$, $j = 0, 1$, and it acts on B^{2k}_ϵ by multiplication with u_0. Then clearly, q is $\mathrm{U}(1)^2$–equivariant, so by the gluing theorem there is a compact, codimension 0 submanifold $K' \subset X$ containing K and a $\mathrm{U}(1)^2$–equivariant open subset $V' \subset \mathcal{B}^*_{x_0,x_1}(K') \times B^{2k}_\epsilon$ containing $R_{K'}(G)$ and satisfying $R_K(V') \subset V$ and such that for all sufficiently large T the space

$$ G^{(T)} = \{(\omega, z) \in {}_\epsilon M^{(T)}_{x_0,x_1} \,:\, (\omega|_{K'}, z) \in V'\} $$

consists only of regular points, and the map

$$ G^{(T)} \to \mathrm{U}(1) \times {}_\epsilon M_{x_0,x_1}(X), \quad (\omega, z) \mapsto (\mathrm{Hol}(\omega), (\omega_{\mathrm{red}}, q_1(\omega|_{K_1}))) \tag{6} $$

is a $\mathrm{U}(1)^2$–equivariant diffeomorphism. But it follows from Lemma 7 that $G^{(T)} = {}_\epsilon M^{(T)}_{x_0,x_1}$ for $T \gg 0$, and dividing out by the action of $\mathrm{U}(1) \times \{1\}$ in (6) we see that (5) is a $\mathrm{U}(1)$–equivariant diffeomorphism.

We now discuss orientations. Given $\delta = \pm 1$ we will say a map is δ–*preserving* if it changes orientations by the factor δ. Set

$$ c := b_1(X) + b_2^+(X). $$

By [6, Proposition 4.3 (ii) and Theorem 5.1] the map (6) is $(-1)^{c+1}$–preserving. Using [6, Proposition 4.4] it is a simple exercise to show that ${}_\epsilon M_{x_0,x_1}(X) \to M_{x_1}(X_1)$ is $(-1)^c$–preserving. On the other hand, $(u_0, 1) \in \mathrm{U}(1) \times \{1\}$ acts on U(1) in (6) by multiplication with u_0^{-1}. Thus we have got three signs, which cancel each other since $(c+1)+c+1$ is even. Therefore, the map (5) does preserve orientations. $\square$

Proof of Theorem 1: For large T let $\mathbb{L} \to {}_\epsilon M^{(T)}$ be the complex line bundle associated to the base-point x_1 as in Section 1. For $j = 1, \ldots, k$ the map

$$s_j : {}_\epsilon M^{(T)}_{x_1} \to \mathbb{C}, \quad [A, \Phi, z] \mapsto \mathrm{Hol}(A) \cdot z_j$$

is U(1)–equivariant in the sense of (1) and therefore defines a smooth section of $\mathbb{L}$. The sections s_j together form a section s of the bundle $\mathbb{E} = \oplus^k \mathbb{L}$ whose zero set is the unparametrized moduli space $M^{(T)}$. It is easy to see that s is a regular section precisely when $M^{(T)}$ is a regular moduli space, which by Corollary 2 and the choice of μ_1 holds at least when T is a sufficiently large natural number. In that case $s^{-1}(0) = M^{(T)}$ as oriented manifolds. Set

$$\ell = \frac{1}{2} \dim M^{(T)} \geq 0,$$

so that $\dim M(X_1) = 2(k + \ell)$. If ℓ is not integral then $\mathrm{SW}(Z_1) = 0 = \mathrm{SW}(Z)$ and we are done. Now suppose ℓ is integral and let T be a large natural number. Choose a smooth section s' of $\mathbb{E}' = \oplus^\ell \mathbb{L}$ such that $\sigma = s'|_{M^{(T)}}$ is a regular section of $\mathbb{E}'|_{M^{(T)}}$, or equivalently, such that $s \oplus s'$ is a regular section of $\mathbb{E} \oplus \mathbb{E}' = \oplus^{k+\ell} \mathbb{L}$. Then

$$\mathrm{SW}(Z_1) = \#(s \oplus s')^{-1}(0) = \#\sigma^{-1}(0) = \mathrm{SW}(Z),$$

where the first equality follows from Proposition 1, and # as usual means a signed count. $\square$

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