Density of extremal measures in parabolic potential theory

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Abstract

It is shown that, for the heat equation on $\mathbb{R}^d \times \mathbb{R}$, $d \geq 1$, any convex combination of harmonic (= caloric) measures $\mu_x^{U_1}, \dots, \mu_x^{U_k}$, where $U_1, \dots, U_k$ are relatively compact open neighborhoods of a given point x, can be approximated by a sequence $(\mu_x^{W_n})_{n\in\mathbb{N}}$ of harmonic measures such that each W_n is an open neighborhood of x in $U_1 \cup \cdots \cup U_k$. Moreover, it is proven that, for every open set U in $\mathbb{R}^{d+1}$ containing x, the extremal representing measures for x with respect to the convex cone of potentials on U (these measures are obtained by balayage, with respect to U, of the Dirac measure at x on Borel subsets of U) are dense in the compact convex set of all representing measures. Since essential ingredients for a proof of corresponding results in the classical case (or more general elliptic situations; see [13]) are not available

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for the heat equation, an approach heavily relying on the transit character of the hyperplanes $\mathbb{R}^d \times \{c\}$, $c \in \mathbb{R}$, is developed.

In fact, the new method is suitable to obtain convexity results for limits of harmonic measures and the density of extremal representing measures on $X := X' \times \mathbb{R}$ for practically every space-time structure which is given by a sub-Markov semigroup $(P_t)_{t>0}$ on a space X' such that there are strictly positive continuous densities $(t,x,y) \mapsto p_t(x,y)$ with respect to a (non-atomic) measure on X'. In particular, this includes many diffusions and corresponding symmetric processes given by heat kernels on manifolds and fractals.

Moreover, the results may be applied to restrictions of the space-time structure on arbitrary open subsets.

Keywords: Heat equation, parabolic potential theory, balayage space, semigroup, resolvent, excessive function, heat kernel, space-time process, stable process, balayage, harmonic measure, caloric measure, extremal measure, Harnack's inequality, Green function,

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1 Introduction

In [13] it is shown that any convex combination of harmonic measures $\mu_x^{U_1}, \dots, \mu_x^{U_k}$, where $U_1, \dots, U_k$ are relatively compact open neighborhoods of a given point $x \in \mathbb{R}^d$, $d \geq 2$, can be approximated by a sequence $(\mu_x^{W_n})_{n\in\mathbb{N}}$ of harmonic measures such that each W_n is an open neighborhood of x in $U_1 \cup \dots \cup U_k$. Moreover, it is proven that, for every Green domain X containing x, the extremal representing measures for x with respect to the convex cone of potentials on X (these measures are obtained by balayage, with respect to X, of the Dirac measure at x on Borel subsets of X) are dense in the compact convex set of all representing measures. These results were established simultaneously for the classical potential theory, for the theory of Riesz potentials, and for fairly general Brelot spaces. The methods used in [13] rely heavily on Harnack's inequalities in a form which is valid only for elliptic structures. It remained entirely questionable, if analogous results would hold for the heat equation on open sets in $\mathbb{R}^d \times \mathbb{R}$ or even in more general parabolic situations.

The purpose of this paper is to show that, nevertheless, the main convexity and density statements hold for the heat equation and similar space-time structures, given by strong Feller semigroups such as, for example, the symmetric α-stable semigroup or semigroups for diffusions on fractals.

The clue to these results is the observation that, for the heat equation, say, potentials of measures, which are supported by a compact set K in $\mathbb{R}^d \times (-\infty, 0)$, still satisfy local Harnack's inequalities on $\mathbb{R}^d \times (0, \infty)$. Having this in mind, the "uniform motion" structure in the time coordinate can be exploited to obtain a suitable approximation of balayage on Borel sets by sweeping on unions of small compact sets contained in finitely many hyperplanes of the form $H_c := \mathbb{R}^d \times \{c\}$, $c \in \mathbb{R}$. In fact, we shall use only two properties of the underlying space.

- The sets H_c, c rational, are pairwise disjoint and their union A is finely dense.
- Each H_c is a transit set.

In probabilistic terms, the first condition expresses the fact that the corresponding process will immediately hit the set A, whereas the second condition means that it may hit each H_c once, but then never again (for an analytic definition, see (1.3)).

Therefore we shall present our results in the general setting of balayage spaces (see [4]). The reader, who is not interested in this generality, may assume that we are only dealing with the heat equation or the space-time structure given by the symmetric α-stable semigroup (or by more general strong Feller semigroups; see Section 7).

So let X be a locally compact space with countable base and let $\mathcal{W}$ be a convex cone of positive lower semicontinuous functions on X (the positive hyperharmonic functions) such that $(X,\mathcal{W})$ is a balayage space, where all points are polar. For convenience, we shall suppose that $1 \in \mathcal{W}$ (which can always be achieved by normalization).

Let $\mathcal{C}(X)$ denote the set of all finite continuous functions on X, and let $\mathcal{K}(X)$ and $\mathcal{C}_0(X)$ be the set of all functions $f \in \mathcal{C}(X)$ having compact support $S(f)$ and vanishing at ∞, respectively. Let $\mathcal{B}(X)$ denote the set of Borel measurable functions on X. We shall write $A \in \mathcal{B}(X)$, if the characteristic function 1_A is Borel measurable.

Let us recall that, for all $A \subset X$, $v \in \mathcal{W}$, and $x \in X$,

$$R_v^A := \inf\{w \in \mathcal{W} \colon w \ge v \text{ on } A\} \qquad \text{and} \qquad \hat{R}_v^A(x) := \liminf_{y \to x} R_v^A(y).$$

It is known that $\hat{R}_v^A = R_v^A$ on $X \setminus A$ (see [4, VI.2.4]).

Let $\mathcal{P}$ denote the convex cone of finite continuous potentials on X, that is, $\mathcal{P}$ is the set of all $p \in \mathcal{W} \cap \mathcal{C}(X)$ such that $p/v \in \mathcal{C}_0(X)$ for some strictly positive $v \in \mathcal{W} \cap \mathcal{C}(X)$. The superharmonic support $C(p)$ of $p \in \mathcal{P}$ is the smallest closed set A in X such that $R_p^A = p$.

Let $\mathcal{M}(\mathcal{P}(X))$ denote the set of all positive Radon measures ν on X such that $\nu(p_0) < \infty$ for some $p_0 \in \mathcal{P}$, $p_0 > 0$. We note that every finite measure is contained in $\mathcal{M}(\mathcal{P}(X))$. For every $A \in \mathcal{B}(X)$ and every $\nu \in \mathcal{M}(\mathcal{P}(X))$, there exists a unique measure $\nu^A \in \mathcal{M}(\mathcal{P}(X))$ such that

$$\nu^A(v) = \int R_v^A \, d\nu \qquad \text{for every } v \in \mathcal{W}.$$

In particular,

$$\|\varepsilon_x^A\| = R_1^A(x) \qquad \text{for all } x \in X. \tag{1.1}$$

If U is an open set in X and $x \in U$, then $\varepsilon_x^{U^c}$ is the harmonic measure μ_x^U (where $U^c := X \setminus U$). Moreover, let us recall the following facts on iterated balayage (see [13, Lemma 2.5]). For all $\nu \in \mathcal{M}(\mathcal{P}(X))$ and $A, B \in \mathcal{B}(X)$,

$$\nu^B = (\nu^{A \cup B})^B = \nu^{A\cup B}|_B + (\nu^{A \cup B}|_{B^c})^B \qquad \text{and} \qquad \|(\nu^A)^B\| \le \|\nu^B\|. \tag{1.2}$$

For every $\nu \in \mathcal{M}(\mathcal{P}(X))$, the set

$$\mathcal{M}_\nu(\mathcal{P}(X)) := \big\{\mu \in \mathcal{M}(\mathcal{P}(X)) \colon \mu(p) \le \nu(p) \text{ for every } p \in \mathcal{P}\big\}$$

is a convex set which is compact in the topology of weak convergence of measures (that is, the topology given by the seminorms $\mu \mapsto |\mu(\varphi)|$, $\varphi \in \mathcal{K}(X)$). By [22] and [4, VI.12.4], its set of extreme points is the set

$$\big(\mathcal{M}_\nu(\mathcal{P}(X))\big)_e = \{\nu^A \colon A \in \mathcal{B}(X)\}.$$

Let us fix a metric ρ on X which is compatible with the topology of X. The main results below will be established under the following assumptions (A): There exist pairwise disjoint sets H_n, $n \in \mathbb{N}$, such that the following holds.

(i) Each H_n is a *transit set*, that is, H_n is closed and

$$\hat{R}_1^{H_n} = 0 \quad \text{on } H_n. \tag{1.3}$$

(ii) The union $\bigcup_{n\in\mathbb{N}} H_n$ is finely dense in X.

(iii) For every finite set $M \subset \mathbb{N}$, all $\varepsilon > 0$ and relatively compact open sets W in X, there exists $\delta > 0$ such that, for every $m \in M$, for all $x, y \in H_m \cap W$ and compact sets K in $\bigcup_{n\in M\setminus\{m\}} H_n \cap W$,

$$\|\varepsilon_x^K\| \le \big(1+\varepsilon\big)\|\varepsilon_y^K\|, \qquad \text{if } \rho(x,y) < \delta. \tag{1.4}$$

THEOREM 1.1. *For every $\nu \in \mathcal{M}(\mathcal{P}(X))$, the set $\{\nu^A \colon A \in \mathcal{B}(X)\}$ of extreme points is dense in $\mathcal{M}_\nu(\mathcal{P}(X))$.*

In view of the Krein-Milman theorem, this is a diluted version of Theorem 6.2. The next result is a weakened version of Corollary 6.4, which can be sharpened if $(X, \mathcal{W})$ is a harmonic space (see Corollary 6.5).

For every $k \in \mathbb{N}$, let

$$\Lambda_k := \big\{\lambda \in [0,1]^k \colon \lambda_1 + \cdots + \lambda_k = 1\big\}.$$

THEOREM 1.2. *Let $\nu \in \mathcal{M}(\mathcal{P}(X))$, let $U_1, \dots, U_k$ be open sets in X, and $\lambda \in \Lambda_k$. Then there exist relatively compact open sets W_n in X, $n \in \mathbb{N}$, such that*

$$\lim_{n\to\infty} \nu^{W_n^c} = \sum_{j=1}^k \lambda_j \nu^{U_j^c}.$$

Let us observe that the open sets W_n in Theorem 1.2 may become very large. Hence it is important to notice that for the heat equation a choice of W_n *within* the union of $U_1, \dots, U_k$ is possible (for a more general statement see Corollary 6.5).

THEOREM 1.3. *Let us suppose that $(\mathbb{R}^{d+1}, \mathcal{W})$ is the harmonic space associated with the heat equation, $U_1, \dots, U_k$ are open sets in $\mathbb{R}^{d+1}$, $\nu \in \mathcal{M}(\mathcal{P}(\mathbb{R}^{d+1}))$ is supported by $U_1 \cap \cdots \cap U_k$, and $\lambda \in \Lambda_k$.*

Then, for every exhaustion (V_n) of $(U_1 \cup \cdots \cup U_k) \setminus (\partial U_1 \cup \cdots \cup \partial U_k)$, there exist relatively compact open sets W_n such that $V_n \subset W_n \subset U_1 \cup \cdots \cup U_k$ and

$$\lim_{n\to\infty} \nu^{W_n^c} = \sum_{j=1}^k \lambda_j \nu^{U_j^c}.$$

Moreover, there is a corresponding corollary to Theorem 1.1 (where ${}^U\nu^A$ denotes the reduced measure with respect to U).

COROLLARY 1.4. *Let U be an open set in $\mathbb{R}^{d+1}$, let U be equipped with the harmonic structure given by the heat equation, and $\nu \in \mathcal{M}(\mathcal{P}(U))$. Then the set $\{{}^U\nu^A \colon A \in \mathcal{B}(X)\}$ of extreme points is dense in $\mathcal{M}_\nu(\mathcal{P}(U))$.*

The material in the subsequent sections is organized as follows. In Section 2, we shall prove that already the existence of an associated Green function in a very weak sense implies the Harnack property (iii). In particular, we shall see in Section 3 that all assumptions are satisfied for the heat equation on arbitrary open sets.

In Section 4, we will use properties (i) and (iii) to obtain that, given $\varepsilon > 0$ and a compact set K which is contained in a finite union of sets H_n, every finite family of pairwise disjoint compact sets in K is a *weak* $(1+\varepsilon)$*-Harnack family* provided their diameters are small enough.

It is this fact which allows us to use the technique of continuous growth of balayage on compact sets, developed in [13], in order to establish, in Section 5, a convexity result for balayage on weak Harnack families in our setting. Then all tools are available for proofs of the main theorems, which will be given in Section 6.

In Section 7, it is shown that our results can be applied not only to the heat equation, but as well to the space-time balayage space given by the symmetric α-stable semigroup or semigroups on fractals and, in fact, to practically all right continuous strong Feller semigroups $\mathbb{P}$ on a locally compact space with countable base. In particular, we cover a wide class of parabolic differential equations

$$Lu = \partial u/\partial t,$$

where L may be a very general second order elliptic operator (see Remark 7.2).

In Section 8, we discuss the extension of our results to the restriction of space-time structures on arbitrary open subsets.

2 Sufficient Greenish conditions

In this section, we shall see that, having a sequence (H_n) of pairwise disjoint sets which are just closed (and not necessarily transit sets), mild continuity properties of an associated Green function (in a weak sense) imply the Harnack property (iii). This will show, in particular, that our assumptions hold for the heat equation.

So let us suppose that we have a function $G\colon X \times X \to [0,\infty)$ such that the following is true:

- The set $\{G > 0\}$ is open and G is continuous on $\{G > 0\}$.
- For every $p \in \mathcal{P}$ and open neighborhood U of $C(p)$, there exist measures μ_n *on* U, $n \in \mathbb{N}$, such that

$$G^{\mu_n} := \int G(\cdot, y)\, d\mu_n(y) \;\to\; p \qquad \text{as } n \to \infty.^1 \tag{2.1}$$

[1] The proof of Proposition 2.1 will show that we only need that, given any two points $x_1, x_2 \in X$, we can find a sequence (μ_n) of measures on U such that $G^{\mu_n}(x) \to p(x)$ for $x = x_1, x_2$.

PROPOSITION 2.1. *Let (H_n) be a sequence of pairwise disjoint closed sets such that, for each choice of different $m, n \in \mathbb{N}$, $G > 0$ on $H_m \times H_n$ or $R_1^{H_n} = 0$ on H_m. Then* (iii) *holds.*

Proof. Let $M \subset \mathbb{N}$ be finite, $\varepsilon > 0$, and let W be a relatively compact open set in X. Let us fix $m \in M$, and let M^+ denote the set of all $n \in M \setminus \{m\}$ such that $G > 0$ on $H_m \times H_n$. There exist $\gamma > 0$ and pairwise disjoint compact neighborhoods C_k of $\overline{W} \cap H_k$, $k \in M^+ \cup \{m\}$, such that $G \geq \gamma$ on $C_m \times C_n$ for all $n \in M^+$. Let C be the union of the sets C_n, $n \in M^+$. Since G is finite and continuous on $C_m \times C$, there exists $\delta_m > 0$ such that, for all $x, y \in C_m$ with $\rho(x, y) < \delta_m$ and every $z \in C$,

$$|G(x,z) - G(y,z)| < \varepsilon\gamma \tag{2.2}$$

and hence

$$G(x,z) \leq G(y,z) + \varepsilon\gamma < \big(1+\varepsilon\big)G(y,z). \tag{2.3}$$

Next we fix a compact set K in $\bigcup_{n \in M \setminus \{m\}} H_n \cap W$. Let $L := K \cap C$ and $M_0 := M \setminus (\{m\} \cup M^+)$. By assumption, $R_1^{H_n} = 0$ on H_m for all $n \in M_0$. Since $K \setminus L$ is contained in the union of all H_n, $n \in M_0$, we hence conclude that

$$R_1^L \leq R_1^K \leq R_1^L + \sum_{n \in M_0} R_1^{H_n} = R_1^L \quad \text{ on } H_m,$$

that is $R_1^K = R_1^L$ on H_m.

Now let $x, y \in H_m \cap W$ with $\rho(x,y) < \delta_m$ and let $\eta > 0$. By [4, VI.1.2], there exists an open neighborhood U of L in C such that $R_1^U(y) < R_1^L(y) + \eta$. Choosing $\varphi \in \mathcal{K}(X)$ such that $1_L \leq \varphi \leq 1_U$, we know that $p := R_\varphi \in \mathcal{P}$, $C(p) \subset S(\varphi) \subset C$ (see [4, III.5.6]), and hence $R_p^C = p$ (see [4, II.6.3]). Further, $R_1^L \leq p$ and $p(y) \leq R_1^U(y) < R_1^L(y) + \eta$. By assumption on G, there exists a measure μ on U such that $p(x) - \eta < G^\mu(x)$ and $G^\mu(y) < p(y) + \eta$. Then, by (2.3),

$$\begin{aligned} R_1^L(x) - \eta \leq p(x) - \eta < G^\mu(x) &= \int G(x,z)\, d\mu(z) \\ \leq \ \big(1+\varepsilon\big) \int G(y,z)\, d\mu(z) &= \big(1+\varepsilon\big) G^\mu(y) \leq (1+\varepsilon)(R_1^L(y) + 2\eta). \end{aligned}$$

This shows that $R_1^L(x) \leq (1+\varepsilon) R_1^L(y)$, that is, $R_1^K(x) \leq (1+\varepsilon) R_1^K(y)$.[2]

In view of (1.1) the proof of (iii) is finished taking $\delta := \min\{\delta_m \colon m \in M\}$. □

3 Heat equation on arbitrary open sets

Let us recall that, for the harmonic space associated with the heat equation

$$\frac{1}{2}\Delta u = \frac{\partial}{\partial t} u$$

[2] Let us note that, in contrast to the reduced function R_1^K, the balayage $\hat{R}_1^K$ need not be harmonic on $X \setminus K$ (see [4, VI.2.7]) so that we would not be able to approximate $\hat{R}_1^K$ by potentials G^μ with support in U!

on $\mathbb{R}^{d+1}$, $d \geq 1$, the function G, defined for $(x', r), (y', s) \in \mathbb{R}^d \times \mathbb{R}$ by

$$G\big((x', r), (y', s)\big) := \begin{cases} \dfrac{1}{(2\pi(r-s))^{d/2}} \exp\left(-\dfrac{|x'-y'|^2}{2(r-s)}\right), & \text{if } r > s, \\ 0, & \text{if } r \leq s, \end{cases}$$

is finite, lower semicontinuous on $\mathbb{R}^{d+1} \times \mathbb{R}^{d+1}$, and continuous off the diagonal. If $-\infty < a < b < \infty$, then obviously $G = 0$ on $H_a \times H_b$ and $G > 0$ on $H_b \times H_a$ (where, as before, $H_c = \mathbb{R}^d \times \{c\}$).

The functions $1_{\mathbb{R}^d \times (c,\infty)}$, $c \in \mathbb{R}$, are hyperharmonic. So it is clear that

$$(3.1) \qquad R_1^{\mathbb{R}^d \times [c,\infty)} = 1_{\mathbb{R}^d \times [c,\infty)} \quad \text{and} \quad \hat{R}_1^{\mathbb{R}^d \times [c,\infty)} = 1_{\mathbb{R}^d \times (c,\infty)}$$

(for the first equality consider the functions $1_{\mathbb{R}^d \times (c-1/n,\infty)}$, $n \in \mathbb{N}$). Moreover, since $H_c \subset \mathbb{R}^d \times (-\infty, c]$, we conclude from (3.1) that

$$(3.2) \qquad R_1^{H_c} = 0 \ \text{ on } \mathbb{R}^d \times (-\infty, c) \quad \text{and} \quad \hat{R}_1^{H_c} = 0 \ \text{ on } H_c.$$

So the sets H_c, $c \in \mathbb{R}$, are transit sets, and $R_1^{H_c} = 0$ on $H_{c'}$ if $c' < c$.

Further, for every $p \in \mathcal{P}$, there exists a measure μ on X such that $p = G^\mu$, and then the support of μ is $C(p)$ (see, for example, [9, Theorem 1.XVII.7] or [28, Theorem 22] and [29]).

Finally, for *every* dense set D in $\mathbb{R}$, the union of the hyperplanes H_c, $c \in D$, is finely dense in $\mathbb{R}^{d+1}$ (see [4, VI.4.7.5]).

Thus, because of Proposition 2.1, the assumptions (A) from Section 1 are satisfied for the heat equation on $\mathbb{R}^{d+1}$.

This is already fine, but we can do much better. Let U be an arbitrary open set in $\mathbb{R}^{d+1}$ (relatively compact or not) and let G_U denote the corresponding Green function. For each $c \in \mathbb{R}$, the intersection $H_c \cap U$ consists of (at most) countably many connected components. They are open in H_c and closed in U. Let $C \subset H_c$ and $C' \subset H_{c'}$ be any two different connected components obtained this way (where $c = c'$ or not). We claim that

$$(3.3) \qquad G_U > 0 \text{ on } C \times C' \quad \text{or} \quad G_U = 0 \text{ on } C \times C'.$$

This could be deduced from a remark on *strict positivity sets* of G_U in [9, p. 300] using an additional geometric argument. For the convenience of the reader and to see what will be needed in a more general setting, let us give a complete proof.

So let us suppose that there exist $x_0 \in C$ and $y_0 \in C'$ such that $G_U(x_0, y_0) = 0$. Let $h := G_U(\cdot, y_0)$ and $A := C \cap \{h = 0\}$. Since h is harmonic on $U \setminus \{y_0\}$, and hence continuous on C, the set A is closed in C. We claim that A is open in C as well. Indeed, let $x = (x_1, \dots, x_{d+1}) \in A$. There exists $\varepsilon > 0$ such that the closure of the cube Q obtained as the product of the intervals $(x_j - \varepsilon, x_j + \varepsilon)$, $1 \leq j \leq d+1$, is contained in U. By [7, p. 83, Lemma 1], $h = 0$ on $Q \cap H_c$, that is, $Q \cap H_c \subset A$. So A is open in C. Since $x_0 \in A$, we hence obtain that $A = C$, that is, $G(\cdot, y_0) = 0$ on C.

Next let $x \in C$, $h' := G_U(x, \cdot)$, and $A' := C' \cap \{h' = 0\}$. Then $y_0 \in A'$ and A' is closed in C'. Applying the arguments above to the adjoint harmonic structure, we obtain that A' is open in C'. Thus $A' = C'$, that is, $G(x, \cdot) = 0$ on C', and the proof of (3.3) is finished.

Let us note, in addition, that ${}^U R_1^{C'} = 0$ on C, if $G_U = 0$ on $C \times C'$. To see this, it suffices to recall that ${}^U R_1^{C'} = {}^U \hat{R}_1^{C'}$ on C and to observe that

$$ {}^U \hat{R}_1^{C'} = \sup\{ {}^U \hat{R}_1^K : K \text{ compact in } C' \} $$

(see [4, VI.1.7]), where, for each compact K in C', there exists a measure μ on K such that ${}^U \hat{R}_1^K = G_U^\mu$ (see [9, Theorem 1.XVII.7] or [28, Theorem 22] and [29]).

Every connected component C of $H_c \cap U$ is a transit set, since ${}^U R_1^C \le R_1^{H_c}$ on C. If D is a dense set in $\mathbb{R}$, then the union of the connected components of the sets $H_c \cap U$, $c \in D$, is obviously finely dense in U.

Thus we know that the assumptions (A) from Section 1 are satisfied for the heat equation even on arbitrary open sets in $\mathbb{R}^{d+1}$.

4 Harnack properties of unions of transit sets

Let us observe first that every transit set H in X has the following simple property. For all $A \subset X$ and $B \subset H$,

$$ \hat{R}_1^A = \hat{R}_1^{A \cup B} \quad \text{on } H. \tag{4.1} $$

Indeed, $\hat{R}_1^A \le \hat{R}_1^{A\cup B} \le \hat{R}_1^A + \hat{R}_1^B$, where $\hat{R}_1^B \le \hat{R}_1^H = 0$ on H.

The following results will allow us to approximate balayage on arbitrary Borel sets by balayage on *weak Harnack families* of small compact pieces, which are pairwise disjoint and contained in finitely many transit sets, where (in contrast to elliptic situations) the pieces within one of these sets may be uncontrollably close to each other.

In this section, let (H_n) denote a sequence of pairwise disjoint transit sets in X, let M be a finite set in $\mathbb{N}$, and

$$ A := \bigcup_{n \in M} H_n. $$

PROPOSITION 4.1. *If* (iii) *holds, then, for all $\varepsilon > 0$ and relatively compact open sets W in X, there exists $\delta > 0$ such that, for every compact set K in $A \cap W$ and all $x, y \in (A \cap W) \setminus K$,*

$$ \|\varepsilon_x^K\| \le (1+\varepsilon)\|\varepsilon_y^K\|, \qquad \textit{if } \rho(x,y) < \delta. \tag{4.2} $$

Proof. Let us fix $\varepsilon > 0$, a relatively compact open set W in X, and choose a corresponding $\delta > 0$ according to (iii). Of course, we may assume that δ is strictly smaller than the distances between the compact sets $H_n \cap \overline{W}$, $n \in M$.

Let K be a compact set in $A \cap W$ and $x, y \in (A \cap W) \setminus K$ such that $\rho(x,y) < \delta$. Then there exists $n \in M$ such that $x, y \in H_n$. Then $L := K \setminus H_n = K \cap (A \setminus H_n)$ is compact and we obtain, by (iii), that

$$ \|\varepsilon_x^L\| \le (1+\varepsilon)\|\varepsilon_y^L\|. \tag{4.3} $$

By (4.1), $\hat{R}_1^K = \hat{R}_1^L$ on H_n. In particular, for $z \in \{x, y\}$,

$$\|\varepsilon_z^K\| = R_1^K(z) = \hat{R}_1^K(z) = \hat{R}_1^L(z) = R_1^L(z) = \|\varepsilon_z^L\|.$$

So (4.2) follows from (4.3). □

If $\varepsilon > 0$ and $K_1, \dots, K_m$ are pairwise disjoint compact sets in X, we shall say that $(K_i)_{1 \le i \le m}$ is a *weak* $(1+\varepsilon)$*-Harnack family* provided that, for every $1 \le i \le m$ and all compact sets K in $\bigcup_{j \ne i} K_j$,

$$\|\varepsilon_x^K\| \le (1+\varepsilon)\,\|\varepsilon_y^K\| \qquad \text{for all } x, y \in K_i. \tag{4.4}$$

COROLLARY 4.2. *If* (iii) *holds, then, for all* $\varepsilon > 0$ *and compact sets* K *in* A, *there exists* $\delta > 0$ *such that every family* $K_1, \dots, K_m$ *of pairwise disjoint compact sets in* K *with* $\operatorname{diam}(K_i) < \delta$, $1 \le i \le m$, *is a weak* $(1+\varepsilon)$*-Harnack family.*

Proof. It suffices to choose a relatively compact open neighborhood W of K and a strictly positive δ, which is smaller than the distance between the sets $\overline{W} \cap H_n$, $n \in M$, such that the statement of Proposition 4.1 hold. □

Finally, we observe the following.

LEMMA 4.3. *Transit sets do not have finely interior points.*

In particular, if the union of all H_n, $n \in \mathbb{N}$, *is finely dense, then, for every* $N \in \mathbb{N}$, *the union of* $H_{N+1}, H_{N+2}, H_{N+3}, \dots$ *is finely dense in* X.

Proof. Let H be a transit set, $x \in H$. Then $\text{f-}\liminf_{y \to x} R_1^H(y) = \hat{R}_1^H(x) = 0$ (see [4, II.4.3]). Since $R_1^H = 1$ on H, we see that H cannot be a fine neighborhood of x. □

5 Downsizing of weak Harnack families

Throughout this section let $K_1, \dots, K_m$ be pairwise disjoint compact sets in X and let ν be a measure in $\mathcal{M}(\mathcal{P}(X))$ not charging points of $K_1 \cup \dots \cup K_m$.

By Proposition 7.1 in [13], for every $1 \le i \le m$, there is an increasing family $(K_i^t)_{0 \le t \le 1}$ of compact sets in K_i such that $K_i^1 = K_i$ and the following holds:

- The family $(K_i^t)_{0 \le t \le 1}$ is increasing and right continuous, that is, $K_i^s \subset K_i^t$ if $s \le t$, and each K_i^t, $t \in [0,1)$, is the intersection of all K_i^s, $s > t$.
- The mapping $t \mapsto \nu^{K_i^t}$ is continuous on $[0,1]$ and $\nu^{K_i^0} = 0$.

The following result generalizes [13, Lemma 4.2] in a way which already was used in [13, Section 7] (without giving all details explicitly). For the convenience of the reader we include the complete proof.

LEMMA 5.1. *Let* $\gamma_1, \dots, \gamma_m \in [0, \infty)$ *and let* Γ *denote the set of all* $t \in [0,1]^m$ *such that, for every* $1 \le i \le m$,

$$\nu^{A_t}(K_i) \le \gamma_i, \qquad \textit{where } A_t := K_1^{t_1} \cup \dots \cup K_m^{t_m}.$$

Then there exists $s \in \Gamma$ *such that* $s \ge t$ *for every* $t \in \Gamma$. *Moreover,* $\nu^{A_s}(K_i) = \gamma_i$ *for every* $i \in \{1, \dots, m\}$ *such that* $s_i < 1$.

Proof. Let us note first that $\nu^{A_t}(K_i) = \nu^{A_t}(K_i^{t_i})$ for every $t \in \Gamma$ and for every $1 \le i \le m$, since ν^{A_t} is supported by the subset A_t of A.

0. Of course, $(0, \dots, 0) \in \Gamma$, since $\nu^{K_i^0} = 0$ for every $1 \le i \le m$.

1. If $t, \tilde{t} \in \Gamma$, then $t \vee \tilde{t} \in \Gamma$. Indeed, let us fix $1 \le i \le m$. We may assume without loss of generality that $t_i \ge \tilde{t}_i$. Since $A_t \subset A_{t\vee\tilde{t}}$, we conclude by (1.2) that

$$\nu^{A_{t\vee\tilde{t}}}(K_i^{t_i\vee\tilde{t}_i}) = \nu^{A_{t\vee\tilde{t}}}(K_i^{t_i}) \le \nu^{A_t}(K_i^{t_i}) \le \gamma_i.$$

By [13, Proposition 7.4], for every $\varphi \in \mathcal{K}(X)$, the mapping $t \mapsto \nu^{A_t}(\varphi)$ is continuous.

Since the compact sets $K_1, \dots, K_m$ are disjoint, we obtain that the mapping

$$t \mapsto \big(\nu^{A_t}(K_1), \dots, \nu^{A_t}(K_m)\big)$$

is continuous on $[0,1]^m$. Therefore Γ is closed.

2. Combining the previous two parts of the proof, we see that

$$s := (\sup_{t\in\Gamma} t_1, \dots, \sup_{t\in\Gamma} t_m) \in \Gamma.$$

Of course, $s \ge t$ for every $t \in \Gamma$.

To finish the proof, let us consider $i \in \{1, \dots, m\}$ such that $s_i < 1$ and suppose that $\nu^{A_s}(K_i) < \gamma_i$. Let us define $\tilde{s} := (s_1, \dots, s_{i-1}, b, s_{i+1}, \dots, s_m)$, where $s_i < b \le 1$. By continuity, we may choose b in such a way that $\nu^{A_{\tilde{s}}}(K_i) < \gamma_i$. Since $A_s \subset A_{\tilde{s}}$, we obtain, by (1.2), that $\nu^{A_{\tilde{s}}}(K_j^{s_j}) \le \nu^{A_s}(K_j^{s_j}) \le \gamma_j$ for every $j \in \{1, \dots, m\}$, $j \ne i$. Thus $\tilde{s} \in \Gamma$, $\tilde{s} \le s$, $b = \tilde{s}_i \le s_i$, a contradiction. □

In addition, we shall need the following simple fact.

LEMMA 5.2. *Let $A, B \in \mathcal{B}(X)$ and $c > 1$ such that $\|\varepsilon_x^A\| \le c\,\|\varepsilon_y^A\|$ for all $x, y \in B$. Then, for all measures $\sigma, \tau \in \mathcal{M}(\mathcal{P}(X))$ such that $\sigma(B) \le \tau(B)$,*

$$\|(\sigma|_B)^A\| \le c\,\|(\tau|_B)^A\|. \tag{5.1}$$

Proof. We may suppose that $\tau(B) > 0$. Fixing $x \in B$, we have

$$R_1^A(x) = \|\varepsilon_x^A\| \le c\,\|\varepsilon_y^A\| = c\,R_1^A(y) \qquad \text{for every } y \in B.$$

An integration with respect to $\tau|_B$ yields that that $\tau(B)R_1^A(x) \le c\,\|(\tau|_B)^A\|$. Hence $\tau(B)R_1^A \le c\,\|(\tau|_B)^A\|$ on B. Integrating next with respect to $\sigma|_B$, we see that

$$\tau(B)\|(\sigma|_B)^A\| \le c\,\sigma(B)\|(\tau|_B)^A\| \le c\,\tau(B)\|(\tau|_B)^A\|.$$

□

THEOREM 5.3. *Let us suppose that $\varepsilon > 0$ and $(K_i)_{1\le i\le m}$ is a weak $(1+\varepsilon)$-Harnack family. Moreover, let $L_j := \bigcup_{i\in I_j} K_i$, where $I_1, \dots, I_k$ is a partition of $\{1, \dots, m\}$, let $\beta_1, \dots, \beta_m \in [0, (1+\varepsilon)^{-1}]$, $\lambda \in \Lambda_k$, and let ν be a measure in $\mathcal{M}(\mathcal{P}(X))$ not charging points of $K_1 \cup \dots \cup K_m$.*

Then there are $s_1, \dots, s_m \in [0,1]$ such that $K := K_1^{s_1} \cup \dots \cup K_m^{s_m}$ satisfies

$$\nu^K(K_i) = \beta_i \sum_{j=1}^{k} \lambda_j \nu^{L_j}(K_i) \qquad \text{for every } 1 \le i \le m. \tag{5.2}$$

Proof. Since the measures ν^{L_j} are supported by L_j, the sum on the right side of (5.2) reduces to the term $\lambda_j \nu^{L_j}(K_i)$ if $i \in I_j$. By Lemma 5.1, there exists $s \in [0,1]^m$ such that

$$K := K_1^{s_1} \cup \cdots \cup K_m^{s_m}$$

satisfies

$$\nu^K(K_i) \le \beta_i \lambda_j \nu^{L_j}(K_i) \qquad \text{for all } i \in I_j,\ 1 \le j \le k, \tag{5.3}$$

with equality whenever $s_i < 1$. We claim that we have

$$\nu^K(K_i) \ge \lambda_j \nu^{L_j}(K_i), \qquad \text{if } s_i = 1,\ i \in I_j,\ 1 \le j \le k, \tag{5.4}$$

and this will clearly finish the proof, since $\beta_i < 1$ (in fact, it even shows that s_i cannot be equal to 1 for $i \in I_j$, unless $\lambda_j \nu^{L_j}(K_i) = 0$).

Indeed, let us suppose, for example, that $s_n = 1$ for some $n \in I_1$ and let $I_1' := I_1 \setminus \{n\}$. Then $A := K_n = K_n^{s_n} \subset K$. Hence, by (1.2),

$$\nu^A = \nu^K|_A + (\nu^K|_{K\setminus A})^A, \tag{5.5}$$

where

$$\nu^K|_{K\setminus A} = \sum_{i \in I_1'} \nu^K|_{K_i} + \sum_{j=2}^{k} \sum_{i \in I_j} \nu^K|_{K_i}. \tag{5.6}$$

Since $\beta_i \le (1+\varepsilon)^{-1}$, (5.3), (4.4), and Lemma 5.2 imply that, for all $i \in I_1'$,

$$\|(\nu^K|_{K_i})^A\| \le (1+\varepsilon)\beta_i \lambda_1 \|(\nu^{L_1}|_{K_i})^A\| \le \lambda_1 \|(\nu^{L_1}|_{K_i})^A\|.$$

Similarly, $\|(\nu^K|_{K_i})^A\| \le \lambda_j \|(\nu^{L_j}|_{K_i})^A\|$ for all $i \in I_j$, $2 \le j \le k$. Taking sums we see that

$$\sum_{i \in I_1'} \|(\nu^K|_{K_i})^A\| \le \lambda_1 \|(\nu^{L_1}|_{L_1\setminus A})^A\| \quad \text{and} \quad \sum_{i \in I_j} \|(\nu^K|_{K_i})^A\| \le \lambda_j \|(\nu^{L_j})^A\|$$

for every $2 \le j \le k$. Therefore, by (5.5) and (5.6),

$$\|\nu^A\| \le \nu^K(A) + \lambda_1 \|(\nu^{L_1}|_{L_1\setminus A})^A\| + \sum_{j=2}^{k} \lambda_j \|(\nu^{L_j})^A\|, \tag{5.7}$$

where $\|(\nu^{L_j})^A\| \le \|\nu^A\|$ by (1.2). Thus

$$\lambda_1 \|\nu^A\| \le \nu^K(A) + \lambda_1 \|(\nu^{L_1}|_{L_1\setminus A})^A\|.$$

Since $\nu^A = \nu^{L_1}|_A + (\nu^{L_1}|_{L_1\setminus A})^A$ by (1.2), we obtain the inequality $\lambda_1 \nu^{L_1}(A) \le \nu^K(A)$, and the proof is finished. $\square$

6 Main convexity and density results

Throughout this section, let us suppose that we have pairwise disjoint sets H_n in X, $n \in \mathbb{N}$, satisfying the assumptions (A) from Section 1. The following convexity result will easily lead to Theorem 1.1 and Theorem 1.2.

THEOREM 6.1. *Let $U_1, \dots, U_k$ be relatively compact open sets in X, $\nu \in \mathcal{M}(\mathcal{P}(X))$, and $\lambda \in \Lambda_k$. Then there exist compact sets K_n in $U_1 \cup \dots \cup U_k$, $n \in \mathbb{N}$, such that*

$$\lim_{n\to\infty} \nu^{K_n} = \sum_{j=1}^{k} \lambda_j \nu^{U_j}.$$

Proof. Let $\mathcal{Q}$ be a finite set in $\mathcal{P}$ such that $\nu(q) < \infty$ for every $q \in \mathcal{Q}$ and let p denote the sum of all $q \in \mathcal{Q}$. We may assume without loss of generality that $p \geq 1$ on $U_1 \cup \dots \cup U_k$. Let $\varepsilon > 0$. It suffices to construct a compact set K in $U_1 \cup \dots \cup U_k$ such that, for every $q \in \mathcal{Q}$,

$$\Big|\nu^K(q) - \sum_{j=1}^{k} \lambda_j \nu^{U_j}(q)\Big| < (4 + 3\nu(p))\varepsilon \tag{6.1}$$

(having chosen (q_m) according to Lemma 2.1 in [13], then, for every $n \in \mathbb{N}$, we may consider $\mathcal{Q} := \{q_1, \dots, q_n\}$ and $\varepsilon := (4 + 3\nu(p))^{-1}(1/n))$.

Since $\nu(p) < \infty$ and the sets H_n, $n \in \mathbb{N}$, are pairwise disjoint, there exists $N \in \mathbb{N}$ such that

$$\sum_{n=N+1}^{\infty} \int_{H_n} p \, d\nu < \varepsilon.$$

Let

$$\mu := 1_{H_1 \cup H_2 \cup \dots \cup H_N}\, \nu \quad \text{and} \quad \tau := \nu - \mu.$$

Then $\tau(p) < \varepsilon$. Hence, for all $q \in \mathcal{Q}$ and for every compact set K in X,

$$\tau^K(q) + \sum_{j=1}^{k} \lambda_j \tau^{U_j}(q) \leq 2\tau(q) \leq 2\tau(p) < 2\varepsilon. \tag{6.2}$$

By Lemma 4.3 and [4, VI.1.7, 1.9], we may choose pairwise disjoint finite sets M_j in $\{N+1, N+2, \dots\}$ and compact sets F_j, $1 \leq j \leq k$, such that,

$$F_j \subset \bigcup_{n \in M_j} H_n \cap U_j \quad \text{and} \quad \big|\mu^{F_j}(q) - \mu^{U_j}(q)\big| < \varepsilon \tag{6.3}$$

for every $q \in \mathcal{Q}$ (see Figure 1).

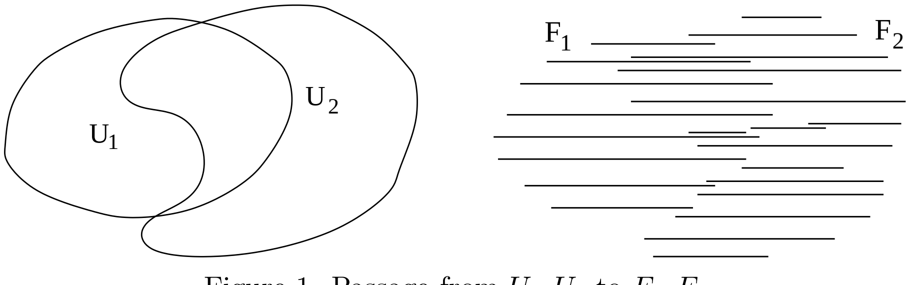

Figure 1. Passage from U_1, U_2 to F_1, F_2

Let us define

$$M := M_1 \cup \cdots \cup M_k, \qquad F := F_1 \cup \cdots \cup F_k,$$

and choose $\delta > 0$ according to Corollary 4.2 with F in place of K. We may assume, in addition, that

$$|q(x) - q(y)| < \varepsilon, \quad \text{whenever } q \in \mathcal{Q} \text{ and } x, y \in F,\ \rho(x,y) < \delta. \tag{6.4}$$

For the moment, let us fix $1 \le j \le k$. Of course, the compact set F_j is a finite union of pairwise disjoint K_σ-sets having a diameter strictly less than δ. Replacing each of these sets by a sufficiently large compact subset, we get finitely many pairwise disjoint compact sets in F_j, having diameter strictly less than δ, such that their union L_j satisfies

$$\left|\mu^{L_j}(q) - \mu^{F_j}(q)\right| < \varepsilon \qquad (q \in \mathcal{Q}). \tag{6.5}$$

Collecting the compact sets obtained this way for $F_1, \dots, F_k$, we get pairwise disjoint compact sets $K_1, \dots, K_m$ in F such that $\operatorname{diam} K_i < \delta$, $1 \le i \le m$, and, for some partition $I_1, \dots, I_k$ of $\{1, \dots, m\}$,

$$L_j = \bigcup\nolimits_{i \in I_j} K_i$$

(see Figure 2). The union L of $L_1, \dots, L_k$ is the union of $K_1, \dots, K_m$. It clearly suffices to consider the case, where all the sets $K_1, \dots, K_m$ are non-empty. By Corollary 4.2, $K_1, \dots, K_m$ is a weak $(1+\varepsilon)$-Harnack family. Let

$$\sigma := \lambda_1 \mu^{L_1} + \cdots + \lambda_k \mu^{L_k}.$$

Obviously,

$$\sigma(p) = \sum_{j=1}^{k} \lambda_j \mu^{L_j}(p) \le \sum_{j=1}^{k} \lambda_j \mu(p) = \mu(p) \le \nu(p).$$

By Theorem 5.3, there exists a compact set K in L (see Figure 2) such that

$$\mu^K(K_i) = (1+\varepsilon)^{-1}\sigma(K_i) \qquad \text{for every } 1 \le i \le m. \tag{6.6}$$

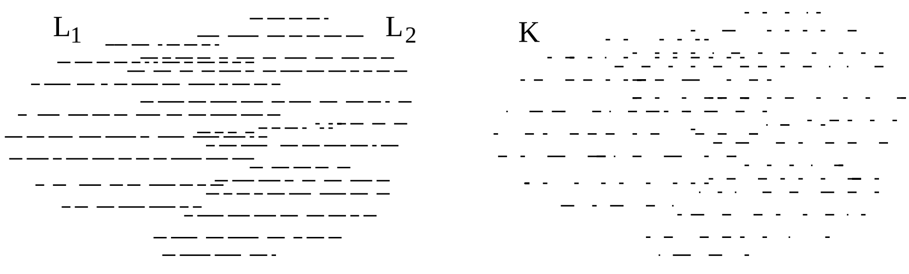

Figure 2. The sets L_1, L_2, and the set K

We now fix $q \in \mathcal{Q}$, points $x_i \in K_i$, $1 \le i \le m$, and define $g := \sum_{i=1}^m q(x_i) 1_{K_i}$. By (6.6),

$$\mu^K(g) = (1+\varepsilon)^{-1}\sigma(g). \tag{6.7}$$

By (6.4), $|g-q| \le \varepsilon \le \varepsilon p$ on L. Therefore

$$\begin{aligned} (6.8) \qquad \left|\mu^K(g) - \mu^K(q)\right| &\le \varepsilon\mu^K(p) \le \nu(p)\varepsilon, \\ (6.9) \qquad |\sigma(g) - \sigma(q)| &\le \varepsilon\sigma(p) \le \nu(p)\varepsilon. \end{aligned}$$

Combining (6.7), (6.8), (6.9), and using the inequality $1 \le (1+\varepsilon)^{-1} + \varepsilon$, we see that

$$\left|\mu^K(q) - \sigma(q)\right| \le \left|\mu^K(q) - (1+\varepsilon)^{-1}\sigma(q)\right| + \varepsilon\sigma(q) \le 3\nu(p)\varepsilon.$$

Taking into account the estimates (6.3) and (6.5), we hence conclude that

$$\left|\mu^K(q) - \sum_{j=1}^k \lambda_j \mu^{U_j}(q)\right| < 2\varepsilon + 3\nu(p)\varepsilon.$$

Because of (6.2) we finally obtain (6.1). □

THEOREM 6.2. *Let $\nu \in \mathcal{M}(\mathcal{P}(X))$, let $A_1, \dots, A_k \in \mathcal{B}(X)$, let (V_n) be a sequence of open neighborhoods of $A_1 \cup \dots \cup A_k$, and $\lambda \in \Lambda_k$. Then there exist compact sets K_n in V_n, $n \in \mathbb{N}$, such that*

$$\lim_{n\to\infty} \nu^{K_n} = \sum_{j=1}^k \lambda_j \nu^{A_j}.$$

Proof. Consequence of Theorem 6.1 and the fact that, for every $1 \le j \le k$, there are relatively compact open $U_{j,n}$ in V_n such that $\nu^{A_j} = \lim_{n\to\infty} \nu^{U_{j,n}}$ (see [4, VI.1.6,1.2]). □

REMARK 6.3. Theorem 6.2 shows that, in fact, Theorem 6.1 holds for *arbitrary* open sets $U_1, \dots, U_k$ (take $V_n := U_1 \cup \dots \cup U_k$, $n \in \mathbb{N}$).

Let us say that a sequence (V_n) of open sets is an *exhaustion of an open set V* in X provided that, for every $n \in \mathbb{N}$, the closure of V_n is a compact set in V_{n+1} and the union of all V_n is V.

COROLLARY 6.4. *Let $\nu \in \mathcal{M}(\mathcal{P}(X))$, let $U_1, \dots, U_k$ be open sets in X, let (L_n) be a sequence of compact sets in $U_1 \cap \dots \cap U_k$, and $\lambda \in \Lambda_k$. Then there exist relatively compact open neighborhoods W_n of L_n in X, $n \in \mathbb{N}$, such that*

$$\lim_{n\to\infty} \nu^{W_n^c} = \sum_{j=1}^k \lambda_j \nu^{U_j^c}.$$

Proof. Let (X_n) be an exhaustion of X such that $L_n \subset X_n$, $n \in \mathbb{N}$. For every compact set K in X, $\lim_{n\to\infty} \nu^{K \cup X_n^c} = \nu^K$, since $\lim_{n\to\infty} \nu^{X_n^c} = 0$. Applying Theorem 6.2, we hence may find compact sets K_n in L_n^c, $n \in \mathbb{N}$, such that

$$\lim_{n\to\infty} \nu^{K_n \cup X_n^c} = \sum_{j=1}^k \lambda_j \nu^{U_j^c}.$$

Defining $W_n := K_n^c \cap X_n$ the proof is finished. □

Let us stress that the open sets W_n in Corollary 6.4 might become very large. As the following result shows, we obtain a better control of the sequence (W_n), if $(X, \mathcal{W})$ is a harmonic space and the restriction of the harmonic structure on some open subset W of X, which contains the sets $U_1, \dots, U_k$, satisfies the conditions (A) imposed in Section 1. Let us recall that, for the harmonic space given by the heat equation, we may even take $W = U_1 \cup \dots \cup U_k$ (see Section 3).

COROLLARY 6.5. *Let us suppose that $(X, \mathcal{W})$ is a harmonic space, $U_1, \dots, U_k$ are open sets in X, $\nu \in \mathcal{M}(\mathcal{P}(X))$ is supported by $U_1 \cap \dots \cap U_k$, and $\lambda \in \Lambda_k$. Further, let (V_n) be an exhaustion of $(U_1 \cup \dots \cup U_k) \setminus (\partial U_1 \cup \dots \cup \partial U_k)$ and W be an open set containing $U_1, \dots, U_k$ such that the restriction of $(X, \mathcal{W})$ on W satisfies* (A).

Then there exist compact sets K_n in $(U_1 \cup \dots \cup U_k) \setminus V_n$, $n \in \mathbb{N}$, such that

$$\lim_{n \to \infty} \nu^{(W \setminus K_n)^c} = \sum_{j=1}^{k} \lambda_j \nu^{U_j^c}. \tag{6.10}$$

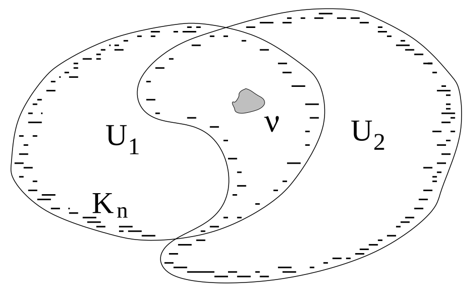

Figure 5. The sets K_n for $W = U_1 \cup U_2$

Proof. Let us choose $p \in \mathcal{P}$, $p > 0$, such that $\nu(p) < \infty$. Let (φ_m) be a sequence of functions in $\mathcal{K}(X)$ with support in W such that $0 \le \varphi_m \le p$, $m \in \mathbb{N}$, and, for all $\sigma_1, \sigma_2, \dots, \sigma_\infty \in \mathcal{M}(W)$, $\lim_{n\to\infty} \sigma_n = \sigma_\infty$ if $\lim_{n\to\infty} \sigma_n(\varphi_m) = \sigma_\infty(\varphi_m)$ for every $m \in \mathbb{N}$.

Let us fix $n \in \mathbb{N}$. There exists $N \in \mathbb{N}$, $N \ge n$, such that, defining

$$\mu := 1_{V_N} \nu,$$

we have $(\nu - \mu)(p) \le 1/n$ and hence $(\nu - \mu)^B(p) \le 1/n$ for every $B \in \mathcal{B}(X)$.

For every $1 \le j \le k$, there exists an open neighborhood $\tilde{U}_j$ of $\overline{V}_N \cap U_j$ which is relatively compact in U_j and satisfies

$$\left|\mu^{\tilde{U}_j^c}(\varphi_m) - \mu^{U_j^c}(\varphi_m)\right| < \frac{1}{n} \qquad \text{for all } 1 \le m \le n. \tag{6.11}$$

Since $(X, \mathcal{W})$ is a harmonic space and μ is supported by $\tilde{U}_j$, we know that

$$\mu^{\tilde{U}_j^c} = \mu^{\partial \tilde{U}_j} = \mu^{\partial \tilde{U}_j \cup W^c}|_W = {}^W\!\mu^{\partial \tilde{U}_j} \qquad (1 \le j \le k) \tag{6.12}$$

(see [4, VI.2.9]). By Theorem 6.2, there exists a compact K_n in $W \setminus V_N$ such that

$$\left|{}^W\!\mu^{K_n}(\varphi_m) - \sum_{j=1}^{k} \lambda_j \, {}^W\!\mu^{\partial \tilde{U}_j}(\varphi_m)\right| < \frac{1}{n} \qquad \text{for all } 1 \le m \le n. \tag{6.13}$$

Moreover, there exists a relatively compact open neighborhood Y of $K_n \cup V_N$ in W such that, for all $1 \le m \le n$,

$$(6.14) \quad \left|{}^W\!\mu^{K_n\cup(W\setminus Y)}(\varphi_m) - {}^W\!\mu^{K_n}(\varphi_m)\right| < \frac{1}{n}, \quad \left|\mu^{K_n\cup Y^c}(\varphi_m) - \mu^{K_n\cup W^c}(\varphi_m)\right| < \frac{1}{n},$$

where, similarly as in (6.12), ${}^W\!\mu^{K_n\cup(W\setminus Y)}(\varphi_m) = \mu^{K_n\cup Y^c}(\varphi_m)$. So, by (6.12), (6.13), and (6.14),

$$\left|\mu^{K_n\cup W^c}(\varphi_m) - \sum_{j=1}^k \lambda_j \mu^{U_j^c}(\varphi_m)\right| < \frac{4}{n} \qquad \text{for all } 1 \le m \le n.$$

Finally, by our definition of μ, we conclude that

$$\left|\nu^{K_n\cup W^c}(\varphi_m) - \sum_{j=1}^k \lambda_j \nu^{U_j^c}(\varphi_m)\right| < \frac{6}{n} \qquad \text{for all } 1 \le m \le n.$$

Thus (6.10) holds. □

7 Application to space-time structures given by semigroups

Let X' be a locally compact space with countable base, let $\mathbb{P} := (P_t)_{t>0}$ be a sub-Markov semigroup on X' and let $\mathbb{V} = (V_\lambda)_{\lambda>0}$ denote the resolvent of $\mathbb{P}$ (that is, $V_\lambda := \int_0^\infty e^{-\lambda t} P_t\, dt$). We suppose that the following holds:

(i) There exist a measure m' on X', not charging points, and a strictly positive finite continuous function $(x', y', t) \mapsto p_t(x', y')$ on $X' \times X' \times (0, \infty)$ such that

$$P_t(x', \cdot) = p_t(x', \cdot)\, m' \qquad \text{for all } t > 0 \text{ and } x' \in X'.$$

(ii) For every $t > 0$, $P_t 1 \in \mathcal{C}(X')$.

(iii) For every $\varphi \in \mathcal{K}(X')$, $\lim_{t\to 0} P_t\varphi = \varphi$ locally uniformly on X'.

(iv) There exist $\beta > 0$ and strictly positive $\mathbb{V}_\beta$-supermedian $u, v \in \mathcal{C}(X')$ such that $u/v \in \mathcal{C}_0(X')$ (where $\mathbb{V}_\beta := (V_{\beta+\lambda})_{\lambda>0}$ with potential kernel V_β).

Let us note that (iv) certainly holds with $v = 1$, if V_β maps $\mathcal{C}_0(X')$ into $\mathcal{C}_0(X')$ (or even $P_t(\mathcal{C}_0(X')) \subset \mathcal{C}_0(X')$ for every $t > 0$). Indeed, then, for every strictly positive $f \in \mathcal{C}_0(X')$, the function $u := V_\beta f \in \mathcal{C}_0(X')$ is $\mathbb{V}_\beta$-supermedian and strictly positive.

For sufficient conditions, which are expressed solely in terms of the function $(x', y', t) \mapsto p_t(x', y')$ and the measure m', see Section 8.

We define $X := X' \times \mathbb{R}$, $m := m' \otimes \lambda_{\mathbb{R}}$, and $G\colon X \times X \to [0, \infty)$ by

$$(7.1) \qquad G\big((x', r), (y', s)\big) := \begin{cases} p_{r-s}(x', y'), & \text{if } r > s,\\ 0, & \text{if } r \le s.\end{cases}$$

Let $\mathbb{T} := (T_t)_{t>0}$ denote the semigroup of uniform translation (to the left) on $\mathbb{R}$, that is, $T_t(r,\cdot) := \varepsilon_{r-t}$, $r \in \mathbb{R}$, and let $\mathbb{Q} := \mathbb{P} \otimes \mathbb{T}$ (see [4, Section V.5]). In other words, $\mathbb{Q} := (Q_t)_{t>0}$, where, for every $f \in \mathcal{B}^+(X)$,

$$Q_t f(x',r) = P_t f(\cdot, r-t)(x') = \int G((x',r),(y',r-t)) f(y',r-t)\, dm'(y').$$

Finally, let $E_{\mathbb{Q}}$ denote the set of all $\mathbb{Q}$-excessive functions, that is,

$$E_{\mathbb{Q}} := \big\{ v \in \mathcal{B}^+(X) \colon\ \sup_{t>0} Q_t v = v \big\}.$$

By (iii), the function 1 is excessive with respect to $\mathbb{P}$, and hence $1 \in E_{\mathbb{Q}}$.

With these ingredients we shall obtain the following.

THEOREM 7.1. $(X, E_{\mathbb{Q}})$ *is a balayage space, where the statements of Theorem 6.1, Theorem 6.2, and Corollary 6.4 hold.*

REMARK 7.2. Let us note that, under very general assumptions, heat kernels on manifolds, fractals, and the infinite-dimensional torus, as well as the corresponding transition functions obtained by subordination using the one-sided stable semigroup $(\eta_t^\alpha)_{t>0}$, $\alpha \in (0,2)$, satisfy our assumptions (see [8, 1, 2, 5]).

In particular, this holds for fundamental solutions of equations $Lu = \partial u/\partial t$ on open subsets X' of $\mathbb{R}^d$, where

$$L := \sum_{i,j=1}^{d} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i \frac{\partial}{\partial x_i} - c$$

is an elliptic differential operator on X' with Hölder continuous coefficients (see [26, pp. 86-88]). Further relevant results on parabolic partial differential operators can be found, for instance, in [6, 11, 14, 19, 20, 24, 25].

The proof of Theorem 7.1 will be given in several steps.

LEMMA 7.3. $\mathbb{P}$ *is a strong Feller semigroup.*

Proof. Let $t > 0$. By (i) and Lebesgue's theorem, $P_t f \in C(X')$, if $f \in \mathcal{B}_b(X')$ has compact support. Let (φ_n) be an increasing sequence in $\mathcal{K}^+(X')$ such that $\sup \varphi_n = 1$ and the interiors of the sets $\{\varphi_n = 1\}$, $n \in \mathbb{N}$, cover X'. Then the functions $P_t(1-\varphi_n) = P_t 1 - P_t \varphi_n$ are continuous and decreasing to 0 as $n \to \infty$. By Dini's lemma, the convergence is locally uniform on X'.

Finally, let $f \in \mathcal{B}(X')$, $0 \le f \le 1$. Then $\lim_{n\to\infty} P_t((1-\varphi_n)f) = 0$ locally uniformly on X'. Knowing that $P_t(\varphi_n f) \in \mathcal{C}(X')$ for every $n \in \mathbb{N}$, we hence see that $P_t f \in \mathcal{C}(X')$. □

LEMMA 7.4. *There exist strictly positive* $u_0, v_0 \in E_{\mathbb{Q}} \cap \mathcal{C}(X)$ *such that* u_0/v_0 *vanishes at infinity.*

Proof. Let us fix $\beta > 0$ and $\mathbb{V}_\beta$-supermedian functions $u, v \in \mathcal{C}(X')$ according to (iv). We may assume without loss of generality that $u \le 1$ and $v \ge 1$ (it suffices to replace u by $u \wedge 1$ and v by $v+1$). By (iii), u and v are even $\mathbb{V}_\beta$-excessive.

For the moment, let w be any $\mathbb{V}_\beta$-excessive function on X'. Then w is excessive with respect to $(e^{-\beta t}P_t)_{t>0}$, that is, $\sup_{t>0} e^{-\beta t}P_t w = w$ (see, for example, [4, II.3.13]). Then, for every $\lambda \in [\beta, \infty)$, the function

$$w_\lambda \colon (x', r) \mapsto e^{\lambda r} w(x'), \qquad (x', r) \in X,$$

is contained in $E_{\mathbb{Q}}$. Indeed, for every $t > 0$,

$$Q_t w_\lambda(x', r) = P_t w_\lambda(\cdot, r - t)(x') = e^{\lambda(r-t)} P_t w(x') = e^{\lambda r} e^{(\beta-\lambda)t} e^{-\beta t} P_t w(x'),$$

where $e^{(\beta-\lambda)t} \le 1$. Hence $\sup_{t>0} Q_t w_\lambda(x', r) = e^{\lambda r} w(x') = w_\lambda(x', r)$.

For all $x' \in X'$ and $r \in \mathbb{R}$, let

$$u_0(x', r) := e^{\beta r} u(x') \quad \text{and} \quad v_0(x', r) := e^{2\beta r} v(x') + 1.$$

By the considerations above, $u_0, v_0 \in E_{\mathbb{Q}} \cap C(X)$. Further, $u_0(\cdot, r)/v_0(\cdot, r) \le e^{-\beta r} u/v$ and, for all $x' \in X'$ and $r \in \mathbb{R}$,

$$\frac{u_0}{v_0}(x', r) \le \frac{e^{\beta r}}{e^{2\beta r} + 1} \le e^{-\beta |r|}.$$

Given $\varepsilon > 0$, we may choose $R > 0$ such that $e^{-\beta R} < \varepsilon$ and a compact set K' in X' such that $u/v < e^{-\beta R}\varepsilon$ on $X' \setminus K'$. Then $u_0(\cdot, r)/v_0(\cdot, r) < \varepsilon$, if $|r| > R$. Moreover, $u_0(\cdot, r)/v_0(\cdot, r) \le e^{\beta R} u/v < \varepsilon$ on $X' \setminus K'$, if $|r| \le R$. Hence $u_0/v_0 < \varepsilon$ outside the compact set $K' \times [-R, R]$. Thus $u_0/v_0 \in C_0(X)$. □

Let W be the potential kernel of $\mathbb{Q}$, that is, for every $f \in \mathcal{B}^+(X)$,

$$\begin{aligned} Wf(x', r) &= \int_0^\infty Q_t f(x', r)\, dt = \int_0^\infty P_t f(\cdot, r - t)(x')\, dt \\ &= \int G((x', r), (y', s)) f(y', s)\, dm(y', s). \end{aligned}$$

In particular, W is proper, since for every bounded interval (a, b) in $\mathbb{R}$,

$$W 1_{X' \times (a,b)}(x', r) = \int_{(r-b)^+}^{(r-a)^+} P_t 1(x')\, dt \le b - a.$$

Thus, by [4, V.5.6], $(X, E_{\mathbb{Q}})$ is a balayage space.

For every $c \in \mathbb{R}$, let $H_c := X' \times \{c\}$. Since $P_t 1 \le 1$, the functions $1_{X' \times (c, \infty)}$, $c \in \mathbb{R}$, are $\mathbb{Q}$-supermedian. Being lower semicontinuous, they are even $\mathbb{Q}$-excessive. So, for every $c \in \mathbb{R}$,

$$R_1^{X' \times [c,\infty)} = 1_{X' \times [c,\infty)} \quad \text{and} \quad \hat{R}_1^{X' \times [c,\infty)} = 1_{X' \times (c,\infty)}$$

(for the first equality consider the functions $1_{X' \times (c - 1/n, \infty)}$), and hence

$$R_1^{H_c} = 0 \ \text{ on } X' \times (-\infty, c) \quad \text{and} \quad \hat{R}_1^{H_c} = 0 \ \text{ on } H_c. \tag{7.2}$$

This shows, in particular, that H_c is a transit set. Moreover, we see that, given $c, c' \in \mathbb{R}$, $c \ne c'$, we have $R_1^{H_c} = 0$ on $H_{c'}$ if $c > c'$, whereas $G > 0$ on $H_{c'} \times H_c$, if $c < c'$.

Next let D be a dense set in $\mathbb{R}$ and let A denote the union of all H_c, $c \in D$. Given $x = (x', r) \in X$ and $\varepsilon > 0$, there exists $t \in (0, \varepsilon)$ such that $c := r - t \in D$. Then the measure $Q_t(x, \cdot)$ is supported by H_c, and hence $Q_t(x, X \setminus A) = 0$. Therefore $X \setminus A$ is not a fine neighborhood of x (see [4, II.4.9]). Thus A is finely dense in X.

Now we shall convince ourselves that every point in X is polar. To that end let us fix $x' \in X'$, $c \in \mathbb{R}$, and consider the point $x := (x', c)$. We have to show that $\hat{R}_1^{\{x\}} = 0$. By (7.2), it suffices to show that $\hat{R}_1^{\{x\}} = 0$ on $X' \times (c, \infty)$. So let $s > c$, $y' \in X'$, $y := (y', s)$, and $\varepsilon > 0$. By assumption, $m'(\{x'\}) = 0$. Hence there exists $\varphi \in \mathcal{K}^+(X')$ such that $\varphi(x') > 1$ and $P_{s-c}\varphi(y') < \varepsilon$. Since $\lim_{t\to 0} P_t\varphi(x') = \varphi(x') > 1$ and the function $t \mapsto P_t\varphi(y')$ is continuous, there exists $\delta > 0$ such that $P_\delta\varphi(x') > 1$ and $P_{s-c+\delta}\varphi(y') < \varepsilon$. We define a function v on X by

$$v(z', r) := \begin{cases} P_{r-c+\delta}\varphi(z'), & \text{if } z' \in X',\ r > c - \delta, \\ 0, & \text{otherwise.} \end{cases}$$

Then $v \in E_{\mathbb{Q}}$, $v(x) > 1$, and $v(y) < \varepsilon$. So $R_1^{\{x\}}(y) \le v(y) < \varepsilon$, and the proof for the polarity of $\{x\}$ is finished.

Finally, let $p \in \mathcal{P}$ and let U be an open neighborhood of the superharmonic support $C(p)$. Then $R_p^U = p$ (see [4, II.6.3]). So, by [3, Theorem 1.3.8], there exists a sequence (g_n) in $\mathcal{B}_b^+(X)$ such that $W(1_U g_n) \uparrow p$. Therefore the measures $\mu_n := 1_U g_n m$, $n \in \mathbb{N}$, have the properties required for Proposition 2.1 in Section 2.

This finishes the proof of Theorem 7.1.

8 Extension of the results to restrictions on open subsets

In order to apply our results to open sets in $X' \times \mathbb{R}$, as we did in the case of the heat equation, we shall need stronger assumptions (which are still satisfied by the heat equation and a wide class of more general parabolic second order partial differential equations).

As before let X' be a locally compact space with countable base and $X := X' \times \mathbb{R}$. We suppose that we have a measure m' on X', not charging points, and a strictly positive Borel measurable real function $(x', y', t) \mapsto p_t(x', y')$ on $X' \times X' \times (0, \infty)$ satisfying the Chapman-Kolmogorov equations:

(i) For all $s, t \in (0, \infty)$ and $x', y' \in X'$,

$$p_{s+t}(x', y') = \int p_s(x', z') p_t(z', y')\, m'(dz').$$

For $t > 0$, we define kernels P_t and ${}^*P_t$ on X' by

$$P_t(x', \cdot) := p_t(x', \cdot)\, m' \quad \text{and} \quad {}^*P_t(y', \cdot) := p_t(\cdot, y')\, m'.$$

Then both $\mathbb{P} := (P_t)_{t>0}$ and ${}^*\mathbb{P} := ({}^*P_t)_{t>0}$ are semigroups on X'. For $t \in \mathbb{R}$ and $x', y' \in X'$, let

$$p_t(x', y') = 0, \qquad \text{if } t \le 0,$$

$$\tilde{p}_t(x', y') := \frac{1}{2}(p_t(x', y') + p_t(y', x')).$$

Moreover, let (X'_n) be an exhaustion of X' and $L'_n := \overline{X'_n}$, $n \in \mathbb{N}$. We suppose that, in addition to the Chapman-Kolmogorov equations (i), the following holds (where, of course, (iv) does not depend on the particular choice of the sequence (X'_n)).

(ii) $1 \in E_{\mathbb{P}} \cap E_{^*\mathbb{P}}$.

(iii) Outside the diagonal in $X' \times X' \times \{0\}$, the function $(x', y', t) \mapsto p_t(x', y')$ is continuous on $X' \times X' \times [0, \infty)$.

(iv) Locally uniformly for $(y', s) \in X' \times [0, \infty)$,

$$\|1_{X' \setminus L'_n} \tilde{p}_s(\cdot, y')\|_{L^1(m')} + \|1_{X' \setminus L'_n} \tilde{p}_s(\cdot, y')\|_\infty \ \to \ 0 \qquad \text{as } n \to \infty.$$

The following lemma shows that the assumptions of Section 7 are satisfied.

LEMMA 8.1. *For every $t > 0$, the kernel P_t is strong Feller and maps $\mathcal{C}_0(X')$ into $\mathcal{C}_0(X')$. Moreover, for every $\varphi \in \mathcal{K}(X')$, $\lim_{t \to 0} P_t \varphi = \varphi$ locally uniformly on X'.*

Proof. Let K' be a compact set in X', $t > 0$, and $\varepsilon > 0$. By (iv), there exists $n \in \mathbb{N}$ such that, for all $0 \le s \le t$ and $y' \in K'$,

$$P_s(y', X' \setminus L'_n) + \sup\{p_s(z', y') \colon z' \in X' \setminus L'_n\} \cdot m'(K') < \varepsilon. \tag{8.1}$$

Let $f \in \mathcal{B}(X')$, $0 \le f \le 1$. By (8.1), $P_t(1_{X' \setminus L'_n} f) \le \varepsilon$ on K'. Moreover, by (iii) and Lebesgue's theorem, $P_t(1_{L'_n} f)$ is continuous on K'. This implies that $P_t f \in \mathcal{C}(X')$.

Let us suppose now that $\varphi \in \mathcal{K}(X')$, $0 \le \varphi \le 1_{K'}$. Then, by (8.1), for every $x' \in X' \setminus L'_n$,

$$\begin{aligned} P_t \varphi(x') &\le \int_{K'} p_t(x', y') \, dm'(y') \\ &\le \sup\{p_t(z', y') \colon z' \in X' \setminus L'_n, y' \in K'\} \cdot m'(K') \le \varepsilon. \end{aligned}$$

This shows that $P_t \varphi \in \mathcal{C}_0(X')$. Since $P_t 1 \le 1$, we conclude that $P_t(\mathcal{C}_0(X')) \subset \mathcal{C}_0(X')$.

Next let V', U' be open sets in X' such that $\overline{V'} \subset U'$. Then, by (8.1), for all $0 < s \le t$ and $y' \in V' \cap K'$,

$$P_s(y', X' \setminus U') \le \varepsilon + \int_{L'_n \setminus U'} p_s(y', z') \, dm'(z'),$$

where, by (iii), the last term tends to 0 as $s \to 0$, uniformly for $y' \in V' \cap K'$. Since $\sup_{s>0} P_s 1 = 1$ by (ii), we hence obtain that $\lim_{s \to 0} P_s(y', U') = 1$, uniformly for $y' \in V' \cap K'$. It is now easily seen that, for every $\varphi \in \mathcal{K}(X')$, $\lim_{s \to 0} P_s \varphi = \varphi$ locally uniformly on X'. $\square$

Let us define

$$\mathbb{Q} := \mathbb{P} \otimes \mathbb{T} \quad \text{ and } \quad {}^*\mathbb{Q} := {}^*\mathbb{P} \otimes {}^*\mathbb{T},$$

where ${}^*\mathbb{T}$ is the semigroup of translation to the right on $\mathbb{R}$ (that is, ${}^*T_t(r, \cdot) := \varepsilon_{r+t}$). Due to the symmetry of our assumptions, we know from Section 7 that we have two balayage spaces $(X, E_{\mathbb{Q}})$ and $(X, E_{*\mathbb{Q}})$, and all statements for the balayage space $(X, E_{\mathbb{Q}})$ have a counterpart for $(X, E_{*\mathbb{Q}})$. Functions which are harmonic with respect to $(X, E_{*\mathbb{Q}})$ will be called $*$-harmonic.

Our aim is to obtain the results of Section 6 for the restriction of the balayage space $(X, E_{\mathbb{Q}})$ on an arbitrary open set U in X (see Theorem 8.9). To that end, we shall assume later on that, in addition, zero sets of positive harmonic and $*$-harmonic functions have a property (Z) (which in most cases is an immediate consequence of parabolic Harnack's inequalities).

We define $G\colon X \times X \to [0, \infty)$ by

$$G((x', r), (y', s)) := p_{r-s}(x', y')$$

(see (7.1)). Then, for every $y \in X$, the function $G(\cdot, y)$ is finite and continuous on $X \setminus \{y\}$. By (i), it is easily verified that, for all $(x', r), (y', s) \in X$, and $t > 0$,

$$Q_t G(\cdot, (y', s))(x', r) = 1_{(s+t,\infty)}(r)\, G((x', r), (y', s)). \tag{8.2}$$

In particular, $G(\cdot, (y', s)) \in E_{\mathbb{Q}}$.

If $a, b \in \mathbb{R}$, $a < b$, and $x = (x', r)$, $y = (y', s)$ with $x', y' \in X'$ and $r, s \in [a, b]$, then, for every open set U in X,

$$H_U G(\cdot, y)(x) = H_U(1_{X' \times [a,b]} G(\cdot, y))(x). \tag{8.3}$$

Indeed, it suffices to recall that $G((z', t), y) = 0$, if $t < s$, and to observe that the measure $H_U(x, \cdot)$ is supported by the absorbing set $X' \times (-\infty, r]$.

LEMMA 8.2. *For every open set U in X, the function $(x, y) \mapsto H_U G(\cdot, y)(x)$ is finite and continuous on $U \times U$.*

Proof. Let K' be a compact set in X' and $I = [a, b]$, $a < b$, such that the set $K := K' \times I$ is contained in U. Let $\varepsilon > 0$. By (iv), there exists $n \in \mathbb{N}$ such that

$$p_t(z', y') < \varepsilon \quad \text{for all } t \in [0, b-a],\ y' \in K',\ \text{and } z' \in X' \setminus L_n'.$$

Then $H_U(1_{(X' \setminus L_n') \times I} G(\cdot, y))(x) \le \varepsilon H_U 1(x) \le \varepsilon$ for all $x, y \in K$. By (iii), the function $(x, y) \mapsto H_U(1_{L_n' \times I} G(\cdot, y))(x)$ is finite and continuous on $K \times K$. Thus, in view of (8.3), the function $(x, y) \mapsto H_U G(\cdot, y)(x)$ is finite and continuous on $K \times K$. □

Using the potential kernel W of $\mathbb{Q}$ and the corresponding resolvent $(W_\lambda)_{\lambda > 0}$ which is defined by $W_\lambda := \int_0^\infty e^{-\lambda t} Q_t\, dt$, $\lambda > 0$, we shall prove the following.

PROPOSITION 8.3. *For all $y \in X$, the function $G(\cdot, y)$ is a potential which is harmonic on $X \setminus \{y\}$.*

Proof. Let $y = (y', s) \in X$ and $u := G(\cdot, y)$. If $x \in X' \times (-\infty, s]$ and U is any relatively compact open neighborhood of x, then $H_U u(x) \le u(x) = 0$ whence $H_U u(x) = u(x)$.

Next let U be an open set such that $\overline{U}$ is a compact set in $X' \times (s, \infty)$. We intend to show that $H_U u = u$. There exists $\varepsilon > 0$ such that $\overline{U} \subset S := X' \times (s + \varepsilon, \infty)$. By (8.2), for every $(x', r) \in X$,

$$Wu(x', r) = (r - s)^+ u(x', r) < \infty.$$

Let $f_n := n(u - nW_n u)$, $n \in \mathbb{N}$. By the resolvent equation,

$$Wf_n = nW_n u \qquad \text{for every } n \in \mathbb{N},$$

where $nW_n u \uparrow u$ as $n \to \infty$. By (8.2), for every $t \in (0, \varepsilon)$, $Q_t u = u$ on S. Since $nW_n u = n \int_0^\infty e^{-nt} Q_t u\, dt$ and $u = n \int_0^\infty e^{-nt} u\, dt$, we obtain that, for all $x \in S$,

$$f_n(x) = n^2 \int_\varepsilon^\infty e^{-nt}(u - Q_t u)(x)\, dt \le ne^{-n\varepsilon} u(x). \tag{8.4}$$

For $n \in \mathbb{N}$, let us define

$$v_n := W(1_S f_n) \quad \text{and} \quad w_n := W(1_{S^c} f_n).$$

Then $H_U v_n \le v_n \le ne^{-n\varepsilon} Wu$. If $w \in E_{\mathbb{Q}}$ such that $w \ge w_n$ on U^c, then $w \ge w_n$ on $\{1_{S^c} f_n > 0\}$. Therefore $H_U w_n = R_{w_n}^{U^c} = w_n$ (see [4, II.7.1]). Since $Wf_n = v_n + w_n$ and hence $H_U W f_n = H_U v_n + H_U w_n$, we obtain that

$$H_U u = \lim_{n \to \infty} H_U W f_n = \lim_{n \to \infty} w_n = \lim_{n \to \infty} W f_n = u.$$

By [4, III.4.4], we now see that $u \in \mathcal{H}^+(X \setminus \{y\})$, that is, $H_U u = u$ for every relatively compact open set U in X such that $y \notin \overline{U}$.

Now let U be a relatively compact open set such that $y \in \overline{U}$. If $y \in U$, then the function $H_U u$ is finite and continuous on U by Lemma 8.2. If $y \in \partial U$ and $x \in U$, we may choose a neighborhood $\tilde{U}$ of x in U such that the closure of $\tilde{U}$ is contained in U, and then $H_{\tilde{U}} u$ is continuous on $\tilde{U}$. Since $H_U u \le u$, we hence obtain, by [4, III.1.2], that $H_U u = H_{\tilde{U}} H_U u$ is finite and continuous on $\tilde{U}$.

So u is superharmonic on X. To show that u is a potential, let $h \ge 0$ be a harmonic minorant of u and $x = (x', r) \in X$. Let I denote the interval $[r \wedge s, r \vee s]$ and let $\varepsilon > 0$. By (iv), there exists $n \in \mathbb{N}$ such that $p_t(z', y') < \varepsilon$ for all $0 \le t \le |r - s|$ and $z' \in X' \setminus L'_n$. Of course, we may assume that $x', y' \in L'_n$. Let U be a relatively compact open neighborhood of $L'_n \times I$. Then $H_U(x, L'_n \times I) = 0$ and hence, by (8.3),

$$H_U u(x) = H_U(1_{X' \times I} u)(x) = H_U(1_{(X' \setminus L'_n) \times I} u)(x) \le \varepsilon H_U 1(x) \le \varepsilon.$$

Therefore $h(x) = H_U h(x) \le H_U u(x) \le \varepsilon$. Thus $h(x) = 0$. □

¿From now on let us fix an open set U in X. Then $(U, {}^*\mathcal{H}^+(U)|_U)$ is a balayage space called the *restriction of* $(X, E_{\mathbb{Q}})$ *on* U (see [4, V.1.1]). The following result is well known for Brelot spaces (see [15]).

LEMMA 8.4. *For all* $x, y \in X$, $R^U_{G(\cdot, y)}(x) = {}^*R^U_{G(x, \cdot)}(y)$.

Proof. By symmetry, it suffices to show that $R^U_{G(\cdot, y)}(x) \ge {}^*R^U_{G(x, \cdot)}(y)$. By Proposition 8.3 and the minimum principle (see [4, III.6.6]),

$$R^U_{G(\cdot, y)} = G(\cdot, y), \qquad \text{if } y \in U. \tag{8.5}$$

Let $x \in X$ and $\nu := \varepsilon^U_x$. Then, for every $y \in X$,

$$R^U_{G(\cdot, y)}(x) = \int G(z, y)\, d\nu(z) = {}^*G^\nu(y). \tag{8.6}$$

By (8.5) and (8.6), ${}^*G^\nu = G(x, \cdot)$ on U. Since ${}^*G^\nu \in E_{^*\mathbb{Q}}$, we see that ${}^*G^\nu \ge {}^*R^U_{G(x, \cdot)}$. So, for every $y \in X$, $R^U_{G(\cdot, y)}(x) = {}^*G^\nu(y) \ge {}^*R^U_{G(x, \cdot)}(y)$. □

We define $G_U\colon U\times U\to[0,\infty)$ by

$$G_U(x,y):=G(x,y)-H_UG(\cdot,y)(x)\qquad(x,y\in U).$$

Then $G_U\ge 0$ and, for every $y\in U$, the function $G_U(\cdot,y)$ is harmonic on $U\setminus\{y\}$. Moreover, by Lemma 8.2, G_U is continuous outside the diagonal in $U\times U$. Further, by Lemma 8.4, $H_UG(\cdot,y)(x)={}^*H_UG(x,\cdot)(y)$ for all $x,y\in U$. Hence, for every $x\in U$, the function $G_U(x,\cdot)$ is $*$-harmonic.

LEMMA 8.5. *Let $q\in\mathcal{P}(U)$ have compact superharmonic support K and let $\tilde U$ be an open neighborhood of K in U. Then there exist measures μ_n on $\tilde U$, $n\in\mathbb{N}$, such that $\lim_{n\to\infty}G_U^{\mu_n}=q$.*

Proof. By [12, Theorem 10.1], there exists a unique $p\in\mathcal{P}$ such that $C(p)=K$ and $p-q$ is harmonic on U. Then $p-H_Up=q$. Indeed, if $U_n\uparrow U$ such that each $\overline{U}_n$, $n\in\mathbb{N}$, is a compact subset of U, then $H_{U_n}(p-q)=p-q$ for every $n\in\mathbb{N}$, where $H_{U_n}p\downarrow H_Up$ and $H_{U_n}q\downarrow 0$, since q is a potential on U. There exist measures μ_n on $\tilde U$, $n\in\mathbb{N}$, such that $G^{\mu_n}\uparrow p$ (see the end of Section 7). Consequently,

$$G_U^{\mu_n}=G^{\mu_n}-H_UG^{\mu_n}\to p-H_Up=q\qquad\text{as } n\to\infty.$$

$\square$

REMARK 8.6. Let us note that, having established Proposition 8.3, potentials on X can be represented as Green potentials of measures (see, for example, [10, 16, 17, 18, 21, 23, 26, 27]). So the proof of Lemma 8.5 shows that, in fact, for every $q\in\mathcal{P}(U)$ with compact superharmonic support in U, there exists a measure μ on U such that even $G_U^{\mu}=q$.

For an application of property (Z), to be introduced below, we have to know that balayage on open cylinders is not affected, if its bottom is added.

LEMMA 8.7. *Let $A:=A'\times(a,b)$, and $B:=A'\times[a,b)$, where A' is a relatively compact open set in X' and $a,b\in\mathbb{R}$, $a<b$. Further, let $x=(x',r)\in X\setminus(\overline{A}\cup\overline{U})$. Then $\varepsilon_x^{A\cup U}=\varepsilon_x^{B\cup U}$.*

Proof. Let $p\in\mathcal{P}$. Of course, $R_p^{A\cup U}\le R_p^{B\cup U}$. To prove the reverse inequality at x, let us fix $\varepsilon>0$. There exists a compact set K in U such that $R_p^{B\cup U}(x)<R_p^{B\cup K}(x)+\varepsilon$. Being harmonic outside $\overline{A}\cup\overline{U}$, the function $R_p^{A\cup U}$ is continuous at x. There exists $\delta\in(0,b-a)$ such that, defining $\tilde B:=A'\times[a,b-\delta)$, the following holds:

$$\begin{aligned}
&R_p^{A\cup U}(x',r+\delta)<R_p^{A\cup U}(x',r),\quad R_p^{B\cup K}(x)<R_p^{\tilde B\cup K}(x)+\varepsilon,\\
&p(y',s-\delta)\le p(y',s)+\varepsilon\quad\text{ for all } (y',s)\in\overline{A}\cup K,\\
&(y',s+\delta)\in U\quad\text{ for all } (y',s)\in K.
\end{aligned}$$

We define $u\colon X\to\mathbb{R}$ by

$$u(y',s):=R_p^{A\cup U}(y',s+\delta).$$

Then $u \in E_{\mathbb{Q}}$. Further, if $(y', s) \in \tilde{B} \cup K$, then $(y', s + \delta) \in A \cup U$ and hence

$$p(y', s) \leq p(y', s + \delta) + \varepsilon = u(y', s) + \varepsilon.$$

Thus $R_p^{\tilde{B} \cup K} \leq u + \varepsilon$. In particular,

$$R_p^{B \cup U}(x) < R_p^{\tilde{B} \cup K}(x) + 2\varepsilon \leq u(x) + 3\varepsilon < R_p^{A \cup U}(x) + 4\varepsilon.$$

Thus $R_p^{A \cup U}(x) = R_p^{B \cup U}(x)$ for every $p \in \mathcal{P}$, that is, $\varepsilon_x^{A \cup U} = \varepsilon_x^{B \cup U}$. □

PROPOSITION 8.8. *Let A' be a relatively compact open set in X' and $a, b \in \mathbb{R}$, $a < b$, such that the closure of $A := A' \times (a, b)$ is contained in U. Moreover, let $x \in U \setminus \overline{A}$ and $B := A' \times [a, b)$. Then ${}^U\varepsilon_x^A = {}^U\varepsilon_x^B$.*

Proof. If $\tilde{U}$ is an open neighborhood of U^c such that x is not contained in the closure of $\tilde{U}$, then $\varepsilon_x^{A \cup \tilde{U}} = \varepsilon_x^{B \cup \tilde{U}}$ by Lemma 8.7. Therefore $\varepsilon_x^{A \cup U^c} = \varepsilon_x^{B \cup U^c}$ (see [4, VI.1.2]). Thus, by [4, VI.2.9], ${}^U\varepsilon_x^A = \varepsilon_x^{A \cup U^c}|_U = \varepsilon_x^{B \cup U^c}|_U = {}^U\varepsilon_x^B$. □

Finally, let us assume that zero sets of harmonic and $*$-harmonic functions have the following property.

(Z) For every $x = (x', r) \in U$, there exists a fundamental system of open neighborhoods N of x in U such that, for every function $h \geq 0$ on X which vanishes at x and is harmonic on N ($*$-harmonic on N, respectively), there exists a neighborhood $\tilde{N}$ of x in N such that

$$h = 0 \quad \text{on } \tilde{N} \cap (X' \times (-\infty, r)) \text{ (on } \tilde{N} \cap (X' \times (r, \infty)), \text{ respectively)}.$$

We now may proceed in a similar way as for the heat equation (see Section 3). Let C and $\tilde{C}$ be two connected components of the intersections of U with sets $X' \times \{a\}$ and $X' \times \{b\}$, respectively, such that $C \neq \tilde{C}$ (but possibly $a = b$). In view of property (Z) (which replaces the use of [7, p.83, Lemma 1]) we obtain that

$$G_U > 0 \text{ on } C \times \tilde{C} \quad \text{or} \quad G_U = 0 \text{ on } C \times \tilde{C}. \tag{8.7}$$

Next let us suppose that $G_U = 0$ on $C \times \tilde{C}$. We claim that

$$ {}^U R_1^{\tilde{C}} = 0 \quad \text{on } C. \tag{8.8}$$

To that end let us fix $x = (x', a) \in C$ and define $h := G_U(x, \cdot)$. Then h is $*$-harmonic on $U \setminus \{x\}$. Let A' be an open set in X' such that the closure of $A' \times \{b\}$ is a compact set in $\tilde{C}$. To prove (8.8) it suffices to show that ${}^U R_1^{A' \times \{b\}}(x) = 0$.

By (Z), there exists $\delta > 0$ such that, for $A := A' \times (b, b+\delta)$ and $B := A' \times [b, b+\delta)$, the closure of A is contained in U and

$$G_U(x, \cdot) = 0 \quad \text{on } A. \tag{8.9}$$

Then ${}^U R_1^{A' \times \{b\}} \leq {}^U R_1^B$, where ${}^U R_1^B(x) = {}^U R_1^A(x)$ by Proposition 8.8. By (8.9), $G_U^\mu(x) = 0$ for every measure μ on A. Hence, by Lemma 8.5, $p(x) = 0$ for every potential $p \in \mathcal{P}(U)$ with $C(p) \subset A$. Since ${}^U R_1^A$ is a limit of such potentials, we get that ${}^U R_1^{A' \times \{b\}}(x) = 0$.

So we finally obtain the following result.

THEOREM 8.9. *For every open set U in X, the statements of Theorem 6.1, Theorem 6.2, and Corollary 6.4 hold for the restriction of $(X, E_{\mathbb{Q}})$ on U.*

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