

ASYMPTOTICS OF CHARACTERISTIC POLYNOMIALS OF WIGNER MATRICES AT THE EDGE OF THE SPECTRUM

H. KÖSTERS

ABSTRACT. We investigate the asymptotic behaviour of the second-order correlation function of the characteristic polynomial of a Hermitian Wigner matrix at the edge of the spectrum. We show that the suitably rescaled second-order correlation function is asymptotically given by the Airy kernel, thereby generalizing the well-known result for the Gaussian Unitary Ensemble (GUE). Moreover, we obtain similar results for real-symmetric Wigner matrices.

1. INTRODUCTION AND STATEMENT OF THE MAIN RESULTS

Let Q be a fixed probability distribution on the real line such that

$$\int x Q(dx) = 0, \quad a := \int x^2 Q(dx) = 1/2, \quad b := \int x^4 Q(dx) < \infty, \quad (1.1)$$

and for any $N = 1, 2, 3, \dots$, let $X_N := (X_{ij})_{i,j=1,\dots,N}$ denote the associated Hermitian Wigner matrix of size N . This means that

$$X_{ij} := \begin{cases} X_{ij}^{\text{Re}} + \mathbf{i} X_{ij}^{\text{Im}} & \text{for } i < j, \\ \sqrt{2} X_{ii}^{\text{Re}} & \text{for } i = j, \\ X_{ji}^{\text{Re}} - \mathbf{i} X_{ji}^{\text{Im}} & \text{for } i > j, \end{cases}$$

where $\{X_{ij}^{\text{Re}} \mid i \leq j\} \cup \{X_{ij}^{\text{Im}} \mid i < j\}$ is a collection of i.i.d. real random variables with distribution Q . The second-order correlation function of the characteristic polynomial of the random matrix X_N is defined by

$$f_N(\mu, \nu) := \mathbb{E}(D_N(\mu) D_N(\nu)) \quad (\mu, \nu \in \mathbb{R}),$$

where $D_N(\lambda) := \det(X_N - \lambda)$. We are interested in the asymptotic behaviour of $f_N(\mu_N, \nu_N)$ as $N \rightarrow \infty$, for certain sequences $(\mu_N), (\nu_N)$ which will be specified below. Furthermore, the correlation coefficient of the characteristic polynomial of the random matrix X_N is defined by

$$\sigma_N(\mu, \nu) := \frac{\mathbb{E}((D_N(\mu) - \mathbb{E}D_N(\mu)) (D_N(\nu) - \mathbb{E}D_N(\nu)))}{\sqrt{\mathbb{E}(D_N(\mu) - \mathbb{E}D_N(\mu))^2} \sqrt{\mathbb{E}(D_N(\nu) - \mathbb{E}D_N(\nu))^2}} \quad (\mu, \nu \in \mathbb{R}).$$

In the special case where Q is the Gaussian distribution with mean 0 and variance $\frac{1}{2}$, the distribution of the random matrix X_N is the so-called Gaussian Unitary Ensemble (GUE) (see e.g. FORRESTER [Fo] or MEHTA [Me] but note that we work with a different variance). In this case, it is well-known that

$$f_N(\mu, \nu) = \sqrt{2\pi} N! e^{(\mu^2 + \nu^2)/4} K_{N+1}(\mu, \nu), \quad (1.2)$$

where

$$K_N(x, y) := e^{-(x^2 + y^2)/4} \sum_{k=1}^N \frac{p_{k-1}(x)p_{k-1}(y)}{\sqrt{2\pi}(k-1)!}$$

and the p_k are the monic orthogonal polynomials with respect to the weight function $e^{-x^2/2}$ (see e.g. Chapter 4.1 in FORRESTER [Fo]). Thus, up to scaling, the p_k coincide with the Hermite polynomials (as defined in SZEGÖ [Sz]), and it is possible to derive the asymptotics of the second-order correlation function f_N from the well-known asymptotics of the Hermite polynomials (see e.g. Theorem 8.22.9 in SZEGÖ [Sz]). More precisely, one obtains the following (well-known) results (see also Chapter 4.2 in FORRESTER [Fo]): For $\xi \in (-2, +2)$ and any $\mu, \nu \in \mathbb{R}$,

$$\lim_{N \rightarrow \infty} c'_N f_N \left(\sqrt{N}\xi + \mu/\sqrt{N}\varrho(\xi), \sqrt{N}\xi + \mu/\sqrt{N}\varrho(\xi) \right) = \mathbb{S}(\mu, \nu), \quad (1.3)$$

where $c'_N := (\sqrt{2\pi} N! N^{1/2} \varrho(\xi) \exp(\frac{1}{2}N\xi^2 + \frac{1}{2}(\mu + \nu)\xi/\varrho(\xi)))^{-1}$, $\varrho(\xi) := \frac{1}{2\pi} \sqrt{4 - \xi^2}$, and

$$\mathbb{S}(\mu, \nu) := \frac{\sin \pi(\mu - \nu)}{\pi(\mu - \nu)}. \quad (1.4)$$

For $\xi = +2$ and any $\mu, \nu \in \mathbb{R}$,

$$\lim_{N \rightarrow \infty} c''_N f_N \left(2\sqrt{N} + \mu/N^{1/6}, 2\sqrt{N} + \nu/N^{1/6} \right) = \mathbb{A}(\mu, \nu), \quad (1.5)$$

where $c''_N := (\sqrt{2\pi} N! N^{1/6} \exp(2N + (\mu + \nu)N^{1/3}))^{-1}$,

$$\mathbb{A}(\mu, \nu) := \frac{\text{Ai}(\mu) \text{Ai}'(\nu) - \text{Ai}'(\mu) \text{Ai}(\nu)}{\mu - \nu}, \quad (1.6)$$

and Ai denotes the Airy function (see e.g. ABRAMOWITZ and STEGUN [AS]). By symmetry, a similar result holds for $\xi = -2$. The functions in (1.4) and (1.6) are also called the sine kernel and the Airy kernel, respectively. Furthermore, it is well-known that the eigenvalues of a random matrix X_N from the GUE are distributed roughly over the interval $[-2\sqrt{N}, +2\sqrt{N}]$. That is why the results (1.3) and (1.5) are also said to refer to the bulk and the edge of the spectrum, respectively.

Recently, GÖTZE and KÖSTERS [GK] have shown that the result (1.3) for the bulk is (almost) “universal” in the sense that it holds not only for the GUE, but also (with minor modifications) for more general Hermitian Wigner matrices as introduced at the beginning of this section. More precisely, under the assumption (1.1), we have

$$\lim_{N \rightarrow \infty} c'_N f_N \left(\sqrt{N}\xi + \mu/\sqrt{N}\varrho(\xi), \sqrt{N}\xi + \mu/\sqrt{N}\varrho(\xi) \right) = \exp(b - \frac{3}{4}) \mathbb{S}(\mu, \nu) \quad (1.7)$$

for all $\xi \in (-2, +2)$ and all $\mu, \nu \in \mathbb{R}$, where c'_N , $\varrho(\xi)$ and $\mathbb{S}(\mu, \nu)$ are the same as in (1.3).

Therefore, it seems natural to ask whether the result (1.5) for the edge can also be generalized to more general Hermitian Wigner matrices. The main purpose of this paper is to answer this question in the affirmative. More precisely, our first result is as follows:

Theorem 1.1. *Under (1.1), we have*

$$\lim_{N \rightarrow \infty} c_N'' f_N \left(2\sqrt{N} + \mu/N^{1/6}, 2\sqrt{N} + \nu/N^{1/6} \right) = \exp(b - \frac{3}{4}) \mathbb{A}(\mu, \nu) \quad (1.8)$$

for all $\mu, \nu \in \mathbb{R}$, where c_N'' and $\mathbb{A}(\mu, \nu)$ are the same as in (1.5).

Corollary 1.2. *Under (1.1), we have*

$$\lim_{N \rightarrow \infty} \sigma_N \left(2\sqrt{N} + \mu/N^{1/6}, 2\sqrt{N} + \nu/N^{1/6} \right) = \frac{\mathbb{A}(\mu, \nu)}{\sqrt{\mathbb{A}(\mu, \mu)} \sqrt{\mathbb{A}(\nu, \nu)}} \quad (1.9)$$

for all $\mu, \nu \in \mathbb{R}$.

Moreover, it turns out that similar results hold for real-symmetric Wigner matrices. Let \tilde{Q} be a fixed probability distribution on the real line such that

$$\int x \tilde{Q}(dx) = 0, \quad \tilde{a} := \int x^2 \tilde{Q}(dx) = 1, \quad \tilde{b} := \int x^4 \tilde{Q}(dx) < \infty, \quad (1.10)$$

and for any $N = 1, 2, 3, \dots$, let $\tilde{X}_N := (\tilde{X}_{ij})_{i,j=1,\dots,N}$ denote the associated real-symmetric Wigner matrix of size N . This means that

$$\tilde{X}_{ij} := \begin{cases} \tilde{X}_{ij}^{\text{Re}} & \text{for } i < j, \\ \sqrt{2} \tilde{X}_{ii}^{\text{Re}} & \text{for } i = j, \\ \tilde{X}_{ji}^{\text{Re}} & \text{for } i > j, \end{cases}$$

where $\{\tilde{X}_{ij}^{\text{Re}} \mid i \leq j\}$ is a collection of i.i.d. real random variables with distribution \tilde{Q} . Then, similarly as above, the second-order correlation function and the correlation coefficient of the characteristic polynomial of the random matrix \tilde{X}_N are defined by

$$\tilde{f}_N(\mu, \nu) := \mathbb{E}(\tilde{D}_N(\mu) \tilde{D}_N(\nu)) \quad (\mu, \nu \in \mathbb{R})$$

and

$$\tilde{\sigma}_N(\mu, \nu) := \frac{\mathbb{E}((\tilde{D}_N(\mu) - \mathbb{E}\tilde{D}_N(\mu)) (\tilde{D}_N(\nu) - \mathbb{E}\tilde{D}_N(\nu)))}{\sqrt{\mathbb{E}(\tilde{D}_N(\mu) - \mathbb{E}\tilde{D}_N(\mu))^2} \sqrt{\mathbb{E}(\tilde{D}_N(\nu) - \mathbb{E}\tilde{D}_N(\nu))^2}} \quad (\mu, \nu \in \mathbb{R}),$$

respectively, where now $\tilde{D}_N(\lambda) := \det(\tilde{X}_N - \lambda)$.

Following the approach by GÖTZE and KÖSTERS [GK], KÖSTERS [Kö] recently showed that under the assumption (1.10), we have

$$\lim_{N \rightarrow \infty} d_N' \tilde{f}_N \left(\sqrt{N}\xi + \mu/\sqrt{N}\varrho(\xi), \sqrt{N}\xi + \nu/\sqrt{N}\varrho(\xi) \right) = \exp(\frac{\tilde{b}-3}{2}) \mathbb{T}(\mu, \nu) \quad (1.11)$$

for all $\xi \in (-2, +2)$ and all $\mu, \nu \in \mathbb{R}$, where $\varrho(\xi)$ is the same as in (1.3), $d_N' := (\sqrt{2\pi} N! N^{3/2} \varrho(\xi)^3 \exp(\frac{1}{2}N\xi^2 + \frac{1}{2}(\mu + \nu)\xi/\varrho(\xi)))^{-1}$, and

$$\mathbb{T}(\mu, \nu) := \frac{2 \sin \pi(\mu - \nu)}{\pi(\mu - \nu)^3} - \frac{2 \cos \pi(\mu - \nu)}{(\mu - \nu)^2}. \quad (1.12)$$

Thus, we also have universality (in the same sense as above) in the bulk of real-symmetric Wigner matrices. In the special case where \tilde{Q} is the Gaussian distribution with mean 0 and variance 1, the distribution of the random matrix \tilde{X}_N is the so-called Gaussian Orthogonal Ensemble (GOE) (see e.g. FORRESTER [Fo] or MEHTA [Me] but, again, note that we work with a different variance). In this case, (1.11) had been obtained previously by BRÉZIN and HIKAMI [BH2].

Our second result shows that the result (1.11) admits an analogue for the edge of the spectrum, too:

Theorem 1.3. *Under (1.10), we have*

$$\lim_{N \rightarrow \infty} d_N'' \tilde{f}_N \left(2\sqrt{N} + \mu/N^{1/6}, 2\sqrt{N} + \nu/N^{1/6} \right) = \exp\left(\frac{\tilde{b}-3}{2}\right) \mathbb{B}(\mu, \nu) \quad (1.13)$$

for all $\mu, \nu \in \mathbb{R}$, where $d_N'' := (\sqrt{2\pi} N! N^{1/2} \exp(2N + (\mu + \nu)N^{1/3}))^{-1}$, and

$$\mathbb{B}(\mu, \nu) := \frac{(\mu + \nu) \operatorname{Ai}(\mu) \operatorname{Ai}(\nu) - 2 \operatorname{Ai}'(\mu) \operatorname{Ai}'(\nu)}{(\mu - \nu)^2} + \frac{2 \operatorname{Ai}(\mu) \operatorname{Ai}'(\nu) - 2 \operatorname{Ai}'(\mu) \operatorname{Ai}(\nu)}{(\mu - \nu)^3}. \quad (1.14)$$

Corollary 1.4. *Under (1.10), we have*

$$\lim_{N \rightarrow \infty} \tilde{\sigma}_N \left(2\sqrt{N} + \mu/N^{1/6}, 2\sqrt{N} + \nu/N^{1/6} \right) = \frac{\mathbb{B}(\mu, \nu)}{\sqrt{\mathbb{B}(\mu, \mu)} \sqrt{\mathbb{B}(\nu, \nu)}} \quad (1.15)$$

for all $\mu, \nu \in \mathbb{R}$.

In the special case of the GOE, (1.13) can already be found in BRÉZIN and HIKAMI [BH3].

It seems interesting to note that the functions \mathbb{S} and \mathbb{T} arising for the bulk of the spectrum are related by the identity

$$\mathbb{T}(x, y) = \left(\frac{1}{x - y} \left(\frac{\partial}{\partial y} - \frac{\partial}{\partial x} \right) \right) \mathbb{S}(x, y)$$

and that the functions \mathbb{A} and \mathbb{B} arising for the edge of the spectrum are related by the analogous identity

$$\mathbb{B}(x, y) = \left(\frac{1}{x - y} \left(\frac{\partial}{\partial y} - \frac{\partial}{\partial x} \right) \right) \mathbb{A}(x, y),$$

see also BRÉZIN and HIKAMI [BH3]. (To check the latter identity, use the fact that the Airy function $\operatorname{Ai}(z)$ satisfies the differential equation $\operatorname{Ai}''(z) = z \operatorname{Ai}(z)$.)

Also, observe that in all the cases previously mentioned, the precise choice of the underlying distribution Q or \tilde{Q} enters into the asymptotic behaviour of the second-order correlation function of the characteristic polynomial only as a multiplicative factor depending on the fourth cumulant. Thus, the results are essentially “universal”.

Let us mention some related results from the literature. It is well-known that at the edge of the spectrum of Wigner matrices, we have universality also for the correlation function of the eigenvalues themselves (see SOSHIKOV [So]). In contrast to that, for the bulk of the spectrum of Wigner matrices, only partial results are available in this direction (see JOHANSSON [Jo]). Furthermore, in the special cases of the GUE and the GOE, several authors have investigated the averages of more

general products and even ratios of characteristic polynomials (see e.g. BRÉZIN and HIKAMI [BH1, BH2, BH3], FYODOROV and STRAHOV [FS], BAIK, DEIFT and STRAHOV [BDS], STRAHOV and FYODOROV [SF], AKEMANN and FYODOROV [AF], VANLESSEN [Va], BORODIN and STRAHOV [BS]). Even more, at least in the Hermitian setting, some of these results have been shown to be “universal” in that they continue to hold (with some modifications) for the class of unitary-invariant ensembles. For Wigner matrices, however, less seems to be known in this respect.

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2. OUTLINE OF THE PROOFS

To prove Theorems 1.1 and 1.3, we will start from the fact that for fixed $\mu, \nu \in \mathbb{R}$, the exponential generating functions of the sequences $f_N(\mu, \nu)$ and $\tilde{f}_N(\mu, \nu)$ are given explicitly by

$$\sum_{N=0}^{\infty} f_N(\mu, \nu) \frac{x^N}{N!} = \frac{\exp\left(\mu\nu \cdot \frac{x}{1-x^2} - \frac{1}{2}(\mu^2 + \nu^2) \cdot \frac{x^2}{1-x^2} + (b - \frac{3}{4})x^2\right)}{(1-x)^{3/2} \cdot (1+x)^{1/2}} \quad (|x| < 1)$$

(see Lemma 2.3 in GÖTZE and KÖSTERS [GK]) and

$$\sum_{N=0}^{\infty} \tilde{f}_N(\mu, \nu) \frac{x^N}{N!} = \frac{\exp\left(\mu\nu \cdot \frac{x}{1-x^2} - \frac{1}{2}(\mu^2 + \nu^2) \cdot \frac{x^2}{1-x^2} + (\frac{\tilde{b}-3}{2})x^2\right)}{(1-x)^{5/2} \cdot (1+x)^{1/2}} \quad (|x| < 1)$$

(see Lemma 2.3 in KÖSTERS [Kö]), respectively. This fact opens up the possibility to study the asymptotic behaviour of the second-order correlation functions by evaluating appropriate contour integrals of their exponential generating functions. In fact, this strategy was already used by GÖTZE and KÖSTERS [GK] and KÖSTERS [Kö] to obtain the above-mentioned results for the bulk of the spectrum. Here we carry out a similar analysis for the edge of the spectrum.

Since it does not require any additional efforts, it seems convenient to evaluate the values

$$\frac{f_N^{(\alpha)}(\mu, \nu)}{N!} := \frac{1}{2\pi i} \int_{\gamma} \frac{\exp\left(\mu\nu \cdot \frac{z}{1-z^2} - \frac{1}{2}(\mu^2 + \nu^2) \cdot \frac{z^2}{1-z^2} + b^* z^2\right)}{(1-z)^{\alpha+(1/2)} \cdot (1+z)^{1/2}} \frac{dz}{z^{N+1}} \quad (2.1)$$

for arbitrary $\alpha > 0$ and $b^* \in \mathbb{R}$, where γ denotes a contour around the origin. By the foregoing, the case $\alpha = 1$ corresponds to the Hermitian case and the case $\alpha = 2$ corresponds to the real-symmetric case.

We remark in passing that for a general parameter $\alpha > 0$, it is not hard to see that the exponential generating function under consideration can be interpreted as that of the second-order correlation function of the characteristic polynomial of a random matrix from (a rescaled version of) the tridiagonal beta ensemble introduced by DUMITRIU and EDELMAN [DE] with $\alpha = 2/\beta$. For this interpretation, one should set $b^* := 0$.

For $\alpha > 0$ and $\mu, \nu \in \mathbb{R}$, put

$$I^{(\alpha)}(\mu, \nu) := \frac{1}{4\pi^{3/2}} \int_{-\infty}^{+\infty} \frac{\exp\left(\frac{1}{12}(1-iu)^3 - \frac{1}{2}(\mu+\nu)(1-iu) - \frac{1}{4}(\mu-\nu)^2/(1-iu)\right)}{(1-iu)^{\alpha+(1/2)}} du. \quad (2.2)$$

In the next section, we will prove the following results:

Proposition 2.1. *For any $\alpha > 0$, $b^* \in \mathbb{R}$ and $\mu, \nu \in \mathbb{R}$, we have*

$$\lim_{N \rightarrow \infty} \left(\sqrt{2\pi} N! N^{(2\alpha-1)/6} \exp(2N + (\mu+\nu)N^{1/3}) \right)^{-1} \cdot f_N^{(\alpha)}(2\sqrt{N} + \mu N^{-1/6}, 2\sqrt{N} + \nu N^{-1/6}) = \exp(b^*) I^{(\alpha)}(\mu, \nu).$$

Proposition 2.2. *For any $\alpha \in \mathbb{N}$ and $\mu, \nu \in \mathbb{R}$, we have*

$$I^{(\alpha)}(\mu, \nu) = \left(\frac{1}{\mu - \nu} \left(\frac{\partial}{\partial \nu} - \frac{\partial}{\partial \mu} \right) \right)^{(\alpha)} \left(\text{Ai}(\mu) \text{Ai}(\nu) \right),$$

where $(\cdot)^{(\alpha)}$ denotes the α -fold application of the given differential operator. (For $\mu = \nu$, consider the continuous extension of the right-hand side.)

It is straightforward to check that

$$I^{(1)}(x, y) = \left(\frac{1}{x - y} \left(\frac{\partial}{\partial y} - \frac{\partial}{\partial x} \right) \right)^{(1)} \left(\text{Ai}(x) \text{Ai}(y) \right) = \mathbb{A}(x, y)$$

and

$$I^{(2)}(x, y) = \left(\frac{1}{x - y} \left(\frac{\partial}{\partial y} - \frac{\partial}{\partial x} \right) \right)^{(2)} \left(\text{Ai}(x) \text{Ai}(y) \right) = \mathbb{B}(x, y).$$

(For the second identity, use the fact that the Airy function $\text{Ai}(z)$ satisfies the differential equation $\text{Ai}''(z) = z \text{Ai}(z)$.) Thus, in view of the preceding comments, it is immediate that Theorems 1.1 and 1.3, which correspond to the special cases $\alpha = 1$ and $\alpha = 2$, follow from Propositions 2.1 and 2.2.

To prove Corollaries 1.2 and 1.4, observe that

$$\sigma_N(\mu, \nu) = \frac{f_N(\mu, \nu) - \mathbb{E}D_N(\mu) \mathbb{E}D_N(\nu)}{\sqrt{f_N(\mu, \mu) - (\mathbb{E}D_N(\mu))^2} \sqrt{f_N(\nu, \nu) - (\mathbb{E}D_N(\nu))^2}}$$

in the Hermitian case and

$$\tilde{\sigma}_N(\mu, \nu) = \frac{\tilde{f}_N(\mu, \nu) - \mathbb{E}\tilde{D}_N(\mu) \mathbb{E}\tilde{D}_N(\nu)}{\sqrt{\tilde{f}_N(\mu, \mu) - (\mathbb{E}\tilde{D}_N(\mu))^2} \sqrt{\tilde{f}_N(\nu, \nu) - (\mathbb{E}\tilde{D}_N(\nu))^2}}$$

in the real-symmetric case. Moreover, it is not difficult to see that

$$\mathbb{E}D_N(\lambda) = \mathbb{E}\tilde{D}_N(\lambda) = g_N(\lambda) := (-1)^N 2^{-N/2} H_N(\lambda/\sqrt{2}) \quad (\lambda \in \mathbb{R}),$$

where $H_N(x)$ is the N th Hermite polynomial (see e.g. Section 5.5 in SZEGÖ [Sz]). Thus, in both cases, the expectation of the characteristic polynomial is given by the *same* function $g_N(\lambda)$. Therefore, to deduce Corollaries 1.2 and 1.4 from Theorems 1.1 and 1.3, respectively, it will be sufficient to show that $g_N(\mu) g_N(\nu)$ is asymptotically negligible in comparison to $f_N(\mu, \nu)$ and $\tilde{f}_N(\mu, \nu)$.

Slightly more generally, we will consider the case of an arbitrary parameter $\alpha > 0$ and investigate the asymptotic behaviour of

$$\sigma_N^{(\alpha)}(\mu, \nu) := \frac{f_N^{(\alpha)}(\mu, \nu) - g_N(\mu)g_N(\nu)}{\sqrt{f_N^{(\alpha)}(\mu, \mu) - g_N(\mu)^2} \sqrt{f_N^{(\alpha)}(\nu, \nu) - g_N(\nu)^2}} \quad (2.3)$$

for any $\mu, \nu \in \mathbb{R}$. In the next section, we will show that Proposition 2.1 entails the following result:

Proposition 2.3. *For any $\alpha > 0$, $b^* \in \mathbb{R}$ and $\mu, \nu \in \mathbb{R}$ such that $I^{(\alpha)}(\mu, \mu) > 0$, $I^{(\alpha)}(\nu, \nu) > 0$, we have*

$$\lim_{N \rightarrow \infty} \sigma_N^{(\alpha)}(2\sqrt{N} + \mu N^{-1/6}, 2\sqrt{N} + \nu N^{-1/6}) = \frac{I^{(\alpha)}(\mu, \nu)}{\sqrt{I^{(\alpha)}(\mu, \mu)} \sqrt{I^{(\alpha)}(\nu, \nu)}}.$$

Furthermore, we will prove the following:

Proposition 2.4. *For any $\alpha \in \mathbb{N}$, we have $I^{(\alpha)}(x, x) > 0$ for all $x \in \mathbb{R}$.*

In view of the preceding comments, it is obvious that Corollaries 1.2 and 1.4, which correspond to the special cases $\alpha = 1$ and $\alpha = 2$, follow from Propositions 2.3 and 2.4.

We remark in passing that for a general parameter $\alpha > 0$, $\sigma_N^{(\alpha)}(\mu, \nu)$ can be interpreted as the correlation coefficient of the characteristic polynomial of a random matrix from (the rescaled version of) the tridiagonal beta ensemble (with $\alpha = 2/\beta$), since in this setting, the average of the characteristic polynomial is also given by the Hermite polynomial (see Theorem 4.1 in DUMITRIU and EDELMAN [DE]).

3. THE PROOFS

Proof of Proposition 2.1. Fix $\mu, \nu \in \mathbb{R}$. We have to evaluate

$$\frac{f_N^{(\alpha)}(\mu_N, \nu_N)}{N!} = \frac{1}{2\pi i} \int_{\gamma} \frac{\exp\left(\mu_N \nu_N \cdot \frac{z}{1-z^2} - \frac{1}{2}(\mu_N^2 + \nu_N^2) \cdot \frac{z^2}{1-z^2} + b^* z^2\right)}{(1-z)^{\alpha+(1/2)} \cdot (1+z)^{1/2}} \frac{dz}{z^{N+1}}, \quad (3.1)$$

where $\mu_N := 2N^{1/2} + \mu N^{-1/6}$, $\nu_N := 2N^{1/2} + \nu N^{-1/6}$, and γ denotes a contour around the origin (which will be chosen further below).

Setting $\xi_N := (\mu_N + \nu_N)/2$ and $\eta_N := (\mu_N - \nu_N)/2$, (3.1) may be written as

$$\exp\left(\frac{1}{2}\xi_N^2 + \eta_N^2\right) \cdot \frac{1}{2\pi i} \int_{\gamma} \frac{\exp\left(-\frac{1}{2}\xi_N^2 \cdot \frac{1-z}{1+z} - \eta_N^2 \cdot \frac{1}{1-z} + b^* z^2\right)}{(1-z)^{\alpha+(1/2)} \cdot (1+z)^{1/2}} \frac{dz}{z^{N+1}} \quad (3.2)$$

with

$$\xi_N^2 = 4N + 2(\mu + \nu)N^{1/3} + \frac{1}{4}(\mu + \nu)^2 N^{-1/3} \quad \text{and} \quad \eta_N^2 = \frac{1}{4}(\mu - \nu)^2 N^{-1/3}.$$

In particular, the leading exponential factor in (3.2) is asymptotically the same as that in Proposition 2.1.

Thus, to complete the proof of Proposition 2.1, it suffices to show that

$$\begin{aligned} \lim_{N \rightarrow \infty} \frac{N^{-(2\alpha-1)/6}}{(2\pi)^{3/2}i} \int_{\gamma} \frac{\exp\left(-\frac{1}{2}\xi_N^2 \cdot \frac{1-z}{1+z} - \eta_N^2 \cdot \frac{1}{1-z} + b^* z^2\right)}{(1-z)^{\alpha+(1/2)} \cdot (1+z)^{1/2}} \frac{dz}{z^{N+1}} \\ = \frac{\exp(b^*)}{4\pi^{3/2}} \int_{-\infty}^{+\infty} \frac{\exp(h_\infty(1-iu))}{(1-iu)^{\alpha+(1/2)}} du, \end{aligned} \quad (3.3)$$

where

$$h_\infty(z) := \frac{1}{12}z^3 - \frac{1}{2}(\mu + \nu)z - \frac{1}{4}(\mu - \nu)^2/z. \quad (3.4)$$

Similarly as in the proof of the main theorem in [GK], the basic idea is that the main contribution to the integral in (3.3) comes from a small neighborhood of the point $z = 1$. Let

$$h_N(z) := -\frac{1}{2}\xi_N^2 \cdot \frac{1-z}{1+z} - \eta_N^2 \cdot \frac{1}{1-z} + b^* z^2 - (\alpha + \frac{1}{2}) \log(1-z) - \frac{1}{2} \log(1+z) - N \log z$$

(where \log denotes the principal branch of the logarithm) and

$$\gamma_N(t) := (1 - N^{-1/3}) \exp(it), \quad -\pi \leq t \leq +\pi.$$

Then the left-hand side in (3.3) can be rewritten as

$$\begin{aligned} \frac{N^{-(2\alpha-1)/6}}{(2\pi)^{3/2}i} \int_{\gamma_N} \frac{\exp\left(-\frac{1}{2}\xi_N^2 \cdot \frac{1-z}{1+z} - \eta_N^2 \cdot \frac{1}{1-z} + b^* z^2\right)}{(1-z)^{\alpha+(1/2)} \cdot (1+z)^{1/2}} \frac{dz}{z^{N+1}} \\ = \frac{N^{-(2\alpha-1)/6}}{(2\pi)^{3/2}i} \int_{\gamma_N} \exp(h_N(z)) \frac{dz}{z} = \frac{N^{-(2\alpha-1)/6}}{(2\pi)^{3/2}} \int_{-\pi}^{+\pi} \exp(h_N(\gamma_N(t))) dt. \end{aligned}$$

Put $I_{N,1}(a) := (-aN^{-1/3}; +aN^{-1/3})$ and $I_{N,2}(a) := (-\pi, +\pi) \setminus (-aN^{-1/3}; +aN^{-1/3})$, where $a > 0$. We shall prove the following:

Claim 1: For any fixed $a > 0$,

$$\lim_{N \rightarrow \infty} \frac{N^{-(2\alpha-1)/6}}{(2\pi)^{3/2}} \int_{I_{N,1}(a)} \exp(h_N(\gamma_N(t))) dt = \frac{\exp(b^*)}{4\pi^{3/2}} \int_{-a}^{+a} \frac{\exp(h_\infty(1-iu))}{(1-iu)^{\alpha+(1/2)}} du.$$

Claim 2: For any $\delta > 0$, there exists a constant $a_0 > 0$ such that for all $a \geq a_0$,

$$\left| \frac{N^{-(2\alpha-1)/6}}{(2\pi)^{3/2}} \int_{I_{N,2}(a)} \exp(h_N(\gamma_N(t))) dt \right| \leq \delta$$

for all $N \in \mathbb{N}$ sufficiently large.

Before turning to the proofs, let us show that Claims 1 and 2 imply (3.3). Observe that

$$\int_{-\infty}^{+\infty} \frac{\exp(h_\infty(1-iu))}{(1-iu)^{\alpha+(1/2)}} du < \infty.$$

Thus, for $a > 0$ sufficiently large, we have not only the conclusion of Claim 2, but also the inequality

$$\left| \frac{\exp(b^*)}{4\pi^{3/2}} \int_{\mathbb{R} \setminus (-a, +a)} \frac{\exp(h_\infty(1-iu))}{(1-iu)^{\alpha+(1/2)}} du \right| < \delta.$$

Hence, in combination with Claim 1, it follows that

$$\begin{aligned} & \left| \frac{N^{-(2\alpha-1)/6}}{(2\pi)^{3/2}} \int_{-\pi}^{+\pi} \exp(h_N(\gamma_N(t))) dt - \frac{\exp(b^*)}{4\pi^{3/2}} \int_{-\infty}^{+\infty} \frac{\exp(h_\infty(1-iu))}{(1-iu)^{\alpha+(1/2)}} du \right| \\ & \leq \left| \frac{N^{-(2\alpha-1)/6}}{(2\pi)^{3/2}} \int_{I_{N,2}(a)} \exp(h_N(\gamma_N(t))) dt \right| \\ & \quad + \left| \frac{N^{-(2\alpha-1)/6}}{(2\pi)^{3/2}} \int_{I_{N,1}(a)} \exp(h_N(\gamma_N(t))) dt - \frac{\exp(b^*)}{4\pi^{3/2}} \int_{-a}^{+a} \frac{\exp(h_\infty(1-iu))}{(1-iu)^{\alpha+(1/2)}} du \right| \\ & \quad + \left| \frac{\exp(b^*)}{4\pi^{3/2}} \int_{\mathbb{R} \setminus (-a, +a)} \frac{\exp(h_\infty(1-iu))}{(1-iu)^{\alpha+(1/2)}} du \right| \\ & \leq \delta + \delta + \delta = 3\delta \end{aligned}$$

for all $N \in \mathbb{N}$ sufficiently large. Since $\delta > 0$ is arbitrary, this proves (3.3).

Proof of Claim 1: First of all, substituting $t = uN^{-1/3}$, we have

$$\frac{N^{-(2\alpha-1)/6}}{(2\pi)^{3/2}} \int_{-aN^{-1/3}}^{+aN^{-1/3}} \exp(h_N(\gamma_N(t))) dt = \frac{N^{-(2\alpha+1)/6}}{(2\pi)^{3/2}} \int_{-a}^{+a} \exp(h_N(\gamma_N(uN^{-1/3}))) du.$$

We will determine the asymptotics of $h_N(\gamma_N(uN^{-1/3}))$ as $N \rightarrow \infty$, for $u \in [-a, +a]$. The \mathcal{O} -bounds occurring in the sequel hold uniformly in this region. To begin with, note that

$$\begin{aligned} z := \gamma_N(uN^{-1/3}) &= (1 - N^{-1/3})e^{iuN^{-1/3}} \\ &= (1 - N^{-1/3})(1 + iuN^{-1/3} - \frac{1}{2}u^2N^{-2/3} - \frac{1}{6}iu^3N^{-1} + \mathcal{O}(N^{-4/3})) \\ &= 1 - (1 - iu)N^{-1/3} - (iu + \frac{1}{2}u^2)N^{-2/3} + (\frac{1}{2}u^2 - \frac{1}{6}iu^3)N^{-1} + \mathcal{O}(N^{-4/3}). \end{aligned}$$

Therefore, for N sufficiently large, we have the following approximations:

$$\begin{aligned} & -\frac{1}{2}\xi_N^2 \cdot \frac{1-z}{1+z} \\ &= -\frac{1}{2}\xi_N^2 \left(\frac{1}{2}(1-z) + \frac{1}{4}(1-z)^2 + \frac{1}{8}(1-z)^3 + \mathcal{O}((1-z)^4) \right) \\ &= -\frac{1}{2} \left(4N + 2(\mu + \nu)N^{1/3} + \mathcal{O}(N^{-1/3}) \right) \\ & \quad \cdot \left(\frac{1}{2}(1-iu)N^{-1/3} + \frac{1}{4}N^{-2/3} + \left(\frac{1}{8} + \frac{1}{8}iu + \frac{1}{8}u^2 - \frac{1}{24}iu^3 \right)N^{-1} + \mathcal{O}(N^{-4/3}) \right) \\ &= -(1-iu)N^{2/3} - \frac{1}{2}N^{1/3} + \left(\frac{1}{12}(1-iu)^3 - \frac{1}{2}(\mu + \nu)(1-iu) - \frac{1}{3} \right) + \mathcal{O}(N^{-1/3}) \\ &= -\eta_N^2 \cdot \frac{1}{1-z} = - \left(\frac{1}{4}(\mu - \nu)^2 N^{-1/3} \right) \cdot \frac{1}{(1-iu)N^{-1/3}} \cdot \left(1 + \mathcal{O}(N^{-1/3}) \right) \\ &= -\frac{1}{4}(\mu - \nu)^2 / (1-iu) \cdot \left(1 + \mathcal{O}(N^{-1/3}) \right) \end{aligned}$$

$$\begin{aligned}
b^* z^2 &= b^* + \mathcal{O}(N^{-1/3}) \\
-(\alpha + \frac{1}{2}) \log(1 - z) &= +\frac{1}{3}(\alpha + \frac{1}{2}) \log N - (\alpha + \frac{1}{2}) \log(1 - iu) + \mathcal{O}(N^{-1/3}) \\
-\frac{1}{2} \log(1 + z) &= -\frac{1}{2} \log 2 + \mathcal{O}(N^{-1/3}) \\
-N \log z &= -N \left(iuN^{-1/3} + \log(1 - N^{-1/3}) \right) \\
&= -N \left(iuN^{-1/3} - N^{-1/3} - \frac{1}{2}N^{-2/3} - \frac{1}{3}N^{-1} + \mathcal{O}(N^{-4/3}) \right) \\
&= (1 - iu)N^{2/3} + \frac{1}{2}N^{1/3} + \frac{1}{3} + \mathcal{O}(N^{-1/3})
\end{aligned}$$

Putting these approximations together, the terms of highest order cancel out, and we end up with

$$\begin{aligned}
h_N(\gamma_N(uN^{-1/3})) &= \frac{1}{3}(\alpha + \frac{1}{2}) \log N + \left(\frac{1}{12}(1 - iu)^3 - \frac{1}{2}(\mu + \nu)(1 - iu) \right. \\
&\quad \left. - \frac{1}{4}(\mu - \nu)^2/(1 - iu) - (\alpha + \frac{1}{2}) \log(1 - iu) + b^* - \frac{1}{2} \log 2 \right) + \mathcal{O}(N^{-1/3}).
\end{aligned}$$

Hence, by an application of the dominated convergence theorem, it follows that

$$\begin{aligned}
&\lim_{N \rightarrow \infty} \frac{N^{-(2\alpha+1)/6}}{(2\pi)^{3/2}} \int_{-a}^{+a} \exp(h_N(\gamma_N(uN^{-1/3}))) du \\
&= \frac{\exp(b^*)}{4\pi^{3/2}} \int_{-a}^{+a} \frac{\exp\left(\frac{1}{12}(1 - iu)^3 - \frac{1}{2}(\mu + \nu)(1 - iu) - \frac{1}{4}(\mu - \nu)^2/(1 - iu)\right)}{(1 - iu)^{\alpha+(1/2)}} du,
\end{aligned}$$

and Claim 1 is proved.

Proof of Claim 2: By symmetry, it suffices to consider the interval $(aN^{-1/3}, \pi)$. Write the integral as

$$\frac{N^{-(2\alpha-1)/6}}{(2\pi)^{3/2}} \int_{aN^{-1/3}}^{\pi} \frac{\exp\left(-\frac{1}{2}\xi_N^2 \cdot \frac{1-\gamma_N(t)}{1+\gamma_N(t)} - \eta_N^2 \cdot \frac{1}{1-\gamma_N(t)} + b^* \gamma_N(t)^2\right)}{(1 - \gamma_N(t))^{\alpha+(1/2)} \cdot (1 + \gamma_N(t))^{1/2}} \frac{dt}{\gamma_N(t)^N}.$$

Since ξ_N and η_N are real and $|\gamma_N(t)| \leq 1$, this is clearly bounded by

$$\frac{N^{-(2\alpha-1)/6}}{(2\pi)^{3/2}} \int_{aN^{-1/3}}^{\pi} \frac{\exp\left(-\frac{1}{2}\xi_N^2 \cdot \operatorname{Re}\left(\frac{1-\gamma_N(t)}{1+\gamma_N(t)}\right) - \eta_N^2 \cdot \operatorname{Re}\left(\frac{1}{1-\gamma_N(t)}\right) + |b^*|\right)}{|1 - \gamma_N(t)|^{\alpha+(1/2)} \cdot |1 + \gamma_N(t)|^{1/2}} \frac{dt}{|\gamma_N(t)|^N}.$$

In the following, let $K, K_1, K_2 > 0$ denote constants which depend only on α, b^*, μ, ν (which are regarded as fixed) but which may change from occurrence to occurrence. Then we have

$$\xi_N^2 \geq 4N - KN^{1/3}$$

and

$$\operatorname{Re}\left(\frac{1 - \gamma_N(t)}{1 + \gamma_N(t)}\right) = \frac{2N^{-1/3} - N^{-2/3}}{(2 + 2 \cos t)(1 - N^{-1/3}) + N^{-2/3}}$$

(as follows from a straightforward calculation) and therefore

$$\frac{1}{2}\xi_N^2 \operatorname{Re}\left(\frac{1 - \gamma_N(t)}{1 + \gamma_N(t)}\right) \geq \frac{4N^{2/3} - 2N^{1/3} - K}{(2 + 2 \cos t)(1 - N^{-1/3}) + N^{-2/3}}.$$

Since

$$\begin{aligned} |\gamma_N(t)|^N &= \exp(N \log(1 - N^{-1/3})) \\ &\geq \exp(N(-N^{-1/3} - \frac{1}{2}N^{-2/3} - KN^{-1})) \\ &= \exp(-N^{2/3} - \frac{1}{2}N^{1/3} - K), \end{aligned}$$

it follows that

$$\begin{aligned} &\frac{\exp\left(-\frac{1}{2}\xi_N^2 \operatorname{Re}\left(\frac{1-\gamma_N(t)}{1+\gamma_N(t)}\right)\right)}{|\gamma(t)|^N} \\ &\leq \exp\left(-\frac{4N^{2/3} - 2N^{1/3} - K_1}{(2 + 2\cos t)(1 - N^{-1/3}) + N^{-2/3}} + N^{2/3} + \frac{1}{2}N^{1/3} + K_2\right) \\ &\leq \exp\left(-\frac{(2 - 2\cos t)N^{2/3} - (1 - \cos t)N^{1/3} - K^*}{(2 + 2\cos t)(1 - N^{-1/3}) + N^{-2/3}}\right) \end{aligned}$$

for some constant $K^* > 0$, say. Now let $\varepsilon > 0$ denote a constant such that $\varepsilon^2 t^2 \leq 1 - \cos t \leq \frac{1}{2}t^2$ for all $t \in [0, \pi]$. Then, for $a \geq \varepsilon^{-1}\sqrt{2K^*}$, $N^{1/3} \geq \max\{\varepsilon^{-2}, a\pi^{-1}\}$ and $t \in [aN^{-1/3}, \pi]$, we have

$$\begin{aligned} &(2 - 2\cos t)N^{2/3} - (1 - \cos t)N^{1/3} - K^* \\ &\geq \varepsilon^2 t^2 N^{2/3} + (\frac{1}{2}\varepsilon^2 t^2 N^{2/3} - \frac{1}{2}t^2 N^{1/3}) + (\frac{1}{2}\varepsilon^2 t^2 N^{2/3} - K^*) \geq N^{2/3}\varepsilon^2 t^2 \end{aligned}$$

and therefore

$$\begin{aligned} &\frac{\exp\left(-\frac{1}{2}\xi_N^2 \operatorname{Re}\left(\frac{1-\gamma_N(t)}{1+\gamma_N(t)}\right)\right)}{|\gamma(t)|^N} \\ &\leq \exp\left(-\frac{N^{2/3}\varepsilon^2 t^2}{(2 + 2\cos t)(1 - N^{-1/3}) + N^{-2/3}}\right) \leq \exp\left(-N^{2/3}\varepsilon^2 t^2/4\right). \end{aligned}$$

Thus, since $|1 \pm \gamma_N(t)| \geq N^{-1/3}$, the integral under consideration is bounded by

$$\begin{aligned} &K N^{-(2\alpha-1)/6} \int_{aN^{-1/3}}^{\pi} \frac{\exp(-N^{2/3}\varepsilon^2 t^2/4)}{|1 - \gamma_N(t)|^{\alpha+1/2} \cdot |1 + \gamma_N(t)|^{1/2}} dt \\ &\leq K N^{-(2\alpha-1)/6} \int_{aN^{-1/3}}^{\pi} \frac{\exp(-N^{2/3}\varepsilon^2 t^2/4)}{N^{-(2\alpha+1)/6}} dt \\ &= K N^{1/3} \int_{aN^{-1/3}}^{\pi} \exp(-N^{2/3}\varepsilon^2 t^2/4) dt \\ &\leq K \int_a^{\infty} \exp(-\varepsilon^2 u^2/4) du. \end{aligned}$$

Obviously, this upper bound can be made arbitrarily small by picking a and N large enough. This proves Claim 2.

The proof of Proposition 2.1 is complete now. \square

The basic ingredient for the proof of Proposition 2.2 will be the following integral representation for the product of two Airy functions:

Lemma 3.1. *For any $x, y \in \mathbb{C}$,*

$$\text{Ai}(x) \text{Ai}(y) = \frac{1}{4\pi^{3/2}i} \int_{\mathcal{L}} \frac{\exp\left(\frac{1}{12}z^3 - \frac{1}{2}(x+y)z - \frac{1}{4}(x-y)^2/z\right)}{z^{1/2}} dz, \quad (3.5)$$

where \mathcal{L} denotes some (unbounded) contour from $\infty e^{-\pi i/3}$ to $\infty e^{+\pi i/3}$.

In the special case where $y = \pm x$, this result can already be found in REID [Re]. For the convenience of the reader, we give a detailed proof of Lemma 3.1:

Proof. We start from the following well-known integral representation of the Airy function:

$$\text{Ai}(x) = \frac{1}{2\pi i} \int_{\mathcal{L}} \exp\left(\frac{1}{3}z^3 - xz\right) dz.$$

A standard application of Cauchy's theorem shows that the contour \mathcal{L} can be deformed into the contour $t \mapsto 1 + it$, $t \in \mathbb{R}$. Thus, we obtain

$$\text{Ai}(x) = \frac{1}{2\pi i} \int_{1-i\infty}^{1+i\infty} \exp\left(\frac{1}{3}z^3 - xz\right) dz. \quad (3.6)$$

Observe that the resulting integral exists in the Lebesgue sense, since we have

$$\left| \exp\left(\frac{1}{3}(1+it)^3 - x(1+it)\right) \right| \leq \exp\left(\frac{1}{3} - t^2 + |x|(1+|t|)\right)$$

for any $t \in \mathbb{R}$.

It follows from (3.6) that

$$\text{Ai}(x) \text{Ai}(y) = \frac{1}{4\pi^2} \iint \exp\left(\frac{1}{3}(1+it)^3 - x(1+it)\right) \exp\left(\frac{1}{3}(1+iu)^3 - y(1+iu)\right) du dt.$$

Substituting $(t, u) = (\frac{1}{2}(v+w), \frac{1}{2}(v-w))$ and doing a small calculation, we find that

$$\text{Ai}(x) \text{Ai}(y) = \frac{1}{8\pi^2} \iint \exp\left(\frac{1}{12}(2+iv)^3 - \frac{1}{4}(2+iv)w^2 - \frac{1}{2}(x+y)(2+iv) - \frac{1}{4}(x-y)iw\right) dw dv.$$

Using the well-known relation

$$\int_{-\infty}^{+\infty} \exp(-aw^2 - bw) dw = \frac{\sqrt{\pi}}{\sqrt{a}} e^{b^2/4a}$$

(where $\text{Re}(a) > 0$), it follows that

$$\text{Ai}(x) \text{Ai}(y) = \frac{1}{4\pi^{3/2}} \int \frac{\exp\left(\frac{1}{12}(2+iv)^3 - \frac{1}{2}(x+y)(2+iv) - \frac{1}{4}(x-y)^2/(2+iv)\right)}{(2+iv)^{1/2}} dv$$

or

$$\text{Ai}(x) \text{Ai}(y) = \frac{1}{4\pi^{3/2}i} \int_{2-i\infty}^{2+i\infty} \frac{\exp\left(\frac{1}{12}z^3 - \frac{1}{2}(x+y)z - \frac{1}{4}(x-y)^2/z\right)}{z^{1/2}} dz.$$

By another application of Cauchy's theorem, the contour $v \mapsto 2 + iv$, $v \in \mathbb{R}$, may be deformed back into the contour \mathcal{L} . \square

Proof of Proposition 2.2. Replacing the contour \mathcal{L} in Lemma 3.1 by the contour $t \mapsto 1 + it$ and substituting $t = -u$, we have

$$\text{Ai}(x) \text{Ai}(y) = \frac{1}{4\pi^{3/2}} \int_{-\infty}^{+\infty} \frac{\exp\left(\frac{1}{12}(1-iu)^3 - \frac{1}{2}(x+y)(1-iu) - \frac{1}{4}(x-y)^2/(1-iu)\right)}{(1-iu)^{1/2}} du.$$

By means of abbreviation, write $E(x, y, u)$ for the numerator inside the integral. Then, for any $\alpha > 0$, we have

$$\frac{\partial}{\partial y} \left(\int \frac{E(x, y, u)}{(1-iu)^\alpha} du \right) = \int \frac{E(x, y, u)}{(1-iu)^\alpha} \left(-\frac{1}{2}(1-iu) + \frac{1}{2}(x-y)/(1-iu)\right) du,$$

$$\frac{\partial}{\partial x} \left(\int \frac{E(x, y, u)}{(1-iu)^\alpha} du \right) = \int \frac{E(x, y, u)}{(1-iu)^\alpha} \left(-\frac{1}{2}(1-iu) - \frac{1}{2}(x-y)/(1-iu)\right) du,$$

and therefore

$$\left(\frac{1}{x-y} \left(\frac{\partial}{\partial y} - \frac{\partial}{\partial x} \right) \right) \left(\int \frac{E(x, y, u)}{(1-iu)^\alpha} du \right) = \int \frac{E(x, y, u)}{(1-iu)^{\alpha+1}} du.$$

The assertion of Proposition 2.2 now follows by induction. \square

Proof of Proposition 2.3. Fix $\mu, \nu \in \mathbb{R}$, and put $\mu_N := 2N^{1/2} + \mu N^{-1/6}$, $\nu_N := 2N^{1/2} + \nu N^{-1/6}$. Using well-known results about the asymptotic properties of the Hermite polynomials (see e.g. Theorem 8.22.9 (c) in SZEGÖ [Sz]), we find that the function $g_N(\lambda)$ given by $g_N(\lambda) := (-1)^N 2^{-N/2} H_N(\lambda/\sqrt{2})$ ($\lambda \in \mathbb{R}$) satisfies

$$\begin{aligned} & g_N(2\sqrt{N} + \mu N^{-1/6}) g_N(2\sqrt{N} + \nu N^{-1/6}) \\ &= 2^{-N} H_N(\sqrt{2N} + \mu N^{-1/6}/\sqrt{2}) H_N(\sqrt{2N} + \nu N^{-1/6}/\sqrt{2}) \\ &= \mathcal{O}\left(2^{-N} \exp(N + \mu N^{1/3}) 2^{N/2} N!^{1/2} N^{-1/12} \exp(N + \nu N^{1/3}) 2^{N/2} N!^{1/2} N^{-1/12}\right) \\ &= \mathcal{O}\left(N! N^{-1/6} \exp(2N + (\mu + \nu)N^{1/3})\right) \\ &= o\left(N! N^{(2\alpha-1)/6} \exp(2N + (\mu + \nu)N^{1/3})\right) \end{aligned}$$

for any $\alpha > 0$. Setting $c_N^{(\alpha)}(\mu, \nu) := (\sqrt{2\pi} N! N^{(2\alpha-1)/6} \exp(2N + (\mu + \nu)N^{1/3}))^{-1}$, we therefore obtain

$$\begin{aligned} & \lim_{N \rightarrow \infty} \sigma_N^{(\alpha)}(2\sqrt{N} + \mu N^{-1/6}, 2\sqrt{N} + \nu N^{-1/6}) \\ &= \lim_{N \rightarrow \infty} \frac{f_N^{(\alpha)}(\mu_N, \nu_N) - g_N(\mu_N) g_N(\nu_N)}{\sqrt{f_N^{(\alpha)}(\mu_N, \mu_N) - g_N(\mu_N)^2} \sqrt{f_N^{(\alpha)}(\nu_N, \nu_N) - g_N(\nu_N)^2}} \\ &= \lim_{N \rightarrow \infty} \frac{c_N^{(\alpha)}(\mu, \nu) \left(f_N^{(\alpha)}(\mu_N, \nu_N) - g_N(\mu_N) g_N(\nu_N)\right)}{\sqrt{c_N^{(\alpha)}(\mu, \mu) \left(f_N^{(\alpha)}(\mu_N, \mu_N) - g_N(\mu_N)^2\right)} \sqrt{c_N^{(\alpha)}(\nu, \nu) \left(f_N^{(\alpha)}(\nu_N, \nu_N) - g_N(\nu_N)^2\right)}} \\ &= \frac{\lim_{N \rightarrow \infty} c_N^{(\alpha)}(\mu, \nu) f_N^{(\alpha)}(\mu_N, \nu_N)}{\sqrt{\lim_{N \rightarrow \infty} c_N^{(\alpha)}(\mu, \mu) f_N^{(\alpha)}(\mu_N, \mu_N)} \sqrt{\lim_{N \rightarrow \infty} c_N^{(\alpha)}(\nu, \nu) f_N^{(\alpha)}(\nu_N, \nu_N)}} \\ &= \frac{I^{(\alpha)}(\mu, \nu)}{\sqrt{I^{(\alpha)}(\mu, \mu)} \sqrt{I^{(\alpha)}(\nu, \nu)}}, \end{aligned}$$

where we have used Proposition 2.1 as well as the assumptions $I^{(\alpha)}(\mu, \mu) > 0$, $I^{(\alpha)}(\nu, \nu) > 0$. This completes the proof of Proposition 2.3. \square

Proof of Proposition 2.4. First of all, note that the definition (2.2) may be extended to the case $\alpha = 0$ and that $I^{(0)}(x, x) = \text{Ai}(x)^2$ for any $x \in \mathbb{R}$ by Lemma 3.1. Thus, $I^{(0)}(x, x) \geq 0$ for any $x \in \mathbb{R}$, with strict inequality for $x > 0$ (since it is well-known that the Airy function does not have any zeroes on the positive half-axis). Moreover, note that for any $\alpha > 0$,

$$\begin{aligned} \frac{\partial}{\partial x} \left(I^{(\alpha)}(x, x) \right) &= \frac{\partial}{\partial x} \left(\frac{1}{4\pi^{3/2}} \int_{-\infty}^{+\infty} \frac{\exp\left(\frac{1}{12}(1-iu)^3 - x(1-iu)\right)}{(1-iu)^{\alpha+(1/2)}} du \right) \\ &= -\frac{1}{4\pi^{3/2}} \int_{-\infty}^{+\infty} \frac{\exp\left(\frac{1}{12}(1-iu)^3 - x(1-iu)\right)}{(1-iu)^{\alpha-(1/2)}} du \\ &= -I^{(\alpha-1)}(x, x) \end{aligned}$$

for any $x \in \mathbb{R}$. Since $\lim_{x \rightarrow \infty} I^{(\alpha)}(x, x) = 0$, this implies that for any $\alpha > 0$,

$$I^{(\alpha)}(x, x) = \int_x^{\infty} I^{(\alpha-1)}(y, y) dy$$

for any $x \in \mathbb{R}$. Proposition 2.4 now follows by a straightforward induction on α . \square

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HOLGER KÖSTERS, FAKULTÄT FÜR MATHEMATIK, UNIVERSITÄT BIELEFELD, POSTFACH 100131, 33501 BIELEFELD, GERMANY

E-mail address: hkoesters@math.uni-bielefeld.de