ASYMPTOTIC EXPANSIONS IN THE FREE CENTRAL LIMIT THEOREM.

G. P. CHISTYAKOV[1,3] AND F. GÖTZE[2,4]

Abstract. Based on an analytical approach to the definition of additive free convolution on probability measures on the real line we prove free analogs of Esseen's expansions for sums for independent identically distributed random variables in classical Probability Theory.

1. Introduction

In recent years a number of papers are investigating limit theorems for the free convolution of probability measures (p-measures) defined by D. Voiculescu. The key concept of this definition is the notion of freeness, which can be interpreted as a kind of independence for non-commutative random variables. As in the classical probability where the concept of independence gives rise to the classical convolution, the concept of freeness leads to a binary operation on the p-measures on the real line, the free convolution. Many classical results in the theory of addition of independent random variables have their counterpart in this theory, such as the law of large numbers, the central limit theorem, the Lévy-Khintchine formula and others. We refer to Voiculescu, Dykema and Nica [39] and Hiai and Petz [22] for introduction to these topics. Bercovici and Pata [8] established the distributional behavior of sums of free identically distributed random variables and described explicitly the correspondence between limits laws for free and classical additive convolution. Using subordination functions for the definition of the additive free convolution Chistyakov and Götze [18] generalized the results of Bercovici and Pata to the case of free non-identically distributed random variables. It was shown that the parallelism found by Bercovici and Pata holds in the common case of free non-identically distributed random variables (see [10] as well). This approach allowed to obtain estimates of the rate of convergence of distribution functions of free sums. An analog of the Berry-Esseen inequality was proved for the semi-circle approximation in [18]. For related results see [23].

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1) Institute for Low Temperature Physics and Engineering, Kharkov, Ukraine.

3) Research supported by SFB 701 and by the join Ukraine-French collaboration Dnipro.

2) Faculty of Mathematics , University of Bielefeld, Germany.

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In this paper we obtain an analogue of Esseen's expansion in the Central Limit Theorem (CLT for short) for free identically distributed random variables, based on the method of subordination functions. In addition we shall give a bound for the remainder term in this expansion.

The paper is organized as follows. In Section 2 we formulate and discuss the main results of the paper. In Section 3 we formulate auxiliary results. In Section 4 we give a formal expansion in the Free CLT and in Sections 5 and 6 we prove Esseen's expansion in the CLT for free identically distributed random variables.

2. Results

Denote by $\mathcal{M}$ the family of all Borel p-measures defined on the real line $\mathbb{R}$. On $\mathcal{M}$ define the compositions laws denoted $*$ and $\boxplus$ as follows. For $\mu_1, \mu_2 \in \mathcal{M}$, let the p-measure $\mu_1 * \mu_2$ denote the classical convolution of μ_1 and μ_2. In probabilistic terms, $\mu_1 * \mu_2$ is the probability distribution of $X+Y$, where X and Y are independent random variables with distributions μ_1 and μ_2 respectively. The measure $\mu_1 \boxplus \mu_2$ on the other hand denotes the free (additive) convolution of μ_1 and μ_2 introduced by Voiculescu [37] for compactly supported measures. Free convolution was extended by Maassen [27] to measures with finite variance and by Bercovici and Voiculescu [6] to the class $\mathcal{M}$. Thus, $\mu_1 \boxplus \mu_2$ is the probability distribution of $X+Y$, where X and Y are free random variables with distributions μ_1 and μ_2, respectively. There are free analogues of multiplicative convolutions as well; these were first studied in Voiculescu [38].

Let μ be a p-measure. Define $m_k(\mu) := \int_{\mathbb{R}} u^k\,\mu(du)$ and $\beta_q(\mu) := \int_{\mathbb{R}} |u|^q\,\mu(du)$, where $k = 0, 1, \dots$ and $q > 0$.

The classical CLT says that if $X_1, X_2, \dots$ are independent and identically distributed random variables with a probability distribution μ such that $m_1(\mu) = 0$ and $m_2(\mu) = 1$, then the distribution function $F_n(x)$ of

$$Y_n := \frac{X_1 + X_2 + \dots + X_n}{\sqrt{n}} \tag{2.1}$$

tends to the standard Gaussian law $\Phi(x)$ as $n \to \infty$ uniformly in x.

A free analogue of this classical result was proved by Voiculescu [36] for bounded free random variables and later generalized by Maassen [27] to unbounded random variables. Other generalizations can be found in [7], [8], [18], [23]–[25], [32], [41]. When the assumption of independence is replaced by the freeness of the non-commutative random variables $X_1, X_2, \dots, X_n$, the limit distribution function of (2.1) is the semicircle law w which has the distribution function with the density $p_w(x) := \frac{1}{2\pi}\sqrt{(4-x^2)_+}$, $x \in \mathbb{R}$, where $a_+ := \max\{a, 0\}$ for $a \in \mathbb{R}$.

Denote by $\varphi(x) := \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$ and by $H_m(x) := (-1)^m e^{x^2/2} \frac{d^m}{dx^m} e^{-x^2/2}$ the Hermite polynomial of degree m.

Let $\{X_j,\ j=1,\dots\}$ have moments of arbitrary order. For the distribution function $F_n(x)$ there exists a formal expansion in a power series in $1/\sqrt{n}$ [20], [33]:

$$F_n(x) = \Phi(x) + \varphi(x)\sum_{p=1}^{\infty}\frac{Q_p(x)}{n^{p/2}}, \tag{2.2}$$

where

$$Q_p(x) = -\sum H_{p+2s-1}(x)\prod_{m=1}^{p}\frac{1}{k_m!}\Big(\frac{\gamma_{m+2}}{(m+2)!}\Big)^{k_m}$$

and γ_m is the cumulant of order m of random variable X_1. In the last equality the summation on the right-hand side is carried out over all non-negative integer solutions $(k_1,\dots,k_m)$ of the equalities $k_1+2k_2+\dots+pk_p=p$ and $s=k_1+\dots+k_p$. Note that $Q_1(x) = -m_3(\mu)H_2(x)/6$.

In terms of characteristic functions (2.2) has the form

$$\int_{-\infty}^{\infty} e^{itx}\,dF_n(x) = e^{-t^2/2} + \sum_{m=1}^{\infty}\frac{P_m(t)}{n^{m/2}}e^{-t^2/2}, \tag{2.3}$$

where

$$\int_{-\infty}^{\infty} e^{itx}\,dQ_m(x) = P_m(t)e^{-t^2/2}.$$

If $\beta_3(\mu)<\infty$, Esseen [19] proved that $F_n(x)$ has a valid asymptotic expansion described as follows.

If the independent random variables $\{X_j,\ j=1,\dots\}$ have the same non-lattice distribution μ with finite third moment, then

$$F_n(x) = \Phi(x) - \frac{m_3(\mu)}{6\sqrt{n}}H_2(x)\varphi(x) + o(1/\sqrt{n}) \tag{2.4}$$

uniformly in x.

If the independent random variables $\{X_j,\ j=1,\dots\}$ take values in an arithmetic progression $\{a+kh; k=0,\pm1,\dots\}$ (h being maximal), then, uniformly in x,

$$F_n(x) = \Phi(x) + \frac{1}{\sqrt{n}}\varphi(x)\Big(-\frac{m_3(\mu)}{6}H_2(x) + hT\Big(\frac{x\sqrt{n}}{h} - \frac{an}{h}\Big)\Big) + o(1/\sqrt{n}), \tag{2.5}$$

where $T(x) := [x] - x + 1/2$.

If $\beta_k(\mu)<\infty$ for $k>3$, then there are generalizations of the asymptotic expansions (2.4) and (2.5) under additional conditions on the characteristic functions of X_1 [33].

An analytical approach using subordination functions allowed us to give explicit estimates for the rate of convergence of distribution functions of (2.1) in the case of free random variables. We demonstrated this (see [18]) by proving a semicircle approximation theorem (an analogue of the Berry-Esseen inequality [33], p. 111). In this paper we shall prove Esseen's expansion in the semicircle approximation theorem.

We now formulate the main results of the paper. As before we denote by $F_n(x)$ the distribution function of Y_n where X_j are free random variables with the same distribution μ. Assume as well that X_j have moments of arbitrary order and $m_1(\mu)=0$, $m_2(\mu)=1$. We denote by μ_n the distribution of Y_n. Denote by $U_m(x)$ the Chebyshev polynomials of the second kind of degree m, i. e.,

$$U_m(x)=U_m(\cos\theta):=\frac{\sin(m+1)\theta}{\sin\theta},\quad m=1,2,\dots.$$

It is easy to see $U_1(x)=2x$, $U_2(x)=4x^2-1$, $U_3(x)=4x(2x^2-1)$.

It turns out that there exists an analogue of the formal expansion (2.3) for $F_n(x)$. To formulate it we need the following notation. For $\mu\in\mathcal{M}$, define its Cauchy transform by

$$G_\mu(z)=\int\limits_{\mathbb{R}}\frac{\mu(dx)}{z-x},\qquad z\in\mathbb{C}^+,\tag{2.6}$$

where $\mathbb{C}^+$ denotes the open upper half of the complex plane. A formal expansion has the form

$$G_{\mu_n}(z)=G_w(z)+\sum_{k=1}^{\infty}\frac{B_k(G_w(z))}{n^{k/2}},\tag{2.7}$$

where

$$B_k(z)=\sum c_{p,m}\frac{z^p}{(1/z-z)^m}\tag{2.8}$$

with coefficients $c_{p,m}$ which depend on the free cumulants $\alpha_3(\mu),\dots,\alpha_{k+2}(\mu)$ and do not depend on n. Free cumulants will be defined in Section 3. Here we note that $\alpha_3(\mu)=m_3(\mu)$ and $\alpha_4(\mu)=m_4(\mu)-2$. The summation on the right-hand side of (2.8) is carried out over a finite set of non-negative integer pairs (p,m). The coefficients $c_{p,m}$ can be calculated explicitly. For the case $k=1,2$ we have

$$B_1(z)=\alpha_3(\mu)\frac{z^3}{1/z-z}$$

and

$$B_2(z)=\Big(\alpha_4(\mu)-\frac{5\alpha_3^2(\mu)}{8}\Big)\frac{z^4}{1/z-z}+\frac{\alpha_3^2(\mu)}{8}\Big(z^3\cdot\frac{3+13z^2}{(1/z-z)^2}+2z^2\cdot\frac{(1+z^2)^2}{(1/z-z)^3}\Big).\tag{2.9}$$

Note that

$$B_1(G_w(z))=\frac{\alpha_3(\mu)}{\sqrt{z^2-4}}G_w^3(z)=-\alpha_3(\mu)\int\limits_{-2}^{2}\frac{1}{z-x}\,d\Big(\frac{1}{3}U_2(x/2)p_w(x)\Big),\quad z\in\mathbb{C}^+.\tag{2.10}$$

If $\alpha_3(\mu) = 0$, then

$$B_2(G_w(z)) = \frac{\alpha_4(\mu)}{\sqrt{z^2-4}} G_w^4(z) = -\alpha_4(\mu) \int_{-2}^{2} \frac{1}{z-x} \, d\Big(\frac{1}{4} U_3(x/2) p_w(x)\Big), \quad z \in \mathbb{C}^+. \tag{2.11}$$

To obtain an analogue of Esseen's expansion for $F_n(x)$ for free identically distributed random variables $X_1, \dots, X_n$ we need the following notations.

Consider the three-parameter family of p-measures $\{\mu_{a,b,d} : a \in \mathbb{R}, b < 1, d < 1\}$ with the reciprocal Cauchy transform

$$\frac{1}{G_{\mu_{a,b,d}}(z)} = a + \frac{1}{2}\Big((1+b)(z-a) + \sqrt{(1-b)^2(z-a)^2 - 4(1-d)}\Big), \quad z \in \mathbb{C}, \tag{2.12}$$

which we will call the free centered (i.e. with mean zero) Meixner measures. In this formula the branch of the analytic square root should be determined by the condition that $\Im z > 0 \implies \Im(1/G_{\mu_{a,b,d}}(z)) \geq 0$. These p-measures are an analogue of the classical p-measures discovered by Meixner [31]. The p-measures of free Meixner type occured in many places in the literature, see [3], [13], [14], [15], [26], [30], [34]. Saiton and Yoshida [34] have proved that the absolutely continuous part of $\mu_{a,b,d}$ is

$$\frac{\sqrt{4(1-d) - (1-b)^2(x-a)^2}}{2\pi f(x)} \tag{2.13}$$

on $a - 2\sqrt{1-d}/(1-b) \leq x \leq a + 2\sqrt{1-d}/(1-b)$, where $f(x) := bx^2 + a(1-b)x + 1 - d$; the measure may also have a discrete part μ_D in the following cases:

1. if $f(x)$ has two real roots $y_1 \neq y_2$, then

$$\mu_D := \lambda_1 \delta_{y_1} + \lambda_2 \delta_{y_2}, \tag{2.14}$$

where

$$\lambda_j := \frac{1}{\sqrt{a^2(1-b)^2 - 4b(1-d)}} \Big(\frac{1-d}{|y_j|} - |y_j|\Big)_+, \quad j = 1, 2, \tag{2.15}$$

2. if $b = 0$ and $a \neq 0$, then

$$\mu_D := \Big(1 - \frac{1-d}{a^2(1-b)^2}\Big)_+ \delta_y, \quad \text{where} \quad y := -\frac{1-d}{a(1-b)}. \tag{2.16}$$

Saiton and Yoshida proved as well that for $0 \leq b < 1$ the (centered) free Meixner measure $\mu_{a,b,d}$ is $\boxplus$-infinitely divisible, i.e., for every integer n there exists a p-measure ν_n such that $\mu_{a,b,d} = \nu_n \boxplus \nu_n \boxplus \cdots \boxplus \nu_n$ (n times). Note (see Bożejko and Bryc [13]) that $\mu_{a,b,d} = w$ if $a = b = d = 0$; $\mu_{a,b,d}$ is the free Poisson type measure, which is also known as Marchenko-Pastur measure [28], if $b = d = 0$ and $a \neq 0$; $\mu_{a,b,d}$ is the free Pascal (negative binomial) type measure if $b > 0$ and $a^2(1-b)^2 > 4b(1-d)$; $\mu_{a,b,d}$ is the free gamma type measure if $b > 0$ and $a^2(1-b)^2 = 4b(1-d)$; $\mu_{a,b,d}$ is the pure free Meixner type measure if $b > 0$ and $a^2(1-b)^2 < 4b(1-d)$.

Introduce the quantities

$$q_1 := \min\{q,4\}, \quad q_2 := q_1 - 11/3, \quad q_3 := \min\{q,6\}, \quad q_4 := q_3 - 16/3.$$

Note that $q_2 > 0$ for $q > 11/3$ and $q_4 > 0$ for $q > 16/3$, respectively. In the sequel we denote by c different positive absolute constants in different (or even in the same) formulae.

We denote as well

$$a_n := \frac{m_3(\mu)}{\sqrt{n}}, \quad b_n := \frac{m_4(\mu) - m_3^2(\mu) - 1}{n}, \quad d_n := \frac{m_4(\mu) - m_3^2(\mu)}{n}.$$

By the well-known moment inequality

$$\begin{vmatrix} 1 & m_1(\mu) & m_2(\mu) \\ m_1(\mu) & m_2(\mu) & m_3(\mu) \\ m_2(\mu) & m_3(\mu) & m_4(\mu) \end{vmatrix} \geq 0$$

(see [1]), we conclude that $m_4(\mu) - 1 - m_3^2(\mu) \geq 0$ in the case where $m_1(\mu) = 0$ and $m_2(\mu) = 1$. Therefore the lower bounds $b_n \geq 0$ and $d_n > 0$ hold. In addition we have $b_n < 1$ and $d_n < 1$ for $n > m_4(\mu) - m_3^2(\mu)$. Consider the p-measures $\mu_{a_n,0,0}$ and μ_{a_n,b_n,d_n}. Thus the p-measures μ_{a_n,b_n,d_n} may be the free Pascal, the free gamma and the pure free Meixner type measures.

Let $b_n > 0$. Note that the polynomial $f_n(x) = b_n x^2 + a_n(1-b_n)x + 1 - d_n$ has two real roots y_{1n} and y_{2n} in the case $a_n^2(1-b_n)^2 - 4b_n(1-d_n) > 0$ and these roots satisfy the inequalities $|y_{jn}| > 1$, $j = 1,2$, for $n \geq 4(m_4(\mu) + m_3^2(\mu) + 1)$. Using (2.14) and (2.15) we see that the discrete part of μ_{a_n,b_n,d_n} is equal to zero for $n \geq 4(m_4(\mu) + m_3^2(\mu) + 1)$.

Let $b_n = 0$ and $a_n \neq 0$. We see from (2.16) that in this case the discrete part of μ_{a_n,b_n,d_n} is equal to zero for $n \geq 2(m_4(\mu) + m_3^2(\mu))$.

Thus, it follows from Saiton and Yoshida's results that the p-measures $\mu_{a_n,0,0}$ and $\mu_{a_n,\beta_n,\gamma_n}$ are $\boxplus$-infinitely divisible and they are absolutely continuous with a density of the form (2.13) for $n \geq 4(m_4(\mu) + m_3^2(\mu) + 1)$.

Now we can formulate an analogue of Esseen's expansion in the Free CLT.

Theorem 2.1. *Assume that the free random variables X_j, $j = 1, \dots,$ have the same distribution μ such that $\beta_q(\mu) < \infty$ with some $q > 11/3$ and $m_1(\mu) = 0$, $m_2(\mu) = 1$. Then there exists a positive absolute constant c such that*

$$\sup_{x\in\mathbb{R}} |F_n(x) - \mu_{a_n,0,0}((-\infty,x))| \leq c\frac{(10^2\beta_{q_1}(\mu))^{1+1/q_2}}{n^{(1+3q_2/5)/2}}, \quad n \in \mathbb{N}, \tag{2.17}$$

where $a_n := m_3(\mu)/\sqrt{n}$.

Note that $q_2 > 0$ may be arbitrary small which explains why we can not include $10^{2(1+1/q_2)}$ into the absolute constant c.

Corollary 2.2. *Under the assumptions of Theorem* 2.1 *there exists a positive absolute constant c such that*

$$F_n(x) = \mu_w((-\infty, x)) - \frac{1}{3} a_n U_2(x/2) p_w(x) + \rho_{n1}(x), \tag{2.18}$$

where the remainder term $\rho_{n1}(x)$ *admits the bound*

$$|\rho_{n1}(x)| \le c \frac{(10^2 \beta_{q_1}(\mu))^{1+1/q_2}}{n^{(1+3q_2/5)/2}}, \qquad x \in \mathbb{R}, \quad n \in \mathbb{N}. \tag{2.19}$$

Before formulating the second result denote by ν a signed measure with the density

$$p_\nu(x) := \frac{1}{2\pi}(x^2 - 1)\sqrt{(4 - x^2)_+}, \quad x \in \mathbb{R}. \tag{2.20}$$

Theorem 2.3. *Assume that the free random variables* X_j, $j = 1, \dots,$ *have the same distribution* μ *such that* $\beta_q(\mu) < \infty$ *with some* $q > 16/3$ *and* $m_1(\mu) = 0$, $m_2(\mu) = 1$. *Then there exists a positive absolute constant c such that*

$$\sup_{x \in \mathbb{R}} |F_n(x) - \mu_{a_n, b_n, d_n}((-\infty, x)) - \frac{1}{n}\nu((-\infty, x))| \le c \frac{(10^2 \beta_{q_3}(\mu))^{1/2+2/q_4}}{n^{1+q_4/4}}, \quad n \in \mathbb{N}. \tag{2.21}$$

The quanity $q_4 > 0$ may be arbitrary small therefore we can not include $10^{2(1/2+2/q_4)}$ into the absolute constant c.

Corollary 2.4. *Assume that the assumptions of Theorem* 2.3 *are satisfied. Then there exists a positive absolute constant c such that, for all real x,*

$$F_n(x + a_n) = \mu_w((-\infty, x)) + \Big(-\frac{a_n^2}{2} U_1\big(\frac{x}{2}\big) + \frac{a_n}{3}\big(3 - U_2\big(\frac{x}{2}\big)\big) - \frac{b_n - a_n^2 - 1/n}{4} U_3\big(\frac{x}{2}\big)\Big) p_w(x) + \rho_{n2}(x), \tag{2.22}$$

where

$$|\rho_{n2}(x)| \le c \frac{(10^2 \beta_{q_3}(\mu))^{1/2+2/q_4}}{n^{1+q_4/4}}, \quad x \in \mathbb{R}, \quad n \in \mathbb{N}. \tag{2.23}$$

If $m_3(\mu) = 0$ *this formula has the following simple form*

$$F_n(x) = \mu_w((-\infty, x)) - \frac{m_4(\mu) - 2}{4n} U_3\big(\frac{x}{2}\big) p_w(x) + \rho_{n2}(x). \tag{2.24}$$

We see from (2.7) and (2.10) that the first two summands on the right-hand side of (2.7) are Cauchy's transform of a finite signed measure from the right-hand side of (2.18). Moreover, in the case $m_3(\mu) = 0$ the first three summands on the right-hand side of (2.7) are Cauchy's transform of a finite signed measure from the right-hand side of (2.24). But in the case $m_3(\mu) \neq 0$ the third summand on the right-hand side of (2.7) can not be a Cauchy transform of a signed measure ζ which is finite on every bounded interval and $\int_{\mathbb{R}} |\zeta(du)|/(1 + |u|) < \infty$. We will prove this in Section 4. The relation (2.22) gives us the correct expansion in the case $m_3(\mu) \neq 0$.

Remark 2.5. Using our method of the proof of Theorems 2.1 and 2.3 we could not obtain free analogue of the Esseen asymptotic expansions with a remainder term of order $O\big(\frac{1}{n^{3/2+\gamma}}\big)$ with $\gamma > 0$. This problem remains open.

Comparing (2.18) and (2.22) with (2.4) and (2.5), we note that in Free CLT in contrast to the classical CLT the asymptotic expansions (2.18) and (2.22) have the same structure for non-lattice and lattice p-measures μ.

3. Auxiliary results

We need results about some classes of analytic functions (see [1], Section 3, and [2], Section 6, §59).

The class $\mathcal{N}$ (Nevanlinna, R.) is the class of analytic functions $f(z) : \mathbb{C}^+ \to \{z : \Im z \geq 0\}$. For such functions there is the integral representation

$$f(z) = a + bz + \int\limits_{\mathbb{R}} \frac{1+uz}{u-z}\,\tau(du) = a + bz + \int\limits_{\mathbb{R}} \Big(\frac{1}{u-z} - \frac{u}{1+u^2}\Big)(1+u^2)\,\tau(du), \quad z \in \mathbb{C}^+, \tag{3.1}$$

where $b \geq 0$, $a \in \mathbb{R}$, and τ is a nonnegative finite measure. Moreover, $a = \Re f(i)$ and $\tau(\mathbb{R}) = \Im f(i) - b$. From this formula it follows that

$$f(z) = (b + o(1))z \tag{3.2}$$

for $z \in \mathbb{C}^+$ such that $|\Re z|/\Im z$ stays bounded as $|z|$ tends to infinity (in other words $z \to \infty$ non-tangentially to $\mathbb{R}$). Hence if $b \neq 0$, then f has a right inverse $f^{(-1)}$ defined on the region

$$\Gamma_{\alpha,\beta} := \{z \in \mathbb{C}^+ : |\Re z| < \alpha \Im z,\, \Im z > \beta\}$$

for any $\alpha > 0$ and some positive $\beta = \beta(f, \alpha)$.

A function $f \in \mathcal{N}$ admits the representation

$$f(z) = \int\limits_{\mathbb{R}} \frac{\sigma(du)}{u-z}, \quad z \in \mathbb{C}^+, \tag{3.3}$$

where σ is a finite nonnegative measure, if and only if $\sup_{y \geq 1} |yf(iy)| < \infty$.

For $\mu \in \mathcal{M}$, consider its Cauchy transform $G_\mu(z)$ (see (2.6)). Following Maassen [27] and Bercovici and Voiculescu [6], we shall consider in the following the *reciprocal Cauchy transform*

$$F_\mu(z) = \frac{1}{G_\mu(z)}. \tag{3.4}$$

The corresponding class of reciprocal Cauchy transforms of all $\mu \in \mathcal{M}$ will be denoted by $\mathcal{F}$. This class coincides with the subclass of Nevanlinna functions f for which $f(z)/z \to 1$ as $z \to \infty$ non-tangentially to $\mathbb{R}$. Indeed, reciprocal Cauchy transforms of p-measures have obviously such property. Let $f \in \mathcal{N}$ and $f(z)/z \to 1$ as $z \to \infty$ non-tangentially to $\mathbb{R}$. Then, by (3.2), f admits the representation (3.1) with $b = 1$. By (3.2) and (3.3), $-1/f(z)$ admits the representation (3.3) with $\sigma \in \mathcal{M}$.

The functions f of the class $\mathcal{F}$ satisfy the inequality

$$\Im f(z) \geq \Im z, \qquad z \in \mathbb{C}^+. \tag{3.5}$$

The function $\phi_\mu(z) = F_\mu^{(-1)}(z) - z$ is called the Voiculescu transform of μ and $\phi_\mu(z)$ is an analytic function on $\Gamma_{\alpha,\beta}$ with the property $\Im\phi_\mu(z) \leq 0$ for $z \in \Gamma_{\alpha,\beta}$, where $\phi_\mu(z)$ is defined. On the domain $\Gamma_{\alpha,\beta}$, where the functions $\phi_{\mu_1}(z)$, $\phi_{\mu_2}(z)$, and $\phi_{\mu_1 \boxplus \mu_2}(z)$ are defined, we have

$$\phi_{\mu_1 \boxplus \mu_2}(z) = \phi_{\mu_1}(z) + \phi_{\mu_2}(z). \tag{3.6}$$

This relation for the distribution $\mu_1 \boxplus \mu_2$ of $X + Y$, where X and Y are free random variables, is due to Voiculescu [37] for the case of compactly supported measures. The result was extended by Maassen [27] to measures with finite variance; the general case was proved by Bercovici and Voiculescu [6].

Assume that $\beta_k(\mu) < \infty$ for some $k \in \mathbb{N}$. Then

$$G_\mu(z) = \frac{1}{z} + \frac{m_1(\mu)}{z^2} + \cdots + \frac{m_k(\mu)}{z^{k+1}} + o\Big(\frac{1}{z^{k+1}}\Big), \quad z \to \infty, \ z \in \Gamma_{\alpha,1}.$$

It follows from this relation (see for example [24]) that

$$\phi_\mu(z) = \alpha_1(\mu) + \frac{\alpha_2(\mu)}{z} + \cdots + \frac{\alpha_k(\mu)}{z^{k-1}} + o\Big(\frac{1}{z^{k-1}}\Big), \quad z \to \infty, \ z \in \Gamma_{\alpha,1}. \tag{3.7}$$

We call the coefficients $\alpha_m(\mu)$, $m = 1, \dots, k$, the free cumulants of the p-measure μ. It is easy to see that $\alpha_1(\mu) = m_1(\mu), \alpha_2(\mu) = m_2(\mu) - m_1^2(\mu)$, $\alpha_3(\mu) = m_3(\mu) - 3m_1(\mu)m_2(\mu) + 2m_1^3(\mu)$. In the case $m_1(\mu) = 0$ and $m_2(\mu) = 1$ we have $\alpha_1(\mu) = 0$, $\alpha_2(\mu) = 1$, $\alpha_3(\mu) = m_3(\mu)$ and $\alpha_4(\mu) = m_4(\mu) - 2$.

If $\mu \in \mathcal{M}$ has moments of any order, that is $\beta_k(\mu) < \infty$ for any $k \in \mathbb{N}$, then there exist cumulants $\alpha_m(\mu)$, $m = 1, \dots$ and we can consider the formal power series

$$\phi_\mu(z) = \sum_{m=1}^{\infty} \frac{\alpha_m(\mu)}{z^{m-1}}. \tag{3.8}$$

In addition $\phi_\mu(z)$ satisfies (3.7) for any fixed $k \in \mathbb{N}$. If μ has a bounded support, $\phi_\mu(z)$ is an analytic function on the domain $|z| > R$ with some $R > 0$ and the series (3.8) converges absolutely and uniformly for such z.

Voiculescu [40] showed for compactly supported p-measures that there exist unique functions $Z_1, Z_2 \in \mathcal{F}$ such that $G_{\mu_1 \boxplus \mu_2}(z) = G_{\mu_1}(Z_1(z)) = G_{\mu_2}(Z_2(z))$ for all $z \in \mathbb{C}^+$. Using Speicher's combinatorial approach [35] to freeness, Biane [12] proved this result in the general case.

Chistyakov and Götze [17] and Bercovici and Belinschi [4] proved, using complex analytic methods, that there exist unique functions $Z_1(z)$ and $Z_2(z)$ in the class $\mathcal{F}$ such that, for $z \in \mathbb{C}^+$,

$$z = Z_1(z) + Z_2(z) - F_{\mu_1}(Z_1(z)) \quad \text{and} \quad F_{\mu_1}(Z_1(z)) = F_{\mu_2}(Z_2(z)). \tag{3.9}$$

The function $F_{\mu_1}(Z_1(z))$ belongs again to the class $\mathcal{F}$ and there exists a p-measure μ such that $F_{\mu_1}(Z_1(z)) = F_\mu(z)$, where $F_\mu(z) = 1/G_\mu(z)$ and $G_\mu(z)$ is the Cauchy transform as in (2.6). We can define the additive free convolution in the following way $\mu_1 \boxplus \mu_2 := \mu$. The measure μ depends on μ_1 and μ_2 only. The relation (3.6) follows immediately from (3.9) and we see that this definition coincides with the Voiculesku, Bercovici, Maassen definition. Hence we have the equivalence of a "characteristic function" approach and a probabilistic approach to the definition of the additive free convolution.

Specializing to $\mu_1 = \mu_2 = \dots = \mu_n = \mu$ write $\mu_1 \boxplus \dots \mu_n = \mu^{n\boxplus}$. As shown in [17], (3.9) admits the following consequence.

Proposition 3.1. *Let $\mu \in \mathcal{M}$. There exists a unique function $Z \in \mathcal{F}$ such that*

$$z = nZ(z) - (n-1)F_\mu(Z(z)), \quad z \in \mathbb{C}^+, \tag{3.10}$$

and $F_{\mu^{n\boxplus}}(z) = F_\mu(Z(z))$.

Let μ and ν be finite signed measures such that $\mu((-\infty, x)) \to 0$ and $\nu((-\infty, x)) \to 0$ as $x \to -\infty$. In the sequel we consider such finite signed measures only. Denote by $\tilde{\nu}$ the total variation of the finite signed measure ν. Denote by $\Delta(\mu, \nu)$ the Kolmogorov distance between the signed measures μ and ν, i.e.,

$$\Delta(\mu, \nu) := \sup_{x \in \mathbb{R}} |\mu((-\infty, x)) - \nu((-\infty, x))|.$$

Define the Cauchy transform $G_\nu(z)$ of the finite signed measure ν in the same way as in (2.6). Furthermore, we shall need the following inequality for the Kolmogorov distance between signed measures in terms of their Cauchy's transforms.

Let $I = [a, b]$, where $-\infty < a < b < \infty$. Given $0 < \varepsilon < (b-a)/2$ introduce the interval $I_\varepsilon = [a + \varepsilon, b - \varepsilon]$.

Lemma 3.2. *Let ν be a finite signed measure with the support on $[a, b]$ and μ be a p-measure such that*

$$\int_{\mathbb{R}} |\nu((-\infty, x)) - \mu((-\infty, x))| \, dx < \infty. \tag{3.11}$$

Then there exists an absolute constant c such that, for any $0 < v < 1$ and $\varepsilon > 10v$,

$$\Delta(\nu, \mu) \le c \int_{\mathbb{R}} |G_\nu(u + i) - G_\mu(u + i)| \, du + c \sup_{x \in I_{\varepsilon/2}} \tilde{\nu}([x - 10v, x + 10v])$$

$$+ c\tilde{\nu}(\mathbb{R} \setminus I_\varepsilon) + c \sup_{x \in I_{\varepsilon/2}} \Big| \int_v^1 (G_\nu(x + iu) - G_\mu(x + iu)) \, du \Big|.$$

This lemma is a simple extension of Corollary 2.3 in Götze and Tikhomirov [21]. Since the proof of the lemma is similar to the proof of Corollary 2.3, we omit it.

4. Formal Asymptotic expansion in the Free CLT

In this section we deduce the formula (2.7).

By Proposition 3.1, there exists $Z(z) \in \mathcal{F}$ such that (3.10) holds, and $F_{\mu^{n\boxplus}}(z) = F_\mu(Z(z)$. Hence $F_{\mu_n}(z) = F_\mu(\sqrt{n}S_n(z))/\sqrt{n}$, $z \in \mathbb{C}^+$, where $S_n(z) := Z(\sqrt{n}z)/\sqrt{n}$. We see from this relation that

$$S_n(z) = F_{\mu_n}(z) + \phi_\mu(\sqrt{n}F_{\mu_n}(z))/\sqrt{n}$$

for $z \in \Gamma_{\alpha,\beta}$ with some $\alpha, \beta > 0$. On the other hand we conclude from (3.10) that

$$S_n(z) = \frac{z}{n} + \frac{n-1}{n}F_{\mu_n}(z), \qquad z \in \mathbb{C}^+.$$

The last two equations give us

$$F_{\mu_n}(z) + \sqrt{n}\phi_\mu(\sqrt{n}F_{\mu_n}(z)) = z, \qquad z \in \Gamma_{\alpha,\beta}.$$

Consider the function $f(z) := z + \sqrt{n}\phi_\mu(\sqrt{n}z)$, $z \in \Gamma_{\alpha,\beta'}$ with some $\beta' \geq \beta$, and define the function

$$g(z) := \frac{1}{2}\Big(f(z) + \sqrt{f^2(z) - 4}\Big), \quad z \in \Gamma_{\alpha,\beta'}, \tag{4.1}$$

where we choose the branch of the analytic square root by the condition $\Im g(z) > 0$ for $z \in \Gamma_{\alpha,\beta'}$. It is easy to see that $g(z) = z(1 + o(1))$ as $z \to \infty$ non-tangentially to $\mathbb{R}$. In addition $g(z)$ satisfies the relation

$$g(F_{\mu_n}(z)) + \frac{1}{g(F_{\mu_n}(z))} = f(F_{\mu_n}(z)) = z, \qquad z \in \Gamma_{\alpha,\beta}. \tag{4.2}$$

We deduce from (4.2) that

$$g(F_{\mu_n}(z)) = F_w(z), \qquad z \in \Gamma_{\alpha,\beta}.$$

Since the function $g(z)$ has a right inverse $g^{(-1)}(z)$ in $\Gamma_{\alpha,\beta''}$ with some $\beta'' \geq \beta'$, we have

$$F_{\mu_n}(z) = g^{(-1)}(F_w(z)), \qquad z \in \Gamma_{\alpha,\beta'''}, \quad \text{where} \quad \beta''' \geq \beta''. \tag{4.3}$$

Let $\mu \in \mathcal{M}$ such that all moments of μ exist. In addition let $m_1(\mu) = 0$ and $m_2(\mu) = 1$. Consider the formal power series in z

$$\sqrt{n}\phi_\mu(\sqrt{n}z) := \sum_{k=1}^{\infty} \frac{\alpha_{k+1}(\mu)}{n^{(k-1)/2}z^k}, \tag{4.4}$$

where $\alpha_k(\mu)$, $k = 1, 2, \dots$, are free cumulants of the measure μ and the formal power series of g:

$$g(z) = z + \sum_{k=0}^{\infty} \frac{a_k}{z^k}. \tag{4.5}$$

In our case $\alpha_1(\mu)=0, \alpha_2(\mu)=1, \alpha_3(\mu)=m_3(\mu)$ and $\alpha_4(\mu)=m_4(\mu)-2$. Using (4.2) and (4.4), (4.5) we obtain the following relation for the considered formal power series

$$z+\sum_{k=0}^{\infty}\frac{a_k}{z^k}+\frac{1}{z}\Big(1-\sum_{k=0}^{\infty}\frac{a_k}{z^{k+1}}+\Big(\sum_{k=0}^{\infty}\frac{a_k}{z^{k+1}}\Big)^2-\dots\Big)=z+\sum_{k=1}^{\infty}\frac{\alpha_{k+1}(\mu)}{n^{(k-1)/2}z^k}.$$

It follows from this relation that $a_0=0$, $a_1=0$ and

$$a_k-a_{k-2}+\sum_{s=0}^{k-3}a_s a_{k-s-3}-\dots+(-1)^{k-1}a_0^{k-1}=\frac{\alpha_{k+1}(\mu)}{n^{(k-1)/2}},\quad k=2,3,\dots. \tag{4.6}$$

We have from (4.6) the relations $a_2=\alpha_3(\mu)/\sqrt{n}$, $a_3=\alpha_4(\mu)/n$. In addition we obtain from (4.6) by induction that

$$a_{2s}=\frac{\alpha_3(\mu)}{\sqrt{n}}+O\Big(\frac{1}{n^{3/2}}\Big),\quad \text{and}\quad a_{2s+1}=\frac{\alpha_4(\mu)}{n}-\frac{(s-1)(s-2)}{2}\frac{\alpha_3^2(\mu)}{n}+O\Big(\frac{1}{n^{3/2}}\Big) \tag{4.7}$$

as $n\to\infty$ for $s=2,\dots$.

Now consider the formal power series for the right inverse $g^{(-1)}(z)$

$$g^{(-1)}(z)=z+\sum_{k=0}^{\infty}\frac{b_k}{z^k}.$$

Rewrite the relation $g(g^{(-1)}(z))=z$ in the form

$$z+\sum_{m=0}^{\infty}\frac{b_m}{z^m}+\sum_{k=2}^{\infty}\frac{a_k}{z^k\Big(1+\sum_{m=0}^{\infty}\frac{b_m}{z^{m+1}}\Big)^k}=z.$$

Using the formula

$$\frac{1}{(1+w)^k}=\sum_{s=0}^{\infty}(-1)^s\binom{k-1+s}{k-1}w^s,$$

we finally get

$$\sum_{m=0}^{\infty}\frac{b_m}{z^m}+\sum_{k=2}^{\infty}\frac{a_k}{z^k}\sum_{s=0}^{\infty}(-1)^s\binom{k-1+s}{k-1}\Big(\sum_{m=0}^{\infty}\frac{b_m}{z^{m+1}}\Big)^s=0.$$

We obtain from this equality that $b_0=b_1=0, b_2=-a_2$ and

$$b_m+a_m+\sum_{k=2}^{m-1}a_k\sum_{s=1}^{m-k}(-1)^s\binom{k-1+s}{k-1}\sum_{m_1+\dots+m_s=m-k-s}b_{m_1}\dots b_{m_s}=0,\quad m=3,\dots. \tag{4.8}$$

Moreover it is easy to deduce from (4.7) and (4.8) that

$$b_{2m}=-a_{2m}+O\Big(\frac{1}{n^{3/2}}\Big)=-\frac{\alpha_3(\mu)}{\sqrt{n}}+O\Big(\frac{1}{n^{3/2}}\Big) \tag{4.9}$$

and

$$b_{2m-1} = -a_{2m-1} + 2\sum_{s=1}^{m-2} s a_{2s} b_{2m-2-2s} + O\Big(\frac{1}{n^{3/2}}\Big)$$
$$= -\frac{\alpha_4(\mu)}{n} - \frac{(m-2)(m+1)}{2}\frac{\alpha_3^2(\mu)}{n} + O\Big(\frac{1}{n^{3/2}}\Big), \quad m = 2, \dots. \tag{4.10}$$

for $m = 2, \dots$. It remains to note that

$$\frac{1}{g^{(-1)}(z)} = \frac{1}{z + \sum_{k=0}^{\infty} \frac{b_k}{z^k}} = \frac{1}{z}\Big(1 - \sum_{k=0}^{\infty} \frac{b_k}{z^{k+1}} + \Big(\sum_{k=0}^{\infty} \frac{b_k}{z^{k+1}}\Big)^2 - \dots\Big)$$

and we can write the formal power series in $1/\sqrt{n}$

$$\frac{1}{g^{(-1)}(z)} = \frac{1}{z} + \sum_{k=1}^{\infty} \frac{B_k(1/z)}{n^{k/2}}.$$

Taking into account the relations (4.9) and (4.10), we easily conclude that

$$B_1(1/z) = \alpha_3(\mu)\sum_{m=2}^{\infty} \frac{1}{z^{2m}} = \alpha_3(\mu)\frac{1}{z^3}\cdot\frac{1}{z-1/z}$$

and

$$B_2(1/z) = \alpha_4(\mu)\sum_{m=2}^{\infty}\frac{1}{z^{2m+1}} + \alpha_3^2(\mu)\Big(\sum_{m=2}^{\infty}\frac{(m-2)(m+1)}{2}\frac{1}{z^{2m+1}}$$
$$+ \frac{1}{z^3}\Big(\sum_{m=1}^{\infty}\frac{1}{z^{2m}}\Big)^2\Big) = \Big(\alpha_4(\mu) - \frac{5}{8}\alpha_3^2(\mu)\Big)\frac{1}{z^4}\cdot\frac{1}{z-1/z}$$
$$+ \frac{1}{8}\alpha_3^2(\mu)\Big(\frac{1}{z^3}\cdot\frac{3+13/z^2}{(z-1/z)^2} + \frac{2}{z^2}\cdot\frac{(1+1/z^2)^2}{(z-1/z)^3}\Big).$$

In view of (4.6) and (4.8), we see as well that $B_k(z)$ are functions of the form

$$B_k(1/z) = \sum c_{p,m}\frac{1}{z^p}\frac{1}{(z-1/z)^m}$$

with coefficients $c_{p,m}$ which depend on the free cumulants $\alpha_3(\mu), \dots, \alpha_{k+2}(\mu)$ and do not depend on n. The summation is carried out over a finite set of non-negative integer pairs (p, m). The coefficients $c_{p,m}$ can be calculated explicitly in the way as for the functions $B_1(1/z)$ and $B_2(1/z)$.

Hence we deduce from (4.3) the formal expansion

$$G_{\mu_n}(z) = G_w(z) + \sum_{k=1}^{\infty}\frac{B_k(G_w(z))}{n^{k/2}}. \tag{4.11}$$

Using the integration by parts, it is not difficult to verify that

$$B_1(G_w(z)) = \frac{\alpha_3(\mu)}{\sqrt{z^2-4}} G_w^3(z) = \frac{\alpha_3(\mu)}{2\pi} \int_{-2}^{2} \frac{x(x^2-3)}{\sqrt{4-x^2}} \frac{dx}{z-x}$$

$$= -\alpha_3(\mu) \int_{-2}^{2} \frac{1}{z-x} \, d\Big(\frac{1}{3} U_2(x/2) p_w(x)\Big), \quad z \in \mathbb{C}^+. \quad (4.12)$$

On the other hand we see that if $\alpha_3(\mu) \neq 0$ then the function $B_2(G_w(z))$ is not the Cauchy transform of some signed measure. Indeed, it is easy to see, using direct calculations, that

$$B_2(G_w(z)) = \frac{\alpha_3^2(\mu)}{2(z^2-4)^{3/2}} + g(z), \quad z \in \mathbb{C}^+, \quad (4.13)$$

where the function $g(z)$ is analytic on $\mathbb{C}^+$ and there exists finite limits, for every $-\infty < t_1 < t_2 < +\infty$,

$$\lim_{y \downarrow 0} \int_{t_1}^{t_2} \Im g(x+iy) \, dx. \quad (4.14)$$

In addition we note that

$$\lim_{y \downarrow 0} \int_{3/2}^{2} \Im \frac{1}{((x+iy)^2-4)^{3/2}} \, dx = \infty. \quad (4.15)$$

Assume now that $B_2(G_w(z))$ is a Cauchy transform of a real-valued function $\omega(x)$ of bounded variation on every bounded interval and such that

$$\int_{-\infty}^{\infty} \frac{|d\omega(x)|}{1+|x|} < \infty.$$

Then, by Stieltjes–Perron's inverse formula [1], we have

$$\frac{\omega(t_2+0)-\omega(t_2-0)}{2} - \frac{\omega(t_1+0)-\omega(t_1-0)}{2} = \lim_{y \downarrow 0} \frac{1}{\pi} \int_{t_1}^{t_2} \Im B_2(G_w(x+iy)) \, dx. \quad (4.16)$$

Assuming in (4.15) $t_1 := 3/2$ and $t_2 := 2$, and taking into account (4.13)–(4.15), we arrive at a contradiction.

If $\alpha_3(\mu)=0$, then

$$B_2(G_w(z))=\frac{\alpha_4(\mu)}{\sqrt{z^2-4}}G_w^4(z)=\frac{\alpha_4(\mu)}{2\pi}\int_{-2}^{2}\frac{x^4-4x^2+2}{\sqrt{4-x^2}}\,\frac{dx}{z-x}$$
$$=-\alpha_4(\mu)\int_{-2}^{2}\frac{1}{z-x}\,d\Big(\frac{1}{4}U_3(x/2)p_w(x)\Big),\quad z\in\mathbb{C}^+. \quad (4.17)$$

5. Esseen's Expansion in Free CLT (the case $\beta_q(\mu)<\infty$, $q>11/3$)

In this section we prove Theorem 2.1. In the sequel we denote by c positive absolute constants and by $c_0,c_1,\dots$ explicit positive absolute constants. In addition we denote by θ real-valued quantity such that $|\theta|\le 1$.

Proof of Theorem 2.1. For $n\in\mathbb{N}$ introduce the following quantities

$$\beta_t(\mu,n):=\int\limits_{|u|\le\frac12\sqrt n}|u|^t\,\mu(du)\quad\text{and}\quad \rho_t(\mu,n):=\int\limits_{|u|>\frac12\sqrt n}|u|^t\,\mu(du),\quad t\ge 0.$$

Let us prove that, for $n\ge n_0:=4(10^2\beta_{q_1}(\mu)/\beta_3(\mu))^{2/(q_1-11/3)}$,

$$\frac12\le\beta_4(\mu,n)\le M_n^{-1}\beta_3(\mu)(n/4)^{1/6}\quad\text{and}\quad\rho_3(\mu,n)\le M_n^{-1}\beta_3(\mu)(n/4)^{-1/3} \quad (5.1)$$

with

$$M_n:=(\beta_3(\mu)/\beta_{q_1}(\mu))(n/4)^{q_1/2-11/6}\ge 10^2. \quad (5.2)$$

To prove (5.1) we see that

$$\int\limits_{|u|\le\frac12\sqrt n}u^4\,\mu(du)=\int\limits_{|u|\le\frac12\sqrt n}|u|^{q_1}|u|^{4-q_1}\,\mu(du)\le\beta_{q_1}(\mu)(n/4)^{(4-q_1)/2}=\frac{\beta_3(\mu)(n/4)^{1/6}}{M_n}.$$

Moreover

$$\int\limits_{|u|>\frac12\sqrt n}|u|^3\,\mu(du)\le(n/4)^{-(q_1-3)/2}\int\limits_{|u|>\frac12\sqrt n}|u|^{q_1}\,\mu(du)\le\beta_{q_1}(\mu)(n/4)^{-(q_1-3)/2}$$
$$=\frac{\beta_3(\mu)(n/4)^{-1/3}}{M_n}. \quad (5.3)$$

Since $M_n\ge 10^2$ for $n\ge n_0$, we have from (5.3) that

$$\int\limits_{|u|\le\frac12\sqrt n}|u|^3\,\mu(du)\ge\frac34,\qquad n\ge n_0.$$

Then, using Lyapunov's inequality $\beta_t^{1/t}(\mu) \le \beta_s^{1/s}(\mu)$ for $0 < t \le s$ and $\mu \in \mathcal{M}$, we have the bound

$$\frac{3}{4} \le \int\limits_{|u|\le\frac{1}{2}\sqrt{n}} |u|^3\,\mu(du) \le (\mu(\{|u| \le \frac{1}{2}\sqrt{n}\}))^{1/4}\Big(\int\limits_{|u|\le\frac{1}{2}\sqrt{n}} u^4\,\mu(du)\Big)^{3/4}$$

from which it follows the lower bound

$$\int\limits_{|u|\le\frac{1}{2}\sqrt{n}} u^4\,\mu(du) \ge \frac{1}{2}, \qquad n \ge n_0.$$

Thus, (5.1) is proved.

Using Lyapunov's inequality $\beta_{q_1}(\mu) \ge \beta_3^{q_1/3}(\mu)$, we easily obtain from the definition of M_n the upper bound

$$M_n \le (\sqrt{n}/\beta_3(\mu))^{q_1-11/3}, \qquad n \in \mathbb{N}. \tag{5.4}$$

Recall that we denote by μ_n the distribution of Y_n in (2.1) for the free random variables X_j. By Proposition 3.1, $G_{\mu_n}(z) = 1/F_{\mu_n}(z)$, $z \in \mathbb{C}^+$, where $F_{\mu_n}(z) := F_\mu(Z(\sqrt{n}z))/\sqrt{n}$. In this formula $Z(z) \in \mathcal{F}$ is the solution of the equation (3.10).

Consider the functions

$$S(z) := \frac{1}{2}(z + \sqrt{z^2-4}), \quad S_n(z) := Z(\sqrt{n}z)/\sqrt{n},$$
$$S_{n1}(z) := a_n + \frac{1}{2}\Big(z - a_n + \sqrt{(z-a_n)^2-4}\Big), \qquad z \in \mathbb{C}^+.$$

Note that $1/S(z) = G_{\mu_w}(z)$, where w denotes Wigner semicircle measure. Since $S_n \in \mathcal{F}$, we saw in Section 3 that there exists a p-measure ν_n such that $1/S_n(z) = G_{\nu_n}(z)$. In addition, it is easy to see, that $1/S_{n1}(z) = G_{\mu_{n1}}(z)$, where $\mu_{n1} := \mu_{a_n,0,0}$.

Using (2.6), we write, for $z \in \mathbb{C}^+$,

$$Z(z)G_\mu(Z(z)) = 1 + \frac{1}{Z^2(z)} + \frac{1}{Z^2(z)}\int\limits_{\mathbb{R}} \frac{u^3\,\mu(du)}{Z(z)-u} = 1 + \frac{1}{Z^2(z)}$$
$$+ \frac{1}{Z^3(z)}\int\limits_{|u|\le\frac{1}{2}\sqrt{n}} u^3\,\mu(du) + \frac{1}{Z^2(z)}\int\limits_{|u|>\frac{1}{2}\sqrt{n}} \frac{u^3\,\mu(du)}{Z(z)-u} + \frac{1}{Z^3(z)}\int\limits_{|u|\le\frac{1}{2}\sqrt{n}} \frac{u^4\,\mu(du)}{Z(z)-u}. \tag{5.5}$$

The equation (3.10) may be rewritten as

$$G_\mu(Z(z))\Big(Z(z) - z\Big) = (n-1)(1 - Z(z)G_\mu(Z(z))), \quad z \in \mathbb{C}^+. \tag{5.6}$$

By (5.5) and the definition of $S_n(z)$, we represent (5.6) in the form

$$\Big(1+\frac{1}{nS_n^2(z)}+\frac{r_{n1}(z)}{nS_n^2(z)}\Big)(S_n(z)-z)$$
$$=-\frac{n-1}{n}\frac{1}{S_n(z)}\Big(1+\frac{m_3(\mu)}{\sqrt{n}S_n(z)}+r_{n2}(z)+\frac{r_{n3}(z)}{\sqrt{n}S_n(z)}\Big), \tag{5.7}$$

for $z\in\mathbb{C}^+$, where

$$r_{n1}(z):=\int_{\mathbb{R}}\frac{u^3\,\mu(du)}{Z(\sqrt{n}z)-u},\qquad r_{n2}(z):=\int_{|u|>\frac12\sqrt{n}}\frac{u^3\,\mu(du)}{Z(\sqrt{n}z)-u}$$

and

$$r_{n3}(z):=\int_{|u|\le\frac12\sqrt{n}}\frac{u^4\,\mu(du)}{Z(\sqrt{n}z)-u}-\int_{|u|>\frac12\sqrt{n}}u^3\,\mu(du).$$

The functions $r_{nj}(z)$, $j=1,2,3$, are analytic on $\mathbb{C}^+$ and with the help of the inequality $\Im Z(\sqrt{n}z)\ge\sqrt{n}\Im z$, $z\in\mathbb{C}^+$, (compare with (3.5)) admit the estimates, for $z\in\mathbb{C}^+$,

$$|r_{n1}(z)|\le\frac{\beta_3(\mu)}{\sqrt{n}\Im z},\qquad |r_{n2}(z)|\le\frac{1}{\sqrt{n}\Im z}\int_{|u|>\frac12\sqrt{n}}|u|^3\,\mu(du)=\frac{\rho_3(\mu,n)}{\sqrt{n}\Im z},$$
$$|r_{n3}(z)|\le\frac{1}{\sqrt{n}\Im z}\int_{|u|\le\frac12\sqrt{n}}u^4\,\mu(du)+\rho_3(\mu,n)=\frac{\beta_4(\mu,n)}{\sqrt{n}\Im z}+\rho_3(\mu,n). \tag{5.8}$$

We assume that $n\ge c_0n_1$, where $n_1:=4(10^2\beta_{q_1}(\mu))^{2/q_2}$ with $q_2:=q_1-11/3$ and c_0 is a sufficiently large positive absolute constant. Denote by $L_{3,n}:=\beta_3(\mu)/\sqrt{n}$ the Lyapunov fraction. We see that $L_{3,n}\le 1/\sqrt{c_0}$ and $M_n\ge 10^2$ for $n\ge c_0n_1$.

We deduce from (5.7) the following relation

$$S_n^3(z)-zS_n^2(z)+(1+\varepsilon_{n1}(z))S_n(z)+\varepsilon_{n2}(z)=0,\quad z\in\mathbb{C}^+, \tag{5.9}$$

where

$$\varepsilon_{n1}(z):=\frac1n r_{n1}(z)+\frac{n-1}{n}r_{n2}(z)\quad\text{and}\quad \varepsilon_{n2}(z):=\frac{m_3(\mu)}{\sqrt{n}}+r_{n4}(z):=a_n+r_{n4}(z)$$

with

$$r_{n4}(z):=\Big(1-\frac1n\Big)\frac{r_{n3}(z)}{\sqrt{n}}-\frac{z}{n}\Big(1+r_{n1}(z)\Big)-\frac{m_3(\mu)}{n\sqrt{n}}.$$

By (5.8), we obtain

$$|r_{n4}(z)|\le\frac{\beta_4(\mu,n)}{n\Im z}+\frac{\rho_3(\mu,n)}{\sqrt{n}}+\frac{|z|}{n}\Big(1+\frac{L_{3,n}}{\Im z}\Big)+\frac{L_{3,n}}{n},\quad z\in\mathbb{C}^+. \tag{5.10}$$

For every $b > 0$ and $d > 0$, introduce $\mathbb{C}_b^+ := \{z \in \mathbb{C} : \Im z > b\}$, $D_{b,d} := \{z \in \mathbb{C} : b \le \Im z \le 1, |\Re z| \le d\}$ and $D_1(R_{n1}) := \{z \in \mathbb{C} : 1 \le \Im z \le R_{n1}, |\Re z| \le R_{n1}\}$, where $R_{n1} := c_1^{-1} L_{3,n}^{-2/3}$ with $c_1 := c_0^{2/3}/10$.

Consider $D_{t_1,4}$ with

$$t_1 := c_1 L_{3,n}/T_{n1}, \quad \text{where} \quad T_{n1} := M_n^{3/5}.$$

By (5.8), (5.10) and

$$M_n \le 2^{2/3} \beta_3(\mu) n^{1/6} = 2^{2/3} L_{3,n} n^{2/3} \tag{5.11}$$

which follows from (5.1), we have, using again (5.1),

$$|\varepsilon_{n1}(z)| \le \frac{T_{n1}}{c_1}\Big(\frac{1}{n} + \frac{\rho_3(\mu,n)}{\beta_3(\mu)}\Big) \le \frac{T_{n1}}{M_n n^{1/3}} = \frac{1}{M_n^{2/5} n^{1/3}} \tag{5.12}$$

and

$$|\varepsilon_{n2}(z)| \le \frac{\beta_4(\mu,n)T_{n1}}{c_1\beta_3(\mu)\sqrt{n}} + \frac{1+T_{n1}}{c_1 L_{3,n}^{2/3} n} + L_{3,n}\Big(1 + \frac{1}{n} + \frac{4^{1/3}}{M_n n^{1/3}}\Big) \le \frac{1}{M_n^{2/5} n^{1/3}} + 2L_{3,n} \tag{5.13}$$

for $z \in D_2 := D_{t_1,4} \cup D_1(R_{n1})$.

For every fixed $z \in \mathbb{C}^+$ consider the cubic equation

$$P(z,w) := w^3 - zw^2 + (1 + \varepsilon_{n1}(z))w + \varepsilon_{n2}(z) = 0.$$

Denote roots of this equation by $w_j = w_j(z)$, $j = 1, 2, 3$.

We shall show that for $z \in D_2$ the equation $P(z,w) = 0$ has a root, say $w_1 = w_1(z)$, such that

$$w_1 = -\varepsilon_{n3}(z) + r_{n5}(z), \tag{5.14}$$

where $\varepsilon_{n3}(z) := \varepsilon_{n2}(z)/(1 + \varepsilon_{n1}(z))$ and the quantity $r_{n5}(z)$ admits the following bound

$$|r_{n5}(z)| \le 5|\varepsilon_{n2}(z)|^2(|z| + 1). \tag{5.15}$$

Indeed, consider the circle $|w + \varepsilon_{n3}(z)| = r$, where $r := 5|\varepsilon_{n3}(z)|^2(|z| + 1)$. Since, by (5.12) and (5.13),

$$5|\varepsilon_{n3}(z)|(|z|+1) \le 10|\varepsilon_{n2}(z)|(|z|+1) \le 10\Big(\frac{1}{M_n^{2/5} n^{1/3}} + 2L_{3,n}\Big)(|z|+1) \le \frac{1}{10},$$

we have $r \le |\varepsilon_{n3}(z)|/2$ for the considered z. Introduce now the polynomials $P_1(w) := w^3 - zw^2$ and $P_2(w) := (1 + \varepsilon_{n1}(z))w + \varepsilon_{n2}(z) = (1 + \varepsilon_{n1}(z))(w + \varepsilon_{n3}(z))$. They admit the following estimates

$$|P_1(w)| \le |w|^3 + |z||w|^2 \le \Big(3|\varepsilon_{n3}(z)|/2\Big)^3 + |z|\Big(3|\varepsilon_{n3}(z)|/2\Big)^2 \le \Big(3|\varepsilon_{n3}(z)|/2\Big)^2(|z|+1)$$

and

$$|P_2(w)| = |1 + \varepsilon_{n1}(z)||w + \varepsilon_{n3}(z)| \ge (1 - |\varepsilon_{n1}(z)|)r \ge \frac{5}{2}(|\varepsilon_{n3}(z)|)^2(|z|+1)$$

on the circle $|w+\varepsilon_{n3}(z)|=r$. Hence $|P_1(w)|<|P_2(w)|$ on this circle and the desired result follows from Rouché's theorem.

In addition we obtain from (5.13)–(5.15) the bound

$$|w_1(z)|\le 10\Big(\frac{1}{M_n^{2/5}n^{1/3}}+L_{3,n}\Big),\qquad z\in D_2. \tag{5.16}$$

Since $P(z,w)=P_3(z,w)(w-w_1)$, where

$$P_3(z,w):=w^2-(z-w_1)w+1+\varepsilon_{n1}(z)-w_1(z-w_1),$$

we see that the other two roots w_2,w_3 of $P(z,w)$ satisfy the relations

$$w_2+w_3=z-w_1,\qquad w_2w_3=1+\varepsilon_{n1}(z)-w_1(z-w_1).$$

By (5.12) and (5.16), $5/6<|w_2||w_3|<7/6$ for $z\in D_2$ and one of these roots, say w_3, satisfies the inequality $|w_3|\ge\sqrt{5/6}$. Then $|w_2|\le 7/\sqrt{30}\le 7/5$ and we have $|w_3|\le|z|+2$. Therefore $|w_2|\ge 5/(6(|z|+2))\ge 2L_{3,n}^{2/3}$ for $z\in D_2$. Hence, by (5.16), $w_1\ne w_2$ and $w_1\ne w_3$ for $z\in D_2$.

Describe a domain in $\mathbb{C}^+$, where $w_2\ne w_3$. Note that $w_2=w_3$ for $z\in D_2$ such that

$$(z-w_1)^2-4(1+\varepsilon_{n1}(z)-w_1(z-w_1))=0. \tag{5.17}$$

We conclude from this relation that $z=\pm2\sqrt{1+\varepsilon_{n1}(z)+w_1^2}-w_1$. Therefore, as it is easy to see, (5.17) does not hold for $z\in D_3$, where $D_3:=D_{t_1,2-\gamma_1}\cup D_1(R_{n1})$ and

$$\gamma_1:=c_2\big(1/(M_n^{2/5}n^{1/3})+L_{3,n}\big),$$

where c_2 is a sufficiently large absolute constant. We choose it later.

Since the equation $P(z,w)=0$ has the unique solution $w_1(z)$ with values in the circle $|w_1|\le 10\big(1/(M_n^{2/5}n^{1/3})+L_{3,n}\big)$ for all $z\in D_3$, we obtain, by the implicit function theorem (see [29], p. 109) and the monodromy theorem, that $w_1(z)$ is an analytic single-valued function on D_3 such that $|w_1(z)|\le 10\big(1/(M_n^{2/5}n^{1/3})+L_{3,n}\big)$ there.

Now we see that the roots w_2 and w_3 have the form

$$w_j:=\frac12\Big(z-w_1+(-1)^{j-1}\sqrt{g(z)}\Big),\quad j=2,3. \tag{5.18}$$

where $g(z):=(z-w_1)^2-4-4\varepsilon_{n1}(z)+4w_1(z-w_1)\ne0$ for $z\in D_3$. In this formula we choose the brunch of the square by the condition $\sqrt{g(i)}\in\mathbb{C}^+$. Since $S_n(z)\in\mathcal{F}$ and satisfies the equation (5.9), we see that $S_n(z)=w_3(z)$ for $z\in D_3$.

Represent w_1 in the form $w_1=-a_n+r_{n6}(z)$, where

$$r_{n6}(z):=\frac{a_n-\varepsilon_{n2}(z)+a_n\varepsilon_{n1}(z)}{1+\varepsilon_{n1}(z)}+r_{n5}(z),$$

and therefore it admits the estimate

$$|r_{n6}(z)|\le 2|r_{n4}(z)|+2\frac{|m_3(\mu)|}{\sqrt n}|\varepsilon_{n1}(z)|+|r_{n5}(z)|,\quad z\in D_2. \tag{5.19}$$

Now we rewrite (5.18) in the following way

$$
\begin{aligned}
w_j &:= \frac{1}{2}\Big(z + a_n + (-1)^{j-1}\sqrt{(z+a_n)^2 - 4 - 4a_n(z+a_n) + r_{n7}(z)}\Big) - \frac{1}{2} r_{n6}(z)\\
&= a_n + \frac{1}{2}\Big(z - a_n + (-1)^{j-1}\sqrt{(z-a_n)^2 - 4 - 4a_n^2 + r_{n7}(z)}\Big) - \frac{1}{2} r_{n6}(z), \quad j = 2,3,
\end{aligned}
\tag{5.20}
$$

where

$$
r_{n7}(z) := -4\varepsilon_{n1}(z) + (2z + 6a_n - 3r_{n6}(z))r_{n6}(z). \tag{5.21}
$$

We need to estimate the quantities $r_{n6}(z)$ and $r_{n7}(z)$ in the domain $D_{t_1,4}$. First we note, by (5.1), (5.8), (5.10) and (5.11), that

$$
|\varepsilon_{n1}(z)| \le \frac{L_{3,n}}{\Im z}\Big(\frac{1}{n} + \frac{2^{2/3}}{M_n n^{1/3}}\Big) \le \frac{2L_{3,n}}{M_n n^{1/3}\Im z} \tag{5.22}
$$

and

$$
|r_{n4}(z)| \le \frac{3L_{3,n}}{M_n n^{1/3}\Im z} + \frac{6}{n}, \qquad |\varepsilon_{n2}(z)| \le \frac{3L_{3,n}}{M_n n^{1/3}\Im z} + 2L_{3,n}. \tag{5.23}
$$

By (5.15) and (5.23), we have

$$
|r_{n5}(z)| \le cL_{3,n}^2\Big(1 + \Big(\frac{1}{M_n n^{1/3}\Im z}\Big)^2\Big). \tag{5.24}
$$

Finally we obtain from (5.19), using (5.22) – (5.24),

$$
|r_{n6}(z)| \le cL_{3,n}^2\Big(\frac{1}{M_n n^{1/3}\Im z}\Big)^2 + \frac{cL_{3,n}}{M_n n^{1/3}\Im z} + cL_{3,n}^2 \le \frac{cL_{3,n}}{M_n n^{1/3}\Im z} + cL_{3,n}^2, \quad z \in D_{t_1,4}. \tag{5.25}
$$

It remains to note that from (5.21), (5.22), and (5.25) it follows that (5.25) holds for the function $r_{n7}(z)$. In addition the constant c in (5.25) does not depend on the constant c_2.

Repeating the previous arguments we arrive at the following estimates for $z \in D_1(R_{n1})$. We easily have (see (5.22) and (5.23))

$$
|\varepsilon_{n1}(z)| \le \frac{2L_{3,n}}{M_n n^{1/3}} \tag{5.26}
$$

and

$$
|r_{n4}(z)| \le \frac{3L_{3,n}}{M_n n^{1/3}} + \frac{2|z|}{n}, \qquad |\varepsilon_{n2}(z)| \le 2L_{3,n} + \frac{2|z|}{n}. \tag{5.27}
$$

Further we deduce

$$
|r_{n5}(z)| \le c\Big(L_{3,n}^2 + \frac{|z|^2}{n^2}\Big)|z|. \tag{5.28}
$$

In addition as before we obtain

$$
|r_{n6}(z)| \le c\Big(L_{3,n}^2 + \frac{|z|^2}{n^2}\Big)|z| + \frac{c|z|}{n} + \frac{cL_{3,n}}{M_n n^{1/3}} \tag{5.29}
$$

and

$$|r_{n7}(z)| \le c\Big(\Big(L_{3,n}^2 + \frac{|z|^2}{n^2}\Big)|z| + \frac{|z|}{n} + \frac{L_{3,n}}{M_n n^{1/3}}\Big)|z|. \tag{5.30}$$

Our next step is to evaluate

$$G_{\mu_{n1}}(z) - G_{\nu_n}(z) \quad \text{and} \quad G_{\nu_n}(z) - G_{\mu_n}(z)$$

for $z \in D_{t_1,2-\gamma_1}$ and $\Im z = 1$.

For $z \in D_3$, using the formula (5.20) with $j = 3$ for $S_n(z)$, we write

$$\frac{1}{S_n(z)} - \frac{1}{S_{n1}(z)} = \frac{S_{n1}(z) - S_n(z)}{S_{n1}(z)S_n(z)}$$
$$= \frac{1}{S_{n1}(z)S_n(z)}\Big(\frac{1}{2}r_{n6}(z) + \frac{4a_n^2 - r_{n7}(z)}{\sqrt{(z-a_n)^2-4} + \sqrt{(z-a_n)^2 - 4 - 4a_n^2 + r_{n7}(z)}}\Big) \tag{5.31}$$

From (5.25) it follows for $r_{n7}(z)$ that

$$|r_{n7}(z)| \le c\Big(\frac{T_{n1}}{M_n n^{1/3}} + L_{3,n}^2\Big) = c\Big(\frac{1}{M_n^{2/5} n^{1/3}} + L_{3,n}^2\Big)$$

for $z \in D_{t_1,2-\gamma_1}$. Therefore, for these z, taking into account that c_2 is sufficiently large, we obtain

$$\Big|\frac{4a_n^2 - r_{n7}(z)}{(z-a_n)^2 - 4}\Big| \le \frac{c}{\gamma_1}\Big(\frac{1}{M_n^{2/5} n^{1/3}} + L_{3,n}^2\Big) \le \frac{1}{10}.$$

Hence we have, for the same z,

$$|\sqrt{(z-a_n)^2-4} + \sqrt{(z-a_n)^2 - 4 - 4a_n^2 + r_{n7}(z)}|$$
$$=|\sqrt{(z-a_n)^2-4}||1 + \sqrt{1 - (4a_n^2 - r_{n7}(z))/((z-a_n)^2-4)}| \ge |\sqrt{(z-a_n)^2-4}|. \tag{5.32}$$

Note, using (5.30), that the bound (5.32) holds for $z \in D_1(R_{n1})$ as well. In addition, as we saw before, $|S_n(z)| = |w_3(z)| \ge \sqrt{5/6}$ for $z \in D_2$. The same estimate holds obviously for $|S_{n1}(z)|$. Therefore we can conclude from (5.31) and (5.32) that

$$\int_{t_1}^{1} \Big|G_{\nu_n}(x+iu) - G_{\mu_{n1}}(x+iu)\Big|\,du = \int_{t_1}^{1} \Big|\frac{1}{S_n(x+iu)} - \frac{1}{S_{n1}(x+iu)}\Big|\,du$$
$$\le c\int_{t_1}^{1} \Big(|r_{n6}(x+iu)| + \frac{a_n^2 + |r_{n7}(x+iu)|}{|\sqrt{(x-a_n+iu)^2-4}|}\Big)\,du \tag{5.33}$$

for $-2+\gamma_1 \le x \le 2-\gamma_1$.

From (5.4) and (5.25) it follows at once

$$\int_{t_1}^{1} |r_{n6}(x+iu)|\,du \le cL_{3,n}\Big(\frac{\log(M_n/L_{3,n})}{M_n n^{1/3}} + L_{3,n}\Big) \le cL_{3,n}\Big(\frac{|\log L_{3,n}|}{M_n n^{1/3}} + L_{3,n}\Big) \tag{5.34}$$

for $-2+\gamma_1 \le x \le 2-\gamma_1$.

From (5.4) and (5.25) for the function $r_{n7}(z)$ we conclude that, for the same x,

$$\int_{t_1}^{1} \frac{a_n^2+|r_{n7}(x+iu)|}{|\sqrt{(x-a_n+iu)^2-4}|}\Big)\,du \le ca_n^2\int_{t_1}^{1}\frac{du}{\sqrt{u}} + c\int_{t_1}^{1}\frac{|r_{n7}(x+iu)|}{(\gamma_1^2+u^2)^{1/4}}\,du$$

$$\le cL_{3,n}^2 + \frac{cL_{3,n}}{M_n n^{1/3}}\int_{t_1}^{1}\frac{du}{u(\gamma_1^2+u^2)^{1/4}} \le cL_{3,n}^2 + \frac{cL_{3,n}}{M_n n^{1/3}}\frac{\log(M_n/L_{3,n})}{\gamma_1^{1/2}}$$

$$\le cL_{3,n}\Big(L_{3,n} + \frac{1}{M_n^{4/5}n^{1/6}}|\log L_{3,n}|\Big). \tag{5.35}$$

It follows from (5.34) and (5.35) that, for $-2+\gamma_1 \le x \le 2-\gamma_1$,

$$\int_{t_1}^{1}\Big|G_{\nu_n}(x+iu) - G_{\mu_{n1}}(x+iu)\Big|\,du \le cL_{3,n}\Big(L_{3,n} + \frac{1}{M_n^{4/5}n^{1/6}}|\log L_{3,n}|\Big). \tag{5.36}$$

By the simple estimate $|S_{n1}(u+i)| \ge c\sqrt{1+u^2}$, $u\in\mathbb{R}$, and (5.31), (5.32), we obtain

$$\int_{-R_{n1}}^{R_{n1}}\Big|\frac{1}{S_n(u+i)} - \frac{1}{S_{n1}(u+i)}\Big|\,du \le c\int_{-R_{n1}}^{R_{n1}}\Big(|r_{n6}(u+i)| + \frac{a_n^2+|r_{n7}(u+i)|}{\sqrt{1+u^2}}\Big)\frac{du}{\sqrt{1+u^2}}. \tag{5.37}$$

From (5.29) we conclude

$$\int_{-R_{n1}}^{R_{n1}}\frac{|r_{n6}(u+i)|}{\sqrt{1+u^2}}\,du \le cL_{3,n}^2L_{3,n}^{-2/3} + \frac{c}{n^2}L_{3,n}^{-2} + \frac{c}{n}L_{3,n}^{-2/3} + \frac{cL_{3,n}|\log L_{3,n}|}{M_n n^{1/3}}$$

$$\le cL_{3,n}^{4/3}\Big(1 + L_{3,n}^{1/3}\frac{|\log L_{3,n}|}{M_n}\Big) \le cL_{3,n}^{4/3}. \tag{5.38}$$

By (5.30), in the same way we deduce

$$\int_{-R_{n1}}^{R_{n1}}\frac{a_n^2+|r_{n7}(u+i)|}{1+u^2}\,du \le cL_{3,n}^{4/3}. \tag{5.39}$$

Hence we finally have from (5.37)–(5.39)

$$\int_{-R_{n1}}^{R_{n1}}\Big|G_{\nu_n}(u+i) - G_{\mu_{n1}}(u+i)\Big|\,du \le cL_{3,n}^{4/3}. \tag{5.40}$$

Now we conclude from (5.5) that

$$G_{\mu_n}(z) - G_{\nu_n}(z) = \frac{r_{n8}(z)}{S_n(z)}, \quad z \in \mathbb{C}^+, \tag{5.41}$$

where

$$r_{n8}(z) := \frac{1}{nS_n^2(z)} + \frac{r_{n1}(z)}{nS_n^2(z)}. \tag{5.42}$$

Since $|S_n(z)| \geq \sqrt{5/6}$ for $z \in D_3$, we see from (5.8) that

$$|r_{n8}(z)| \leq \frac{2}{n} + \frac{2L_{3,n}}{n\Im z}, \qquad z \in D_3. \tag{5.43}$$

Therefore, we deduce from (5.43) and (5.4), for $-2+\gamma_1 \leq x \leq 2-\gamma_1$,

$$\begin{aligned}\int_{t_1}^{1} \Big|G_{\mu_n}(x+iu) - G_{\nu_n}(x+iu)\Big|\, du &\leq \int_{t_1}^{1} \frac{|r_{n8}(x+iu)|}{|S_n(x+iu)|}\, du \\ &\leq \frac{c}{n}\Big(1 + L_{3,n}\log(M_n/L_{3,n})\Big) \leq \frac{c}{n}.\end{aligned} \tag{5.44}$$

In addition

$$\int_{-R_{n1}}^{R_{n1}} \Big|G_{\mu_n}(u+i) - G_{\nu_n}(u+i)\Big|\, du \leq \int_{-R_{n1}}^{R_{n1}} \frac{|r_{n8}(u+i)|}{|S_n(u+i)|}\, du \leq \int_{-R_{n1}}^{R_{n1}} |r_{n8}(u+i)|\, du \leq \frac{c}{n^{2/3}}. \tag{5.45}$$

It remains to estimate $|G_{\mu_n}(u+i) - G_{\mu_{n1}}(u+i)|$ for $|u| > R_{n1}$, using the bound

$$\begin{aligned}|G_{\mu_{n1}}(u+i) - G_{\mu_n}(u+i)| \leq |G_{\mu_{n1}}(u+i) - G_{\mu_w}(u+i)| + |G_{\mu_w}(u+i) - G_{\nu_n}(u+i)| \\ + |G_{\nu_n}(u+i) - G_{\mu_n}(u+i)|.\end{aligned} \tag{5.46}$$

In the first step we note that

$$\begin{aligned}|G_{\mu_{n1}}(u+i) - G_w(u+i)| &= \frac{|S(u+i) - S_{n1}(u+i)|}{|S(u+i)||S_{n1}(u+i)|} \leq \frac{|a_n|}{|S(u+i)||S_{n1}(u+i)|} \\ &+ \frac{a_n^2 + 2|a_n|\sqrt{1+u^2}}{|S(u+i)||S_{n1}(u+i)||\sqrt{(u+i)^2-4} + \sqrt{(u-a_n+i)^2-4}|} \leq \frac{cL_{3,n}}{1+u^2}.\end{aligned} \tag{5.47}$$

Therefore we conclude from (5.47) the upper bound

$$\int_{|u|>R_{n1}} |G_{\mu_{n1}}(u+i) - G_w(u+i)|\, du \leq cL_{3,n}^{5/3}. \tag{5.48}$$

From (5.5) we see that the function $S_n(z)$ satisfies the approximate functional equation

$$S_n(z) - z = -\frac{1 - r_{n9}(z)}{S_n(z)}, \tag{5.49}$$

for $z \in \mathbb{C}^+_{1/2} := \{z \in \mathbb{C} : \Im z > 1/2\}$, where

$$r_{n9}(z) := 1 - \Big(1 - \frac{1}{n}\Big)\frac{1 + r_{n1}(z)}{1 + r_{n8}(z)}.$$

Here $r_{n9}(z)$ is an analytic function on $z \in \mathbb{C}^+_{1/2}$ which, by (5.8) and (5.43), is bounded as follows

$$|r_{n9}(z)| \le 2\Big(\frac{1}{n} + |r_{n1}(z)| + |r_{n8}(z)|\Big) \le 6L_{3,n}, \quad z \in \mathbb{C}^+_{1/2}. \tag{5.50}$$

Solving the equation (5.49), we see that

$$S_n(z) = \frac{1}{2}\Big(z \pm \sqrt{f_n(z)}\Big), \quad z \in \mathbb{C}^+_{1/2},$$

where $f_n(z) := z^2 - 4 + 4r_{n9}(z)$. Note that the function $f_n(z)$ is non-zero on the half-plane $\mathbb{C}^+_{1/2}$. Indeed, let $f_n(w) = 0$ for some $w \in \mathbb{C}^+_{1/2}$. Then, by (5.49), $S_n(w)^2 - wS_n(w) = -w^2/4$ and we have $S_n(w) = w/2$. But the function $S_n(z)$ satisfies the inequality $\Im S_n(z) \ge \Im z$, $z \in \mathbb{C}^+$, a contradiction. We define the function $\sqrt{f_n(z)}$ on $\mathbb{C}^+_{1/2}$, taking the branch of $\sqrt{f_n(z)}$ such that $\sqrt{f_n(i)} \in \mathbb{C}^+$. Since $S_n(z) \in \mathcal{N}$, we see that $S_n(z) = \frac{1}{2}\Big(z + \sqrt{f_n(z)}\Big)$ for $z \in \mathbb{C}^+_{1/2}$.

For $z \in \mathbb{C}^+_{1/2}$, using the previous formula for $S_n(z)$ and $S(z) = \frac{1}{2}(z + \sqrt{z^2 - 4})$, we write

$$\frac{1}{S_n(z)} - \frac{1}{S(z)} = \frac{S(z) - S_n(z)}{S(z)S_n(z)} = \frac{1}{S(z)S_n(z)} \cdot \frac{2r_{n9}(z)}{\sqrt{z^2 - 4} + \sqrt{z^2 - 4 + 4r_{n9}(z)}}. \tag{5.51}$$

Since, for $z = u + i$, $u \in \mathbb{R}$, the inequality $|z^2 - 4| \ge \frac{1}{2}(u^2 + 1)$ holds, we obtain from (5.50) the following upper bound

$$\Big|\frac{r_{n9}(z)}{z^2 - 4}\Big| \le \frac{12L_{3,n}}{1 + (\Re z)^2} \le \frac{1}{10}, \qquad \Im z = 1. \tag{5.52}$$

Hence we get, for $\Im z = 1$,

$$|\sqrt{z^2 - 4} + \sqrt{z^2 - 4 + 4r_{n9}(z)}| = \sqrt{|z^2 - 4|}\Big|1 + \sqrt{1 + 4r_{n9}(z)/(z^2 - 4)}\Big| \ge \sqrt{|z^2 - 4|}.$$

Using this estimate we deduce from (5.50) and (5.51), for $\Im z = 1$,

$$\Big|\frac{1}{S_n(z)} - \frac{1}{S(z)}\Big| \le 2\frac{|r_{n9}(z)|}{|\sqrt{z^2 - 4}|}\frac{1}{|S(z)||S_n(z)|} \le \frac{24L_{3,n}}{\sqrt{1 + (\Re z)^2}}\frac{1}{|S(z)||S_n(z)|}. \tag{5.53}$$

Since, for $u \in \mathbb{R}$, $\max\{1, \sqrt{1 + u^2} - 1\} \le |S(u + i)| \le \sqrt{1 + u^2} + 1$, the relation (5.53) leads to the following estimate, for $u \in \mathbb{R}$,

$$\begin{aligned} \frac{1}{4}\sqrt{1 + u^2} &\le |S(u + i)| - \frac{24L_{3,n}}{\sqrt{1 + u^2}} \le |S_n(u + i)| \\ &\le |S(u + i)| + \frac{24L_{3,n}}{\sqrt{1 + u^2}} \le \sqrt{1 + u^2} + 2. \end{aligned} \tag{5.54}$$

Now we conclude, using (5.53) and (5.54),

$$\int\limits_{|u|>R_{n1}} |G_{\mu_w}(u+i)-G_{\nu_n}(u+i)|\,du \le cL_{3,n}\int\limits_{|u|>R_{n1}} \frac{du}{(1+u^2)^{3/2}} \le c\frac{L_{3,n}}{R_{n1}^2} \le cL_{3,n}^{7/3}. \quad (5.55)$$

Using (5.42) and (5.54), we easily obtain the following inequality

$$|r_{n8}(u+i)| \le \frac{c}{n(1+u^2)}, \qquad u\in\mathbb{R}.$$

From this inequality and (5.41) we have

$$\int\limits_{|u|>R_{n1}} |G_{\mu_n}(u+i)-G_{\nu_n}(u+i)|\,du \le \frac{c}{n}. \quad (5.56)$$

Therefore we obtain finally from (5.46), (5.48), (5.55), and (5.56) the bound

$$\int\limits_{|u|>R_{n1}} |G_{\mu_n}(u+i)-G_{\mu_{n1}}(u+i)|\,du \le cL_{3,n}^{5/3}. \quad (5.57)$$

Apply Lemma 3.2 to the p-measures μ_{n1} and μ_n. Since $\beta_3(\mu)<\infty$, it is well-known that $m_2(\mu_n)<\infty$ and the condition (3.11) holds. We choose $I_\varepsilon=[-2+a_n+\varepsilon, 2+a_n-\varepsilon]$ with $\varepsilon=2\gamma_1$. By the definition of μ_{n1}, we see that

$$\sup_{x\in I_{\varepsilon/2}} \mu_{n1}([x-10t_1, x+10t_1]) \le ct_1$$

and

$$\mu_{n1}(\mathbb{R}\setminus I_\varepsilon) \le c\gamma_1^{3/2}.$$

Using these bounds and (5.36), (5.40), (5.44), (5.45), (5.57), and recalling (5.2), we obtain, by Lemma 3.2, the estimate

$$\Delta(\mu_n,\mu_{n1}) \le ct_1 + c\gamma_1^{3/2} + cL_{3,n}\Big(\frac{|\log L_{3,n}|}{M_n^{4/5}n^{1/6}} + L_{3,n}\Big) + cL_{3,n}^{4/3}$$
$$+\frac{c}{n}+\frac{c}{n^{2/3}}+cL_{3,n}^{5/3} \le c\Big(\big(\beta_3^2(\mu)\beta_{q_1}^3(\mu)\big)^{1/5}\frac{1}{n^{1/2+3q_2/10}} + \frac{\beta_3^{7/6}(\mu)}{n^{7/12}}\Big) \le c\frac{\beta_{q_1}(\mu)}{n^{1/2+3q_2/10}}$$

for $n\ge c_0 n_1$ from which it follows (2.17).

Hence, Theorem 2.1 is proved. □

Proof of Corollary 2.2. It is easy to see that the statement of Corollary 2.2 follows from Theorem 2.1 and from the following simple formula

$$\mu_{n1}((-\infty,x))-\mu_w((-\infty,x)) = -\frac{m_3(\mu)}{3\sqrt{n}}(x^2-1)p_w(x)+c\theta\Big(\frac{|m_3(\mu)|}{\sqrt{n}}\Big)^{3/2}, \quad x\in\mathbb{R}. \ \square$$

6. Esseen's Expansion in Free CLT (the case $\mu_q(\mu) < \infty$, $q > 16/3$)

In this section we prove Theorem 2.3. The proof of the theorem is similar to the proof of Theorem 2.1 but with some essential technical differences. Therefore we describe in details arguments which differ from the proof of Theorem 2.1 and omit arguments which repeat directly the arguments of Section 5. We save all notations of Section 5. Denote as well

$$S_{n2}(z) := a_n + \frac{1}{2}\Big(\big(1+b_n\big)(z-a_n) + \sqrt{\big(1-b_n\big)^2(z-a_n)^2 - 4\big(1-d_n\big)}\Big), \quad z \in \mathbb{C}^+,$$

where a_n, b_n and d_n are defined in Section 2. The function $S_{n2}(z) \in \mathcal{F}$ and $1/S_{n2}(z) = G_{\mu_{n2}}(z)$, where $\mu_{n2} := \mu_{a_n,b_n,d_n}$ is the free Meixner measure with the parameters a_n, b_n and d_n.

If $\beta_q(\mu) < \infty$ for $q > 16/3$, then the following bound holds

$$\rho_4(\mu, n) \le \frac{m_4(\mu)}{N_n n^{2/3}}, \quad \text{where} \quad N_n := \frac{(\frac{1}{2}\sqrt{n})^{q_3-16/3} m_4(\mu)}{\beta_{q_3}(\mu)}. \tag{6.1}$$

Note that $N_n \ge 10^2$ for $n \ge n_2 := 4(10^2\beta_{q_3}(\mu))^{2/q_4}$ with $q_4 := q_3 - 16/3$.

Proof of Theorem 2.3. Using (2.6), we write, for $z \in \mathbb{C}^+$,

$$\begin{aligned} Z(z)G_\mu(Z(z)) &= 1 + \frac{1}{Z^2(z)} + \frac{m_3(\mu)}{Z^3(z)} \\ &\quad + \frac{1}{Z^3(z)} \int\limits_{|u|>\frac{1}{2}\sqrt{n}} \frac{u^4\,\mu(du)}{Z(z)-u} + \frac{\beta_4(\mu,n)}{Z^4(z)} + \frac{1}{Z^4(z)} \int\limits_{|u|\le\frac{1}{2}\sqrt{n}} \frac{u^5\,\mu(du)}{Z(z)-u}. \end{aligned} \tag{6.2}$$

By (6.2) and the definition of $S_n(z)$, the equation (3.10) may be rewritten as

$$\begin{aligned} &\Big(1 + \frac{1}{nS_n^2(z)} + \frac{m_3(\mu)+\eta_{n1}(z)}{n^{3/2}S_n^3(z)} + \frac{m_4(\mu)-\eta_{n2}(z)}{n^2 S_n^4(z)}\Big)(S_n(z)-z) \\ &= -\frac{n-1}{n}\frac{1}{S_n(z)}\Big(1 + \frac{m_3(\mu)+\eta_{n1}(z)}{\sqrt{n}S_n(z)} + \frac{m_4(\mu)-\eta_{n2}(z)}{nS_n^2(z)}\Big) \end{aligned} \tag{6.3}$$

for $z \in \mathbb{C}^+$, where

$$\eta_{n1}(z) := \int\limits_{|u|>\frac{1}{2}\sqrt{n}} \frac{u^4\,\mu(du)}{Z(\sqrt{n}z)-u}, \quad \eta_{n2}(z) := \int\limits_{|u|>\frac{1}{2}\sqrt{n}} u^4\,\mu(du) - \int\limits_{|u\le\frac{1}{2}\sqrt{n}} \frac{u^5\,\mu(du)}{Z(\sqrt{n}z)-u}.$$

The functions $\eta_{n1}(z)$ and $\eta_{n2}(z)$ are analytic on $\mathbb{C}^+$ and with the help of the inequality $\Im Z(\sqrt{n}z) \ge \sqrt{n}\Im z$, $z \in \mathbb{C}^+$, admit the estimates, for $z \in \mathbb{C}^+$,

$$|\eta_{n1}(z)| \le \frac{1}{\sqrt{n}\Im z} \int\limits_{|u|>\frac{1}{2}\sqrt{n}} u^4 \, \mu(du) = \frac{\rho_4(\mu,n)}{\sqrt{n}\Im z} \le \frac{m_4(\mu)}{N_n n^{7/6}\Im z}, \tag{6.4}$$

$$|\eta_{n2}(z)| \le \rho_4(\mu,n) + \frac{\beta_5(\mu,n)}{\sqrt{n}\Im z} \le \frac{m_4(\mu)}{N_n n^{2/3}} + \frac{\beta_5(\mu,n)}{\sqrt{n}\Im z}. \tag{6.5}$$

We deduce from (6.3) the following relation, for $z \in \mathbb{C}^+$,

$$S_n^5(z) - zS_n^4(z) + S_n^3(z) + \frac{\eta_{n3}(z)}{\sqrt{n}} S_n^2(z) + \frac{\eta_{n4}(z)}{n} S_n(z) - \frac{\eta_{n4}(z)z}{n^2} = 0, \tag{6.6}$$

where $\eta_{n3}(z) := m_3(\mu) + \eta_{n1}(z) - z/\sqrt{n}$ and $\eta_{n4}(z)(z) := m_4(\mu) - \eta_{n2}(z)$.

We assume that $n \ge c_3 n_2$ with a sufficiently large positive absolute constant c_3. Then $L_{3,n} + L_{4,n} \le 1/c_3$ and $N_n \ge 10^2$, where $L_{4,n} := m_4(\mu)/n$.

Consider the domains $D_{t_2,4}$ and $D_1(R_{n2})$ with $t_2 := c_4 L_{4,n}/T_{n2}$, where $T_{n2} := m_4^{2/5}(\mu) N_n^{3/5}$, and $R_{n2} := c_4^{-1}(a_n^2 + L_{4,n})^{-1/3}$. Here $c_4 := c_3^{1/3}/10$.

For $z \in D_{t_2,4}$, by (6.4) and (6.5), we have the bounds

$$\begin{aligned}
\frac{|\eta_{n1}(z)|}{\sqrt{n}} &\le \frac{L_{4,n}}{N_n n^{2/3}\Im z} \le \frac{T_{n2}}{c_4 N_n n^{2/3}}, \\
\frac{|\eta_{n2}(z)|}{n} &\le \frac{L_{4,n}}{N_n n^{2/3}} + \frac{\beta_5(\mu)}{n^{3/2}\Im z} \le \frac{L_{4,n}}{N_n n^{2/3}} + \frac{\beta_5(\mu)}{c_4 m_4(\mu)} \frac{T_{n2}}{\sqrt{n}}, \\
\frac{|\eta_{n3}(z)|}{\sqrt{n}} &\le |a_n| + \frac{L_{4,n}}{N_n n^{2/3}\Im z} + \frac{|z|}{n} \le |a_n| + \frac{T_{n2}}{c_4 N_n n^{2/3}} + \frac{5}{n}, \\
\frac{|\eta_{n4}(z)|}{n} &\le \frac{m_4(\mu) + |\eta_{n2}(z)|}{n} \le 2L_{4,n} + \frac{\beta_5(\mu)}{c_4 m_4(\mu)} \frac{T_{n2}}{\sqrt{n}}
\end{aligned} \tag{6.7}$$

and, for $z \in D_1(R_{n2})$,

$$\begin{aligned}
&\frac{|\eta_{n1}(z)|}{\sqrt{n}} \le \frac{L_{4,n}}{N_n n^{2/3}}, \quad \frac{|\eta_{n2}(z)|}{n} \le L_{4,n}\Big(\frac{1}{N_n n^{2/3}} + \frac{\beta_5(\mu)}{m_4(\mu)\sqrt{n}}\Big), \\
&\frac{|\eta_{n3}(z)|}{\sqrt{n}} \le |a_n| + \frac{L_{4,n}}{N_n n^{2/3}} + \frac{R_{n2}}{n}, \quad \frac{|\eta_{n4}(z)|}{n} \le 3L_{4,n}.
\end{aligned} \tag{6.8}$$

Note that

$$\frac{T_{n2}}{N_n n^{2/3}} = \Big(\frac{m_4(\mu)}{N_n}\Big)^{2/5} \frac{1}{n^{2/3}} \le 2\Big(\frac{\beta_{q_3}(\mu)}{n^{(q_3-16/3)/2}}\Big)^{2/5} \frac{1}{n^{2/3}} \le \frac{2\beta_{q_3}^{2/5}(\mu)}{n^{2/3}} < \frac{1}{100}$$

and

$$\frac{\beta_5(\mu)}{m_4(\mu)} \frac{T_{n2}}{\sqrt{n}} = \frac{\beta_5(\mu)}{\sqrt{n}} \Big(\frac{N_n}{m_4(\mu)}\Big)^{3/5} \le \frac{\beta_5(\mu)}{\sqrt{n}} \frac{n^{1/5}}{\beta_{q_3}^{3/5}(\mu)} < \frac{1}{100}.$$

For every fixed $z \in \mathbb{C}^+$ consider the equation

$$Q(z,w) := w^5 - zw^4 + w^3 + \frac{\eta_{n3}(z)}{\sqrt{n}}w^2 + \frac{\eta_{n4}(z)}{n}w - \frac{\eta_{n4}(z)z}{n^2} = 0. \tag{6.9}$$

Denote roots of this equation by $w_j = w_j(z)$, $j = 1, \dots, 5$. Let us show that for every fixed $z \in D_4 := D_{t_2,4} \cup D_1(R_{n2})$ the equation $Q(z,w) = 0$ has three roots, say $w_j = w_j(z)$, $j = 1,2,3$, such that

$$|w_j| \le 5\varepsilon(z) := 5\max\Big\{|a_n| + \frac{|z|+1}{n} + \frac{|\eta_{n1}(z)|}{\sqrt{n}}, \sqrt{\frac{m_4(\mu) + |\eta_{n2}(z)|}{n}}\Big\}, \quad j = 1,2,3, \tag{6.10}$$

where, by (6.7) and (6.8), $\varepsilon(z)$ admits the bound

$$\varepsilon(z) \le 10(|a_n| + \sqrt{L_{4,n}}) + \frac{T_{n2}}{c_4 N_n n^{2/3}} + \sqrt{\frac{\beta_5(\mu)}{c_4 m_4(\mu)}\frac{\sqrt{T_{n2}}}{n^{1/4}}} < \frac{1}{100} \tag{6.11}$$

for $z \in D_{t_2,4}$, and

$$\varepsilon(z) \le 3(|a_n| + \sqrt{L_{4,n}}) < \frac{1}{100}, \quad \text{for} \quad z \in D_1(R_{n2}). \tag{6.12}$$

Indeed, consider the polynomials

$$Q_1(z,w) := w^5 - zw^4 + \frac{\eta_3(z)}{\sqrt{n}}w^2 + \frac{\eta_{n4}(z)}{n}w - \frac{\eta_{n4}(z)z}{n^2} \quad \text{and} \quad Q_2(w) := w^3.$$

We see that $|Q_1(z,w)| \le \frac{1}{2}|Q_2(w)|$ on the circle $|w| = 5\varepsilon(z)$. Therefore, by Rouché's theorem, we obtain that the polynomial $Q_1(z,w) + Q_2(w)$ has three roots which do not exceed $5\varepsilon(z)$, that was to be proved.

Represent $Q(z,w)$ in the form

$$Q(z,w) = (w^2 + q_1 w + q_2)(w^3 + g_1 w^2 + g_2 w + g_3),$$

where $w^3 + g_1 w^2 + g_2 w + g_3 = (w - w_1)(w - w_2)(w - w_3)$. We have from this formula the relations

$$q_1 + g_1 = -z, \quad q_2 + q_1 g_1 + g_2 = 1, \quad q_2 g_1 + q_1 g_2 + g_3 = \frac{\eta_{n3}(z)}{\sqrt{n}},$$
$$q_2 g_2 + q_1 g_3 = \frac{\eta_{n4}(z)}{n}, \quad q_2 g_3 = -\frac{\eta_{n4}(z)z}{n^2}. \tag{6.13}$$

By Vieta's formulae and (6.10), note that

$$|g_1| \le 15\varepsilon(z), \quad |g_2| \le 75\varepsilon^2(z), \quad |g_3| \le 125\varepsilon^3(z). \tag{6.14}$$

Now we obtain from (6.13) and (6.14) the following bounds, for $z \in D_4$,

$$|q_1| \le |z| + 15\varepsilon(z), \quad |1 - q_2| \le 16\varepsilon(z)(|z|+1) \le \frac{1}{2}, \quad |g_3| \le 2\frac{|\eta_{n4}(z)||z|}{n^2}. \tag{6.15}$$

Then we conclude from (6.13) and (6.15) that, for the same z,

$$\Big|g_2-\frac{\eta_{n4}(z)}{n}\Big|\le 32\varepsilon(z)\frac{|\eta_{n4}(z)|(|z|+1)}{n}+4\frac{|\eta_{n4}(z)|(|z|+1)^2}{n^2}$$
$$\le \eta_{n5}(z):=36\varepsilon(z)\frac{|\eta_{n4}(z)|(|z|+1)}{n}. \tag{6.16}$$

From the first three relations in (6.13) it follows the formula

$$g_1+zg_1^2=a_n+\beta_n z+\eta_{n6}(z), \tag{6.17}$$

where $\beta_n:=L_{4,n}-1/n=(m_4(\mu)-1)/n$ and

$$\eta_{n6}(z):=\frac{\eta_{n1}(z)}{\sqrt{n}}-\frac{\eta_{n2}(z)z}{n}+\Big(g_2-\frac{\eta_{n4}(z)}{n}\Big)z-g_1^3+2g_1g_2-g_3.$$

By (6.14) and (6.16), we have the estimate, for $z\in D_4$,

$$|\eta_{n6}(z)|\le\frac{|\eta_{n1}(z)|}{\sqrt{n}}+\frac{|\eta_{n2}(z)z|}{n}+|\eta_{n5}(z)z|+10^4\varepsilon^3(z). \tag{6.18}$$

Rewrite (6.17) in the form

$$g_1(1+a_nz)=a_n+\beta_nz+(a_n+\beta_nz)\Big(\frac{1}{1+g_1z}-1\Big)+a_ng_1z+\frac{\eta_{n6}(z)}{1+g_1z}.$$

Taking into account (6.14), (6.16) and (6.18) we arrive from this relation at the bound, for $z\in D_4$,

$$|g_1-a_n-(\beta_n-a_n^2)z|\le\eta_{n7}(z)$$
$$:=4|a_n|(\beta_n+a_n^2)|z|^2+30\varepsilon(z)|z|^2(15(|a_n|+\beta_n|z|)\varepsilon(z)+\beta_n)+2|\eta_{n6}(z)|. \tag{6.19}$$

To find the roots w_4 and w_5 we need to solve the equation $w^2+q_1w+q_2=0$. Using (6.13), we have, for $j=4,5$,

$$\begin{aligned}w_j&=\frac12\Big(-q_1+(-1)^j\sqrt{q_1^2-4q_2}\Big)\\&=\frac12\Big(z+g_1+(-1)^j\sqrt{(z+g_1)^2-4(1+(z+g_1)g_1-g_2)}\Big)\\&=\frac12\Big(z+g_1+(-1)^j\sqrt{(z-g_1)^2-4-4(g_1^2-g_2)}\Big)=\frac12\eta_{n8}(z)+a_n\\&+\frac12\Big(\big(1+b_n\big)(z-a_n)+(-1)^j\sqrt{\big(1-b_n\big)^2(z-a_n)^2-4\big(1-d_n\big)+\eta_{n9}(z)}\Big),\end{aligned} \tag{6.20}$$

where, as in the definition of the p-measure μ_{n2}, $b_n:=\beta_n-a_n^2$, $d_n:=L_{4,n}-a_n^2$ and

$$\begin{aligned}\eta_{n8}(z)&:=g_1-a_n-b_n(z-a_n),\\\eta_{n9}(z)&:=-4b_n^2(z-a_n)^2-2(z-a_n)\big((1-b_n)\eta_8(z)+4b_n(a_n+\eta_8(z))\big)\\&\quad+4(g_2-L_{4,n})-\eta_{n8}(z)\big(8a_n+3\eta_{n8}(z)\big).\end{aligned} \tag{6.21}$$

We choose the branch of the analytic square root by the condition $\Im w_4(i)\ge 0$.

We obtain, by (6.7), (6.10), (6.11), (6.16), (6.18) and (6.19), for $z \in D_{t_2,4}$,

$$|\eta_{n8}(z)| \le c\Big(|a_n| b_n + |a_n|^3 + |a_n|\varepsilon^2(z) + \beta_n\varepsilon(z) + \frac{|\eta_{n1}(z)|}{\sqrt{n}} + \frac{|\eta_{n2}(z)|}{n}$$
$$+ \varepsilon(z)\Big(L_{4,n} + \frac{|\eta_{n2}(z)|}{n}\Big) + \varepsilon^3(z)\Big) \le c\Big((|a_n| + \sqrt{L_{4,n}})^3 + \frac{|\eta_{n1}(z)|}{\sqrt{n}} + \frac{|\eta_{n2}(z)|}{n}$$
$$+ L_{4,n}\sqrt{\frac{|\eta_{n2}(z)|}{n}}\Big) \le c\Big((|a_n| + \sqrt{L_{4,n}})^3 + \frac{1}{c_4}\frac{T_{n2}}{N_n n^{2/3}} + \frac{1}{c_4}\frac{\beta_5(\mu)}{m_4(\mu)}\frac{T_{n2}}{\sqrt{n}}$$
$$+ \frac{1}{\sqrt{c_4}}L_{4,n}\sqrt{\frac{\beta_5(\mu)}{m_4(\mu)}\frac{T_{n2}}{\sqrt{n}}}\Big) \le \eta_{n10} := c\Big((|a_n| + \sqrt{L_{4,n}})^3 + \frac{1}{\sqrt{c_4}}\frac{\beta_5(\mu)}{m_4(\mu)}\frac{T_{n2}}{\sqrt{n}}\Big), \quad (6.22)$$

and by (6.8), (6.10), (6.12), (6.16), (6.18) and (6.19), for $z \in D_1(R_{n2})$,

$$|\eta_{n8}(z)| \le c\Big((|a_n| + \sqrt{L_{4,n}})^3|z|^2 + \frac{|\eta_{n1}(z)|}{\sqrt{n}} + \frac{|\eta_{n2}(z)|}{n}|z|\Big)$$
$$\le c(|a_n| + \sqrt{L_{4,n}})^3\Big(1 + \frac{\beta_5(\mu)}{m_4^{3/2}(\mu)}\Big)|z|^2. \quad (6.23)$$

The constants c in (6.22) and (6.23) do not depend on c_3. In the same way we obtain the following estimates for $|\eta_{n9}(z)|$

$$|\eta_{n9}(z)| \le c\eta_{n10}, \quad z \in D_{t_2,4}, \quad (6.24)$$

and

$$|\eta_{n9}(z)| \le c(|a_n| + \sqrt{L_{4,n}})^3\Big(1 + \frac{\beta_5(\mu)}{m_4^{3/2}(\mu)}\Big)|z|^3, \quad z \in D_1(R_{n2}). \quad (6.25)$$

The constants c in (6.24) and (6.25) do not depend on c_3 as well.

Using (6.22), (6.23) and (6.24), (6.25), we obtain from (6.20) the bound $|w_4| \ge \frac{3}{4}$, $z \in D_4$, for $n \ge n_2$.

For $n \ge n_2$, $\eta_{n10} \le c_5$, where c_5 depends on c_3 and is suficiently small. Consider the domain

$$D_5 := D_4 \setminus (\{|z - 2 - a_n| \le 10c\eta_{n10}\} \cup (\{|z + 2 - a_n| \le 10c\eta_{n10}\})$$

with an absolute constant $c > 0$ from (6.24) which does not depend on c_3.

Let us show that $w_4 \ne w_s$ for $s = 1, 2, 3, 5$ and $z \in D_5$. By (6.11) and (6.12), $|w_s| < 1/10$ for $s = 1, 2, 3$ and $z \in D_5$. On the other hand, by (6.22)–(6.25), we note that $|\eta_{n8}(z)| + |\eta_{n9}(z)| \le 1/100$ for $z \in D_4$ and we conclude from (6.20) that $|w_4| \ge 3/4$ for $z \in D_4$. The assertion is proved for $s = 1, 2, 3$. It remains to show that $w_4 \ne w_5$. If $w_4 = w_5$, we conclude from (6.20) that

$$\big(1 - b_n\big)^2(z - a_n)^2 - 4\big(1 - d_n\big) = -\eta_{n9}(z).$$

Since $z \in D_5$, we arrive at a contradiction, using (6.24) and (6.25). The assertion is completely proved.

At every point $z_0 \in D_5$ the equation (6.9) has the root $w_4(z_0)$, satisfying the conditions

$$Q(z_0, w_4(z_0)) = 0, \qquad \frac{\partial Q(z_0, , w_4(z_0))}{\partial w} \neq 0.$$

It follows from the implicit function theorem (see [29], p. 109) that z_0 has a neighborhood $\mathcal{N}(z_0)$ in which there can be defined a single-valued analytic function $f(z_0, z)$, called a local solution of (6.9) such that $f(z_0, z_0) = w_4(z_0)$ and $Q(z_0, f(z_0, z)) = 0$, $z \in \mathcal{N}(z_0)$. Then, by the monodromy theorem, there exists a single-valued analytic function $f(z)$ on D_5 such that $f(z) = w_4(z)$ for $z \in D_5$. It is easy to see that $w_4(z) = S_n(z)$, for $z \in D_5$.

Denote by $z_1, \dots, z_q$ zeros of the discriminant of the equation (6.9) for $z \in D_4$. Let G be the domain obtained by deleting these points from D_4. Then $w_4(z)$ can be continued analytically along any curve $L \subset G$. Moreover, $|w_4(z)| \geq 3/4$ for $z \in L$. Therefore $S_n(z) = w_4(z)$ along any curve $L \subset G$ and we conclude that

$$|S_n(z)| \geq 3/4, \qquad z \in D_4. \tag{6.26}$$

The bound (6.26) allows us to improve the estimate (6.5) for $\eta_{n2}(z)$ in $D_{t_2,4}$. By (6.26), we have, for $z \in D_{t_2,4}$,

$$\Big| \int\limits_{|u|\leq \frac{1}{2}\sqrt{n}} \frac{u^5\,\mu(du)}{Z(\sqrt{n}z) - u} \Big| = \Big| \int\limits_{|u|\leq \frac{1}{2}\sqrt{n}} \frac{u^5\,\mu(du)}{\sqrt{n}S_n(z) - u} \Big| \leq \int\limits_{|u|\leq \frac{1}{2}\sqrt{n}} \frac{|u|^5\,\mu(du)}{\sqrt{n}|S_n(z)| - |u|} \leq 4\frac{\beta_5(\mu, n)}{\sqrt{n}}.$$

Hence

$$|\eta_{n2}(z)| \leq \rho_4(\mu, n) + 4\frac{\beta_5(\mu, n)}{\sqrt{n}} \leq \frac{m_4(\mu)}{N_n n^{2/3}} + 4\frac{\beta_5(\mu, n)}{\sqrt{n}}, \qquad z \in D_{t_2,4}, \tag{6.27}$$

and therefore, for the same z,

$$\frac{m_4(\mu) + |\eta_{n2}(z)|}{n} \leq L_{4,n}\Big(1 + \frac{1}{N_n n^{2/3}} + \frac{4\beta_5(\mu, n)}{m_4(\mu)\sqrt{n}}\Big) \leq 6L_{4,n}. \tag{6.28}$$

Compaire (6.28) with (6.7). From (6.28) we conclude that in (6.11)

$$\varepsilon(z) \leq 3(|a_n| + \sqrt{L_{4,n}}) + \frac{T_{n2}}{N_n n^{2/3}} \leq 4(|a_n| + \sqrt{L_{4,n}}), \qquad z \in D_{t_2,4}. \tag{6.29}$$

Now we can improve estimates for $\eta_{n8}(z)$ and $\eta_{n9}(z)$ when $z \in D_{t_2,4}$. Recalling (6.21) and using in (6.16), (6.18) and (6.19) the improved estimates (6.28) and (6.29) for $(m_4(\mu) + |\eta_{n2}(z)|)/n$ and $\varepsilon(z)$, respectively, we deduce, for $z \in D_{t_2,4}$,

$$|\eta_{n8}(z)| + |\eta_{n9}(z)| \leq \frac{cL_{4,n}}{N_n n^{2/3}\Im z} + c(|a_n| + \sqrt{L_{4,n}})^3 \leq \eta_{n11} := \frac{cT_{n2}}{N_n n^{2/3}} + c(|a_n| + \sqrt{L_{4,n}})^3. \tag{6.30}$$

Our next step is to evaluate

$$G_{\mu_{n2}}(z) - G_{\nu_n}(z) \quad \text{and} \quad G_{\nu_n}(z) + \frac{1}{n}(G_{\mu_{n2}}(z))^3 - G_{\mu_n}(z)$$

for $z \in D_{t_2,2-\gamma_2}$ and $\Im z = 1$, where $\gamma_2 := 10\eta_{n11}$.

For $z \in D_{t_2,2-\gamma_2} \cup D_1(R_{n2}))$, using the formula (6.20) with $j=4$ for $S_n(z)$, we write

$$S_{n2}(z)S_n(z)\Big(\frac{1}{S_n(z)}-\frac{1}{S_{n2}(z)}\Big)=S_{n2}(z)-S_n(z)=-\frac{1}{2}\eta_{n8}(z)$$
$$-\frac{\eta_{n9}(z)}{\sqrt{(1+b_n)^2(z-a_n)^2-4(1-d_n)}+\sqrt{(1+b_n)^2(z-a_n)^2-4(1-d_n)+\eta_{n9}(z)}}. \quad (6.31)$$

By (6.25) and (6.30), we have, for the same z,

$$\begin{aligned}&|\sqrt{(1+b_n)^2(z-a_n)^2-4(1-d_n)}+\sqrt{(1+b_n)^2(z-a_n)^2-4(1-d_n)+\eta_{n9}(z)}|\\ =&|\sqrt{(1+b_n)^2(z-a_n)^2-4(1-d_n)}||1+\sqrt{1+\eta_{n9}(z)/((1+b_n)^2(z-a_n)^2-4(1-d_n))}|\\ &\ge|\sqrt{(1+b_n)^2(z-a_n)^2-4(1-d_n)}|. \end{aligned} \quad (6.32)$$

It is easy to see that the bound (6.26) holds for $S_{n2}(z)$ as well. Therefore we can conclude from (6.30)–(6.32) that, for $-2+a_n+\gamma_2 \le x \le 2+a_n-\gamma_2$,

$$\begin{aligned}&\int_{t_2}^{1}\Big|G_{\nu_n}(x+iu)-G_{\mu_{n2}}(x+iu)\Big|\,du=\int_{t_2}^{1}\Big|\frac{1}{S_n(x+iu)}-\frac{1}{S_{n2}(x+iu)}\Big|\,du\\ &\le c\int_{t_2}^{1}\Big(|\eta_{n8}(x+iu)|+\frac{|\eta_{n9}(x+iu)|}{|\sqrt{(1+b_n)^2(x-a_n+iu)^2-4(1-d_n)}|}\Big)\,du\\ &\le c(|a_n|+\sqrt{L_{4,n}})^3+c\frac{L_{4,n}}{N_n n^{2/3}}\frac{\log(T_{n2}/L_{4,n})}{\sqrt{\gamma_2}}\\ &\le c(|a_n|+\sqrt{L_{4,n}})^3\Big(1+L_{4,n}^{1/4}\frac{\log(T_{n2}/L_{4,n})}{T_{n2}}\Big)+cL_{4,n}^{4/3}\frac{\log(T_{n2}/L_{4,n})}{T_{n2}^{4/3}}\\ &\le c(|a_n|+\sqrt{L_{4,n}})^3+cL_{4,n}^{4/3}\frac{\log(T_{n2}/L_{4,n})}{T_{n2}^{4/3}}. \end{aligned} \quad (6.33)$$

Using the estimate: $\min\{|S_n(u+i)|,|S_{n2}(u+i)|\} \ge c\sqrt{1+u^2}$, $u \in \mathbb{R}$, and (6.23), (6.25), (6.31), (6.32), we obtain

$$\begin{aligned}&\int_{-R_{n2}}^{R_{n2}}\Big|G_{\nu_n}(x+iu)-G_{\mu_{n2}}(x+iu)\Big|\,du=\int_{-R_{n2}}^{R_{n2}}\Big|\frac{1}{S_n(u+i)}-\frac{1}{S_{n2}(u+i)}\Big|\,du\\ &\le c\int_{-R_{n2}}^{R_{n2}}\Big(|\eta_{n8}(u+i)|+\frac{|\eta_{n9}(u+i)|}{\sqrt{1+u^2}}\Big)\frac{du}{1+u^2}\le c\Big(1+\frac{\beta_5(\mu)}{m_4^{3/2}(\mu)}\Big)R_{n2}(|a_n|+\sqrt{L_{4,n}})^3. \end{aligned} \quad (6.34)$$

Now we conclude from (5.5) that

$$G_{\mu_n}(z) - G_{\nu_n}(z) - \frac{1}{nS_n^3(z)} = \frac{r_{n1}(z)}{nS_n^3(z)}, \quad z \in \mathbb{C}^+, \tag{6.35}$$

where, by (5.8), we see that

$$\frac{|r_{n1}(z)|}{n|S_n^3(z)|} \le \frac{cL_{3,n}}{n|S_n(z)|^3 \Im z}, \qquad z \in \mathbb{C}^+. \tag{6.36}$$

Since for the same z

$$\Big|\frac{1}{S_n^3(z)} - \frac{1}{S_{n2}^3(z)}\Big| \le 2\Big|\frac{1}{S_n(z)} - \frac{1}{S_{n2}(z)}\Big|\Big(\frac{1}{|S_n(z)|^2} + \frac{1}{|S_{n2}(z)|^2}\Big), \tag{6.37}$$

we obtain, using (6.26), (6.33), (6.36) and (6.37), that, for $-2+a_n+\gamma_2 \le x \le 2+a_n-\gamma_2$,

$$\begin{aligned}
&\int_{t_2}^{1}\Big|G_{\mu_n}(x+iu) - G_{\nu_n}(x+iu) - \frac{1}{n}(G_{\mu_{n2}}(x+iu))^3\Big|\,du \\
&\quad\le \int_{t_2}^{1}\Big|G_{\mu_n}(x+iu) - G_{\nu_n}(x+iu) - \frac{1}{n}(G_{\nu_n}(x+iu))^3\Big|\,du \\
&\quad+ \frac{1}{n}\int_{t_2}^{1}\Big|(G_{\mu_{n2}}(x+iu))^3 - (G_{\nu_n}(x+iu))^3\Big|\,du \\
&\quad\le \frac{c}{n}\Big(L_{3,n}\log(T_{n2}/L_{4,n}) + (|a_n| + \sqrt{L_{4,n}})^3 + L_{4,n}^{4/3}\frac{\log(T_{n2}/L_{4,n})}{T_{n2}^{4/3}}\Big).
\end{aligned} \tag{6.38}$$

In addition, using the same arguments as in the proof of (6.34), we have

$$\begin{aligned}
&\int_{-R_{n2}}^{R_{n2}}\Big|G_{\mu_n}(u+i) - G_{\nu_n}(u+i) - \frac{1}{n}(G_{\mu_{n2}}(u+i))^3\Big|\,du \\
&\qquad\le \frac{c}{n}\Big(L_{3,n} + \Big(1 + \frac{\beta_5(\mu)}{m_4^{3/2}(\mu)}\Big)(|a_n| + \sqrt{L_{4,n}})^3\Big).
\end{aligned} \tag{6.39}$$

Now estimate $|G_{\mu_n}(u+i) - G_{\mu_{n2}}(u+i)|$ for $|u| > R_{n2}$, using the bound

$$\begin{aligned}
|G_{\mu_{n2}}(u+i) - G_{\mu_n}(u+i)| &\le |G_{\mu_{n2}}(u+i) - G_{\mu_w}(u+i)| + |G_{\mu_w}(u+i) - G_{\nu_n}(u+i)| \\
&\quad+ |G_{\nu_n}(u+i) - G_{\mu_n}(u+i)|.
\end{aligned} \tag{6.40}$$

In the first step we note that, for $|u| > R_{n2}$,

$$|G_{\mu_{n2}}(u+i) - G_w(u+i)| = \frac{|S(u+i) - S_{n2}(u+i)|}{|S(u+i)||S_{n2}(u+i)|}$$
$$\le \frac{c}{|S(u+i)||S_{n2}(u+i)|}\Big(\frac{L_{4,n}}{|u|} + \frac{|a_n|}{|u|^2}\Big) \le \frac{c}{|u|^3}\Big(L_{4,n} + \frac{|a_n|}{|u|}\Big). \tag{6.41}$$

Therefore we conclude from (6.41) the upper bound

$$\int\limits_{|u|>R_{n2}} |G_{\mu_{n2}}(u+i) - G_w(u+i)|\, du \le \frac{c}{R_{n2}^2}\Big(L_{4,n} + \frac{|a_n|}{R_{n2}}\Big). \tag{6.42}$$

In the second step we deduce, using (6.35), (6.36), and the bound $|S_n(u+i)| \ge c\sqrt{1+u^2}$, $u \in \mathbb{R}$, that

$$\int\limits_{|u|>R_{n2}} |G_{\mu_n}(u+i) - G_{\nu_n}(u+i)|\, du \le \frac{c}{nR_{n2}^2}. \tag{6.43}$$

From (5.53), we conclude, using the lower bound $\min\{|S(u+i)|, |S_n(u+i)|\} \ge c\sqrt{1+u^2}$, $u \in \mathbb{R}$,

$$\int\limits_{|u|>R_{n2}} |G_{\mu_w}(u+i) - G_{\nu_n}(u+i)|\, du \le cL_{3,n} \int\limits_{|u|>R_{n2}} \frac{du}{(1+u^2)^{3/2}} \le c\frac{L_{3,n}}{R_{n2}^2}. \tag{6.44}$$

Therefore we obtain finally from (6.42)–(6.44) the upper bound

$$\int\limits_{|u|>R_{n2}} |G_{\mu_n}(u+i) - G_{\mu_{n2}}(u+i) - \frac{1}{n}(G_{\mu_{n2}}(u+i))^3|\, du \le c\frac{L_{3,n}+L_{4,n}}{R_{n2}^2}. \tag{6.45}$$

It is easy to see that

$$(G_{\mu_{n2}}(z))^3 = \int\limits_{\mathbb{R}} \frac{\hat{\mu}_{n2}(dx)}{z-x} = \int\limits_{\mathbb{R}} \frac{p_{\hat{\mu}_{n2}}(x)\,dx}{z-x}, \quad z \in \mathbb{C}^+,$$

where

$$p_{\hat{\mu}_{n2}}(x) := \frac{1}{8\pi}\sqrt{(4(1-d_n) - (1-b_n)^2(x-a_n)^2)_+}$$
$$\times \frac{3((1+b_n)x + (1-b_n)a_n)^2 + (1-b_n)^2(x-a_n)^2 - 4(1-d_n)}{(b_nx^2 + (1-b_n)a_nx + 1 - d_n)^3}, \quad x \in \mathbb{R}.$$

Apply Lemma 3.2 to the signed measure $\nu = \mu_{n2} + \frac{1}{n}\hat{\mu}_{n2}$ and to the p-measure μ_n. Since $\beta_3(\mu) < \infty$, it is well-known that $m_2(\mu_n) < \infty$ and the condition (3.11) holds. We choose $I_\varepsilon = [-2 - a_n + \varepsilon, 2 - a_n - \varepsilon]$ with $\varepsilon = 2\gamma_2$. By the definition of ν, we see that

$$\sup_{x \in I_{\varepsilon/2}} \tilde{\nu}([x - 10t_2, x + 10t_2]) \le ct_2$$

and

$$\tilde{\nu}(\mathbb{R} \setminus I_\varepsilon) \le c\gamma_2^{3/2},$$

where $\tilde{\nu}$ is the total variation of ν. Using these bounds and (6.33), (6.34), (6.38), (6.39), and (6.45), we obtain, by Lemma 3.2,

$$\Delta(\mu_n, \mu_{n2} + \frac{1}{n}\hat{\mu}_{n2}) \le ct_2 + c\gamma_2^{3/2} + c(|a_n| + \Big(\frac{L_{4,n}}{T_{n2}}\Big)^{4/3} \log\frac{T_{n2}}{L_{4,n}} + c\frac{L_{3,n}\log(T_{n2}/L_{3,n})}{n}$$
$$+ cR_{n2}(|a_n| + \sqrt{L_{4,n}})^3\Big(1 + \frac{\beta_5(\mu)}{m_4^{3/2}(\mu)}\Big) + c\frac{L_{3,n}+L_{4,n}}{R_{n2}^2}. \tag{6.46}$$

In view of the following relations $T_{n2} = m_4^{2/5}(\mu)N_n^{3/5}$, $R_{n2} = c_4^{-1}(a_n^2 + L_{4,n})^{-1/3}$ and $N_n = (\frac{1}{2}\sqrt{n})^{q_3-16/3}m_4(\mu)/\beta_{q_3}(\mu)$, we get from (6.46) the upper bound, using Lyapunov's inequality as well,

$$\Delta(\mu_n, \mu_{n2} + \frac{1}{n}\hat{\mu}_{n2}) \le c\frac{\beta_{q_3}^2(\mu)}{n^{1+q_4/4}}, \quad n \ge c_3 n_2. \tag{6.47}$$

Direct calculations shows that

$$\Delta(\hat{\mu}_{n2}, \nu) \le c(|a_n| + L_{4,n}), \quad n \ge c_3 n_2, \tag{6.48}$$

where ν is the signed measure with the density (2.20). The statement of Theorem 2.3 follows from (6.47) and (6.48) immediately.

Proof of Corollary 2.4. Recalling the definition of the density $p_{\mu_{n2}}(x)$ of the measure μ_{n2} we see that

$$p_{\mu_{n2}}(x + a_n) = \frac{1}{2\pi}\sqrt{(4(1-d_n) - (1-b_n)^2x^2)_+}(1 + d_n - b_n - a_n x - (b_n - a_n^2)(x^2-1))$$
$$+ c\theta(L_{4,n} + a_n^2)^{3/2}, \qquad x \in \mathbb{R}.$$

In addition we have, for $x \in \mathbb{R}$,

$$\frac{1}{2\pi}\int_{-\infty}^{x} \sqrt{(4(1-d_n) - (1-b_n)^2u^2)_+}\,du = (1 - d_n + b_n)\mu_w((-\infty, x))$$
$$+ \big(\frac{1}{2}d_n - b_n\big)x\frac{1}{2\pi}\sqrt{(4-x^2)_+} + c\theta L_{4,n}^{3/2}$$

and, for $x \in (-l_n, l_n)$, where $l_n := \max\{2, 2\sqrt{1-d_n}/(1-b_n)\}$,

$$\big|\sqrt{(4(1-d_n) - (1-b_n)^2x^2)_+} - \sqrt{(4-x^2)_+}\big|$$
$$\le \frac{cL_{4,n}}{\sqrt{(4(1-d_n) - (1-b_n)^2x^2)_+} + \sqrt{(4-x^2)_+}}.$$

Using these formulae, with the help of simple calculations we obtain from (2.21) the representation (2.22). □

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Gennadii Chistyakov
Institute for Low Temperature Physics and Engineering
National Academy of Sciences of Ukraine
47 Lenin Ave.
61103 Kharkov
Ukraine

E-mail address: chistyakov@ilt.kharkov.ua, chistyak@mathematik.uni-bielefeld.de

Friedrich Götze
Fakultät für Mathematik
Universität Bielefeld
Postfach 100131
33501 Bielefeld
Germany
E-mail address: goetze@mathematik.uni-bielefeld.de