TOPICS ON MEIXNER FAMILIES AND ULTRASPHERICAL TYPE GENERATING FUNCTIONS FOR ORTHOGONAL POLYNOMIALS

Marek Bozejko[1] and Nizar Demni [2]

Abstract. We highlight some connections between different characterizations of the classical Meixner family. While proceeding, we give a partial answer to the characterization of the probability distributions with ultraspherical-type orthogonal polynomials generating functions. Then we investigate the free Meixner family for which we set analogs of both the Sheffer's and the Al-Salam and Chihara's characterizations. Finally, we discuss the q-deformed case, $|q| < 1$.

1. Introduction

A remarkable and exciting feature of Mathematics is the fact that several independent and seemingly unreleated works may lead to a common result. It is often the lack of communication among various schools that prevents the full understanding of the existing inter-connections, thereby arises the problem of unifying the different approaches, which stands for an important, useful and also tricky task. In this spirit, the so-called *Meixner family* of probability distributions, referred to as *Meixner distributions*, admits the following characterizations:

- When it first appeared in [21], the Meixner family was defined as the set of probability measures μ with finite exponential moments in a neighborhood of zero such that:

$$\psi(z,x) := \sum_{n\geq 0} P_n(x)z^n = \frac{e^{xH(z)}}{\mathbb{E}(e^{XH(z)})}, \tag{1}$$

  where H is analytic around $z = 0$ such that $H(0) = 0, H'(0) = 1$, X is a random variable in some probability space $(\Omega, \mathscr{F}, \mathbb{P})$ with law $\mu = \mathbb{P}\circ X^{-1}$ and $(P_n)_{n\geq 0}$ is the set of orthogonal polynomials (OP) with respect to μ. ψ is an OP generating function of exponential-type. Up to translations and dilations, the Meixner family consists of Gaussian, Poisson, Gamma, negative binomial, Meixner and binomial distributions. The latter being non-infinitely-divisible in the classical sense. Another proof of this characterization was given in [17].
- Sheffer characterized $(P_n)_n$ as the set of OP of type zero ([24]).
- Meixner laws are the solutions of a quadratic regression problem ([19]).

Date: June 30, 2008.

[1]Instytut Matematyczny, Uniwersytet Wroclawski, Pl. Grunwaldzki 2/4, 50-384 Wroclaw, Poland.

[2]SFB 701, Faculty of Mathematics, Bielefeld University, Germany.

- Morris showed that the variance of the natural exponential family parametrized by the mean associated with a probability measure μ, is an at most quadratic polynomial in the mean if and only if μ belongs to the Meixner family ([22]).
- OP with respect to Meixner laws form a distinguished subclass of the so-called Al-Salam and Chihara set of OP ([1]).

After the birth of free probability theory, for which miscellaneous analogs of known results in classical probability theory were derived, the *free Meixner family* was defined in [3] and covers six probability distributions called analogously to their classical counterparts free Gaussian or Wigner, free Poisson or Marchenko-Pastur, free Gamma, free negative binomial, free Meixner and free binomial known also as the free stationary Jacobi law ([12],[14],[15]). Their study revealed free versions of some of the above characterizations: the exponential-type OP generating functions are replaced by generating functions of Cauchy-Stieltjes-type. ([3], [9],[18]), *free Meixner distributions* solve a quadratic regression problem in a free setting ([8]) and to them are associated Cauchy-Stieltjes-type families with linear means and quadratic variances ([10]). To our best knowledge, no free analogs of both the Sheffer's and the Al-Salam and Chihara's characterizations exist in literature.

In the present paper, we highlight some intrinsic connections between the different characterizations in both the classical and and the free settings. More precisely, we start by explaining how the Sheffer's characterization leads to Al-Salam and Chihara results and vice-versa. Then we do the same for both the Kubo's and the Morris characterizations. In the free setting, the connection is set between Bryc and Ismail's characterization using Cauchy-Stieljes families of quadratic variances and the one given by the authors via Cauchy-Stieltjes-type generating functions of OP ([9]).

While studying the connection between Kubo's and Morris characterizations, we observed that, to a given OP generating function of a probability measure satisfying suitable integrability conditions, corresponds a family of probability measures parametrized by the mean such that the variance is an at most quadratic polynomial in the mean (we will say for short 'an at most quadratic variance') . This fact is an immediate consequence of the orthogonality or equivalently the three-terms recurrence relation satisfied by the OP. Besides, the coefficients of the variance are easily expressed in terms of the Jacobi-Szegö parameters. This general observation gives a machinery that can be used for instance to adress the following problem: characterize probability distributions μ_λ (of finite all order moments) having *ultraspherical-type* generating functions

$$\psi_\lambda(z,x) = \sum_{n\geq 0} a_{n,\lambda} P_n^\lambda(x) z^n = \frac{1}{u_\lambda(z)(f_\lambda(z)-x)^\lambda}, \quad \lambda > 0$$

for suitable functions u_λ, f_λ and a sequence of real numbers $(a_{n,\lambda})_{n\geq 0}$, where $(P_n^\lambda)_{n\geq 0}$ denotes the set of monic orthogonal polynomials with respect to μ_λ. This problem reduces to the free Meixner case when $\lambda = 1$ and is far from being easy for other values. Hence, we shall impose some restrictions and we solve this problem assuming that f_λ is independent of λ. This independence of λ is equivalent to the fact that the lowering operator of the corresponding OP has coefficients that are also independent of λ. In the last part of the paper, we give the free analogs of both the Sheffer's and the Al-Salam and Chihara's characterizations by means of the free derivative operator.

Remark 1.1. *In order to avoid confusions, the Meixner family will be referred to as the classical Meixner family.*

2. On Sheffer and AL-Salam and Chihara characterizations

Few years after the Meixner classification was given, Sheffer ([24]) provided an easy proof of it while studying the following general situation: let $(Q_n)_n$ be a set of polynomials, that is Q_n is of degree n for each $n \geq 0$, which are not necessarily orthogonal. Let D denote the derivative operator and take $Q_0(x) = 1$, then there exists a unique differential operator $J := \sum_{n\geq 1} c_n D^n$ with constant coefficients $(c_n)_n, c_1 \neq 0$ such that $JQ_n = Q_{n-1}, Q_0 = 1, Q_{-1} = 0$. $(Q_n)_n$ is then said to be a set of *type zero* w.r.t J and this is equivalent to

$$\sum_{n\geq 0} Q_n(x)z^n = A(z)e^{xH(z)} \tag{2}$$

where A is an entire function around zero (called the determining function) and H is the inverse function of J when the latter is viewed as an entire function. It is then obvious that one immediately recovers the Meixner family under the additional orthogonality assumption on $(Q_n)_n$. In the middle 70's, Al-Salam and Chihara ([1]) adressed the following problem: characterize the set of OP $(r_n)_n, (q_n)_n$ such that their convolution

$$s_n(x,y) = \sum_{k=0}^{n} r_k(x)q_{n-k}(y) \tag{3}$$

defines a set of OP in the variable x for all y. $(r_n)_n$ and $(q_n)_n$ belong to the so-called Al-Salam and Chihara set of OP. If one further requires that $s_n(x,y) := p_n(x+y)$ for some polynomial p_n, then the subset of orthogonality probability distributions reduces to the classical Meixner family. More precisely, $r_n = q_n$ for all n.

Remark 2.1. *The stability of the Meixner family under the convolution action was firstly noticed by Lancaster in* [20]*.*

2.1. From Sheffer to Al-Salam and Chihara.

Let us now see how one carries the Al-Salam and Chihara problem into the setting of Sheffer. Let $(r_n)_n, (q_n)_n, (p_n)_n$ three sets of OP related by the convolution below

$$p_n(x+y) = \sum_{k=0}^{n} r_k(x)q_{n-k}(y). \tag{4}$$

Denote J and $\tilde{J}$ the (unique) operators ([24]) such that $J(r_n) = r_{n-1}$ and $\tilde{J}(q_n) = q_{n-1}$. Write

$$J = \sum_{k\geq 1} c_k(x)D^k, \quad \tilde{J} = \sum_{k\geq 1} \tilde{c}_k(x)D^k$$

where $c_k, \tilde{c}_k$ are polynomials of degree at most $k-1$ and $c_1 = \tilde{c}_1 = 1$. Note first that if (4) holds, then $c_n(x) = \tilde{c}_n(x)$ for all n and this follows from the symmetry of $(x,y) \mapsto p_n(x+y)$ in both arguments. More precisely,

$$J(p_n(\cdot + y))(x) = \sum_{k=1}^{n} r_{k-1}(x) q_{n-k}(y) = p_{n-1}(x+y)$$
$$= \sum_{k=0}^{n-1} r_k(y) q_{n-k-1}(x) = \tilde{J}(p_n(\cdot + y))(x)$$

which proves our claim by induction. Again, the symmetry gives

$$p_{n-1}(x+y) = J(p_n(\cdot + y))(x) = J(p_n(x + \cdot))(y)$$

which yields $c_n(x) = c_n(y) = c_n$ for all n. As a result, $(r_n)_n, (q_n)_n$ are of type zero w.r.t. the same operator J (hence the same H) and Sheffer's result applies.

Remark 2.2. *Since the expecation of $\psi(z, X)$ equals 1 when $(P_n)_n$ is a set of OP, then A is entirely determined by the knowledge of H and then so is ψ. Thus, $r_n = q_n$ for all n whenever both sets of OP belong to the same lowering operator. This claim is no more true for sets of not necessarily OP.*

2.2. **From Al-Salam and Chihara to Sheffer.** Consider a set $(r_n)_n$ of OP w.r.t. a probability distribution μ and assume that it is of type zero w.r.t. J:

$$J = \sum_{n \geq 1} c_n D^n, \; c_1 = 1.$$

Form the convolution

$$s_n(x,y) = \sum_{k=0}^{n} r_k(x) r_{n-k}(y),$$

it then follows that $Js_n(\cdot, y)) = J(s_n(x, \cdot))$ which implies (by induction) that $s_n(x,y) = p_n(x+y)$ for a certain polynomial p_n. The orthogonality of $(p_n(\cdot + y))_n$ is shown as follows: let μ be the orthogonality probability distribution of $(r_n)_n$, then

$$\int_{\mathbb{R}} p_n(x+t) p_m(x+t) \mu(dx)\mu(dt) = \sum_{k=0}^{n}\sum_{l=0}^{m} \int_{\mathbb{R}} r_k(x) r_l(x) \mu(dx) \int_{\mathbb{R}} r_{n-k}(t) r_{m-l}(t)\mu(dt)$$
$$= \sum_{k=0}^{n}\sum_{l=0}^{m} ||r_k||^2 ||r_{n-k}||^2 \delta_{kl}\delta_{n-k,m-l} = \sum_{k=0}^{n} ||r_k||^2 ||r_{n-k}||^2 \delta_{n,m},$$

which implies that $(p_n)_n$ is a set of OP w.r.t. the classical convolution $\mu \star \mu$. Thus, for each y, $(p_n(\cdot + y))_n$ is a set of OP w.r.t $\mu \star \mu \star \delta_y$.

Remark 2.3. *Since*

$$p_n(x) = \sum_{k=0}^{n} r_k(x-y) q_{n-k}(y),$$

for all y, then $(p_n)_n$ belongs to J and by orthogonality, $p_n = r_n$ for all n.

3. On Kubo and Morris characterizations

In the early eightees, the classical Meixner family appeared in [22] as the set of probability measures which generate natural exponential families parametrized by the means, such that the variance of a given exponential family is at most quadratic of the mean. More precisely, given a (non-degenerate) probability measure μ with finite exponential moments in some neighbourhood of zero, its natural exponential family is the one-parameter family of probability measures defined by

$$\left\{\mathbb{P}_\theta(dx) := \frac{e^{\theta x}}{\mathbb{E}(e^{\theta X})}\mu(dx),\, \theta \in V(0)\right\}$$

Let z and V denote respectively the mean and the variance of $\mathbb{P}_\theta$ for a given θ. Then $z = \tau'(\theta), V = \tau''(\theta)$ where

$$\tau(\theta) := \log \mathbb{E}(e^{\theta X})$$

is the cumulants generating function of μ. Since $V(\theta) > 0$, then τ' is invertible and $\theta := S(z)$ makes sense for small z. We also have $S(0) = 0, S'(z) = 1/V(z)$ and $S'(0) \neq 0$ which one may suppose to equal 1. This gives rise to a new exponential family parametrized by the mean

$$\left\{\mathbb{P}_z(dx) := \frac{e^{S(z)x}}{\mathbb{E}(e^{S(z)X})}\mu(dx),\, z \in V(0)\right\}$$

of mean z and variance $V(z) = \tau''(S(z))$. It was shown that V is at most quadratic in z if and only if μ belongs to the classical Meixner family.

Few years ago, another proof of the Meixner result via OP generating functions was given in ([17]) using Asai-Kuo-Kubo criterion ([2]). It was shown that $(P_n)_n$ has an OP generating function of the form (2) if and only if

$$aH^{-1}(z) + b = \tau'(z), \tag{5}$$

where $a \in \mathbb{R}, b > 0$ are respectively the mean and the variance of the orthogonality measure and

$$H'(z) = \frac{1}{\beta z^2 + \gamma z + 1}, \quad (\beta \geq 0) \text{ or } (\beta < 0,\, b/\beta \in \mathbb{R} \setminus \mathbb{Z}). \tag{6}$$

The values $(\beta < 0,\, b/\beta \in \mathbb{Z})$ correspond to a signed measure

3.1. **From Kubo to Morris.** It is known that the assumptions on S allow to expand

$$\frac{e^{S(z)x}}{\mathbb{E}(e^{S(z)x})} = \sum_{n\geq 0} P_n(x) z^n, \tag{7}$$

where P_n is a polynomial of exact degree n and $(P_n)_{n\geq 0}$ need not to be orthogonal (see [23] p.45). Then, it suffices to identify S with H for $b = 0, a = 1$ (μ has mean zero and variance one and is referred to as standard probability distribution) and the assumption of quadratic variance with (6) (one further needs to check the sign of the parameters involved in the variance).

Nevertheless, another proof of the orthogonality of $(P_n)_n$ can be derived. Indeed, the fact that $\mathbb{P}_z$ has a total mass equal 1 gives

$$\int_{\mathbb{R}} P_n(x)\mu(dx) = 0,\, n \geq 1.$$

Next, since the mean equals z and V is at most quadratic, then

$$\int_{\mathbb{R}} xP_n(x)\mu(dx) = 0,\, n \geq 2 \quad \text{and} \quad \int_{\mathbb{R}} x^2 P_n(x)\mu(dx) = 0,\, n \geq 3.$$

It follows that P_1 is orthogonal to P_0, P_2 is orthogonal to P_0, P_1 and P_3 is orthogonal to P_0, P_1, P_2. To get the orthogonality for all n, one needs similar vanishing integrals for higher orders, that is for all $l \geq 3$,

$$\int_{\mathbb{R}} x^l P_n(x)\mu(dx) = 0, \quad n \geq l+1.$$

To derive such a result, take the $(l-1)$-th derivative, $l \geq 3$, in both sides of $\tau'(S(z)) = z$ together with $V(z) = 1/S'(z)$ and use the at most quadratic variance assumption to see that $\tau^{(l)}(S(z))$ is at most of degree l. Since the l-th moment of $\mathbb{P}_z$ is written as $\tau^{(l)}(S(z))+$ terms of lower degrees, we are done.

3.2. **From Morris to Kubo.** In order to prove the Kubo's characterization using the results of Morris, observe that an exponential-type OP generating function ψ of a given measure μ defines the exponential family

$$\left\{\mathbb{P}_z(dx) := \frac{e^{H(z)x}}{\mathbb{E}(e^{H(z)X})}\mu(dx),\ z \in V(0)\right\}$$

Indeed, by the orthogonality assumption, $\mathbb{P}_z$ is a probability measure for each $z \in V(0)$. To simplify, take $a = 0, b = 1$, then relations (5) and (6) follows directly by computing respectively both the mean, say $m(z)$, and the variance $V(z)$ of $\mathbb{P}_z$ in two different ways: the first way expresses m and V by means of τ' yielding $m(z) = \tau'(H(z))$ and $V(z) = \tau''(H(z))$ while the second way uses the three-terms recurrence relation satisfied by the OP. More precisely, let $P_n := a_n V_n, V_0 = P_0 = 1 (a_0 = 1)$ where V_n is the n-th monic polynomial, then It is easily seen that (1) may be written as

$$\frac{e^{H(z)x}}{\mathbb{E}(e^{H(z)X})} = \sum_{n\geq 0} a_n V_n(x) z^n = \sum_{n \geq 0} \frac{1}{n!} V_n(x) z^n \tag{8}$$

which follows from substituting x with x/z then letting $z \to 0$ and finally using $H'(0) = 1$. Moreover,

$$xV_n(x) = V_{n+1}(x) + \alpha_n V_n(x) + \omega_n V_{n-1}(x), \quad V_{-1} := 0,\, \omega_0 := 1$$

where $(\alpha_n)_n, (\omega_n)_n$ are the Jacobi-Szegö parameters ([13]). Intertwining the order of integration and using the orthogonality of $(V_n)_n$, it easily follows that

$$m(z) = \sum_{n\geq 0} \int_{\mathbb{R}} xV_n(x)\mu(dx)\frac{z^n}{n!} = \alpha_0 + \omega_1 z = z$$

since μ is standard. Similarly, one gets

$$V(z) = (\omega_2 - 2)\frac{z^2}{2} + \alpha_1 z + 2$$

which fits into the work of Morris and identifies the coefficients β, γ in (6).
What is quite interesting is that every OP generating function of a given measure satisfying suitable integrabiltiy conditions defines a one-parameter family of probability measures parametrized by the mean such that, for a fixed value of the parameter (the mean), the variance is at most quadratic in the mean. When the OP generating function is handable (which is not the case for instance for the q-Meixner family due to the infinite product, [4]), one gets a machinery that can

be used to characterize a family of probability measures via their OP generating functions. This is, as we will see, the case of the free Meixner family with Cauchy-Stieltjes-type OP generating functions and of a natural generalization of ultraspherical polynomials we shall describe later.

Remarks. *1/ The infinitely divisible subfamily is related to Lévy processes and is known as the Lévy-Meixner family* (*see* [23] *and references there in*).
2/Recently, Bryc and Ismail used convolution of Meixner exponential measures to construct exponential kernels with quadratic variances ([10]).

4. Free Meixner family

A parallel definition of Lévy-Meixner processes in free probability theory appeared in [3] after the pioneering work of Biane [6] on free processes with free increments, where the author derived free analogs of exponential martingales involving free cumulants generating functions. Then, the free Meixner family was defined in [8] by proving a free Laha-Lukacs characterization and it consists of six laws mentioned in the introduction. Only the free binomial distribution is not infinitely divisible with respect to the free additive convolution. Similarly to the classical case, one can set the connection between Bryc and Ismail characterization via Cauchy-Stieltjes families with at most quadratic variances and the one recently given by the authors via Cauchy-Stieltjes-type OP generating functions ([8]).

On the one hand, a combination of [3] and [8] allowed us to characterize the free Meixner distributions and Asai-Kuo-Kubo criterion ([2]) was used to derive analogous identities to (5) and (6) involving the free cumulants generating function ([9]). More precisely, Let μ be a standard probability distribution with finite all order moments. Then, μ belongs to the free Meixner family if and only if

$$\psi(z,x) := \sum_{n\geq 0} P_n(x)z^n = \frac{1}{u(z)[f(z)-x]} \tag{9}$$

where

$$f(z) = K(u(z)) = z + \frac{1}{u(z)} \tag{10}$$

$$\frac{u(z)}{z} = \frac{1}{1+az+bz^2} \tag{11}$$

for $a \in \mathbb{R}, b \geq -1$ (parameters of μ and they don't have to be confused with a, b in used the previous sections), where K is the right inverse in a neighborhood of 0, say $V(0)$, of the Cauchy-Stieltjes transform of μ:

$$G(z) := \int \frac{1}{z-x}\mu(dx), \quad z \in \mathbb{C} \setminus \operatorname{supp}(\mu).$$

The first equality in (10) follows from the unit expectation of $\psi(z, X)$, where X is a random variable with law μ, while the second is analogous to (5) if it is written as $R(u(z)) = z$ where $R(z) = K(z) - 1/z$ is the free cumulants generating function. Relation (11) shall be compared with (6) where the classical derivative is replaced by the free derivative defined by

$$D^0(f)(z) := \frac{f(z)-f(0)}{z}, \quad z \neq 0. \tag{12}$$

On the other hand, to a compactly-supported probability measure is associated a Cauchy-Stieltjes family defined by ([11]):

$$\left\{\mathbb{P}_\theta(dx) := \frac{1}{M(\theta)}\frac{1}{1-\theta x}\mu(dx), \qquad \theta \in V(0)\right\},$$

where M is the moment generating function defined by

$$M(\theta) := \int_{\mathbb{R}} \frac{1}{1-\theta x}\mu(dx) = \frac{1}{\theta}G\left(\frac{1}{\theta}\right).$$

If μ is not degenerate, then the mean of $\mathbb{P}_\theta$

$$z(\theta) := \int_{\mathbb{R}} x\mathbb{P}_\theta(dx) = \frac{M(\theta)-1}{\theta M(\theta)}$$

is invertible as a function of small enough θ, thus a new parametrization by the mean is performed on $\mathbb{P}_\theta$ giving rise to

$$\left\{\mathbb{P}_z(dx) := \frac{1}{M(v(z))}\frac{1}{1-v(z)x}\mu(dx),\, z \in V(0)\right\} \qquad \theta = v(z).$$

Let $V(z)$ denote the variance of P_z for z small enough. Then:

$$\mathbb{P}_z(dx) := \frac{V(z)}{V(z)+z(x-z)}\mu(dx) \tag{13}$$

when μ has mean zero. It was shown that V is quadratic if and only if μ belongs to the free Meixner family. To deduce this result using OP generating functions, expand the ratio in (13) as

$$\frac{V(z)}{V(z)+z(x-z)} = \sum_{n\geq 0} P_n(x)z^n$$

with non-necessarily OP. Set $u(z) := z/V(z)$ with $V(z) = 1 + az + bz^2$. Then (13) and (9) are the same. Comparing (2.1) in [11] with (10), we are done.
Conversely, suppose we are given a standard probability measure with finite all order moments and the OP generating function (9) with $u(0) = 0, u'(0) = 1, uf(0) = 1$. Then, one defines a Cauchy-Stieltjes family ($\mathbb{P}_z$) parametrized by the mean for which (10) holds. Since $z \mapsto R(z)$ is defined and analytic in a ball centered at $z = 0$, then μ is compactly-supported ([5]). Moreover, $V(z) = z/u(z)$ is at most quadratic in the mean z (it is more convenient to use G rather than M). Recall that this follows from the three-terms recurrence relation satisfied by (monic) $(P_n)_n$. Hence, $\mathbb{P}_z$ takes the form of (13) so that Bryc and Ismail's results apply.

5. Ultraspherical-type generating functions

Recall from the previous section that a probability measure μ with finite all-order moments is a (standard) free Meixner distribution of parameters $a \in \mathbb{R}, b \geq -1$ if and only if

$$\psi(z,x) = \sum_{n\geq 0} P_n(x)z^n = \frac{1+az+bz^2}{1+(a-x)z+(b+1)z^2}$$

Recall also that the (monic) Gegenbauer or ultraspherical polynomials are defined by

$$\sum_{n\geq 0} 2^n\frac{(\lambda)_n}{n!}C_n^\lambda(x)z^n = \frac{1}{(1-2zx+z^2)^\lambda}, \quad \lambda > -1/2,\, \lambda \neq 0. \tag{14}$$

They are orthogonal w.r.t $\nu_\lambda(dx) = C_\lambda(1-x^2)^{\lambda-1/2}dx$ for $x \in [-1,1]$, where C_λ is a normalizing constant (this distribution is not standard). In the sequel, we suggest the following generalization of the free Meixner family given by (standard) probability measures with finite all-order moments, say μ_λ, such that

$$\psi_\lambda(z,x) := \sum_{n\geq 0} a_{n,\lambda}P_n^\lambda(x)z^n = \frac{1}{u_\lambda(z)(f_\lambda(z)-x)^\lambda}, \quad \lambda > 0$$

for analytic functions u_λ, f_λ around $z = 0$ with

$$u_\lambda(0) = 0, \quad \lim_{z\to 0} zf_\lambda(z) = 1, \quad \lim_{z\to 0} u_\lambda(z)(f_\lambda(z))^\lambda = 1$$

for all $\lambda > 0$. Both limits above imply that $\lim_{z\to 0} u(z)/z^\lambda = 1$. We shall say that ψ_λ is of ultraspherical-type. By orthogonality, the integral of $\psi_\lambda(z,\cdot)$ equals one which yields

$$u_\lambda(z) = \int_{\mathbb{R}} \frac{1}{(f_\lambda(z)-x)^\lambda}\mu_\lambda(dx)$$

Using the binomial Theorem

$$\frac{1}{(1-x)^\lambda} = \sum_{n\geq 0}\frac{(\lambda)_n}{n!}x^n,$$

one gets $a_{n,\lambda} = (\lambda)_n/n!, n \geq 0$. The situation is far from being easy and we shall give a partial answer under the assumption that f_λ is independent of λ. Then, μ_1 belongs to the (standard) free Meixner family so that

$$f_\lambda(z) = f_1(z) = z + \frac{1+az+bz^2}{z} = \frac{1+az+(b+1)z^2}{z} := f(z), \quad a \in \mathbb{R}, b \geq -1.$$

It remains to determine u_λ. To proceed, define a family of probability measures by

$$\nu_\lambda(dx) := \psi_\lambda(z,x)\mu(dx) = \frac{1}{u_\lambda(z)(f(z)-x)^\lambda}\mu_\lambda(dx), \quad \lambda > 0 \tag{15}$$

and compute its mean in two different ways: the first way uses the OP expansion of ψ_λ as well as the three-terms recurrence relation while the second way performs direct computations on ψ_λ. One easily gets

$$\lambda z = f(z) - \frac{u_{\lambda,1}(z)}{u_\lambda(z)}, \quad u_{\lambda,1}(z) := \int_{\mathbb{R}}\frac{1}{(f(z)-x)^{\lambda-1}}\mu_\lambda(dx)$$

Using the relation $(1-\lambda)f'(z)u_\lambda(z) = (u_{\lambda,1})'(z), \lambda \neq 1, z \in V(0)$, one gets:
(16)

$$\frac{u_\lambda'(z)}{u_\lambda(z)} = \lambda\frac{1-f'(z)}{f(z)-\lambda z} = \lambda\frac{1-bz^2}{z(1+az+(b+1-\lambda)z^2)} = \lambda\left[\frac{1}{z} + \frac{(\lambda-1-2b)z-a}{1+az+(b+1-\lambda)z^2}\right]$$

which, together with $u_\lambda(0) = 0, \lim_{z\to 0} u_\lambda(z)/z^\lambda = 1$, uniquely determine u and entirely solve the problem.
One can easily check that the monic Gegenbauer polynomials correspond to $a = 0, b = (\lambda - 1/2) > -1$: on the one hand, the above differential equation simplifies to give $u_\lambda(z) = z^\lambda$ and

$$\psi_\lambda(z,x) = \frac{1}{(1-zx+(\lambda+1)z^2/2)^\lambda}.$$

On the other hand, one first normalizes the ultraspherical Beta distribution ν_λ in order to have variance one, that is one considers

$$\mu_\lambda(dx) = C_\lambda(2(1+\lambda) - x^2)^{\lambda-1/2}\mathbf{1}_{[-\sqrt{2(1+\lambda)},\sqrt{2(1+\lambda)}]}(x)dx.$$

The corresponding monic polynomials are given by

$$[\sqrt{2(1+\lambda)}]^n C_n^\lambda\left(\frac{x}{\sqrt{2(1+\lambda)}}\right), \qquad n \geq 0$$

so that

$$\psi_\lambda(z,x) = \sum_{n\geq 0}\frac{(\lambda)_n}{n!}C_n^\lambda\left(\frac{x}{\sqrt{2(1+\lambda)}}\right)[\sqrt{2(1+\lambda)}z]^n = \frac{1}{(1-zx+(\lambda+1)z^2/2)^\lambda}$$

where the last equality follows from (14).

Remarks. *1/If we would like to set an analog of the Sheffer's characterization for (not necessarily orthogonal) polynomials with an ultraspherical-type (15)-like expansion, we have to look for an operator, possibly depending on λ, wich acts on $x \mapsto (1-zx)^{-\lambda}, \lambda \neq 0$ by multiplication by $z \in V(0)$. Using the binomial Theorem, this operator, say $D(\lambda)$, is defined on monomials by*

$$D(\lambda)1 = 0, \quad D(\lambda)x^n := \frac{n}{n+\lambda-1}x^{n-1},$$

and clearly reduces to D^0 for $\lambda = 1$. Following the lines of Sheffer, a set of polynomials has a generating series of the form (15) if and only if it belongs to some operator J_λ which may be written as

$$J(\lambda) = \sum_{n\geq 1} c_n(\lambda)[D(\lambda)]^n, \quad c_1(\lambda) \neq 0\,\forall\lambda.$$

$J(\lambda)$, viewed as an entire function, is the inverse in the sense of composition of $\rho_\lambda := 1/f_\lambda$. As a result, the assumption that f_λ is independent of λ is equivalent to the fact that $c_n(\lambda) = c_n$ for all $n \geq 1$.
2/ If we drop the independence of f_λ on λ, then (16) takes the form

$$\frac{u'_\lambda(z)}{u_\lambda(z)} = \lambda\frac{1-f'_\lambda(z)}{f_\lambda(z)-\lambda z},$$

and computing the variance in both ways mentioned above, one gets:

$$\frac{\lambda(\lambda+1)}{2}\omega_2^\lambda z^2 + \lambda\alpha_1^\lambda z + 2 = \lambda z f_\lambda(z) - \frac{1}{u_\lambda(z)}\int_{\mathbb{R}}\frac{x}{(f_\lambda(z)-x)^{\lambda-1}}\mu_\lambda(dx)$$

where $\alpha_n^\lambda, \omega_n^\lambda$ are the Jacobi-Szegö parameters of μ_λ.

6. Free-type zero polynomials and free Sheffer's characterization

Let $(U_n)_n$ be the set of monic Tchebycheff polynomials of the second kind ([13]) defined by

$$\sum_{n\geq 0} U_n(x)z^n = \frac{1}{1-zx+z^2}.$$

It is easy to see from the above OP generating function that $(U_n)_n$ can not be a set of finite A, B, C-types as defined in [24] (the A-type includes the type zero). This was already noticed for the Legendre polynomials (see p.621-622 in [24]) and the same reasoning applies for $(U_n)_n$. Easily speaking, this is due to the fact that the above generating function is not of an exponential-type (A-type) and its derivative with respect to either x or z will raise it to the square so that, for each n, the polynomials coefficients in the expansion of U_n' (B-type) and nU_n (C-type) in terms of $(U_{n-1}, \ldots, U_0)$ have unbounded degrees in n. Nevertheless, they will be of type zero in the free sense, that is, when substituting the usual derivative D by its free analog D^0 defined by (12) and the exponential function by $x \mapsto (1-x)^{-1}$. Following the lines of Sheffer, we first define 'free differential operators' by

$$J^0 := \sum_{n \geq 0} c_n(x)(D^0)^n$$

where c_n are polynomials. We will say that a set of polynomials $(P_n)_n$ *belongs to* J if $J^0 P_n = P_{n-1}, P_0 \neq 0, P_{-1} := 0$. Since $(D^0)^k x^n = x^{n-k}$ for $1 \leq k \leq n$ and $D^0 1 = 0$, then the above sum terminates when acting on polynomials. In addition, J^0 maps x^n to a polynomial of degree $\leq n-1$ if and only if $c_0 = 0$ and the degree of $c_k, 1 \leq k \leq n$ does not exeed $k-1$. Writing in that case

$$c_k(x) = c_{k,0} + c_{k,1}x + \ldots c_{k,k-1}x^{k-1}, \quad 1 \leq k \leq n,$$

then J^0 maps x^n to a polynomial of exact degree $n-1$ if and only if there exist at least k such that $c_{k,k-1} \neq 0$.
Second, to a given set of polynomials $(P_n)_n$ correponds a unique free differential operator J^0 such that $J^0 P_n = P_{n-1}$ and $c_1 \neq 0$, for otherwise P_1 will be constant which disagree the fact that $(P_n)_n$ is a set of polynomials. Recall also ([24]) that $(P_n)_n$ and $(Q_n)_n$ belong to the same operator J^0 if and only if there exists a sequence $(a_n)_{n \geq 0}$ such that for each n

$$P_n(x) = a_0 Q_n(x) + \ldots a_{n-1} Q_1(x) + a_n Q_0(x), \, a_0 \neq 0. \tag{17}$$

Finally, $(P_n)_n$ is said to be of *free-type* $k, k \in \mathbb{N}$ w.r.t. J^0 if it belongs to J^0 and, for all $n \geq 1$, the degree of c_n is at most k. Thus, a set of *free-type zero* if and only if $c_n(x) = c_n \in \mathbb{R}$ for all $n \geq 1$.

Let us suppose that $c_1 = 1$ (which holds for instance if $(P_n)_n$ are monic) such that J^0, viewed as an entire series, is invertible near 0 with inverse ρ with $\rho'(0) = 1$. Since D^0 acts on $x \mapsto (1-zx)^{-1}$ by multiplication by z similarly as D^1 does on $x \mapsto e^{zx}$, then we claim that a set of polynomials $(P_n)_n$ is of free type zero if and only if

$$\sum_{n \geq 0} P_n(x) z^n = A(z)(1 - \rho(z)x)^{-1}, \quad A(0) = \sum_{n \geq 0} a_n z_n. \tag{18}$$

As a result, $(P_n)_n$ is a set of monic OP w.r.t a probability measure μ with finite all order moments and of free-type zero w.r.t. a given J^0 if and only μ belongs to the free Meixner family with $\rho(J(z)) = z$. Recall from (10) that:

$$\rho(z) = \frac{1}{f(z)} = \frac{1}{K(u(z))}, \qquad A(z) = \frac{1}{f(z)u(z)} = \frac{f(z) - z}{f(z)} = 1 - z\rho(z)$$

when μ is a standard free Meixner distribution of parameters $a \in \mathbb{R}$, $b \geq -1$. This shows that the knowledge of J is enough to characterize $(P_n)_n$ and implies that $J(z) = R(G(1/z))$ for small enough z (recall that $R(u(z)) = z$). Using the relation

$$G(z) = \frac{1}{z + R(G(z))},$$

it follows that

$$J(z) = \frac{1}{G(1/z)} - \frac{1}{z} = F\left(\frac{1}{z}\right) - \frac{1}{z}.$$

One can derive an analog of formula (2.20) p. 599 in [24] which states that a set of polynomials of free-type zero satisfies

$$P_n(x) = \sum_{k\geq 1}(\gamma_k + \beta_k x)(J^0)^k P_n(x) = \sum_{k=1}^{n}(\gamma_k + \beta_k x)P_{n-k}(x) \tag{19}$$

where

$$\sum_{k\geq 1}\gamma_k z^k = 1 - \frac{A(0)}{A(z)}, \qquad \sum_{k\geq 1}\beta_k z^k = \frac{A(0)}{A(z)}\rho(z).$$

Recall from (18) and (17) that $A(0) = a_0 \neq 0$ and that $\beta_1 = \rho'(z) = 1$ since $\rho(0) = 0$. Now, as for the classical Meixner family, (19) entirely determines the free Meixner family. More precisely, we shall consider a set $(Q_n)_n$ of monic OP and use (19) to see if there exists a sequence $(d_n)_n$ such that $(P_n := d_nQ_n)_n$ is a set of free-type zero. We only have to compare (19) with the three-terms recurrence relation satisfied b P_n:

$$P_n(x) = (x - \alpha_{n-1})P_{n-1}(x) + \omega_{n-1}P_{n-2}(x) \quad n \geq 1. \tag{20}$$

We must assume that $P_0(x) \neq 0$ for otherwise, the type zero property implies that P_1 will be constant. Without loss of generality, take $d_0 = 1$. Comparing both coefficients of the n-th degree terms in (19) and (20), one gets $d_n = d_{n-1} = d_0 = 1$ for all $n \geq 1$. Doing the same for the $(n-1)$-th degree terms for $n \geq 2$, it follows that $\beta_2 + \gamma_1 = -\alpha_{n-1}$ and for $n = 1$ one has $-\alpha_0 = \gamma_1$. Similar computations for the $(n-2)$-th degree terms for $n \geq 3$ using the previous relations on $(\alpha_n)_n$ as well as (20) for Q_{n-1}, one sees that $\omega_{n-1} = \beta_2^2 + \beta_2\gamma_1 - \beta_3$. For $n = 2$, one easily gets $\omega_1 = \beta_2\gamma_1 - \gamma_2$. As a result, $(\alpha_n)_n, (\omega_n)_n$ are stationary sequences from the ranks $n_0 = 1, n_0 = 2$ respectively, a fact that characterizes the free Meixner family ([8], one needs to check when $\omega_n > 0$ for all $n \geq 1$).

Remark 6.1. *For the particular values $a = b = 0$ corresponding to the standard Wigner law, $u(z) = z, \rho(z) = z/(1+z^2)$ and (19) reduces to the three-terms recurrence relation of $(P_n = U_n(\cdot/2))_n$.*

7. Free Al-Salam and Chihara's characterization

The traditional way to set the free version of Al-Salam and Chihara's characterization of the classical Meixner family is to replace the product of exponential families by the product of Cauchy-Stieltjes functions. However, we think of it as the natural way since it will have a similar connection as in the classical case with the free Sheffer's characterization. Indeed, let μ be a standard free Meixner distribution with parameters $a \in \mathbb{R}, b \geq -1$ and $(P_n)_n$ be the set of monic OP with respect to μ. Recall that these polynomials satisfy the three-terms recurrence relation ([3]):

$$xP_n(x) = P_{n+1}(x) + aP_n(x) + (1+b)P_{n-1}(x), \quad n \geq 2$$

with $P_0(x) = 1, P_1(x) = x - a$, and that

$$P_n(x) = (1+b)^{n/2}U_n(x-a, \sqrt{1+b}) + aU_{n-1}(x-a, \sqrt{1+b}) + bU_{n-2}(x-a, \sqrt{1+b}) \tag{21}$$

where

$$U_n(x-a,\sqrt{1+b}) := (1+b)^{n/2}U_n\left(\frac{x-a}{\sqrt{1+b}}\right)$$

is the monic shifted Tchebycheff polynomial of the second kind. Using the elementary relation

$$\frac{x-y}{(z-x)(z-y)} = \frac{1}{z-x} - \frac{1}{z-y}, \quad x \neq y$$

then

$$\sum_{n\geq 0} \frac{P_n(x)-P_n(y)}{x-y} z^n = u(z)\psi(z,x)\psi(z,y)$$

where ψ is given by (9). Since and $P_0(x) = P_0(y) = 1$ and $z/u(z) = 1 + az + bz^2$, then

$$(1+az+bz^2)\sum_{n\geq 0} \frac{P_{n+1}(x)-P_{n+1}(y)}{x-y} z^n = \psi(z,x)\psi(z,y). \tag{22}$$

Set

$$R_n(x,y) = \frac{P_{n+1}(x)-P_{n+1}(y)}{x-y},$$

then $R_0(x) = 1$ and R_n is a linear combination of

$$\frac{x^{k+1}-y^{k+1}}{x-y} = x^k + x^{k-1}y + \cdots + xy^{k-1} + y^k = \sum_{r=0}^{k} y^r (D_x^0)^r(x^k) = \sum_{r=0}^{k} x^r (D_y^0)^r(y^k).$$

as one easily checks. Thus, (22) is equivalent to

(23)

$$\sum_{k=0}^{n} P_k(x)P_{n-k}(y) = R_n(x,y) + aR_{n-1}(x,y) + bR_{n-2}(x,y),\ R_{-1} = R_{-2} := 0.$$

One also easily checks that if a polynomial s_n in two variables (x,y) of total degree n satisfies the Cauchy equation $D_x^0 s_n = D_y^0 s_n$, then it is a linear combination of $x^k + x^{k-1}y + xy^{k-1} + y^k, 0 \leq k \leq n$. Therefore, the free Al-Salam and Chihara's characterization is stated as

Proposition 7.1. *Let $(r_n)_n, (q_n)_n$ be two sets of monic OP and let*

$$s_n(x,y) = \sum_{k=0}^{n} r_n(x)q_{n-k}(y)$$

be their convolution. Then, s_n has the expansion

$$s_n(x,y) = \sum_{k=0}^{n} b_k \frac{x^{k+1}-y^{k+1}}{x-y}$$

if and only if $r_n = q_n$ for all n and $(r_n)_n$ belongs to the free Meixner family.

Remark 7.1. *A remarkable difference with the classical setting is the fact that $(s_n(\cdot,y))_n$ no longer defines a set of OP. Let us for instance consider the case of monic Tchebycheff polynomials $(U_n)_n$, then one easily derives for all $n \geq 1$,*

$$xR_n(x,y) = R_{n+1}(x,y) + R_{n-1}(x,y) - U_{n+1}(y).$$

This also can be noticed from Al-Salam and Chihara paper [1], *where it is shown that, in order to get the desired orthogonality, it is not allowed, except for the classical Meixner family, to take $r_n = q_n, n \geq 1$.*

Proof: the sufficiency was already proved and the necessity can be proved using the free Sheffer's characterization. Indeed, let

$$J = \sum_{k\geq 1} c_k(x)(D^0)^k, \quad \tilde{J} = \sum_{k\geq 1} \tilde{c}_k(x)(D^0)^k,$$

denote the unique operators satisfying $J(r_n) = r_{n-1}, \tilde{J}(q_n) = q_{n-1}$ $(c_1 = \tilde{c}_1 = 1)$. Then, by symmetry of $(x, y) \mapsto s_n(x, y)$ in both variables, we first get that $J = \tilde{J}$. Using,

$$\begin{aligned} D_x^0(x^k + x^{k-1}y + \cdots + xy^{k-1} + y^k) &= x^{k-1} + x^{k-2}y + \cdots + xy^{k-2} + y^{k-1} \\ &= D_y^0(x^k + x^{k-1}y + \cdots + xy^{k-1} + y^k) \end{aligned}$$

for $k \geq 1$, one has $J^0 s_n(\cdot, y) = J^0 s_n(x, \cdot)$. Thus, $c_n(x) = c_n(y) = c_n, n \geq 1$ for some constants $(c_n)_n$. ■

Remarks. *1/ Integrating (23) with respect to μ, one obtains:*

$$Q_n(x) + aQ_{n-1}(x) + bQ_{n-2}(x) = P_n(x), \quad Q_n(x) := \int_{\mathbb{R}} \frac{P_{n+1}(x) - P_{n+1}(y)}{x - y} \mu(dy).$$

Q_{n-1} is the n-th numerator *polynomial of P_n* ([13]) *and Q_n is known to be equal to $U_n(x - a, \sqrt{1 + b})$* ([13] *p. 89*). *Thus, the above equality coincides with (21).*

2/ Let $(\mathscr{H}, < \cdot >)$ be a complex Hilbert space and let $H^{\otimes n}$ be the n-fold tensor product of H with $H^{\otimes 0} = \mathbb{C}\Omega$ for a distinguished vector $\Omega \in H$ called the vaccum. Let $\Gamma_0(H)$ denote the full Fock space associated with H defined by

$$\Gamma_0(H) := \bigoplus_{n\geq 0} H^{\otimes n}$$

and for $f \in H$, consider the annihilation and creation operators on $\Gamma_0(H)$ defined respectively by

$$\begin{aligned} a_f\Omega &= 0, \\ a_f(f_1 \otimes f_2 \cdots \otimes f_n) &= f \otimes f_1 \otimes f_2 \otimes \dots f_n \end{aligned}$$

and

$$\begin{aligned} a_f^\star\Omega &= f, \\ a_f^\star(f_1 \otimes f_2 \cdots \otimes f_n) &= < f, f_1 > f_2 \otimes \cdots \otimes f_n. \end{aligned}$$

These operators satisfy the free commutation *relation*

$$a_f a_f^\star =< f, f > \mathbf{1},$$

where $\mathbf{1}$ is the idendity operator ([7]). *Take $f \in H$ such that $< f, f >= 1$, then it was shown in* ([7]) *that for all n*

$$a_f^n + a_f^{n-1}a_f^\star + \cdots + a_f(a_f^\star)^{n-1} + (a_f^\star)^n = U_n(a_f + a_f^\star)$$

and it is known as the free Wick product. As a result, with the same notations of the above Proposition,

$$s_n(a_f, a_f^\star) = \sum_{k=0}^{n} b_k U_k(a_f + a_f^\star).$$

8. The q-deformed Meixner family, $|q| < 1$

Though both the classical and free settings provide different characterizations of their corresponding Meixner families, the q-deformed setting is rather complicated as we shall explain. Let us first recall that the q-deformed Meixner family was already defined in [4] and its elements are (up to a rescaling) the orthogonality measures of the Al-Salam and Chihara OP characterized by the following property: $(r_n)_n$ and $(q_n)_n$ belong to the Al-Salam and Chihara family of OP if and only if their convolution

$$s_n(x, y) = \sum_{k=0}^{n} r_n(x) q_{n-k}(y)$$

defines a set of OP in x for infinitely many y ([1]). Up to rescaling, their generating function writes ([10], p.14)

$$\psi_q(z, x) = \prod_{k=0}^{\infty} \frac{1 + aq^k z + bq^{2k} z^2}{1 + (a - (1-q)x)q^k z + (b + 1 - q)q^{2k} z^2},$$

for $a \in \mathbb{R}, b \geq -1 + \max(q, 0)$, which may be written as

$$\psi_q(z, x) = A_q(z) \prod_{k=0}^{\infty} \frac{1}{1 - (1-q)q^k H_k(q, z)x} \tag{24}$$

for some analytic function $H(q, \cdot)$ around zero with $H(q, 0) = 0, H'(q, 0) \neq 0$. One may express ψ_q through the q-exponential function e_q defined by

$$e_q(x) := \sum_{k \geq 0} \frac{x^k}{[k]_q!} = \prod_{k=0}^{\infty} \frac{1}{1 - (1-q)q^k x}$$

whenever it makes sense. First, the infinite product form prevents us to derive q-analogs of relations (5), (6), (10), (11) by performing direct computations of the first and the second moment. Second, One may use the characterization via q-exponential families given in [10]: a q-exponential family associated with a compactly-supported measure and parametrized by the mean has an at most quadratic variance if and only if the measure belongs to the q-Meixner family. Thus, regarding relations (6) and (11), one considers a quadratic variance function

$$V(z) = bz^2 + az + 1,\ b \geq -1 + \max(q, 0),\ a \in \mathbb{R}$$

of a q-Meixner distribution and defines f by:

$$D^q f(q, z) = \frac{1}{V(z)}, \quad f(q, 0) = 0,\ z \in V(0),$$

where D^q is the q-derivative defined by

$$D^q f(q, z) := \frac{f(q, z) - f(q, qz)}{(1-q)z}.$$

It is easy to see that

$$f(q,z) = (1-q)\sum_{k\geq 0}\frac{q^k z}{1+aq^k z + bq^{2k}z^2}.$$

Then, f is invertible and we think of its inverse as a q-analog of the free cumulants generating function for $|q|<1$. Unfortunately, the inversion procedure stands for a hard task.
Third, though e_q is uniquely determined by the requirement

$$D_q^x(e_q(zx)) = ze_q(zx),$$

it is not immediate that ψ_q is given by (24) since the function H may depend on both k and q.
Finally, the independence of H on the index k shows that

$$D_q^x[\psi_q(z,x)\psi_q(z,y)] \neq D_q^x[\psi_q(z,x)\psi_q(z,y)]$$

in general (except in the case $a=b=0$ corresponding to the q-Hermite polynomials, [7]). This may also be seen using the fact a polynomial s_n in two variables x,y of degree n satisfies $D_x^q s_n = D_y^q s_n$ if and only if s_n is a linear combination of the homogeneous polynomials

$$h_k(x,y) := x^k + [k]_q x^{k-1}y + \cdots + [k]_q xy^{k-1} + y^k, \quad 1\leq k\leq n.$$

Thus, s_n is far from being the convolution of two identical polynomials from the Al-Salam and Chihara polynomials. Moreover, the Wick product representation $H_q^k(a_f + a_f^\star)$ fails ([7], p.138), where H_k^q is the k-th q-Hermite polynomial and $a_f, a_f^\star$ are the annihilation and creation operators satisfying the q-commutation relation $a_f a_f^\star - q a_f^\star a_f = \mathbf{1}$ for a unit norm function f in an infinite Hilbert space.

Acknowledgments. Authors thank Professor F. Goetze for the fruitful work atmosphere at Bielefeld university. The research of the second author was supported by CRC 701 and he thanks Doctor G. Elsner and Rebecca Reischuk for their hospitatlity and their encouragements during the staying at Bielefeld university.

References

[1] *W. Al-Salam, T. S. Chihara.* Convolutions of orthogonal polynomials. *SIAM J. Funct. Anal.* **7**, 1976, 16-28.
[2] *N. Asai, I. Kubo, H. H. Kuo.* Multiplicative renormalization and generating function I. *Taiwan. J. Math. 7.* **1**. 2003, 89-101.
[3] *M. Anshelevich.* Free martingale polynomials. *J. Funct. Anal.* **201**, 2003, 228-261.
[4] *M. Anshelevich.* Appell polynomials and their relatives. *Int. Math. Res. Not.* **65**, 2004, 3469–3531.
[5] *F. Benaych-Georges.* Taylor expansions of R-transforms: application to supports and moments. *Indiana Univ. Math. J.* **55**. no. 2. 2006, 465-481.
[6] *P. Biane.* Processes with free increments. *Math. Z.* **227**, No 1, 1998, 143-174.
[7] *M. Bozejko, B. Kümmerer, R. Speicher.* q-Gaussian processes: non-commutative and classical aspects. *Comm. Math. Phys.* **185**, no. 1. 1997, 129-154.
[8] *M Bozejko, W. Bryc.* On a class of free Lévy laws related to a regression problem. *J. Funct. Anal.* **236**, no. 1, 2006, 59-77.
[9] *M. Bozejko, N. Demni.* Generating functions of Cauchy-Stieltjes type for orthogonal polynomials. *Submitted in Infinite. Dimen. Anal. Quantum Probab. Rel. Top.*
[10] *W. Bryc, M. E. H. Ismail.* Approximation operators, q-exponential and free exponential families. *Available on arXiv.*

[11] *W. Bryc, M. E. H. Ismail.* Cauchy-Stieltjes kernel families. *Available on arXiv.*

[12] *M. Capitaine, M. Casalis.* Asymptotic freeness by generalized moments for Gaussian and Wishart matrices. Application to beta random matrices. *Indiana Univ. Math. J.* **53**, no. 2, 2004, 397-431.

[13] *T. S. Chihara.* An Introduction to Orthogonal polynomials. *Gordon and Breach Science Publishers*, 1978.

[14] *B. Collins.* Product of random projections, Jacobi ensembles and universality problems arising from free probability. *P. T. R. F.*, **133**, 2005, no. 3, 315-344.

[15] *N. Demni.* Free Jacobi process. *To appear in J. Theo. Probab.*

[16] *M. E. H. Ismail.* Classical and Quantum orthogonal polynomials in one variable. *Cambridge University Press.* 2005.

[17] *I. Kubo.* Generating functions of exponential-type for orthogonal polynomials. *Inf. Dimens. Anal, Quantum Probab. Related Topics.* **7**, no 1, 2004, 155-159.

[18] *I. Kubo, H. H. Kuo, S. Namli.* The characterization of a class of probability measures by multiplicative renormalization. *Communications on Stochastic Analysis.* **1**, no. 3. 2007, 455–472.

[19] *R. G. Laha, E. Lukacs.* On a problem connected with quadratic regression. *Biometrika.* **47**, 1960, 335-343.

[20] *H. O. Lancaster.* Joint probability distributions in the Meixner classes. *J. Roy. Stat. Soc. Series B.* **37**, no. 3. 1975, 434-443.

[21] *J. Meixner.* Orthogonale polynomsysteme mit einer besonderern der erzeugenden funktion. *J. London. Math. Soc.* **9**, 1934, 6-13.

[22] *C. N. Morris.* Natural exponential families with quadratic variance function. *Ann. Stat.* **10**, no. 1. 1982, 65-80.

[23] *W. Schoutens.* Stochastic Processes and Orthogonal Polynomials. Lecture notes in Statistics. Springer, 2000.

[24] *I. M. Sheffer.* Some properties of polynomial set of type zero. *Duke Math. J.* **5**, 1939, 590-622.