CANONICAL TILTING MODULES OVER SHOD ALGEBRAS ARE REGULAR IN CODIMENSION ONE

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Abstract. We show that for a class of modules over shod algebras, including the canonical tilting modules, the closures of the corresponding orbits in module varieties are regular in codimension one.

Throughout the paper k is a fixed algebraically closed field. By $\mathbb{Z}$, $\mathbb{N}$, and $\mathbb{N}_+$, we denote the sets of integers, nonnegative integers, and positive integers, respectively. If $i, j \in \mathbb{Z}$, then $[i, j]$ denotes the set of all $l \in \mathbb{Z}$ such that $i \leq l \leq j$.

Introduction and the main result

Given a finite dimensional k-algebra Λ and an element $\mathbf{d}$ of the Grothendieck group $K_0(\Lambda)$ of the category of Λ-modules, one defines the variety $\operatorname{mod}_\Lambda(\mathbf{d})$ of Λ-modules of dimension vector $\mathbf{d}$. A product $\mathrm{GL}(\mathbf{d})$ of general linear groups acts on $\operatorname{mod}_\Lambda(\mathbf{d})$ in such a way that the $\mathrm{GL}(\mathbf{d})$-orbits correspond to the isomorphism classes of Λ-modules of dimension vector $\mathbf{d}$. The study of properties of the module varieties is an important direction of research in the representation theory of algebras (see for example [15, 17, 23, 25]). It is particularly interesting how properties of points of $\operatorname{mod}_\Lambda(\mathbf{d})$, viewed as Λ-modules, correspond to their geometric properties. For example, if $M \in \operatorname{mod}_\Lambda(\mathbf{d})$, then $\operatorname{End}_\Lambda(M)$ measures the size of its $\mathrm{GL}(\mathbf{d})$-orbit, $\operatorname{Ext}^1_\Lambda(M, M)$ measures the size of the tangent space to $\operatorname{mod}_\Lambda(\mathbf{d})$ at M [20], and, roughly speaking, $\operatorname{Ext}^2_\Lambda(M, M)$ says how far is M from being a regular point of $\operatorname{mod}_\Lambda(\mathbf{d})$ [16]. Using these observations a series of results describing properties of varieties of module over quasitilted algebras was obtained (see for example [3–5, 8–10]). Recall, that an algebra Λ is called quasitilted if $\operatorname{gl.dim} \Lambda \leq 2$ and either $\operatorname{pd}_\Lambda X \leq 1$ or $\operatorname{id}_\Lambda X \leq 1$ for each indecomposable Λ-module X.

The shod (small homological dimensions) algebras introduced by Coelho and Lanzilotta [14] are natural generalization of the quasitilted algebras. An algebra Λ is called shod if either $\operatorname{pd}_\Lambda X \leq 1$ or $\operatorname{id}_\Lambda X \leq 1$ for each indecomposable Λ-module X. The shod algebras share an

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important property with the quasitilted algebras, namely for each indecomposable module X over a shod algebra Λ, either $X \in \mathcal{L}_\Lambda$ or $X \in \mathcal{R}_\Lambda$, where $\mathcal{L}_\Lambda$ denotes the class of all indecomposable Λ-modules X such that $\operatorname{pd}_\Lambda Y \leq 1$ for each predecessor Y of X in the module category, and $\mathcal{R}_\Lambda$ is defined dually. The aim of the paper is to make a first step in order to generalize geometric results obtained for quasitilted algebras to shod algebras. Obviously, we may concentrate on strict shod algebras, where a shod algebra Λ is called strict if $\operatorname{gl.dim} \Lambda > 2$ (equivalently, $\operatorname{gl.dim} \Lambda = 3$).

Given a strict shod algebra Λ, Coelho, Happel and Unger [13] defined the canonical tilting module T, which is the direct sum of the indecomposable Ext-injective objects in $\mathcal{L}_\Lambda$ and the indecomposable projective Λ-modules which do not belong to $\mathcal{L}_\Lambda$. Such a module seems to be not far from being directing (in the sense of [19]). Consequently, the following theorem, which is the main result of the paper, should be viewed as an extension of [6, Main Theorem] to the case of shod algebras.

Main Theorem. *Let Λ be a strict shod algebra and T the canonical tilting module over Λ. If M is a direct summand of T^n for some $n \in \mathbb{N}$, then the closure of the orbit of M is regular in codimension one.*

We remark that in the situation of the theorem the closure of the orbit of M is an irreducible component of the corresponding module variety.

The paper is organized as follows. In the first section we recall basic informations about quivers and their representations. We also collect necessary facts about shod algebras there. In the next section, we define module varieties and present geometric tools used in the proof of Main Theorem. In the final section, we prove Main Theorem.

For a basic background on representation theory of algebras we refer to [2]. Basic algebraic geometry used in the article can be found in [22].

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1. Preliminaries on quivers and shod algebras

By a quiver Δ we mean a finite set Δ_0 of vertices and a finite set Δ_1 of arrows together with maps $s, t : \Delta_1 \to \Delta_0$ which assign to $\alpha \in \Delta_1$ the starting vertex s_α and the terminating vertex t_α. By a path of length $n \in \mathbb{N}_+$ we mean a sequence $\sigma = \alpha_1 \cdots \alpha_n$ with $\alpha_1, \ldots, \alpha_n \in \Delta_1$ such that $s_{\alpha_i} = t_{\alpha_{i+1}}$ for each $i \in [1, n-1]$. In the above situation we put $s_\sigma = s_{\alpha_n}$ and $t_\sigma = t_{\alpha_1}$. Additionally, for each $x \in \Delta_0$ we introduce the trivial path of length 0 starting and terminating at x and denoted by x. A path σ of positive length such that $s_\sigma = t_\sigma$ is called an oriented cycle. A full subquiver Δ' of Δ (i.e. a pair (Δ'_0, Δ'_1) such that $\Delta'_0 \subseteq \Delta_0$ and Δ'_1 consists of all $\alpha \in \Delta_1$ such that $s_\alpha, t_\alpha \in \Delta'_0$) is called convex if for every path $\alpha_1 \cdots \alpha_n$ in Δ with $\alpha_1, \ldots, \alpha_n \in \Delta_1$ and $s_{\alpha_n}, t_{\alpha_1} \in \Delta_0'$, $\alpha_i \in \Delta_1'$ for each $i \in [1, n]$.

For a quiver Δ we denote by $k\Delta$ the path algebra of Δ defined as follows. The elements of $k\Delta$ are the formal linear combinations of paths in Δ and for two paths σ_1 and σ_2 the product of σ_1 and σ_2 is either the composition $\sigma_1\sigma_2$ of paths, if $s_{\sigma_1} = t_{\sigma_2}$, or 0, otherwise. Fix $x, y \in \Delta_0$ and let $\rho = \lambda_1\sigma_1 + \cdots + \lambda_n\sigma_n$ for $n \in \mathbb{N}_+$, $\lambda_1, \ldots, \lambda_n \in k \setminus \{0\}$, and pairwise different paths $\sigma_1, \ldots, \sigma_n$. If $s_{\sigma_i} = x$ and $t_{\sigma_i} = y$ for each $i \in [1, n]$, then we write $s_\rho = x$ and $t_\rho = y$. If, additionally, the length of σ_i is bigger than 1 for each $i \in [1, n]$, then ρ is called a relation. A set $\mathfrak{R}$ of relations is called minimal if for each $\rho \in \mathfrak{R}$, ρ does not belong to the ideal $\langle \mathfrak{R} \setminus \{\rho\} \rangle$ generated by $\mathfrak{R} \setminus \{\rho\}$. If $\mathfrak{R}$ is a minimal set of relations such that there exists $n \in \mathbb{N}_+$ with the property $\sigma \in \langle \mathfrak{R} \rangle$ for each path σ of length at least n, then the pair $(\Delta, \mathfrak{R})$ is called a bound quiver. If $(\Delta, \mathfrak{R})$ is a bound quiver, then $k\Delta/\langle \mathfrak{R} \rangle$ is called the path algebra of $(\Delta, \mathfrak{R})$.

Gabriel proved (see for example [2, Corollaries I.6.10 and II.3.7]) that for each finite dimensional algebra Λ there exists a bound quiver $(\Delta, \mathfrak{R})$ such that the category $\operatorname{mod} \Lambda$ of Λ-modules is equivalent to the category of modules over the path algebra of $(\Delta, \mathfrak{R})$. In addition, Δ is uniquely (up to isomorphism) determined by Λ and we call it the Gabriel quiver of Λ. An algebra Λ is called triangular if its Gabriel quiver contains no oriented cycles. From now on we assume that all considered algebras are path algebras of bound quivers.

Let Δ be a quiver. A collection $M = (M_x, M_\alpha)_{x \in \Delta_0, \alpha \in \Delta_1}$ of finite dimensional vector spaces M_x, $x \in \Delta_0$, and linear maps $M_\alpha : M_{s_\alpha} \to M_{t_\alpha}$, $\alpha \in \Delta_1$, is called a representation of Δ. If M and N are representations of Δ, then the morphism space $\operatorname{Hom}_\Delta(M, N)$ consists of the collections $f = (f_x)_{x \in \Delta_0}$ of linear maps $f_x : M_x \to N_x$, $x \in \Delta_0$, such that $N_\alpha f_{s_\alpha} = f_{t_\alpha} M_\alpha$ for each $\alpha \in \Delta_1$. If $\sigma = \alpha_1 \cdots \alpha_n$, for $n \in \mathbb{N}_+$ and $\alpha_1, \ldots, \alpha_n \in \Delta_1$, is a path, then for a representation M of Δ we put $M_\sigma = M_{\alpha_1} \cdots M_{\alpha_n}$. Similarly, if $\rho = \lambda_1\sigma_1 + \cdots + \lambda_n\sigma_n$, for $n \in \mathbb{N}_+$, $\lambda_1, \ldots, \lambda_n \in k \setminus \{0\}$, and pairwise different paths σ_1, $\ldots$, σ_n in Δ, is a relation, then for a representation M of Δ we put $M_\rho = \lambda_1 M_{\sigma_1} + \cdots + \lambda_n M_{\sigma_n}$.

Let Λ be the path algebra of a bound quiver $(\Delta, \mathfrak{R})$. By $\operatorname{rep}(\Delta, \mathfrak{R})$ we denote the full subcategory of the category of representations of Δ consisting of the representations M such that $M_\rho = 0$ for each $\rho \in \mathfrak{R}$. The assignment which assigns to a Λ-module M the representation $(M_x, M_\alpha)_{x \in \Delta_0, \alpha \in \Delta_1}$, where $M_x = xM$ for $x \in \Delta_0$ and $M_\alpha(m) = \alpha m$ for $\alpha \in \Delta_1$ and $m \in M_{s_\alpha}$, induces an equivalence between $\operatorname{mod} \Lambda$ and $\operatorname{rep}(\Delta, \mathfrak{R})$ (see for example [2, Theorem III.1.6]). We will treat this equivalence as identification. With every $M \in \operatorname{rep}(\Delta, \mathfrak{R})$ (hence also with every Λ-module) we associate its dimension vector $\mathbf{dim}\, M \in \mathbb{N}^{\Delta_0}$

defined by $(\mathbf{dim}\, M)_x = \dim_k M_x$ for $x \in \Delta_0$. We call the elements of $\mathbb{N}^{\Delta_0}$ dimension vectors.

Let Λ be an algebra of finite global dimension with the Gabriel quiver Δ. We define the homomological bilinear form $\langle -, - \rangle_\Lambda : \mathbb{Z}^{\Delta_0} \times \mathbb{Z}^{\Delta_0} \to \mathbb{Z}$ and the Euler quadratic form $\chi_\Lambda : \mathbb{Z}^{\Delta_0} \to \mathbb{Z}$ by

$$\langle \mathbf{d}', \mathbf{d}'' \rangle_\Lambda = \sum_{n \in \mathbb{N}} \sum_{x,y \in \Delta_0} (-1)^n d'_x d''_y \dim_k \operatorname{Ext}^n_\Lambda(S(x), S(y))$$

and $\chi_\Lambda(\mathbf{d}) = \langle \mathbf{d}, \mathbf{d} \rangle_\Lambda$ for $\mathbf{d}', \mathbf{d}'', \mathbf{d} \in \mathbb{Z}^{\Delta_0}$, where for $x \in \Delta_0$ we put $P(x) = \Lambda x$ and $S(x) = P(x)/\operatorname{rad} P(x)$. It follows by easy induction that

$$\langle \mathbf{dim}\, M, \mathbf{dim}\, N \rangle_\Lambda = \sum_{n \in \mathbb{N}} (-1)^n \dim_k \operatorname{Ext}^n_\Lambda(M, N)$$

for all Λ-modules M and N. Note that $P(x)$, $x \in \Delta_0$, form a complete set of pairwise nonisomorphic indecomposable projective Λ-modules.

Let Λ be the path algebra of a bound quiver $(\Delta, \mathfrak{R})$. By a convex subalgebra of Λ we mean every algebra of the form $k\Delta'/\langle \mathfrak{R}' \rangle$, where Δ' is a convex subquiver of Δ and $\mathfrak{R}' = \mathfrak{R} \cap k\Delta'$. If Λ' is a convex subalgebra of Λ determined by a subquiver Δ' of Δ, then we may identify the Λ'-modules with the Λ-modules M such that $\operatorname{Hom}_\Lambda(P(x), M) = 0$ for each $x \in \Delta_0 \setminus \Delta'_0$. Moreover, $\operatorname{Ext}^n_{\Lambda'}(M, N) = \operatorname{Ext}^n_\Lambda(M, N)$ for all Λ'-modules M and N and $n \in \mathbb{N}$. Consequently, $\operatorname{gl.dim} \Lambda' < \infty$ provided $\operatorname{gl.dim} \Lambda < \infty$ and, if this is the case, $\langle \mathbf{d}', \mathbf{d}'' \rangle_{\Lambda'} = \langle \mathbf{d}', \mathbf{d}'' \rangle_\Lambda$ for all $\mathbf{d}', \mathbf{d}'' \in \mathbb{Z}^{\Delta'_0}$.

An algebra Λ is called shod if either $\operatorname{pd}_\Lambda X \leq 1$ or $\operatorname{id}_\Lambda X \leq 1$ for each indecomposable Λ-module X. This condition implies in particular that $\operatorname{gl.dim} \Lambda \leq 3$ [18, Proposition II.1.1]. A shod algebra Λ is called strict shod if $\operatorname{gl.dim} \Lambda = 3$.

Let Λ be a strict shod algebra with the Gabriel quiver Δ. By $\mathcal{L}_\Lambda$ we denote the class of indecomposable Λ-modules X such that $\operatorname{pd}_\Lambda Y \leq 1$ for each indecomposable Λ-module Y such that there exists a sequence $Y = X_0, \ldots, X_n = X$ such that $\operatorname{Hom}_\Lambda(X_{i-1}, X_i) \neq 0$ for each $i \in [1, n]$. Obviously, $\operatorname{pd}_\Lambda X \leq 1$ for each $X \in \mathcal{L}_\Lambda$. An indecomposable Λ-module X if called Ext-injective in $\mathcal{L}_\Lambda$ if $X \in \mathcal{L}_\Lambda$ and $\operatorname{Ext}^1_\Lambda(Y, X) = 0$ for each $Y \in \mathcal{L}_\Lambda$. By J_Λ we denote the direct sum of a complete set of pairwise nonisomorphic Ext-injective modules in $\mathcal{L}_\Lambda$. Let Δ' denote the full subquiver of Δ such that $x \in \Delta'_0$ if and only if $P(x) \in \mathcal{L}_\Lambda$. Then Δ' is a convex subquiver of Δ and we denote by Λ_λ the convex subalgebra of Λ determined Δ'. Dually, we define $\mathcal{R}_\Lambda$, Q_Λ and Λ_ρ, and we put $\mathcal{P}_\Lambda = \mathcal{R}_\Lambda \setminus \mathcal{L}_\Lambda$. We know from [14, Theorem] that if X is an indecomposable Λ-module, then either $X \in \mathcal{L}_\Lambda$ or $X \in \mathcal{R}_\Lambda$.

For a module M over an algebra Λ let $\operatorname{add} M$ denotes the full subcategory of $\operatorname{mod} \Lambda$ formed by all direct sums of direct summands of M. We define $\operatorname{add} \mathcal{C}$ for a class $\mathcal{C}$ of Λ-modules similarly. The following proposition collects the properties of the strict shod algebras which will be needed in our proofs.

Proposition 1.1. *Let Λ be a strict shod algebra.*

(1) $\operatorname{Hom}_\Lambda(Y,X)=0$ *and* $\operatorname{Ext}^n_\Lambda(X,Y)=0$ *for all* $n\in\mathbb{N}_+$, $X\in\mathcal{L}_\Lambda$, *and* $Y\in\mathcal{P}_\Lambda$.
(2) *If* $\operatorname{Hom}_\Lambda(J_\Lambda\oplus Q_\Lambda,X)\neq 0\neq \operatorname{Hom}_\Lambda(X,J_\Lambda\oplus Q_\Lambda)$ *for an indecomposable* Λ*-module* X, *then* $X\in\operatorname{add}(J_\Lambda\oplus Q_\Lambda)$.
(3) *If* $X\in\operatorname{add}(J_\Lambda\oplus Q_\Lambda)$, *then* $\operatorname{End}_\Lambda(X)\simeq k$ *and* $\operatorname{Ext}^n_\Lambda(X,X)=0$ *for each* $n\in\mathbb{N}_+$.
(4) $\operatorname{gl.dim}\Lambda_\lambda\leq 2$ *and* $\operatorname{gl.dim}\Lambda_\rho\leq 2$.
(5) *If* $M\in\operatorname{mod}\Lambda$, *then* $M\in\operatorname{add}\mathcal{L}_\Lambda$ *if and only if* $\operatorname{Ext}^1_\Lambda(M,J_\Lambda)=0$ *and* $\operatorname{Hom}_\Lambda(P,M)=0$ *for each indecomposable projective* Λ*-module* P *such that* $P\notin\mathcal{L}_\Lambda$.
(6) *If* $M\in\operatorname{mod}\Lambda$, *then* $M\in\operatorname{add}\mathcal{P}_\Lambda$ *if and only if* $\operatorname{Hom}_\Lambda(M,J_\Lambda)=0$.

Proof. (1) Obvious (one uses Auslander–Reiten formula, see for example [2, Theorem 2.13]).

(2) It follows from [1, Proposition 3.4] and its dual.

(3) The first part follows for example from [13, Proposition 2.4], the rest is clear.

(4) See [26, Lemma 3.1].

(5) The latter part of the condition says that every $X\in\mathcal{L}_\Lambda$ is a Λ_λ-module, which is obvious. Now the claim follows from the dual of [2, Theorem VI.2.5], since J_Λ is a cotilting $\mathcal{L}_\Lambda$-module and $\operatorname{add}\mathcal{L}_\Lambda$ coincides with the class of Λ-modules (equivalently, Λ_λ-modules) cogenerated by J_Λ [1, Theorem 4.2].

(6) Since $X\in\mathcal{P}_\Lambda$ if and only if $X\notin\mathcal{L}_\Lambda$, this follows from [1, Theorem 4.2]. □

2. Preliminaries on module varieties

Let Λ be the path algebra of a bound quiver $(\Delta,\mathfrak{R})$ and $\mathbf{d}\in\mathbb{N}^{\Delta_0}$. By the variety of Λ-modules of dimension vector $\mathbf{d}$ we mean the set $\operatorname{mod}_\Lambda(\mathbf{d})$ consisting of all $M\in\operatorname{rep}(\Delta,\mathfrak{R})$ such that $M_x=k^{d_x}$. By forgetting the spaces M_x, $x\in\Delta$, we identify $\operatorname{mod}_\Lambda(\mathbf{d})$ with a Zariski-closed subset of $\prod_{\alpha\in\Delta_1}\mathbb{M}_{d_{t_\alpha}\times d_{s_\alpha}}(k)$. Observe that for each Λ-module N of dimension vector $\mathbf{d}$ there exists $M\in\operatorname{mod}_\Lambda(\mathbf{d})$ such that $M\simeq N$. We will usually assume that all considered Λ-modules are points of module varieties.

Let Λ be an algebra, Δ its Gabriel quiver, and $\mathbf{d}\in\mathbb{N}^{\Delta_0}$. Then $\operatorname{GL}(\mathbf{d})=\prod_{x\in\Delta_0}\operatorname{GL}_{d_x}(k)$ acts on $\operatorname{mod}_\Lambda(\mathbf{d})$ by conjugation: $(gM)_\alpha=g_{t_\alpha}M_\alpha g_{s_\alpha}^{-1}$ for $g\in\operatorname{GL}(\mathbf{d})$, $M\in\operatorname{mod}_\Lambda(\mathbf{d})$, and $\alpha\in\Delta_1$. Observe that $M\simeq N$ for $M,N\in\operatorname{mod}_\Lambda(\mathbf{d})$ if and only if $\mathcal{O}(M)=\mathcal{O}(N)$, where for $M\in\operatorname{mod}_\Lambda(\mathbf{d})$ we denote by $\mathcal{O}(M)$ its $\operatorname{GL}(\mathbf{d})$-orbit. One easily calculates that $\dim \mathcal{O}(M) = \dim \mathrm{GL}(\mathbf{d}) - \dim_k \mathrm{End}_\Lambda(M)$ for each $M \in \mathrm{mod}_\Lambda(\mathbf{d})$ [21, 2.2]. Moreover, we have a canonical injection

$$T_M \mathrm{mod}_\Lambda(\mathbf{d})/T_M\mathcal{O}(M) \hookrightarrow \mathrm{Ext}^1_\Lambda(M, M) \tag{2.1}$$

where for a point x of a variety $\mathcal{V}$ we denote by $T_x\mathcal{V}$ the tangent space to $\mathcal{V}$ at x [20, II.2.7, Satz 4]. Consequently,

$$\dim_k T_M \mathrm{mod}_\Lambda(\mathbf{d}) \leq \dim \mathrm{GL}(\mathbf{d}) - \dim_k \mathrm{End}_\Lambda(M) + \dim_k \mathrm{Ext}^1_\Lambda(M, M). \tag{2.2}$$

Let $M, N \in \mathrm{mod}_\Lambda(\mathbf{d})$ for $\mathbf{d} \in \mathbb{N}^{\Delta_0}$. We call N a degeneration of M and write $M \leq_{\deg} N$ if $N \in \overline{\mathcal{O}(M)}$. If $M \leq_{\deg} N$ and $M \not\simeq N$, then we write $M <_{\deg} N$ and call N a proper degeneration of M. A degeneration $M \leq_{\deg} N$ is called minimal if $M <_{\deg} N$ and there is no $L \in \mathrm{mod}_\Lambda(\mathbf{d})$ such that $M <_{\deg} L <_{\deg} N$. The formula for the dimension of an orbit implies that $\dim_k \mathrm{End}_\Lambda(M) < \dim_k \mathrm{End}_\Lambda(N)$ provided $M <_{\deg} N$. More generally, if $M \leq_{\deg} N$, then $\dim_k \mathrm{Hom}_\Lambda(M, X) \leq \dim_k \mathrm{Hom}_\Lambda(N, X)$ and $\dim_k \mathrm{Hom}_\Lambda(X, M) \leq \dim_k \mathrm{Hom}_\Lambda(X, N)$ for all Λ-modules X [24, Proposition 2.1]. On the other hand, if there exists an exact sequence $0 \to N_1 \to M \to N_2 \to 0$ such that $N_1 \oplus N_2 \simeq N$, then $M \leq_{\deg} N$ (see for example [12, Lemma 1.1]).

Let $\mathcal{V}$ be a variety and $x \in \mathcal{V}$. We say that x is a regular point of $\mathcal{V}$ if $\dim_k T_x\mathcal{V} = \dim_x \mathcal{V}$ (equivalently, $\dim_k T_x\mathcal{V} \leq \dim_x \mathcal{V}$), where $\dim_x \mathcal{V}$ denotes the maximum of the dimensions of the irreducible components of $\mathcal{V}$ passing through x. In particular, if $\mathcal{V}$ is irreducible, then x is a regular point of $\mathcal{V}$ if and only if $\dim_k T_x\mathcal{V} = \dim \mathcal{V}$ (equivalently, $\dim_k T_x\mathcal{V} \leq \dim \mathcal{V}$). We say that $\mathcal{V}$ is regular in codimension one if the codimension of the complement of the set of regular points of $\mathcal{V}$ is at least 2.

Now we show a special case of a more general fact proved by Geiß [16]. We start with the following lemma, where for a dimension vector $\mathbf{d}$ we put $a_\Lambda(\mathbf{d}) = \dim \mathrm{GL}(\mathbf{d}) - \chi_\Lambda(\mathbf{d})$.

Lemma 2.3. *Let Λ be an algebra of finite global dimension. If N is a Λ-module of dimension vector $\mathbf{d}$ such that $\mathrm{Ext}^n_\Lambda(N, N) = 0$ for each $n \in \mathbb{N}_+$ with $n \geq 2$, then $\dim T_N \mathrm{mod}_\Lambda(\mathbf{d}) \leq a_\Lambda(\mathbf{d})$.*

Proof. It follows immediately from (2.2). □

Observe that if Λ is an algebra of finite global dimension and M a Λ-module such that $\mathrm{Ext}^n_\Lambda(M, M) = 0$ for each $n \in \mathbb{N}_+$, then $\dim \mathcal{O}(M) = a_\Lambda(\mathbf{d})$. Together with the above lemma this immediately implies the following.

Corollary 2.4. *Let Λ be an algebra of finite global dimension, and M and N Λ-modules such that $\mathrm{Ext}^n_\Lambda(M, M) = 0$ for each $n \in \mathbb{N}_+$ and $\mathrm{Ext}^n_\Lambda(N, N) = 0$ for each $n \in \mathbb{N}_+$ with $n \geq 2$. If $M \leq_{\deg} N$, then N is a regular point of $\overline{\mathcal{O}(M)}$.* □

Let Λ be an algebra and $\mathbf{d}'$, $\mathbf{d}''$ dimension vectors. For $d \in \mathbb{N}$ we denote by $\mathcal{E}_d^{\mathbf{d}',\mathbf{d}''}$ the set of all $(U, V) \in \operatorname{mod}_\Lambda(\mathbf{d}') \times \operatorname{mod}_\Lambda(\mathbf{d}'')$ such that $\operatorname{Hom}_\Lambda(V, U) = 0$ and $\dim_k \operatorname{Ext}^1_\Lambda(V, U) = d$. By extending the arguments from [7, (2.7)] in an obvious way one shows that if $\operatorname{gl.dim} \Lambda < \infty$ and $(U, V) \in \mathcal{E}_d^{\mathbf{d}',\mathbf{d}''}$, then

$$(2.5) \qquad \dim_k T_{U,V} \mathcal{E}_d^{\mathbf{d}',\mathbf{d}''} \leq a_\Lambda(\mathbf{d}') + a_\Lambda(\mathbf{d}'') - \dim_k \operatorname{Ext}^2_\Lambda(V, U)$$

provided the following conditions are satisfied: $\operatorname{Ext}^n_\Lambda(U, U) = 0 = \operatorname{Ext}^n_\Lambda(V, V)$ for each $n \in \mathbb{N}_+$ with $n \geq 2$ and there exists an exact sequence $0 \to U \to M \to V \to 0$ such that either $\operatorname{Ext}^2_\Lambda(V, M) = 0$ or $\operatorname{Ext}^2_\Lambda(M, U) = 0$.

For the rest of the section we assume that Λ is a strict shod algebra, $\mathcal{L} = \mathcal{L}_\Lambda$, and $\mathcal{P} = \mathcal{P}_\Lambda$. For a dimension vector $\mathbf{d}$ we denote by $\mathcal{L}(\mathbf{d})$ the set of all $M \in \operatorname{mod}_\Lambda(\mathbf{d})$ such that $M \in \operatorname{add} \mathcal{L}$. We define $\mathcal{P}(\mathbf{d})$ similarly. More generally, if $\mathbf{d}'$ and $\mathbf{d}''$ are dimensions vectors, then $\mathcal{L}(\mathbf{d}') \oplus \mathcal{P}(\mathbf{d}'')$ denotes the set of all $M \in \operatorname{mod}_\Lambda(\mathbf{d}' + \mathbf{d}'')$ such that $M \simeq U \oplus V$ for some $U \in \mathcal{L}(\mathbf{d}')$ and $V \in \mathcal{P}(\mathbf{d}'')$.

Let $\mathbf{d}$ be a dimension vector. It follows from Proposition 1.1 (5) and (6) that $\mathcal{L}(\mathbf{d})$ and $\mathcal{P}(\mathbf{d})$ are open subsets of $\operatorname{mod}_\Lambda(\mathbf{d})$. Moreover, if $\mathcal{L}(\mathbf{d}) \neq \varnothing$, then it is irreducible of dimension $a_\Lambda(\mathbf{d})$ [3, Proposition 3.1] (for the last statement one uses that $a_\Lambda(\mathbf{d}) = a_{\Lambda_\lambda}(\mathbf{d})$ and Proposition 1.1 (4)). This implies that $\dim_k T_M \operatorname{mod}_\Lambda(\mathbf{d}) = a_\Lambda(\mathbf{d})$ for each $M \in \mathcal{L}(\mathbf{d})$, according to Lemma 2.3. The analogous statement holds for $\mathcal{P}(\mathbf{d})$. Finally, if $\mathbf{d}'$ and $\mathbf{d}''$ are dimension vectors such that $\mathcal{L}(\mathbf{d}')$ and $\mathcal{P}(\mathbf{d}'')$ are nonempty, then $\mathcal{L}(\mathbf{d}') \oplus \mathcal{P}(\mathbf{d}'')$ is an irreducible constructible subset of $\operatorname{mod}_\Lambda(\mathbf{d}' + \mathbf{d}'')$.

3. Proof of the main result

This section is devoted for proving the main result of the paper. The line of the proof mainly follows that of the proof of [6, Main Theorem], however some arguments have to be adapted to the setting of strict shod algebras.

Throughout this section Λ is a fixed strict shod algebra, T the canonical tilting Λ-module, $\mathcal{L} = \mathcal{L}_\Lambda$, $J = J_\Lambda$, $\mathcal{P} = \mathcal{P}_\Lambda$, and $Q = Q_\Lambda$. Finally, we fix a Λ-module $M \in \operatorname{add} T$. Observe that $M \in \operatorname{add}(J \oplus Q)$. We write $M = L \oplus R$ with $L \in \operatorname{add} \mathcal{L}$ and $R \in \operatorname{add} \mathcal{R}$, and put $\mathbf{d} = \mathbf{dim}\, M$.

The main aim of this section is to prove the following extension of Main Theorem.

Theorem 3.1. *$\overline{\mathcal{O}(M)}$ is regular in codimension one.*

We start with the following.

Lemma 3.2. *If N is a minimal degeneration of M, then there exists an exact sequence $0 \to U \to M \to V \to 0$ such that $N \simeq U \oplus V$.*

Proof. If there is not such a sequence, then the minimality of the degeneration $M <_{\deg} N$ and [27, Theorem 4] imply that there exists an indecomposable direct summand X of N such that $M' <_{\deg} X$ for a direct summand M' of M. Then

$$\dim_k \operatorname{Hom}_\Lambda(M', X) \geq \dim_k \operatorname{Hom}_\Lambda(M', M') > 0$$

and

$$\dim_k \operatorname{Hom}_\Lambda(X, M') \geq \dim_k \operatorname{Hom}_\Lambda(M', M') > 0,$$

hence $X \in \operatorname{add}(J \oplus Q)$ by Proposition 1.1 (2). Thus $\operatorname{End}_\Lambda(X) \simeq k$ according to Proposition 1.1 (3), and $\dim_k \operatorname{End}_\Lambda(M') = 0$, a contradiction. □

Proposition 3.3. *Let $U \in \operatorname{add} \mathcal{L}$ and $V \in \operatorname{add} \mathcal{P}$ be such that $U \oplus V$ is a minimal degeneration of M.*

1. *If $\mathbf{dim}\, U \neq \mathbf{dim}\, L$, then there exists an exact sequence $0 \to U \to M \to V \to 0$.*
2. *If $\mathbf{dim}\, U = \mathbf{dim}\, L$, then either $U \simeq L$ or $V \simeq R$.*

Proof. (1) Assume that $\mathbf{dim}\, U \neq \mathbf{dim}\, L$. According to the above lemma there exists an exact sequence $\sigma : 0 \to U' \to M \to V' \to 0$ such that $U' \oplus V' \simeq U \oplus V$. Write $U' = L_1 \oplus R_1$ and $V' = L_2 \oplus R_2$, where $U \simeq L_1 \oplus L_2$ and $V \simeq R_1 \oplus R_2$. We first show that we may assume that either $R_1 = 0$ or $L_2 = 0$.

Let $p : U' \to R_1$ be the canonical projection and $\sigma' : 0 \to R_1 \to M' \to V' \to 0$ be the push-out exact sequence $p \circ \sigma$. If σ' splits, then we easily get an exact sequence $0 \to L_1 \to M \to V' \oplus R_1 \to 0$, thus the claim follows in this case. On the other hand, if σ' does not split, then $M \simeq M' \oplus L_1$ by the minimality of the degeneration $M <_{\deg} U \oplus V$. Moreover, the pull-back sequence exact $\sigma' \circ i$, where $i : L_2 \to V'$ is the canonical injection, splits by Proposition 1.1 (1) and we get an exact sequence $0 \to L_2 \oplus R_1 \to M' \to R_2 \to 0$, which in turn gives rise to an exact sequence $0 \to U' \oplus L_2 \to M \to R_2 \to 0$, as desired.

Without loss of generality we assume that $L_2 = 0$, i.e. $U \simeq L_1$. If σ' is as before, then $M' \in \operatorname{add} \mathcal{P}$ (one may use for example Proposition 1.1 (6)). Moreover, we have a short exact sequence $0 \to U \to M \to M' \to 0$. This sequence does not split, since $\mathbf{dim}\, U \neq \mathbf{dim}\, L$. Consequently, σ' splits, i.e. $M' \simeq V$, and we get an exact sequence $0 \to U \to M \to V \to 0$, thus the claim follows.

(2) Assume that $\mathbf{dim}\, U = \mathbf{dim}\, L$. Consequently, $\mathbf{dim}\, V = \mathbf{dim}\, R$. Observe that both U and L are modules of projective dimension at most 1 and $\operatorname{Ext}^1_\Lambda(L, L) = 0$. Together with (2.1) this implies that $\overline{\mathcal{O}(L)}$ is an irreducible component of $\operatorname{mod}_\Lambda(\mathbf{dim}\, L)$, hence $L \leq_{\deg} U$, since $\mathcal{L}(\mathbf{dim}\, L)$ is irreducible. Analogously, $R \leq_{\deg} V$. If the both degenerations are proper, then we have the following sequence of proper

degenerations

$$M = L \oplus R <_{\deg} U \oplus R <_{\deg} U \oplus V$$

which contradicts the minimality of the degeneration $M <_{\deg} U \oplus V$. □

Proof of Theorem 3.1. Let $\mathcal{X}$ be an irreducible component of $\overline{\mathcal{O}(M)} \setminus \mathcal{O}(M)$. Our aim is to show that there exists an open subset $\mathcal{U}$ of $\mathcal{X}$ such that all points of $\mathcal{U}$ are regular points of $\overline{\mathcal{O}(M)}$. Observe that there exist $\mathbf{d}', \mathbf{d}'' \in \mathbb{N}^{\Delta_0}$ such that $(\mathcal{L}(\mathbf{d}') \oplus \mathcal{P}(\mathbf{d}'')) \cap \mathcal{X}$ contains an open subset $\mathcal{U}'$ of $\mathcal{X}$. Put

$$d_0 = \min\{\dim_k \operatorname{End}_\Lambda(N) \mid N \in \mathcal{X}\}$$

and

$$d_1 = \min\{\dim_k \operatorname{Ext}^1_\Lambda(N, N) \mid N \in \mathcal{X}\}.$$

Let $\mathcal{U}$ be the set of all $N \in \mathcal{U}'$ such that the dimensions of $\operatorname{End}_\Lambda(N)$ and $\operatorname{Ext}^1_\Lambda(N, N)$ equal d_0 and d_1, respectively, and N does not belong to an irreducible component of $\overline{\mathcal{O}(M)} \setminus \mathcal{O}(M)$ different from $\mathcal{X}$. Obviously, $\mathcal{U}$ is a nonempty open subset of $\mathcal{X}$ consisting of minimal degenerations of M. Fix $N \in \mathcal{U}$, and $U \in \operatorname{add} \mathcal{L}$ and $V \in \operatorname{add} \mathcal{R}$ such that $N \simeq U \oplus V$.

First assume that $\mathbf{d}' = \mathbf{dim}\, L$. According to Proposition 3.3 (2) we may assume without loss of generality that $U \simeq L$. Moreover, since $\operatorname{id}_\Lambda V \leq 1$, V is a regular point of $\overline{\mathcal{O}(R)}$ by Corollary 2.4. Finally,

$$\dim_k \operatorname{Hom}_\Lambda(L, R) = \langle \mathbf{d}', \mathbf{d}'' \rangle = \dim_k \operatorname{Hom}_\Lambda(L, V)$$

and

$$\dim_k \operatorname{Hom}_\Lambda(R, L) = 0 = \dim_k \operatorname{Hom}_\Lambda(V, L)$$

hence the claim follows from [11, Theorem 2].

Now assume that $\mathbf{d}' \neq \mathbf{dim}\, L$. According to Proposition 3.3 (1) there exists an exact sequence $\xi : 0 \to U \to M \to V \to 0$, which in particular implies that $\operatorname{pd}_\Lambda V \leq 2$. Put $\mathcal{V} = \operatorname{mod}_\Lambda(\mathbf{d}') \times \operatorname{mod}_\Lambda(\mathbf{d}'')$. We identify $\mathcal{V}$ with a subset of $\operatorname{mod}_\Lambda(\mathbf{d})$ in an obvious way. Without loss of generality we may assume that $N \in \mathcal{V}$, i.e. $N = U \oplus V$. Since $\dim_k T_N \mathcal{V} = a_\Lambda(\mathbf{d}') + a_\Lambda(\mathbf{d}'')$, it is sufficient to show that

$$(*) \qquad \dim_k T_N(\mathcal{V} \cap \overline{\mathcal{O}(M)}) \leq a_\Lambda(\mathbf{d}') + a_\Lambda(\mathbf{d}'') - \dim_k \operatorname{Ext}^2_\Lambda(V, U).$$

Indeed, this will imply that

$$\dim_k T_N \overline{\mathcal{O}(M)} \leq \dim_k T_N \operatorname{mod}_\Lambda(\mathbf{d}) - \dim_k \operatorname{Ext}^2(V, U),$$

hence we may apply (2.2) and use that $\operatorname{Ext}^2_\Lambda(V, U) = \operatorname{Ext}^2_\Lambda(N, N)$ and $\operatorname{pd}_\Lambda N \leq 2$.

We prove $(*)$. Let $\mathcal{U}_0$ be the intersection of $\mathcal{U}$ with $\mathcal{V}$. Observe that $\mathcal{U}_0$ is an open subset of $\mathcal{V} \cap \overline{\mathcal{O}(M)}$. Moreover, $\mathcal{U}_0 \subset \mathcal{E}_d^{\mathbf{d}', \mathbf{d}''}$, where

$$d = d_1 - d_0 + \chi_\Lambda(\mathbf{d}') + \chi_\Lambda(\mathbf{d}'') + \langle \mathbf{d}', \mathbf{d}'' \rangle_\Lambda.$$

Consequently, the claim follows from (2.5). □

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