

RADIAL DUNKL PROCESSES : EXISTENCE AND UNIQUENESS, HITTING TIME AND RANDOM MATRICES

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ABSTRACT. This paper is essentially concerned with the study of the radial Dunkl process associated with a reduced root system. It is shown that this diffusion is the unique strong solution for all $t \geq 0$ of a stochastic differential equation (SDE) with singular drift. Then we prove, via stochastic calculus, that the first hitting time T_0 of the positive Weyl chamber is finite for small values of the multiplicity function. Once we did, we compute the tail distribution of T_0 and we highlight how radial Dunkl processes generalize the eigenvalues of well known selfadjoint matrix-valued processes which is twofold. On the one hand, existence and uniqueness results for the eigenvalue process of some matrix diffusions are improved. On the other hand, the tail distributions of the first collision time and the first time the smallest eigenvalue hits zero are recovered. Next, we use determinantal representations of multivariate special functions to recover results by Grabiner on Brownian motions in Weyl chambers, to write down the generalized Bessel function for the D -type root system and to deal with the particular case of the Brownian motion reflected on the boundary of the Weyl chamber. Finally, we investigate the trigonometric analog which involves the non-reduced root system of type BC and is related to the eigenvalues processes of the real and complex matrix Jacobi processes. Our existence and uniqueness result is extended to this setting and the semi-group density is given.

1. PRELIMINARIES

We begin by pointing out some facts on root systems and radial Dunkl processes. We refer to [36] for the Dunkl theory, to both [7] and [26] for a background on root systems and [11] and references therein for facts on radial Dunkl processes. Let $(V, \langle \cdot, \cdot \rangle)$ be a finite real Euclidean space of dimension m . A *reduced* root system R is a finite set of non zero vectors in V such that :

$$1 \quad R \cap \mathbb{R}\alpha = \{\alpha, -\alpha\} \text{ for all } \alpha \in R,$$

$$2 \quad \sigma_\alpha(R) = R,$$

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where σ_α is the reflection with respect to (w.r.t.) the hyperplane H_α orthogonal to α :

$$\sigma_\alpha(x) = x - 2 \frac{\langle \alpha, x \rangle}{\langle \alpha, \alpha \rangle} \alpha, \quad x \in V.$$

A simple system Δ is a basis of $\text{span}(R)$ which induces a total ordering in R . A root α is positive if it is a positive linear combination of elements of Δ . The set of positive roots is called a positive subsystem and is denoted by R_+ . The (finite) reflection group W is the group generated by all the reflections σ_α for $\alpha \in R$. Given a root system R with positive and simple systems R_+, Δ , define the *positive Weyl chamber* C by:

$$C := \{x \in V, \langle \alpha, x \rangle > 0 \forall \alpha \in R_+\} = \{x \in V, \langle \alpha, x \rangle > 0 \forall \alpha \in \Delta\}$$

and $\bar{C}, \partial C$ its closure and boundary respectively. One of the most important properties is that the convex cone \bar{C} is a fundamental domain, that is, each $\lambda \in V$ is conjugate to one and only one $\mu \in \bar{C}$.

The radial Dunkl process X is defined as the \bar{C} -valued continuous paths Markov process whose generator is given by :

$$\mathcal{L}_k u(x) = \frac{1}{2} \Delta u(x) + \sum_{\alpha \in R_+} k(\alpha) \frac{\langle \alpha, \nabla u(x) \rangle}{\langle \alpha, x \rangle}$$

where $u \in C_c^2(\bar{C})$ satisfying the boundary conditions $\nabla u(x) \cdot \alpha = 0$ for all $x \in H_\alpha, \alpha \in R_+$, and $k(\alpha) \geq 0$ is a multiplicity function (a W -invariant function). The semi-group density of X is given by :

$$(1) \quad p_t^k(x, y) = \frac{1}{c_k t^{\gamma+m/2}} e^{-(|x|^2+|y|^2)/2t} D_k^W \left(\frac{x}{\sqrt{t}}, \frac{y}{\sqrt{t}} \right) \prod_{\alpha \in R_+} \langle \alpha, y \rangle^{2k(\alpha)}$$

for $x, y \in C$, where $\gamma = \sum_{\alpha \in R_+} k(\alpha)$,

$$D_k^W(x, y) := \sum_{w \in W} D_k(x, wy)$$

where D_k denotes the Dunkl kernel and c_k is given by the McDonald-Mehta integral ([36]). Indeed, as $D_k(0, y) = 1$ ([36]), one gets

$$t^{\gamma+m/2} c_k = |W| \int_C e^{-|y|^2/2t} \prod_{\alpha \in R_+} \langle \alpha, y \rangle^{2k(\alpha)} dy = \int_V e^{-|y|^2/2t} \prod_{\alpha \in R_+} |\langle \alpha, y \rangle|^{2k(\alpha)} dy$$

since $V = \cup_{w \in W} w\bar{C}$. $D_k^W(x, y)$ is known as *the generalized Bessel function* (up to the constant $|W|$) since it reduces to a Bessel function in the so-called *rank-one* case. The latter corresponds to $V = \mathbb{R}, R = B_1 = \{\pm 1\}$. Hence $k(\alpha) := k_1 \geq 0$ and X is a *Bessel process* ([35]) of *index* $\nu = k_1 - 1/2$. When $k_1 > 0$ and $X_0 = x \geq 0$, it is the unique strong solution of:

$$dX_t = dB_t + \frac{k_1}{X_t} dt, \quad t \geq 0,$$

where B is a standard Brwonian motion (hereafter BM). A multivariate well known example is given by the A_{m-1} -type root system defined by:

$$A_{m-1} = \{\pm(e_i - e_j), 1 \leq i < j \leq m\},$$

with positive and simple systems given by :

$$R_+ = \{e_i - e_j, 1 \leq i < j \leq m\}, \quad \Delta = \{e_i - e_{i+1}, 1 \leq i \leq m\},$$

where $(e_i)_{1 \leq i \leq m}$ is the standard basis of \mathbb{R}^m . In this case, $V = \mathbb{R}^m$, the span of R is the hyperplane of \mathbb{R}^m consisting of vectors whose coordinates sum to zero and $C = \{x \in \mathbb{R}^m, x_1 > \dots > x_m\}$. Besides, there is only one orbit and $k(\alpha) := k_1 \geq 0$. Thus, X satisfies :

$$(2) \quad dX_t^i = d\nu_t^i + k_1 \sum_{j \neq i} \frac{dt}{X_t^i - X_t^j} \quad 1 \leq i \leq m, \quad t < \tau$$

with $X_0^1 > \dots > X_0^m$, where $(\nu^i)_i$ are independent BMs and τ is *the first collision time* . For strictly positive k_1 , this process was deeply studied ([8], [9], [10]): it evolves as m -interacting particles on the real line with electrostatic repulsions proportional to the inverse of the distance separating them. It was shown in [8] that (2) has a unique strong solution for all $t \geq 0$ and $X_0^1 > \dots > X_0^m$ provided that $k_1 > 0$. When reading the proof in [8], one hopes to extend this result for any root system since the ingredients used there are not typical for the A_{m-1} -type. This was the original motivation of this work. Our first result claims that

$$dX_t = dB_t - \nabla \Phi(X_t)dt, \quad X_0 \in \overline{C}$$

where $\Phi(x) = -\sum_{\alpha \in R_+} k(\alpha) \ln(\langle \alpha, x \rangle)$, $k > 0$, has a unique strong solution for all $t \geq 0$. Meanwhile and independently, Chybiryakov and Schapira provide two other proofs: both authors used well posed martingale problems associated respectively with the \mathbb{R}^m -valued Dunkl and the radial Heckman-Opdam processes as well as geometric arguments ([11], [38]). However, Chybiryakov's result does not cover all the root systems since the author assumed that the simple system is a basis of V which is not valid for $R = A_{m-1}$. Next, we are mainly interested in the first hitting time T_0 of ∂C . We prove, via stochastic calculus, of the fact that $T_0 < \infty$ when $k(\alpha) < 1/2$ for at least one $\alpha \in R_+$. Though this was proved in [11] using local martingales, our proof provide more information in the sense that, for such an α , we prove that $\langle \alpha, X \rangle$ hits zero almost surely (a.s.). At this level, other proofs exist for the above results. To our best knowledge, the content of the remainder of the paper is new and is outlined as follows:

- Using the absolute-continuity relations derived in [11], we derive the tail distribution of T_0 when the process starts from $X_0 = x \in C$. A W -invariant analytic x -dependent integral, whose value at 0 is given by a Selberg integral, is involved. It will be shown that this integral, viewed as

a function in the variable x , is an eigenfunction of

$$\mathcal{L}_k - E_1 := \mathcal{L}_k - \sum_{i=1}^m x_i \partial_i$$

associated with the eigenvalue $m + |R_+|$. For the irreducible root systems of types A, B, C, D , this eigenfunction is identified with a multivariate hypergeometric series and therefore the tail distribution is explicitly written. A surprising fact is that the above eigenoperator can be expressed in terms of a Schrödinger operator \mathcal{H} and its minimal eigenvalue E_{min} (minimal energy) ([36]):

$$-\mathcal{L}_k + E_1 = -e^{|x|^2/4}(\mathcal{H} - E_{min})e^{-|x|^2/4}.$$

- We specialize our formulae to the particular case of the so-called BM reflected on ∂C . This process was defined in [11] as the projection of the m -dimensional BM and thus corresponds to a radial Dunkl process with zero multiplicity function. The tail distribution is written as a determinant for the A, B, C and D -types and this follows from the determinantal representations of some hypergeometric series of two arguments.
- We explain how $(X_t)_{t \geq 0}$ specializes for some values of k to eigenvalues processes of self-adjoint matrix processes. More precisely, the radial Dunkl process of type A generalizes the eigenvalues processes of both the symmetric and Hermitian Brownian matrices ([21], [29]). When $R = B_m$, X is related to the process of eigenvalues of both the Wishart and Laguerre processes ([5], [30], [13]). As a matter of fact, our uniqueness and existence result applies and even improves well known results from matrix theory. Besides, We recover the tail distribution of the first collision time of two particles when $R = A_{m-1}$ and the first time the smallest eigenvalue hits zero when $R = B_m$.
- When the multiplicity function $k \equiv 1$, X fits the Brownian motion in the Weyl chamber. In that case, the determinantal representation of multivariate hypergeometric series allow to recover Grabiner's results ([22]) and to write down the generalized Bessel function for the root system of type D .
- In the last part, we investigate the trigonometric version of X which generalizes the eigenvalues processes of both the real and complex matrix Jacobi processes ([16]). This case involves the *non reduced affine* root system of type BC and the underlying process lives in the so-called *principal Weyl alcove* ([26]). It is a convex and bounded domain and the process may be interpreted as particles in an interval. We extend our existence and uniqueness results to this setting. Then, we similarly prove that $T_0 < \infty$ a.s. for

small values of k and briefly visit the Brownian motion in the Weyl alcove. Finally, we write down the semi-group density using the multivariate Jacobi polynomials defined in [31].

2. RADIAL DUNKL PROCESS : EXISTENCE AND UNIQUENESS OF A STRONG SOLUTION

Theorem 1. *Let R be a reduced root system and define:*

$$\Phi(x) = - \sum_{\alpha \in R_+} k(\alpha) \ln(\langle \alpha, x \rangle) := \sum_{\alpha \in R_+} k(\alpha) \theta(\langle \alpha, x \rangle), \quad x \in C,$$

where $k(\alpha) > 0$ for all $\alpha \in R_+$. Then the SDE

$$(3) \quad dX_t = dB_t - \nabla \Phi(X_t) dt, \quad X_0 \in \bar{C}$$

where X is an adapted continuous process valued in \bar{C} and B is a Brownian motion in \mathbb{R}^m , has a unique strong solution.

Proof: From Theorem 2. 2 in [9], we deduce that:

$$(4) \quad dX_t = dB_t - \nabla \Phi(X_t) dt + n(X_t) dL_t, \quad X_0 \in \bar{C}$$

where $n(x)$ is a (unitary) inward normal vector to C at x , L is the boundary process satisfying:

$$dL_t = \mathbf{1}_{\{X_t \in \partial C\}} dL_t,$$

has a unique strong solution for all $t \geq 0$. Moreover:

$$(5) \quad \mathbb{E} \left[\int_0^T \mathbf{1}_{\{X_t \in \partial C\}} dt \right] = 0,$$

$$(6) \quad \mathbb{E} \left[\int_0^T |\nabla \Phi(X_t)| dt \right] < \infty$$

for all $T > 0$. Thus, it remains to prove that the boundary process vanishes. To proceed, we need two Lemmas.

Remark. Both Lemmas below discard the reducedness of R . In fact, this assumption figures in the definition of the Dunkl process and originates from analytic purposes like the commutativity of Dunkl operators ([19]).

Lemma 1. *Set $dG_t := n(X_t) dL_t$. Then, $\forall \alpha \in R_+$,*

$$\mathbf{1}_{\{\langle X_t, \alpha \rangle = 0\}} < dG_t, \alpha \rangle = 0.$$

Proof: The proof is roughly an extension to arbitrary root systems of the one given in [8] for $R = A_{m-1}$. In order to convince the reader, we provide an outline. Using the occupation density formula, we may write:

$$\int_0^\infty L_t^\alpha(\langle \alpha, X \rangle) \theta'(a) da = \langle \alpha, \alpha \rangle \int_0^t \theta'(\langle \alpha, X_s \rangle) ds$$

where $L_t^a(\langle \alpha, X \rangle)$ is the local time up to time t at the level $a \geq 0$ of the real continuous semimartingale $\langle \alpha, X \rangle \geq 0$ ([35]). On the other hand, the following inequality holds (instead of (2.5) in [8]) for all $a \in C$:

$$\begin{aligned}
\langle \nabla \Phi(x), x - a \rangle &= \sum_{\alpha \in R_+} k(\alpha) \theta'(\langle \alpha, x \rangle) \langle \alpha, x - a \rangle \\
&\stackrel{(1)}{\geq} \sum_{\alpha \in R_+} k(\alpha) [b_\alpha \theta'(\langle \alpha, x \rangle) - c_\alpha \langle \alpha, x - a \rangle - d_\alpha] \\
&\geq \min_{\alpha \in R_+} (b_\alpha k(\alpha)) \sum_{\alpha \in R_+} \theta'(\langle \alpha, x \rangle) - |x - a| \sum_{\alpha \in R_+} k(\alpha) c_\alpha |\alpha| - \sum_{\alpha \in R_+} k(\alpha) d_\alpha \\
&:= A \sum_{\alpha \in R_+} \theta'(\langle \alpha, x \rangle) - B|x - a| - C
\end{aligned}$$

by Cauchy-Schwarz inequality, where in (1), we used eq. (2.1) in [8]: let g be a convex C^1 -function on an open convex set $D \subset \mathbb{R}^m$, then $\forall a \in D$, there exist $b, c, d > 0$ such that for all $x \in D$:

$$\langle \nabla g(x), x - a \rangle \geq b|\nabla g(x)| - c|x - a| - d.$$

Note also that $A > 0$ since $b_\alpha k(\alpha) > 0$ for all $\alpha \in R_+$. Then, the continuity of X , (6) and the above inequality yield:

$$\int_0^t \theta'(\langle \alpha, X_s \rangle) ds < \infty$$

which implies that:

$$\int_0^\infty L_t^a(\langle \alpha, X \rangle) \theta'(a) da < \infty$$

Thus, $L_t^0(\langle \alpha, X \rangle) = 0$ since the function $a \mapsto \theta'(a)$ is not integrable at 0. The next step consists in using Tanaka formula to compute

$$dZ_t := d[\langle \alpha, X_t \rangle - (\langle \alpha, X_t \rangle)^+]$$

$$= \mathbf{1}_{\{\langle \alpha, X_t \rangle = 0\}} \langle \alpha, dB_t \rangle - \mathbf{1}_{\{\langle \alpha, X_t \rangle = 0\}} \langle \alpha, \nabla \Phi(X_t) \rangle dt + \mathbf{1}_{\{\langle \alpha, X_t \rangle = 0\}} \langle \alpha, dG_t \rangle$$

for $\alpha \in \Delta$. It is obvious that the second term vanishes. The first vanishes too since it is a continuous local martingale with null bracket (occupation density formula). As $X_t \in \overline{C}$, then $dZ_t = 0$ a.s. which gives the result. \blacksquare

Lemma 2. *Let $x \in \partial C$. Then $\langle n(x), \alpha \rangle \neq 0$ for some $\alpha \in \Delta$ such that $\langle x, \alpha \rangle = 0$.*

Proof: Let us assume that $\langle n(x), \alpha \rangle = 0$ for all $\alpha \in \Delta$ such that $\langle x, \alpha \rangle = 0$. Then, our assumption implies that $\langle x, \alpha \rangle > 0$ for all $\alpha \in \Delta$ such that $\langle n(x), \alpha \rangle \neq 0$. If $\langle n(x), \alpha \rangle < 0$ for these simple roots, then $x - n(x) \in \overline{C}$. By the virtue of the definition of the inward normal $n(x)$ to C at x , i. e.,

$$(7) \quad \langle x - a, n(x) \rangle \leq 0, \quad \forall a \in \overline{C},$$

it follows that $n(x)$ is the null vector which is not possible. Otherwise, choosing:

$$0 < \epsilon < \min_{\alpha < n(x), \alpha > > 0} \frac{\langle x, \alpha \rangle}{\langle n(x), \alpha \rangle},$$

we claim that $a := x - \epsilon n(x) \in \partial C$. Arguing as before, we are done. \blacksquare

Now we proceed to end the proof of Theorem (1). If $X_t \in H_\alpha$ for one and only one $\alpha \in \Delta$. Then, $n(X_t) = \alpha/|\alpha|$ and Lemma 1 gives

$$\mathbf{1}_{\langle X_t, \alpha \rangle = 0} \langle dG_t, \alpha \rangle = \mathbf{1}_{\langle X_t, \alpha \rangle = 0} |\alpha| dL_t = 0.$$

Hence, $(L_t)_{t \geq 0}$ vanishes a.s.. More generally, Lemma 2 asserts that

$$\{X_t \in \partial C\} \subset \cup_{\alpha \in \Delta} \{\langle X_t, \alpha \rangle = 0, \langle n(X_t), \alpha \rangle \neq 0\}$$

for all t . It follows that

$$\begin{aligned} 0 \leq L_t &\leq \sum_{\alpha \in \Delta} \int_0^t \mathbf{1}_{\langle X_s, \alpha \rangle = 0, \langle n(X_s), \alpha \rangle \neq 0} dL_s \\ &= \sum_{\alpha \in \Delta} \int_0^t \frac{1}{\langle n(X_s), \alpha \rangle} \mathbf{1}_{\langle X_s, \alpha \rangle = 0, \langle n(X_s), \alpha \rangle \neq 0} \langle n(X_s), \alpha \rangle dL_s = 0 \end{aligned}$$

by Lemma 1. \blacksquare

Remark. When $m = 1$, $(X_t)_{t \geq 0}$ is a Bessel process of dimension $\delta = 2k_0 + 1$ and $k_0 > 0 \Leftrightarrow \delta > 1$. It is well known that the local time vanishes (see Ch. XI in [35]).

3. FINITENESS OF THE FIRST HITTING TIME OF THE WEYL CHAMBER

Let $T_0 := \inf\{t > 0, X_t \in \partial C\}$ be the first hitting time of the Weyl chamber. It was shown in [11] that $T_0 = \infty$ a. s. when $k(\alpha) \geq 1/2$ for all $\alpha \in R_+$. In [9], where $R = A_{m-1}$ and $T_0 = \inf\{t > 0, X_t^i = X_t^j \text{ for some } (i, j)\}$, authors showed that $T_0 < \infty$ a.s. if and only if $0 < k_1 < 1/2$. More generally, the following holds:

Proposition 1. *Let $\alpha_0 \in \Delta$ and $T_{\alpha_0} := \inf\{t > 0, \langle \alpha_0, X_t \rangle = 0\}$ such that $T_0 = \inf_{\alpha_0 \in \Delta} T_{\alpha_0}$. If $0 < k(\alpha_0) < 1/2$, then $(\langle \alpha_0, X_t \rangle)_{t \geq 0}$ hits a.s. 0. In particular, $T_0 < T_{\alpha_0} < \infty$ a.s.*

Proof: Assume $k(\alpha) > 0$ for all $\alpha \in R$ and let $\alpha_0 \in \Delta$. Our scheme is roughly the same as that used in [9], thus we shall show that the process $\langle \alpha_0, X \rangle$ is a.s. less than or equal to a Bessel process of dimension $2k(\alpha_0) + 1$. The result follows from the fact that $2k(\alpha_0) + 1 < 2$ when $k(\alpha) < 1/2$ so that the Bessel process hits zero a.s.. Using (3), one has for all $t \geq 0$

$$\begin{aligned} d \langle \alpha_0, X_t \rangle &= |\alpha_0| d\gamma_t + \sum_{\alpha \in R_+} k(\alpha) \frac{\langle \alpha, \alpha_0 \rangle}{\langle \alpha, X_t \rangle} dt \\ &= |\alpha_0| d\gamma_t + k_0 \frac{|\alpha_0|^2}{\langle \alpha_0, X_t \rangle} dt + \sum_{\alpha \in R_+ \setminus \alpha_0} k(\alpha) \frac{\langle \alpha, \alpha_0 \rangle}{\langle \alpha, X_t \rangle} dt \end{aligned}$$

where k_0 is the value of $k(\alpha_0)$ corresponding to the conjugacy class of α_0 . Set

$$R = \cup_{j=1}^p R^j,$$

where R^j , $1 \leq j \leq p$ denote the conjugacy classes of R under the W -action, then

$$R_+ = \cup_{i=1}^p R_+^j$$

so that:

$$d \langle \alpha_0, X_t \rangle = \|\alpha_0\| d\gamma_t + k_0 \frac{\|\alpha_0\|^2}{\langle \alpha_0, X_t \rangle} dt + \sum_{j=0}^p k_j \sum_{\alpha \in R_+^j \setminus \alpha_0} \frac{\langle \alpha, \alpha_0 \rangle}{\langle \alpha, X_t \rangle} dt.$$

For a conjugacy class R^j and $\alpha \in R^j$, if $\langle \alpha, \alpha_0 \rangle = a(\alpha) > 0$ then, it is easy to check that $\langle \sigma_0(\alpha), \alpha_0 \rangle = -a(\alpha)$ where σ_0 is the reflection with respect to the orthogonal hyperplane H_{α_0} . Note that $\sigma_0(\alpha)$ belongs to the same conjugacy class of α and that $\sigma_0(\alpha) \in R_+ \setminus \alpha_0$ for $\alpha \in R_+ \setminus \alpha_0$ (see Proposition 1. 4 in [26]). Hence,

$$d \langle \alpha_0, X_t \rangle = \|\alpha_0\| d\gamma_t + k_0 \frac{\|\alpha_0\|^2}{\langle \alpha_0, X_t \rangle} dt - \sum_{j=0}^p k_j \sum_{\substack{\alpha \in R_+^j \setminus \alpha_0 \\ a(\alpha) > 0}} \frac{a(\alpha) \langle \alpha - \sigma_0(\alpha), X_t \rangle}{\langle \alpha, X_t \rangle \langle \sigma_0(\alpha), X_t \rangle} dt.$$

Furthermore,

$$\alpha - \sigma_0(\alpha) = 2 \frac{\langle \alpha, \alpha_0 \rangle}{\langle \alpha_0, \alpha_0 \rangle} \alpha_0 \quad \Rightarrow \quad \langle \alpha - \sigma_0(\alpha), X_t \rangle = 2a(\alpha) \frac{\langle \alpha_0, X_t \rangle}{\|\alpha_0\|^2}.$$

Consequently:

$$d \langle \alpha_0, X_t \rangle = \|\alpha_0\| d\gamma_t + k_0 \frac{\|\alpha_0\|^2}{\langle \alpha_0, X_t \rangle} dt + F_t dt$$

where $F_t < 0$ on $\{T_{\alpha_0} = \infty\}$. Using the comparison Theorem in [28] (Proposition 2. 18. p. 293 and Exercice 2. 19. p. 294), one claims that $\langle \alpha_0, X_t \rangle \leq Y_{\|\alpha_0\|^2 t}^x$ for all $t \geq 0$ on $\{T_{\alpha_0} = \infty\}$, where Y^x is a Bessel process defined on the same probability space w.r.t. the same Brownian motion, of dimension $2k_0 + 1$ and starting at $Y_0 = x \geq \langle \alpha_0, X_0 \rangle > 0$. This is not possible since a Bessel process of dimension < 2 hits 0 a.s. ([35] Chap. XI). ■

4. THE LAW OF T_0

In this paragraph, we focus on the tail distribution of T_0 deduced from absolute-continuity relations derived in ([11]). Recall that the *index* of X is defined by $l(\alpha) := k(\alpha) - 1/2$ for $\alpha \in R$. Then, Proposition 1 asserts that if $-1/2 \leq l(\alpha) < 0$ for some $\alpha \in \Delta$, then $T_0 < \infty$ a.s.. Besides, if $l(\alpha) \geq 0$ for all $\alpha \in \Delta$ then $T_0 = \infty$ a.s. ([11]). Taking into account both statements, two major parts are considered:

- $l(\alpha) \geq 0$ for all $\alpha \in R$ with at least one α such that $0 < l(\alpha) \leq 1/2$, so that the process with index $-l$ hits ∂C a.s..

- $-1/2 \leq l(\alpha) < 0$ for at least one α and the process X itself hits ∂C .

The tail distribution involves a W -invariant x -dependent integral, where $X_0 = x$. Our strategy relies on showing that this integral, viewed as a function in x , is an eigenfunction of an appropriate differential operator. Then, using uniqueness results for some hypergeometric systems, the tail distribution is written down for the irreducible A , B and D -types root systems by means of multivariate hypergeometric functions. For the type B , we recover known results from matrix theory concerning Wishart and Laguerre processes. However, we find it better to postpone this in the next section where links with eigenvalues of matrix-valued processes are detailed.

4.1. First formulae. Let us denote by P_x^l the law of $(X_t)_{t \geq 0}$ starting from $x \in C$. Let E_x^l be the corresponding expectation. Recall that ([11], Proposition 2.15.c), if $l(\alpha) \geq 0 \forall \alpha \in R_+$, then:

$$\begin{aligned} P_x^{-l}(T_0 > t) &= E_x^l \left[\left(\prod_{\alpha \in R_+} \frac{\langle \alpha, X_t \rangle}{\langle \alpha, x \rangle} \right)^{-2l(\alpha)} \right] \\ &= \prod_{\alpha \in R_+} \langle \alpha, x \rangle^{2l(\alpha)} \frac{e^{-|x|^2/2t}}{c_k t^{\gamma+m/2}} \int_C e^{-|y|^2/2t} D_k^W \left(\frac{x}{\sqrt{t}}, \frac{y}{\sqrt{t}} \right) \prod_{\alpha \in R_+} \langle \alpha, y \rangle dy \\ &= \prod_{\alpha \in R_+} \langle \alpha, \frac{x}{\sqrt{t}} \rangle^{2l(\alpha)} \frac{e^{-|x|^2/2t}}{c_k} \int_C e^{-|y|^2/2} D_k^W \left(\frac{x}{\sqrt{t}}, y \right) \prod_{\alpha \in R_+} \langle \alpha, y \rangle dy \\ &= \prod_{\alpha \in R_+} \langle \alpha, \frac{x}{\sqrt{t}} \rangle^{2l(\alpha)} \frac{e^{-|x|^2/2t}}{c_k} g \left(\frac{x}{\sqrt{t}} \right) \end{aligned}$$

where

$$g(x) := \int_C e^{-|y|^2/2} D_k^W(x, y) \prod_{\alpha \in R_+} \langle \alpha, y \rangle dy.$$

Our key result is formulated as follows:

Theorem 2. *Let T_i be the i -th difference Dunkl operator and $\Delta_k = \sum_{i=1}^m T_i^2$ the Dunkl Laplacian ([36]). Define:*

$$\mathcal{J}_k^x := -\Delta_k^x + \sum_{i=1}^m x_i \partial_i^x := -\Delta_k^x + E_1^x$$

where $E_1^x := \sum_{i=1}^m x_i \partial_i^x$ is the Euler operator and the superscript indicates the derivative action. Then

$$\mathcal{J}_k^x \left[e^{-|y|^2/2} D_k^W(x, y) \right] = E_1^y \left[e^{-|y|^2/2} D_k^W(x, y) \right].$$

Proof: Recall that if f is W -invariant then $T_i^x f = \partial_i^x f$ and that $T_i^x D_k(x, y) = y_i D_k(x, y)$ (see [36]). On the one hand ([36]):

$$\Delta_k^x D_k^W(x, y) = \sum_{i=1}^m y_i^2 \sum_{w \in W} D_k(x, wy) := p_2(y) D_k^W(x, y).$$

On the other hand,

$$\begin{aligned} E_1^x D_k^W(x, y) &= \sum_{w \in W} \sum_{i=1}^m x_i T_i^x D_k(x, wy) = \sum_{w \in W} \sum_{i=1}^m (x_i)(wy)_i D_k(x, wy) \\ &= \sum_{w \in W} \langle x, wy \rangle D_k(x, wy) = \sum_{w \in W} \langle w^{-1}x, y \rangle D_k(x, wy) \\ &= E_1^y D_k^W(x, y) \end{aligned}$$

where the last equality follows from $D_k(x, wy) = D_k(w^{-1}x, y)$ since $D_k(wx, wy) = D_k(x, y)$ for all $w \in W$. The result follows from an easy computation.

Corollary 1. g is an eigenfunction of $-\mathcal{J}_k$ corresponding to the eigenvalue $m + |R_+|$.

Proof: Theorem 2 and an integration by parts give :

$$\begin{aligned} -\mathcal{J}_k^x g(x) &= -\int_C E_1^y \left[e^{-|y|^2/2} D_k^W(x, y) \right] \prod_{\alpha \in R_+} \langle \alpha, y \rangle dy \\ &= -\sum_{i=1}^m \int_C y_i \prod_{\alpha \in R_+} \langle \alpha, y \rangle \partial_i^y \left[e^{-|y|^2/2} D_k^W(x, y) \right] dy \\ &= \sum_{i=1}^m \int_C e^{-|y|^2/2} D_k^W(x, y) \partial_i \left[y_i \prod_{\alpha \in R_+} \langle \alpha, y \rangle \right] dy \\ &= \int_C e^{-|y|^2/2} D_k^W(x, y) \prod_{\alpha \in R_+} \langle \alpha, y \rangle \sum_{i=1}^m \left[1 + \sum_{\alpha \in R_+} \frac{\alpha_i y_i}{\langle \alpha, y \rangle} \right] dy \end{aligned}$$

and the proof ends by summing over i . ■

4.1.1. *The A-type.* As mentioned in the introduction, the A_{m-1} -type root system is characterized by:

$$\begin{aligned} R &= \{\pm(e_i - e_j), 1 \leq i < j \leq m\} & R_+ &= \{e_i - e_j, 1 \leq i < j \leq m\} \\ \Delta &= \{e_i - e_{i+1}, 1 \leq i \leq m\} & C &= \{y \in \mathbb{R}^m, y_1 > y_2 > \cdots > y_m\}. \end{aligned}$$

The reflection group W is the permutations group S_m and there is one orbit so that $k = k_1 > 0$ thereby $\gamma = k_1 m(m-1)/2$. Moreover, the generalized Bessel

function ² is given by ([2] p. 212-214, [12]):

$$\frac{1}{|W|} D_k^W(x, y) = {}_0F_0^{(1/k_1)}(x, y) := \sum_{p=0}^{\infty} \sum_{\tau} \frac{J_{\tau}^{(1/k_1)}(x) J_{\tau}^{(1/k_1)}(y)}{J_{\tau}^{(1/k_1)}(1) p!}$$

where $\tau = (\tau_1, \dots, \tau_m)$ is a partition of weight $|\tau| = p$ and length m , $J_{\tau}^{(1/k_1)}$ is the Jack polynomial of Jack parameter $1/k_1$ ³, (see [2], [33]). Hence, letting $y \mapsto V(y)$ be the Vandermonde function, one writes:

$$P_x^{-l}(T_0 > t) = V(x)^{2k_0-1} \frac{|W| e^{-|x|^2/2t}}{c_k t^{k_0 m(m-1)/2}} \int_C e^{-|y|^2/2} {}_0F_0^{(1/k_0)}\left(\frac{x}{\sqrt{t}}, y\right) V(y) dy.$$

Besides, \mathcal{J}_k writes on W -invariant functions

$$- \mathcal{J}_k^x = D_0^x - E_1^x := \sum_{i=1}^m \partial_i^{2,x} + 2k_1 \sum_{i \neq j} \frac{1}{x_i - x_j} \partial_i - \sum_{i=1}^m x_i \partial_i^x.$$

Finally, since g is W -invariant, then

$$\begin{aligned} (D_0^x - E_1^x) g(x) &= m \frac{m+1}{2} g(x), \\ g(0) = m! \int_C e^{-|y|^2/2} V(y) dy &= \int_{\mathbb{R}^m} e^{-|y|^2/2} |V(y)| dy. \end{aligned}$$

Let us recall that the Gauss hypergeometric function

$${}_2F_1^{(1/k_1)}(e, b, c, z) = \sum_{p=0}^{\infty} \sum_{\tau} \frac{(e)_{\tau} (b)_{\tau}}{(c)_{\tau}} \frac{J_{\tau}^{(1/k_1)}(z)}{p!}$$

is the unique symmetric eigenfunction that equals to 1 at 0 of ([4] p. 585)

$$(8) \quad \sum_{i=1}^m z_i (1-z_i) \partial_i^{2,z} + 2k_1 \sum_{i \neq j} \frac{z_i (1-z_i)}{z_i - z_j} \partial_i^z + \sum_{i=1}^m [c - k_1(m-1) - (e+b+1 - k_1(m-1)) z_i] \partial_i^z$$

associated with the eigenvalue meb . Letting $z = (1/2)(1 - x/\sqrt{b})$, $e = (m+1)/2$ and

$$c = k_1(m-1) + \frac{1}{2}[e+b+1 - k_1(m-1)] = \frac{b}{2} + \frac{k_1}{2}(m-1) + \frac{m+3}{4},$$

²Authors use the change of variable $x \mapsto \sqrt{2}x$, $y \mapsto \sqrt{2}y$ to fit the hypergeometric function obtained when deriving the generating function for Hermite polynomials. This amounts to multiply the eigenoperator by a constant (see p. 183).

³With the same notations in [2], $k_1 = 2/\alpha$. This can be seen either from the eigenoperator below or from the orthogonality weight function involved in the semi group density.

then, the resulting function is an eigenfunction of

$$\sum_{i=1}^m \left(1 - \frac{x_i^2}{b}\right) \partial_i^{2,x} + 2k_1 \sum_{i \neq j} \frac{(1 - x_i^2/b)}{x_i - x_j} \partial_i^x - \sum_{i=1}^m \left(b + \frac{m+3}{2} - k_1(m-1)\right) \frac{x_i}{b} \partial_i^x$$

and $D_0^x - E_1^x$ is the limiting operator as b tends to infinity. Hence,

Proposition 2. For $1/2 < k_1 \leq 1$,

$$g(x) = g(0)C(m, k_1) \lim_{b \rightarrow \infty} {}_2F_1^{(1/k_1)} \left[\frac{m+1}{2}, b, \frac{b}{2} + \frac{k_1}{2}(m-1) + \frac{m+3}{4}, \frac{1}{2} \left(1 - \frac{x}{\sqrt{b}}\right) \right]$$

where

$$C(m, k_1)^{-1} = \lim_{b \rightarrow \infty} {}_2F_1^{(1/k_1)} \left(\frac{m+1}{2}, b, \frac{b}{2} + \frac{k_1}{2}(m-1) + \frac{m+3}{4}, \frac{1}{2} \right).$$

Remark. One cannot exchange the infinite sum and the limit operation. Indeed, expand the generalized Pochhammer symbol as ([27])

$$(a)_\tau = \prod_{i=1}^m (a - k_1(m-1))_{\tau_i} = \prod_{i=1}^m \frac{\Gamma(a - k_1(m-1) + \tau_i)}{\Gamma(a - k_1(m-1))}$$

and use Stirling formula to see that each term in the above product is equivalent to $(a + k_1(m-1) + \tau_i)^{\tau_i}$ for large enough positive a . Moreover, since $J_\tau^{(1/k_1)}$ is homogeneous, one has

$$J_\tau^{(1/k_1)} \left[\frac{1}{2} \left(1 - \frac{x}{\sqrt{b}}\right) \right] = 2^{-p} J_\tau^{(1/k_1)} \left(1 - \frac{x}{\sqrt{b}}\right), \quad |\tau| = p.$$

It follows that

$$\frac{(b)_\tau}{(b/2 + (m-1)k_1/2 + (m+3)/4)_\tau} J_\tau^{(1/k_1)} \left[\frac{1}{2} \left(1 - \frac{x}{\sqrt{b}}\right) \right] \approx J_\tau^{(1/k_1)}(1)$$

for large positive b . Thus, the above Gauss hypergeometric function reduces to

$${}_1F_0^{(1/k_1)} \left(\frac{m+1}{2}, 1 \right).$$

Unfortunately, the above series diverges for instance for $k_1 = 1/2, 2$ since one has in that cases ([27])

$${}_1F_0^{(1/k_1)}(a, x) = \prod_{i=1}^m (1 - x_i)^{-a}.$$

4.1.2. *The B_m -type.* This root system is defined by

$$\begin{aligned} R &= \{\pm e_i, \pm e_i \pm e_j, 1 \leq i < j \leq m\} & R_+ &= \{e_i, 1 \leq i \leq m, e_i \pm e_j, 1 \leq i < j \leq m\} \\ \Delta &= \{e_i - e_{i+1}, 1 \leq i \leq m, e_m\} & C &= \{y \in \mathbb{R}^m, y_1 > y_2 > \cdots > y_m > 0\}. \end{aligned}$$

The Weyl group is generated by transpositions and sign changes ($x_i \mapsto -x_i$) and there are two orbits so that $k = (k_0, k_1)$ thereby $\gamma = mk_0 + m(m-1)k_1$. The generalized Bessel function⁴ is given by ([2] p. 214):

$$\frac{1}{|W|} D_k^W(x, y) = {}_0F_1^{(1/k_1)}(k_0 + (m-1)k_1 + \frac{1}{2}, \frac{x^2}{2}, \frac{y^2}{2t})$$

where

$${}_0F_1^{(1/k_1)}(c, x, y) = \sum_{p=0}^{\infty} \sum_{\tau} (c)_{\tau} \frac{J_{\tau}^{(k_1)}(x) J_{\tau}^{(1/k_1)}(y)}{J_{\tau}^{(1/k_1)}(1) p!}.$$

Then, one has:

$$g(x) = |W| \int_C e^{-|y|^2/2} {}_0F_1^{(1/k_1)}(k_0 + (m-1)k_1 + \frac{1}{2}, \frac{x^2}{2}, \frac{y^2}{2}) \prod_{i=1}^m (y_i) V(y^2) dy.$$

The eigenoperator writes on W -invariant functions:

$$\begin{aligned} -\mathcal{J}_k^x &= \sum_{i=1}^m \partial_i^{2,x} + 2k_0 \sum_{i=1}^m \frac{1}{x_i} \partial_i^x + 2k_1 \sum_{i \neq j} \left[\frac{1}{x_i - x_j} + \frac{1}{x_i + x_j} \right] \partial_i^x - E_1^x, \\ -\mathcal{J}_k^x g(x) &= m(m+1)g(x), \quad g(0) = \int_{\mathbb{R}^m} e^{-|y|^2} \prod_{i=1}^m |y_i| |V(y^2)| dy. \end{aligned}$$

A change of variable $x_i = \sqrt{2y_i}$ shows that $u(y) := g(\sqrt{2y})$ satisfies

$$\begin{aligned} -\tilde{\mathcal{J}}_k^y u(y) &= m \frac{(m+1)}{2} u(y), \quad u(0) = g(0), \\ -\tilde{\mathcal{J}}_k^y &= \sum_{i=1}^m y_i \partial_i^{2,y} + 2k_1 \sum_{i \neq j} \frac{y_i}{y_i - y_j} \partial_i^y + \left(k_0 + \frac{1}{2} \right) \sum_{i=1}^m \partial_i^y - E_1^y. \end{aligned}$$

which implies that

$$u(y) = u(0) {}_1F_1^{(1/k_1)}\left(\frac{m+1}{2}, k_0 + (m-1)k_1 + \frac{1}{2}, y\right)$$

where

$${}_1F_1^{(1/k_1)}(b, c, z) = \sum_{p=0}^{\infty} \sum_{\tau} \frac{(b)_{\tau}}{(c)_{\tau}} \frac{J_{\tau}^{(1/k_1)}(z)}{p!}.$$

⁴There is an erroneous sign in one of the arguments in [2]. Moreover, to recover this expression in the B_m case from that given in [2], one should make substitutions $a = k_0 - 1/2$, $k_1 = 1/\alpha$, $q = 1 + (m-1)k_1$. We point to the reader that this is different from the one used in [12] p. 121.

This can be seen from the differential equation (8) and using ([2]):

$$\lim_{e \rightarrow \infty} {}_2F_1^{(1/k_1)}(e, b, c, \frac{z}{e}) = {}_1F_1^{(1/k_1)}(b, c, z).$$

Finally

$$g\left(\frac{x}{\sqrt{t}}\right) = g(0) {}_1F_1^{(1/k_1)}\left(\frac{m+1}{2}, k_0 + (m-1)k_1 + \frac{1}{2}, \frac{x^2}{2t}\right).$$

Hence, the tail distribution is given by:

Proposition 3. For $1/2 \leq k_0, k_1 \leq 1$ with either $k_0 > 1/2$ or $k_1 > 1/2$, one has

$$P_x^{-l}(T_0 > t) = C_k \prod_{i=1}^m \left(\frac{x_i^2}{2t}\right)^{l_0} \left(V\left(\frac{x^2}{2t}\right)\right)^{2l_1} e^{-|x|^2/2t} {}_1F_1^{(1/k_1)}\left(\frac{m+1}{2}, k_0 + (m-1)k_1 + \frac{1}{2}, \frac{x^2}{2t}\right)$$

where $l_0 := k_0 - 1/2, l_1 := k_1 - 1/2$ and V stands for the Vandermonde function.

4.1.3. *The D_m -type.* This root system is defined by ([26] p. 42)

$$R = \{\pm e_i \pm e_j, 1 \leq i < j \leq m\}, \quad R_+ = \{e_i \pm e_j, 1 \leq i < j \leq m\},$$

and there is one orbit so that $k(\alpha) = k_1$ thereby $\gamma = m(m-1)k_1$. The Weyl chamber is given by:

$$C = \{x \in \mathbb{R}^m, x_1 > x_2 > \dots > |x_m|\} = C_1 \cup s_m C_1,$$

where C_1 is the Weyl chamber of type B and s_m stands for the reflection w.r.t the vector e_m acting by sign change on the variable x_m .

Proposition 4. For $1/2 < k_1 \leq 1$, the tail distribution writes:

$$\mathbb{P}_x^{-l}(T_0 > t) = C_k \left[V\left(\frac{x^2}{2t}\right)\right]^{2l} e^{-|x|^2/2t} {}_1F_1\left(\frac{m}{2}, (m-1)k_1 + \frac{1}{2}, \frac{x^2}{2}\right).$$

Remark. We warn the reader that the tail distribution for the type D is not a limiting case of the one derived for the B -type root system with $k_0 = 0$. In fact, g is W -invariant and thus depends on the Weyl group (or the Weyl chamber). Nevertheless, we will give later a second proof that uses the formula derived for the root system of type B and this requires the invariance under the Weyl group of type B .

Proof. It follows nearly the one given for the root system of type B with the following modifications: if $x \in C_1$, then we perform the change of variable $x_i = \sqrt{2y_i}, 1 \leq i \leq m$ and for $x \in s_m C_1$ we perform the change of variable $x_i = \sqrt{2y_i}, 1 \leq i \leq m-1$ and $x_m = -\sqrt{2y_m}$. In both cases, one gets that $y \mapsto u(y) = g(x^2/2)$ is a symmetric eigenfunction (invariant under permutation) of

$$(9) \quad \sum_{i=1}^m y_i \partial_i^{2,y} + 2k_1 \sum_{i \neq j} \frac{y_i}{y_i - y_j} \partial_i^y + \frac{1}{2} \sum_{i=1}^m \partial_i^y - \sum_{i=1}^m y_i \partial_i^y.$$

corresponding to the eigenvalue $m^2/2$. This spectral problem with initial value 1 at $y = 0$ has a unique solution given by

$${}_1F_1^{(1/k')}(m/2, (m-1)k' + 1/2, y),$$

and the expression of the tail distribution follows. \blacksquare

4.2. Second formulae. Suppose that $l(\alpha) < 0$ for at least one $\alpha \in R$. Then, by part (b) of Proposition 2.15. in [11], one writes

$$\begin{aligned} P_x^l(T_0 > t) &= E_x^0 \left[\prod_{\alpha \in R_+} \left(\frac{\langle \alpha, X_t \rangle}{\langle \alpha, x \rangle} \right)^{l(\alpha)} \exp \left(-\frac{1}{2} \sum_{\alpha, \zeta \in R_+} \int_0^t \frac{\langle \alpha, \gamma \rangle l(\alpha) l(\zeta)}{\langle \alpha, X_s \rangle \langle \zeta, X_s \rangle} ds \right) \right] \\ &= E_x^r \left[\prod_{\alpha \in R_+} \left(\frac{\langle \alpha, X_t \rangle}{\langle \alpha, x \rangle} \right)^{l(\alpha) - r(\alpha)} \exp \left(-\frac{1}{2} \sum_{\alpha, \zeta \in R_+} \int_0^t \frac{\langle \alpha, \zeta \rangle l(\alpha, \zeta)}{\langle \alpha, X_s \rangle \langle \zeta, X_s \rangle} ds \right) \right], \end{aligned}$$

where the last equality follows from part (a) of the same Proposition, $l(\alpha, \zeta) := l(\alpha)l(\zeta) - r(\alpha)r(\zeta)$ and

$$r(\alpha) = \begin{cases} l(\alpha) & \text{if } l(\alpha) \geq 0 \\ -l(\alpha) & \text{if } l(\alpha) < 0. \end{cases}$$

Then $l(\alpha, \zeta) = 0$ if $l(\alpha)l(\zeta) \geq 0$ and $l(\alpha, \zeta) = -2r(\alpha)r(\zeta)$ otherwise. As a result,

$$P_x^l(T_0 > t) = E_x^r \left[\prod_{\substack{\alpha \in R_+ \\ l(\alpha) < 0}} \left(\frac{\langle \alpha, x \rangle}{\langle \alpha, X_t \rangle} \right)^{2r(\alpha)} \exp \left(\sum_{\substack{\alpha, \zeta \in R_+ \\ l(\alpha)l(\zeta) < 0}} \int_0^t \frac{\langle \alpha, \zeta \rangle r(\alpha)r(\zeta)}{\langle \alpha, X_s \rangle \langle \zeta, X_s \rangle} ds \right) \right].$$

Next, note that the exponential functional in the RHS equals 1 for the irreducible root systems of types A, B, C, D . For both types A and D , it is obvious since R consists of one orbit so that $\{\alpha, \zeta \in R_+, l(\alpha)l(\zeta) < 0\}$ is empty and we fit into the first previous case. For the type B , note that $l(\alpha)l(\zeta) < 0$ implies that α and ζ belong to different orbits. Thus, writing $R_+ = \{e_i, 1 \leq i \leq m\} \cup \{e_j \pm e_k, 1 \leq j < k \leq m\}$ so that $\langle e_i, e_j \pm e_k \rangle = \delta_{ij} \pm \delta_{ik}$ gives

$$\begin{aligned} S &= \sum_{i=1}^m \sum_{i < k} \frac{1}{X_t^i} \left[\frac{1}{X_t^i - X_t^k} + \frac{1}{X_t^i + X_t^k} \right] + \sum_{i=1}^m \sum_{k < i} \frac{1}{X_t^i} \left[\frac{-1}{X_t^k - X_t^i} + \frac{1}{X_t^k + X_t^i} \right] \\ &= \sum_{i=1}^m \sum_{i < k} \frac{2}{(X_t^i)^2 - (X_t^k)^2} - \sum_{i=1}^m \sum_{k < i} \frac{2}{(X_t^k)^2 - (X_t^i)^2} = 0, \end{aligned}$$

where S stands for the sum between brackets (up to a constant). The same obviously holds for the type C since

$$C_m = \{2e_i, 1 \leq i \leq m, \pm e_i \pm e_j, 1 \leq i < j \leq m\}.$$

Let us now investigate the case $k_0 < 1/2$, $k_1 \geq 1/2$ for which $l_0 < 0$, $l_1 \geq 0$. One has to compute:

$$g(x) = \int_C e^{-|y|^2/2} D_k^W(x, y) \prod_{i=1}^m (y_i) V^{2k_1}(y^2) dy.$$

The machinery used before still applies and gives:

$$- \mathcal{J}_k g = m[1 + 2k_1(m - 1)]g,$$

and one finally gets:

Proposition 5. *For the B_m -type root system and when $k_0 < 1/2$, $k_1 \geq 1/2$:*

$$P_x^l(T_0 > t) = C_k \prod_{i=1}^m \left(\frac{x_i^2}{2t} \right)^{l_0} e^{-|x|^2/2t} {}_1F_1^{(1/k_1)}((1/2)+k_1(m-1), k_0+(m-1)k_1 + \frac{1}{2}, \frac{x^2}{2t}).$$

When $k_0 \geq 1/2$, $k_1 < 1/2$, one has

$$g(x) = \int_C e^{-|y|^2/2} D_k^W(x, y) \prod_{i=1}^m (y_i)^{2k_0} V(y^2) dy$$

so that

$$- \mathcal{J}_k g = m[2k_0 + m - 1]g.$$

Thus

Proposition 6. *When $k_0 \geq 1/2$, $k_1 < 1/2$,*

$$P_x^l(T_0 > t) = C_k V \left(\frac{x^2}{2t} \right)^{2l_1} e^{-|x|^2/2t} {}_1F_1^{(1/k_1)}(l_0 + \frac{m}{2}, k_0 + (m - 1)k_1 + \frac{1}{2}, \frac{x^2}{2t}).$$

Remark. In cases when generalized Bessel functions are expressed by means of multivariate hypergeometric functions (for instance, when R is of type A, B, C, D), possible computations may be performed using Corollary 2 p. 1107 in [27]. The latter gives the value of the integral of a Jack polynomial against the Selberg weight which is a particular case of the weight given by powers of the product over the positive roots.

5. BROWNIAN MOTIONS REFLECTED ON THE WALLS OF WEYL CHAMBERS

In the probability scope, the reflected BM is the absolute value of a real BM. It is easy to see that its semi group density writes for $x \geq 0$:

$$(10) \quad q_t(x, y) = [N_t(y - x) + N_t(y + x)] \mathbf{1}_{\{y \geq 0\}}$$

where as before $N_t(x)$ is the heat kernel. The BM reflected on the walls of a given Weyl chamber C was defined in [11] as the projection of a m -dimensional BM (Dunkl process of zero multiplicity function) in \overline{C} , that is, as a radial Dunkl

process of zero multiplicity function (its index function equals $-1/2$). Its semi group density writes

$$p_t^{0,W}(x, y) = \frac{1}{c_0 t^{n/2}} \exp - \left\{ \frac{|x|^2 + |y|^2}{2t} \right\} \sum_{w \in W} \exp \left(\frac{1}{t} \langle x, wy \rangle \right).$$

For $R = B_1$, $W = \{Id, \sigma\}$ where $\sigma(x) = -x$, $x \in \mathbb{R}$ and one recovers (10). The formulae below are particular cases of the previous ones derived in subsection (4.1) for the particular multiplicity function $k \equiv 1$. The involved hypergeometric functions take determinantal forms ([23]) and easy computations yield (C is a constant that may differ from one case to the other and depends on m):

- *A-type root system.* In this case, one has

$$g(x) = C \lim_{b \rightarrow \infty} \frac{\det [(x_i^b)^{m-j} {}_2\mathcal{F}_1 [(m+1)/2 - j + 1, b - j + 1, b/2 + m - j + 5/4, x_i^b]]_{i,j=1}^m}{V(x^b)}$$

where $x^b := (1/2)(1 - x/\sqrt{b})$. Note that

$$V(x^b) = \prod_{i < j} (x_i^b - x_j^b) = (2\sqrt{b})^{-m(m-1)/2} V(-x).$$

Thus,

$$g(x) = \frac{C}{V(x)} \lim_{b \rightarrow \infty} \det \left[b^{(m-1)/4} (x_i^b)^{m-j} {}_2\mathcal{F}_1 \left[\frac{m+1}{2} - j + 1, b - j + 1, \frac{b}{2} + m - j + 5/4, x_i^b \right] \right]_{i,j=1}^m$$

so that the tail distribution writes:

$$P_x^{-\nu}(T_0 > t) = C e^{-|x|^2/2t} \lim_{b \rightarrow \infty} \det \left[b^{(m-1)/4} \left(1 - \frac{x_i}{\sqrt{bt}} \right)^{m-j} {}_2\mathcal{F}_1 \left[\frac{m+1}{2} - j + 1, b - j + 1, \frac{b}{2} + m - j + 5/4, \frac{1}{2} \left(1 - \frac{x_i}{\sqrt{bt}} \right) \right] \right]_{i,j=1}^m$$

- *B-type root system.*

$$\begin{aligned} P_x^{-1/2}(T_0 > t) &= C \det \left[\left(\frac{x_i^2}{2t} \right)^{m-j+1/2} e^{-x_i^2/2t} {}_1\mathcal{F}_1 \left(\frac{m+1}{2} - j + 1, m + \frac{1}{2} - j + 1, \frac{x_i^2}{2t} \right) \right]_{i,j=1}^m \\ &= c_k \det \left[\left(\frac{x_i^2}{2t} \right)^{m-j+1/2} {}_1\mathcal{F}_1 \left(\frac{m}{2}, m - j + \frac{3}{2}, -\frac{x_i^2}{2t} \right) \right]_{i,j=1}^m \end{aligned}$$

where the last line follows from Kummer's transformation ([32]).

- *D-type root system.* Similarly, one gets

$$\mathbb{P}_x^{-1/2}(T_0 > t) = C \det \left[\left(\frac{x_i^2}{2t} \right)^{m-j} {}_1\mathcal{F}_1 \left(\frac{m-1}{2}, m - j + \frac{1}{2}, -\frac{x_i^2}{2t} \right) \right]_{i,j=1}^m$$

Remark. Let B be a m -dimensional BM starting at $x \in C$ and let g be the last hitting time before time $t = 1$ of ∂C . By time inversion property and scaling, one may relate g to T_0 by the following identity:

$$\mathbb{P}_x \left(g < \frac{1}{1+u} \right) = \mathbb{E}_y(e^{-u/2T_0}), \quad y = x/|x|, u \geq 0.$$

5.1. The heat equation. The tail distribution of the first exit time T_0 from C by the BM was considered in [17] and involves pfaffians. One way to show the agreement of both approaches is to use the fact that

$$x \mapsto g(x) = \int_C e^{-|y|^2/2} D_1^W(x, y) \prod_{\alpha \in R_+} \langle \alpha, y \rangle dy$$

satisfies

$$(11) \quad [2\mathcal{L}_1 - \sum_{i=1}^m x_i \partial_i]g = (m + |R_+|)g.$$

in order to prove that $(x, t) \mapsto u(x, t) := \mathbb{P}_x^{-1/2}(T_0 > t)$ solves the heat equation

$$\frac{1}{2}\Delta u(x, t) = \partial_t u(x, t),$$

with boundary values $u(x, t) = 0$ if $x \in \partial C$ and $u(x, 0) = 1$ if $x \in C$ ([17]). To proceed, let us first write the tail distribution as

$$\mathbb{P}_x^{-1/2}(T_0 > t) = C(vhg) \left(\frac{x}{\sqrt{t}} \right), \quad v(x) := e^{-|x|^2/2}, h(x) = \prod_{\alpha \in R_+} \langle \alpha, x \rangle.$$

Now, let us compute the space derivatives: it is easy to see from

$$\nabla h = h \nabla \log h = h \sum_{\alpha \in R_+} \frac{\alpha}{\langle \alpha, \cdot \rangle}$$

that

$$\sum_{i=1}^m x_i \partial_i h(x) = \langle x, \nabla \log h(x) \rangle h(x) = |R_+| h(x).$$

Therefore, since

$$\Delta(gh) = h\Delta g + g\Delta h + 2 \langle \nabla g, \nabla h \rangle,$$

then $2\mathcal{L}_1$ takes the form (h -transform property, [35])

$$2\mathcal{L}_1^W(\cdot) = \frac{1}{h}\Delta(h\cdot)$$

so that (11) is equivalent to

$$\Delta(gh)(x) = m(gh)(x) + \langle x, \nabla(gh)(x) \rangle.$$

Now, since $\nabla v(x) = -xv(x)$ and using (5.1), then

$$\begin{aligned}
\Delta(vgh)(x) &= (gh)(x)\Delta(v)(x) + v(x)\Delta(gh)(x) + 2 \langle \nabla v(x), \nabla(gh)(x) \rangle \\
&= (gh)(x) \sum_{i=1}^m \partial_i[-x_i v(x)] + v(x)[\Delta(gh)(x) - 2 \langle x, \nabla(gh)(x) \rangle] \\
&= (vgh)(x) \sum_{i=1}^m [x_i^2 - 1] + v(x)[m(gh)(x) - \langle x, \nabla(gh)(x) \rangle] \\
&= |x|^2(vgh)(x) - \langle x, \nabla(gh)(x) \rangle v(x) = - \langle x, \nabla(vgh)(x) \rangle .
\end{aligned}$$

Keeping in mind that $u(x, t) = C(vgh)(x/\sqrt{t})$, then

$$\frac{1}{2}\Delta[u(x, t)] = \frac{C}{2t}\Delta(vgh)\left(\frac{x}{\sqrt{t}}\right) = \frac{C}{2t^{3/2}} \langle x, \nabla(vgh)\left(\frac{x}{\sqrt{t}}\right) \rangle .$$

Finally, the derivative w.r.t. the time variable t is easily computed once one writes $t \mapsto u(x, t)$ as a composition of the functions $t \mapsto x/\sqrt{t}$ for fixed $x \in E$ and $y \mapsto (vgh)(y)$. \blacksquare

Remark. The fact that the tail distribution is the unique solution of the heat equation with the corresponding boundary values is valid for all homogeneous W -invariant Markov processes. Particularly, it is true for the radial Dunkl processes and we will not do here it since the computations are tedious. Nevertheless, since $\mathbb{P}_x^{-\nu}(T_0 > t) = (vgh^{2\nu})(x/\sqrt{t})$ and v, g are W -invariant, then we hint the interested reader at the fact that the Dunkl partial derivative T_i ([36]) acts on the product of two functions as a derivation when at least one of them is W -invariant.

6. β -PROCESSES AND RANDOM MATRICES

In the sequel, we will show how eigenvalues of some well known matrix-valued processes and radial Dunkl processes are interrelated. This connection was already checked by physicists throughout eigenvalues probability densities and Fokker-Planck equations for parameter-dependent random matrices ([12]). Here, we use the SDE (3). For $R = A_{m-1}$, (3) reduces to (2) and identifies X with the eigenvalues process of symmetric and Hermitian BMs when $k_1 = 1/2, 1$ respectively ([29]). Set $k_1 := \beta/2$, $\beta > 0$, then such a process shall be called β -Dyson, referring to the Dyson model when $\beta = 2$. The parameter $\beta > 0$ is known as the Dyson index.

6.1. The B_m -type and β -Laguerre processes. The B_m system turns out to be related to the process of eigenvalues of Wishart and Laguerre processes $(\lambda(t))_{t \geq 0}$ satisfying (see [5],[13]):

$$(12) \quad d\lambda_i(t) = 2\sqrt{\lambda_i(t)} d\nu_i(t) + \beta \left[\delta + \sum_{k \neq i} \frac{\lambda_i(t) + \lambda_k(t)}{\lambda_i(t) - \lambda_k(t)} \right] dt, \quad 1 \leq i \leq m$$

for $\beta = 1, 2$ and $\delta \geq m + 1, m$ respectively, where $(\nu_i)_i$ are independent BMs and $\lambda_1(0) > \dots > \lambda_m(0) > 0$. Recall that these processes remain strictly positive if they are initially strictly positive and that the particles never collide. Define the β -Laguerre process as a solution, when it exists, of (12) with arbitrary $\beta, \delta \geq 0$ up to time $t < R_0 \wedge \tau$, where $R_0 = \inf\{t, \lambda_m(t) = 0\}$, τ is the first collision time and $\lambda_1(0) > \dots > \lambda_m(0) > 0$. Set $r_i := \sqrt{\lambda_i}$, then, for $t < \tau \wedge R_0$:

$$\begin{aligned} dr_i(t) &= d\nu_i(t) + \frac{1}{2r_i(t)} \left[\beta\delta - 1 + \beta \sum_{j \neq i} \frac{r_i^2 + r_j^2}{r_i^2 - r_j^2} \right] dt \\ &= d\nu_i(t) + \frac{\beta(\delta - m + 1) - 1}{2r_i(t)} dt + \frac{\beta}{2} \sum_{j \neq i} \left[\frac{1}{r_i(t) - r_j(t)} + \frac{1}{r_i(t) + r_j(t)} \right] dt \\ &= d\nu_i(t) + \frac{k_0}{r_i(t)} dt + k_1 \sum_{j \neq i} \left[\frac{1}{r_i(t) - r_j(t)} + \frac{1}{r_i(t) + r_j(t)} \right] dt \end{aligned}$$

with $2k_0 = \beta(\delta - m + 1) - 1$, $2k_1 = \beta$. Consequently, the process $r = (r_1, \dots, r_m)$ defined for all $t < \tau \wedge R_0$ is a B_m -radial Dunkl process. Using Theorem 1, one claim that the SDE above has a unique strong solution for all $t \geq 0$ provided that $k_0, k_1 > 0$, that is, $\beta > 0, \delta > m - 1 + (1/\beta)$.

This improves results from matrix theory: for Wishart processes, $\beta = 1$ so that the strong uniqueness for the eigenvalues process holds for all $\delta > m$. For Laguerre processes $\beta = 2$ and this holds for $\delta > m - 1/2$. Besides, $T_0 = R_0 \wedge \tau$ and $T_0 < \infty$ a.s. if $0 < \beta < 1/2$ and/or $m - 1 + (1/\beta) < \delta < m - 1 + (2/\beta)$ and $T_0 = \infty$ a.s. for $\beta \geq 1$ and $\delta \geq m - 1 + (2/\beta)$. Moreover, (1) writes ([2])⁵:

(13)

$$p_t^{k_0, k_1}(x, y) = \frac{|W|}{c_k t^{\gamma+m/2}} e^{-(|x|^2 + |y|^2)/2t} {}_0F_1^{(2/\beta)}\left(\frac{\beta\delta}{2}, \frac{x^2}{2t}, \frac{y^2}{2t}\right) \prod_{i=1}^m (y_i)^{2k_0} [V(y^2)]^{2k_1} dy.$$

Using the change of variable $y \mapsto \sqrt{y}$, the semi-group density of the β -Laguerre process writes:

$$q_t^{k_0, k_1}(x, y) = \frac{C_{k_0, k_1}}{t^{\gamma+2k_1+m/2}} e^{-(\sum_{i=1}^m (x_i + y_i)/2t)} {}_0F_1^{(2/\beta)}\left(\frac{\beta\delta}{2}, \frac{x}{2t}, \frac{y}{2t}\right) \prod_{i=1}^m (y_i)^{k_0-1/2} V^{2k_1}(y) dy$$

which reduces the density of the β -Laguerre ensemble ([18]).

Remark. Let $l_0 := k_0 - 1/2$, $l_1 := k_1 - 1/2$ be the indices of B -type radial Dunkl process X . Let us prime the multiplicity function corresponding to the index function $-l$, that is, $-l_0 = k'_0 - 1/2$, $-l_1 = k'_1 - 1/2$. Given a Wishart process with $\beta' = 1$ and $m - 1 \leq \delta' < m + 1$ ([5]), then $k'_1 = 1/2$ ($\beta' = 1$) and $k'_0 = (\delta' - m)/2 \Rightarrow -l_1 = 0$, $-l_0 = (\delta' - m - 1)/2 < 0$. Set $\delta' = m + 1 - 2\nu$, $0 < \nu < 1/2$, then,

⁵With the same notations used in [2], one has $\beta a'/2 = k_0$, $a = k_0 - 1/2$, $\beta = 2/\alpha \Rightarrow a + q = \beta\delta/2$.

$l_1 = 0, l_0 = \nu \Rightarrow k_1 = 1/2 (\beta = 1)$ and $k_0 = \nu + 1/2 (\delta = m + 1 + 2\nu)$. Results of subsection 4.1 write:

$$P_x^{-l}(T_0 > t) = C_{m,\delta} \prod_{i=1}^m \left(\frac{x_i^2}{2t} \right)^\nu e^{-|x|^2/2t} {}_1F_1^{(2)}\left(\frac{m+1}{2}, \frac{\delta}{2}, \frac{x^2}{2t}\right)$$

which fits the expression already derived in [15].

For Laguerre processes ([13]) with $\beta = 2$ and $m - 1/2 \leq \delta < m$, one should apply results derived in section 4.2. Then $k_1 = 1, k_0 = \delta - m + 1/2 \Rightarrow l_1 = 1/2, l_0 = \delta - m := -\nu$ with $0 < \nu < 1/2$. Thus $r_1 = l_1 = 1/2, r_0 = -l_0 = \nu \Rightarrow \tilde{k}_1 = r_1 + 1/2 = 1$ and $\tilde{k}_0 = r_0 + 1/2 = \nu + 1/2$. Denoting (with no confusion) β, δ the parameters corresponding to \tilde{k}_1, \tilde{k}_0 , one recovers ([13]):

$$P_x^l(T_0 > t) = C_{m,\delta} \prod_{i=1}^m \left(\frac{x_i^2}{2t} \right)^\nu e^{-|x|^2/2t} {}_1F_1^{(1)}\left(m, \delta, \frac{x^2}{2t}\right).$$

6.2. BMs in Weyl chambers and D -type generalized Bessel function.

BMs in Weyl chambers were deeply studied in [22] and enjoy many interesting properties: they satisfy (3) with $k \equiv 1$ so that $T_0 = \infty$ a.s. and they are h -transforms in Doob's sense of m independent real BMs killed when they first hit ∂C ([22]). They are then interpreted as m independent particles constrained to stay in the Weyl chamber C . The proofs of those properties use probabilistic arguments. Here, we show how, for both types A and B , these properties follow from determinantal representations of multivariate hypergeometric functions of two arguments involved in the semi group density. Besides, the comparison between the determinantal representations of the BM in the D -type Weyl chamber derived in [22] on the one side and of a multivariate hypergeometric series on the other side hints to what the D -type generalized Bessel function should be.

Recall first that when the Jack parameter $1/k_1 = 1, {}_0F_1^{(1)}$ has a determinantal representation ([13], [23]) by means of the univariate hypergeometric functions ([32]). For instance,

$${}_0F_0^{(1/k_1)}(x, y) = C_m \frac{\det({}_0\mathcal{F}_0(x_i y_j)_{i,j=1}^m)}{V(x)V(y)} = C(m) \frac{\det(e^{x_i y_j})_{i,j=1}^m}{V(x)V(y)}$$

so that (1) writes for $R = A_{m-1}$:

$$(14) \quad p_t^1(x, y) = \frac{V(y)}{V(x)} \det(N_t(y_j - x_i))_{i,j=1}^m,$$

where N_t is the heat kernel. Similarly, since $D_k^W = |W| {}_0F_1^{(1/k_1)}$ for $R = B_m$, then

$$p_t^{k_0,1}(x, y) = C_m \frac{h(y)}{h(x)} \frac{e^{-(|x|^2+|y|^2)/2t}}{t^{m/2}} \prod_{i,j=1}^m \left(\frac{x_i y_j}{t} \right) \det \left[{}_0\mathcal{F}_1 \left(k_0 + \frac{1}{2}, \frac{(x_i y_j)^2}{4t^2} \right) \right]_{i,j=1}^n,$$

where $h(y) = V(y^2) \prod_{i=1}^n y_i$. Taking $k_0 = 1$ and using the identity ([32])

$${}_0\mathcal{F}_1\left(\frac{3}{2}, z\right) = \frac{C}{2\sqrt{z}} \sinh(2\sqrt{z}).$$

for some constant C , one gets for the BM in the B -type Weyl chamber

$$(15) \quad p_t^{1,1}(x, y) = \frac{h(y)}{h(x)} \det [N_t(y_j - x_i) - N_t(y_j + x_i)]_{i,j=1}^m.$$

The h -transform property is then obvious (note that the square of (X_t^i) are the eigenvalues of a Laguerre process and that they are the V -transform of m independent square Bessel processes killed when they first collide, [30]). Now, Grabiner's result reads for the BM in the Weyl chamber of type D ($k_1 = 1$):

$$\begin{aligned} p_t^1(x, y) &= \frac{V(y^2) \det [N_t(y_i - x_j) - N_t(y_i + x_j)]_{i,j=1}^m + \det [N_t(y_i - x_j) + N_t(y_i + x_j)]_{i,j=1}^m}{V(x^2) \cdot 2} \\ &= \frac{C_m}{t^{\gamma+m/2}} e^{-(|x|^2+|y|^2)/2t} \frac{\det [\sinh(x_i y_j / t)]_{i,j=1}^m + \det [\cosh(x_i y_j / t)]_{i,j=1}^m}{V(x^2/4t^2) V(y^2)} V^2(y^2) \end{aligned}$$

where $\gamma = m(m-1)$. With the help of the determinantal representations ([23], [13]), ${}_0\mathcal{F}_1(\frac{3}{2}, z) = C \sinh(2\sqrt{z})/\sqrt{z}$ and ${}_0\mathcal{F}_1(1/2, z) = \cosh(2\sqrt{z})$ ([6]), one writes:

$$p_t^1(x, y) = \frac{e^{-(|x|^2+|y|^2)/2t}}{c_k t^{\gamma+m/2}} \left[\prod_{i=1}^m \left(\frac{x_i y_i}{2t} \right) {}_0F_1^{(1)} \left(m + \frac{1}{2}, \frac{x^2}{2t}, \frac{y^2}{2t} \right) + {}_0F_1^{(1)} \left(m - \frac{1}{2}, \frac{x^2}{2t}, \frac{y^2}{2t} \right) \right] V^2(y^2).$$

With regard to (1) and setting $q = 1 + (m-1)k_1$, it is natural to prove that:

Proposition 7. *For the D -type root system, the generalized Bessel function writes*

$$\frac{1}{|W|} \sum_{w \in W} D_k(x, wy) = \prod_{i=1}^m \left(\frac{x_i y_i}{2} \right) {}_0F_1^{(1/k_1)} \left(q + \frac{1}{2}, \frac{x^2}{2}, \frac{y^2}{2} \right) + {}_0F_1^{(1/k_1)} \left(q - \frac{1}{2}, \frac{x^2}{2}, \frac{y^2}{2} \right).$$

Proof: it uses the so-called *shift principle* which in turn relies on Opdam's characterization of D_k^W ([20]). Let us briefly outline the shift principle: Let E be a conjugacy class of roots of R under the action W . Let k_E be the value of the multiplicity function on this class. Then, the generalized Bessel function associated with the multiplicity function k' defined by:

$$k'(\alpha) = \begin{cases} k(\alpha) + 1 & \text{if } \alpha \in E \\ k(\alpha) & \text{otherwise} \end{cases}$$

is given by:

$$\frac{1}{|W|} D_{k'}^W(x, y) = C \sum_{w \in W} \xi_E(w) \frac{D_k(x, wy)}{p_E(x) p_E(y)}, \quad p_E(x) := \prod_{\alpha \in R_+ \cup E} \langle \alpha, x \rangle$$

where $\xi_E(w)$ is defined by $p_E(wx) = \xi_E(w) p_E(x)$ and C is some constant. Let W^B, W^D denote the reflection groups associated with the root systems of types B, D respectively and k^B, k^D be the corresponding multiplicity functions. Recall

that ([26]) W^B is the semi-direct product of S_m and $(\mathbb{Z}/2\mathbb{Z})^m$ (sign changes) while W^D is the semi-direct product of S_m and $(\mathbb{Z}/2\mathbb{Z})^{m-1}$ (even sign changes). Let $E := \{e_i, 1 \leq i \leq m\}$, then $p_E(x)$ is invariant under permutations and even sign changes and skew-invariant under odd sign changes (Note that $D_k^{W^D}$ is not W^B -invariant). It follows that $\xi_E(w) = 1$ for $w \in W^D$ while $\xi_E(w) = -1$ for $w \in W^B \setminus W^D$. Now, recall that k^D takes only one value while k^B takes two values. Since the Dunkl Laplacian Δ_k^B of B reduces to Δ^D when the value of the multiplicity function k_0 on E is zero ([36]), the (non-symmetric) Dunkl kernel D_k associated with $R = D_m$ is given by the Dunkl kernel D_k associated with $R = B_m$ when $k_E^B = 0$. In fact, this is true since D_k is the solution of a spectral problem which is independent of the reflection group. As a result,

$$\begin{aligned} \frac{1}{|W^D|} D_k^{W^D}(x, y) &= \frac{1}{|W^D|} \sum_{w \in W^D} D_k^{(D)}(x, wy) \\ &= \frac{1}{2|W^D|} \sum_{w \in W^B} (1 + \xi_E(w)) D_k^{(B, k_0=0)}(x, wy) \\ &= \frac{|W^B|}{2|W^D|} \left[\frac{1}{|W^B|} D_k^{W^B(k_0=0)} + \frac{p_E(x)p_E(y)}{C|W^B|} D_{k'}^{W^B(k_0=0)} \right], \end{aligned}$$

where we used the shift principle to derive the last line. Keeping in mind that

$$\frac{1}{|W^B|} D_k^{W^B}(x, y) = {}_0F_1^{(1/k_1)}\left(k_0 + (m-1)k_1 + \frac{1}{2}, \frac{x^2}{2}, \frac{y^2}{2}\right)$$

and using that $|W^B| = 2|W^D|$ ([26] p. 44), we are done with $C = 2^m$. ■

6.3. A second proof of the tail distribution. Here, we give a second way to derive the tail distribution of the first hitting time of the Weyl chamber of type D based on the result derived for the type B . Note that the symmetrical w.r.t. s_m of the generalized Bessel function of type D is the generalized Bessel function of type B for $k_0 = 0$, that is,

$${}_0F_1^{(1/k_1)}\left((m-1)k_1 + \frac{1}{2}, \frac{x^2}{2}, \frac{y^2}{2}\right).$$

It follows from this symmetry that

$$g(x) = 2 \int_{C_1} e^{-|y^2/2} {}_0F_1^{(1/k_1)}\left((m-1)k_1 + \frac{1}{2}, \frac{x^2}{2}, \frac{y^2}{2}\right) V(y^2) dy$$

where C_1 is the Weyl chamber of type B . This shows that g is invariant under the action of the Weyl group associated to the B -type root system which strictly contains the one associated with the D -type root system ([26]). Thus, one may determine the restriction of g on C_1 and we are led to the B -type setting with $k_0 = 0$. Following the scheme used for the B -type case with the eigenvalue $m + m(m-1) = m^2$, we are done. ■

7. ALCOVE-VALUED PROCESS: β -JACOBI PROCESS

7.1. Existence and uniqueness. Recall that the eigenvalues process of the real and the complex Jacobi matrix processes of parameters (p, q) is the unique strong solution of (we write λ_i instead $\lambda_i(t)$, [16]):

$$d\lambda_i = 2\sqrt{\lambda_i(1-\lambda_i)}d\nu_i + \beta \left[(p - (p+q)\lambda_i) + \sum_{j \neq i} \frac{\lambda_i(1-\lambda_j) + \lambda_j(1-\lambda_i)}{\lambda_i - \lambda_j} \right] dt$$

for $\beta = 1, 2, p \wedge q \geq m+1, m$ respectively subject to $0 < \lambda_m(0) < \dots < \lambda_1(0) < 1$. The β -Jacobi process is then defined as a solution, when it exists, of the above SDE with an arbitrary $\beta, p, q > 0$ up to time $t < \inf\{s, \lambda_m(s) = 0 \text{ or } \lambda_1(s) = 1\} \wedge \tau$. It is easy to see that if λ is a β -Jacobi process of parameters (p, q) , then $1 - \lambda$ is a β -Jacobi process of parameters (q, p) . The connection with root systems was already given in ([4]) where authors identified the hypergeometric series associated with the BC -type root system with the multivariate hypergeometric series defined via Jack polynomials. It states that:

$$(16) \quad d\phi_i = d\nu_i + \left[k_0 \cot \phi_i + k_1 \cot 2\phi_i dt + k_2 \sum_{i \neq j} [\cot(\phi_i + \phi_j) + \cot(\phi_i - \phi_j)] \right] dt$$

where $\lambda_i := \sin^2 \phi_i$ and

$$(17) \quad 2k_0 = \beta(p - q), \quad k_1 = \beta(q - (m - 1)) - 1, \quad 2k_2 = \beta.$$

Easy computations show that $\pi/2 - \phi$ satisfies (16) with (p, q) intertwined. Moreover $0 < \phi_1 < \dots < \phi < \pi/2$ so that ϕ evolves like particles in an interval. The involved root system is the non reduced BC_m defined by

$$\begin{aligned} R &= \{\pm e_i, \pm 2e_i, 1 \leq i \leq m, \pm(e_i \pm e_j), 1 \leq i < j \leq m\} \\ R_+ &= \{e_i, 2e_i, 1 \leq i \leq m, (e_i \pm e_j), 1 \leq i < j \leq m\} \\ \Delta &= \{e_i - e_{i+1}, 1 \leq i \leq m - 1, e_m\}. \end{aligned}$$

When $k_0 = 0$ ($p = q$), it reduces to the reduced C_m system

$$\begin{aligned} R &= \{\pm e_i \pm e_j, 1 \leq i < j \leq m, \pm 2e_i, 1 \leq i \leq m\} \\ R_+ &= \{e_i \pm e_j, 1 \leq i < j \leq m, 2e_i, 1 \leq i \leq m\} \\ \Delta &= \{e_i - e_{i+1}, 1 \leq i \leq m - 1, 2e_m\} \end{aligned}$$

and it is known as the *ultraspherical* case. Setting $\tilde{\phi}_i := \phi_i/\pi$, then the process is valued in the principal Weyl alcove ([26]) defined by:

$$\tilde{A} = \{\tilde{\phi} \in \mathbb{R}^m, \langle \alpha, \tilde{\phi} \rangle > 0 \forall \alpha \in \Delta, \langle \tilde{\alpha}, \tilde{\phi} \rangle < 1\}$$

where $\tilde{\alpha} = 2e_1$ is the highest positive root (that is $\tilde{\alpha} - \alpha \in R_+ \forall \alpha \in R$, [26]). The generator of ϕ writes:

$$\mathcal{L}g(\phi) := \frac{1}{2}\Delta g(\phi) - \langle \nabla g(\phi), \nabla \Phi(\phi) \rangle, \quad \Phi(\phi) = - \sum_{\alpha \in R_+} k(\alpha) \log \sin(\langle \alpha, \phi \rangle).$$

Thus, with minor modifications, Theorem 1 states that (16) has a unique strong solution for all $t > 0$ with $\lambda(0) \in \overline{\tilde{A}}$ provided that $k_0 > 0$, $k_1 > 0$, $k_2 > 0 \Leftrightarrow \beta > 0$, $p > q > (m-1) + 1/\beta$. Applying this to $\pi/2 - \phi$, this holds for $\beta > 0$, $q > p > (m-1) + 1/\beta$. Since the ultraspherical case still involves a reduced root system, then (16) has a unique strong solution for $p \wedge q > (m-1) + 1/\beta$ which simplifies to $p \wedge q > m$ in the real case $\beta = 1$ and $p \wedge q > m - 1/2$ in the complex one $\beta = 2$. To convince the reader, we show how Theorem 1 extends to the trigonometric setting: $\partial \tilde{A} = \cup_{\alpha \in \Delta} H_\alpha \cup H_{\alpha,1}$ where

$$H_{\alpha,1} = \{\tilde{\phi}, \langle \tilde{\alpha}, \tilde{\phi} \rangle = 1\} = \{\phi, \pi - \langle \tilde{\alpha}, \phi \rangle = 0\}.$$

Compared with (4), the convex function $x \mapsto -\ln(\langle \alpha, x \rangle)$ should be substituted by $\phi \mapsto -\ln(\sin(\langle \alpha, \phi \rangle))$ and one has to deal with an additional term in the expression of the boundary process $(L_t)_{t \geq 0} : \mathbf{1}_{\{\pi - \langle \tilde{\alpha}, \phi \rangle = 0\}}$. Then the occupation density formula writes:

$$\begin{aligned} \int_0^{\pi/2} L_t^a(\pi - \langle \tilde{\alpha}, \phi \rangle) |\theta'(a)| da &= \langle \tilde{\alpha}, \tilde{\alpha} \rangle \int_0^t |\theta'(\pi - \langle \tilde{\alpha}, X_s \rangle)| ds \\ &= \langle \tilde{\alpha}, \tilde{\alpha} \rangle \int_0^t |\theta'(\langle \tilde{\alpha}, X_s \rangle)| ds \end{aligned}$$

since $\cot(\pi - z) = -\cot(z)$. Hence, the same proof applies and Lemma 1 remains valid for $\tilde{\alpha} \in R_+$. Besides, either it will exist $\alpha \in \Delta$ such that $\langle \alpha, x \rangle = 0$ and Lemma 2 applies, or we will need to prove that $\langle n(x), \tilde{\alpha} \rangle \neq 0$ if x belongs only to $H_{\tilde{\alpha},1}$. Let us first recall that the highest root is the unique positive root such that $\tilde{\alpha} - \alpha \in R_+$ for all $\alpha \in R_+$. Thus it may be written as $\tilde{\alpha} = \sum_{\alpha \in \Delta} a_\alpha \alpha$ where $a_\alpha \geq 1$. Otherwise, if it exists $\alpha_0 \in \Delta$ such that $a_{\alpha_0} < 1$ and since $\tilde{\alpha}$ must be greater than all simple roots (in particular greater than α_0) then

$$\tilde{\alpha} - \alpha_0 = (a_{\alpha_0} - 1)\alpha_0 + \sum_{\alpha_0 \neq \alpha \in \Delta} a_\alpha \alpha = c_{\alpha_0} \alpha_0 + \sum_{\alpha_0 \neq \alpha \in \Delta} c_\alpha \alpha$$

for some $c_\alpha \geq 0$. Our claim follows from the fact that Δ is a basis. Next, it is not difficult to see from the definition of $n(x)$ and the fact that $\langle \alpha, x \rangle > 0$ for all $\alpha \in \Delta$ that $n(x)$ is colinear to $-\sum_{\alpha \in \Delta} \alpha$. It follows that

$$\langle n(x), \tilde{\alpha} \rangle = -c \sum_{\alpha \in \Delta} \langle \alpha, \tilde{\alpha} \rangle = -c \sum_{\alpha \in \Delta} \sum_{\theta \in \Delta} a_\alpha \langle \alpha, \theta \rangle.$$

If $\langle n(x), \tilde{\alpha} \rangle = 0$, then

$$\left\| \sum_{\alpha \in \Delta} \alpha \right\|^2 = \sum_{\alpha \in \Delta} \sum_{\theta \in \Delta} \langle \alpha, \theta \rangle \leq \sum_{\alpha \in \Delta} \sum_{\theta \in \Delta} a_\alpha \langle \alpha, \theta \rangle = 0$$

which implies that $n(x) = 0$. ■

Remark. Another alcove-valued process was studied in [24]: it evolves like particles on the circle and involves an affine root system of type A ([26]).

7.2. Brownian motion in the principal Weyl alcove. Let

$$h_1(\phi) := \prod_{\alpha \in R_+} \sin(\langle \alpha, \phi \rangle).$$

Then, h_1 is strictly positive on \tilde{A} and vanishes for $\phi \in \partial A$. One can also show that $(1/2)\Delta h_1 = ch$ for some strictly negative constant c (see the Appendix in [16]). Let $P_t^{h_1}$ denote the semi group given by:

$$P_t^{h_1} f(\phi) := e^{-ct} \frac{P_t(h_1 f)(\phi)}{h_1(\phi)},$$

where P_t denotes the semi group of m -independent BMs in A killed when it first reaches ∂A (one may derive the density of P_t using the semi group density of a BM killed when it first exits the interval $]0, \pi/2[$ and the Karlin-McGregor's formula, [35]). The corresponding generator writes :

$$\mathcal{L}^{h_1}(f) = \frac{1}{h_1} \left[\frac{1}{2} \Delta - c \right] (h_1 f) = \frac{1}{2} \Delta + \sum_{i=1}^m (\partial_i \log h_1) \partial_i f$$

which fits the generator \mathcal{L} of ϕ for $k_0 = k_1 = k_2 = 1$ corresponding to the parameters $\beta = 2, q = m + 1/2, p = m + 3/2$. Therefore, ϕ evolves like m BMs constrained to stay in the BC -type Weyl alcove. Note that p, q are not integers which means that ϕ or λ cannot be realized as eigenvalues process of a matrix-valued process (this is also the case for the BM in the B_m -Weyl chamber since $\delta = m + (1/2)$).

7.3. The first hitting time \tilde{T}_0 . We define similarly the first hitting time of the alcove's walls by $\tilde{T}_0 = \inf\{t > 0, (\phi(t)/\pi) \in \partial \tilde{A}\} = \tilde{T}_{\tilde{\alpha}} \wedge \inf\{\tilde{T}_\alpha, \alpha \in \Delta\}$, where

$$\begin{aligned} \tilde{T}_\alpha &:= \inf\{t > 0, \langle \alpha, \phi(t) \rangle = 0\}, \\ \tilde{T}_{\tilde{\alpha}} &:= \inf\{t > 0, \langle \tilde{\alpha}, \phi(t) \rangle = 2\phi_1 = \pi\}, \end{aligned}$$

and ϕ is the unique strong solution for all $t \geq 0$ of ⁶:

$$d\phi(t) = d\nu(t) + \sum_{\alpha \in R_+} k(\alpha) \cot(\langle \alpha, \phi(t) \rangle) \alpha dt, \quad \frac{\phi(0)}{\pi} \in \tilde{A},$$

⁶ $k(2e_i) = k_1/2$ for all $1 \leq i \leq m$.

for the non reduced root system $R = BC_m$ with $k(\alpha) > 0$ for all α and $p \wedge q > (m-1) + 1/\beta$. The finiteness of \tilde{T}_{α_0} for small values of the multiplicity is somewhat different from the one of T_{α_0} defined in Proposition 1. Our result summarizes as follows:

Proposition 8. • *If $k_2 < 1/2$ then $\langle \alpha_0, \phi \rangle$ hits 0 a.s. where $\alpha_0 \in \{e_i - e_{i+1}, 1 \leq i \leq m-1\}$.*
 • *If $2k_0 + k_1 < 1$ then ϕ_m and ϕ_1 hit zero and $\pi/2$ a.s. respectively.*

7.3.1. $\alpha_0 = e_i - e_{i+1}, 1 \leq i \leq m-1$. The same scheme described in the proof of Proposition 1 applies here since the main ingredients used there are the SDE (3) and the fact that $\sigma_0(\alpha) \in R_+$ if $\alpha \neq \alpha_0$. The second assertion follows from $\sigma_0(2e_j) = 2\sigma_0(e_j) = 2(\delta_{ij}e_{i+1} + \delta_{(i+1)j}e_i + \mathbf{1}_{\{j \neq i, j \neq i+1\}}e_j) \in R_+$. As a result, one writes for all $t \geq 0$:

$$d \langle \alpha_0, \phi(t) \rangle = \|\alpha_0\| d\gamma_t + k_2 \|\alpha_0\|^2 \cot \langle \alpha_0, \phi(t) \rangle dt + \sum_{\substack{\alpha \in R_+ \\ \alpha \neq \alpha_0}} k(\alpha) a(\alpha) \cot \langle \alpha, \phi(t) \rangle dt$$

where $a(\alpha) = \langle \alpha_0, \alpha \rangle$.

$$d \langle \alpha_0, \phi(t) \rangle = \|\alpha_0\| d\gamma_t + k_2 \|\alpha_0\|^2 \cot \langle \alpha_0, \phi(t) \rangle dt + F_t$$

where

$$F_t = \sum_{\substack{\alpha \in R_+ \setminus \alpha_0 \\ a(\alpha) > 0}} k(\alpha) a(\alpha) [\cot \langle \alpha, \phi(t) \rangle - \cot \langle \sigma_0(\alpha), \phi(t) \rangle],$$

where $\sigma_0 = \sigma_{\alpha_0}$. This drift is strictly negative on $\{\tilde{T}_{\alpha_0} = \infty\}$ since $\phi \mapsto \cot \phi$ is a decreasing function, $\langle \alpha_0, \phi(t) \rangle > 0$ and since :

$$\langle \alpha - \sigma_0(\alpha), \phi(t) \rangle = 2 \frac{a(\alpha)}{\|\alpha\|^2} \langle \alpha_0, \phi(t) \rangle > 0.$$

This implies that $\mathbb{P}_x(\forall t \geq 0, \langle \alpha_0, \phi(t) \rangle \leq Z_t) = 1$ where $\phi(0) = x$ and

$$dZ_t = \|\alpha_0\| d\gamma_t + \|\alpha_0\|^2 k_2 \cot(Z_t) dt, \quad Z_0 = \langle \alpha_0, \phi(0) \rangle = x$$

on the same probability space. Using (16) with $\beta = 1, m = 1$, one easily sees that $(Z_t)_{t \geq 0} = (\arcsin \sqrt{J_{\|\alpha_0\|^2 t}})_{t \geq 0}$ where J is a one dimensional Jacobi process of parameters $d = 2k_2 + 1, d' = 1$ (see [40]):

$$dJ_t = 2\sqrt{J_t(1-J_t)} d\gamma_t + (d - (d+1)J_t) dt, \quad 0 < k_2 < 1/2 \Leftrightarrow 0 < d < 2.$$

As J hits 0 almost surely when $0 < d < 2$ (use the skew product in [40] and properties of squared Bessel processes), then so does Z and therefore $\langle \alpha_0, \phi \rangle$ when $k_2 < 1/2 \Rightarrow \tilde{T}_{\alpha_0} < \infty$ a.s.

7.3.2. $\alpha_0 = e_m$. We will see that the finiteness of \tilde{T}_{e_m} involves both values k_0, k_1 since the root system is non reduced. In fact, compared with the previous case and the non affine setting as well, the difference arises from the fact that $\sigma_0(\alpha) \in R_+$ if $\alpha \in R_+ \setminus \{e_m, 2e_m\}$ and this claim is directly checked since for $\alpha = e_i \pm e_j$, this amounts to consider the reduced root system B_m , otherwise for $\alpha = e_i$ with $i \neq m$, $\sigma_0(e_i) = e_i$. Accordingly:

$$d \langle \alpha_0, \phi(t) \rangle = d\phi_m(t) = d\gamma_t + k_0 \cot(\phi_m(t))dt + k_1 \cot(2\phi_m) + F_t$$

where

$$F_t = \sum_{\substack{\alpha \in R_+ \setminus \{e_m, 2e_m\} \\ a(\alpha) > 0}} k(\alpha)a(\alpha)[\cot(\langle \alpha, \phi(t) \rangle) - \cot(\langle \sigma_0(\alpha), \phi(t) \rangle)]$$

and $R_+^1 = \{e_i - e_j, 1 \leq i < j \leq m\}$. Using once again (16), we shall compare this process with $(\arcsin \sqrt{J_t})_{t \geq 0}$ where

$$dJ_t = 2\sqrt{J_t(1 - J_t)}d\gamma_t + (d - (d + d')J_t)dt, \quad d' = k_1 + 1, \quad d = 2k_0 + k_1 + 1.$$

Hence, $\tilde{T}_{e_m} < \infty$ a.s. if $0 < 2k_0 + k_1 < 1/2 \Leftrightarrow \beta p - (\beta(m - 1)) < 2$. This agrees with the case $m = 1$ for which $p < 2$ (use the skew product in [40]). Finally, note that since $a(\alpha) = 0$ for $\alpha \in \{e_i, 2e_i, 1 \leq i \leq m - 1\}$, F only involves $k_2 = \beta$ which is independent from p, q . Keeping in mind that $\pi/2 - \phi$ is still a β -Jacobi process with (p, q) intertwined which has no effect on the strict negativity of F by the above remark, we conclude that $\tilde{T}_{\tilde{\alpha}} < \infty$ for $0 < \beta q - \beta(m - 1) < 2$. ■

7.4. Semi-group density. We end this paper by giving the semi group density of the β -Jacobi process. Before proceeding, we provide the reader with the explicit expressions derived in the univariate case $m = 1$ and the complex Hermitian one ($m \geq 1, \beta = 2$). Let $P_n^{r,s}$ denote the real Jacobi polynomial of degree n defined by ([1]):

$$P_n^{r,s}(\lambda) := \frac{(r+1)_n}{n!} {}_2F_1 \left(-n, n+r+s+1, r+1; \frac{1-\lambda}{2} \right),$$

for $\lambda \in [-1, 1]$, $r, s > -1$, where ${}_2F_1$ is the univariate Gauss hypergeometric function. These polynomials are orthogonal in

$$L^2([-1, 1], Z^{r,s}(\lambda)d\lambda := (1-\lambda)^r(1+\lambda)^s d\lambda)$$

Moreover, $(P_n^{r,s})_{n \geq 0}$ form a complete set of this Hilbert space and satisfy

$$\left\{ \sqrt{1-\lambda^2} \partial_\lambda^2 + [(s-r) - (s+r+2)\lambda] \partial_\lambda \right\} P_n^{r,s}(\lambda) = -n(n+r+s+1)P_n^{r,s}(\lambda) := d_n P_n^{r,s}(\lambda)$$

The above eigenoperator defines a diffusion which is related to the one we considered with $m = 1$ via the map $\lambda \mapsto (1-\lambda)/2$ and a deterministic time change

($t \mapsto t/2$). The semi group density w.r.t Lebesgue measure is written (see [41])

$$p_t^{r,s}(\theta, \lambda) = \sum_{n=0}^{\infty} e^{-2dnt} \frac{P_n^{r,s}(\theta) P_n^{r,s}(\lambda)}{\|P_n^{r,s}\|^2} W^{r,s}(\lambda)$$

where $p = 2(r + 1)$, $q = 2(s + 1)$ and $W^{r,s}(\lambda)d\lambda$ is the probability measure corresponding to the measure $Z^{r,s}(\lambda)d\lambda$. No closed forms seems to be known for this density, nonetheless an attempt to get a handier expression was tried in [14]. Multivariate analogs appeared in literature ([4], [27], [31] for instance) and are obtained by applying the Gram-Schmidt orthogonalization to the symmetric Jack polynomials w.r.t. measure

$$Z_m^{r,s,\beta}(\lambda)d\lambda := \prod_{i=1}^m \lambda_i^r (1 - \lambda_i)^s \prod_{1 \leq i < j \leq m} |\lambda_i - \lambda_j|^\beta d\lambda_1 \dots d\lambda_m$$

We shall denote them⁷ by $P_\tau^{r,s,\beta}$ for a given partition τ (instead of $G_\tau^{\alpha,\beta}$ used in literature) and recall that $(P_\tau^{r,s,\beta})_\tau$ are the unique symmetric polynomial eigenfunctions of the Laplace Beltrami operator

$$\begin{aligned} \mathcal{L}_{p,q}^\beta &:= 2 \sum_{i=1}^m \lambda_i (1 - \lambda_i) \partial_i^2 + \beta \sum_{i=1}^m \left[p - (p + q) \lambda_i + \sum_{j \neq i} \frac{\lambda_i (1 - \lambda_j) + \lambda_j (1 - \lambda_i)}{\lambda_i - \lambda_j} \right] \partial_i \\ &= 2 \sum_{i=1}^m \lambda_i (1 - \lambda_i) \partial_i^2 + \beta \sum_{i=1}^m [p - (m - 1) - (p + q - 2(m - 1)) \lambda_i] \partial_i + 2\beta \sum_{i \neq j} \frac{\lambda_i (1 - \lambda_j)}{\lambda_i - \lambda_j} \partial_i \end{aligned}$$

with $\beta(p - (m - 1)) = 2(r + 1)$, $\beta(q - (m - 1)) = 2(s + 1)$, associated with the eigenvalues

$$(18) \quad 2d_{n,\tau}^\beta := 2 \left[\sum_{i=1}^m \tau_i (\tau_i - 1 - \beta(i - 1)) + |\tau| (r + s + \beta(m - 1) + 2) \right], \quad |\tau| = n.$$

However, with regard to the strong uniqueness for all $t \geq 0$ previously derived, we shall restrict ourselves to $p \wedge q > (m - 1) + 1/\beta$. $\beta(q - (m - 1)) > 1$ is equivalent to $s > -1/2$ and $\beta(p - (m - 1)) > 1$ is equivalent to $r > -1/2$. As a result, $r, s > -1/2$.

It is known that the eigenvalues process of the complex Hermitian Jacobi process (or 2-Jacobi process) is the h -transform (in Doob's sense) for $h = V$ of a process whose components are real Jacobi processes of parameters $2(p - (m - 1)) = 2(r + 1)$, $2(q - (m - 1)) = 2(s + 1)$ constrained to never collide (or to stay in the A_{m-1} -type Weyl chamber, [16]). Here, V denotes as usual the Vandermonde function and satisfies (see appendix in [16])

$$\mathcal{A}V = cV = -m(m - 1) \left(\frac{2(m - 2)}{3} + \frac{p + q}{2} \right) V$$

⁷The normalization is different from the one used in [4], [31].

where \mathcal{A} is the generator of a process whose components are m independent real Jacobi processes. Noting that the parameters r, s are the same both in the univariate and in the multivariate cases, it follows by Karlin-McGregor formula ([28]) that the semi group density writes on $\{0 < \lambda_m < \dots < \lambda_1 < 1\}$

$$\begin{aligned}
K_t^{r,s,2}(\theta, \lambda) &:= e^{-ct} \frac{V(\lambda)}{V(\theta)} \det \left(\sum_{n=0}^{\infty} e^{-2n(n+r+s+1)t} P_n^{r,s}(\theta_i) P_n^{r,s}(\lambda_j) W^{r,s}(\lambda_j) \right)_{i,j} \\
&= e^{-ct} \det \left(\sum_{n=0}^{\infty} e^{-2n(n+r+s+1)t} P_n^{r,s}(\theta_i) P_n^{r,s}(\lambda_j) \right)_{i,j} \frac{W_m^{r,s,2}(\lambda)}{V(\theta)V(\lambda)} \\
&= e^{-ct} \left[\sum_{\sigma_1 \in S_m} \epsilon(\sigma_1) \sum_{n_1, \dots, n_m \geq 0} e^{-2 \sum_{i=1}^m n_i(n_i+r+s+1)t} \prod_{i=1}^m P_{n_i}^{r,s}(\theta_i) P_{n_i}^{r,s}(\lambda_{\sigma_1(i)}) \right] \frac{W_m^{r,s,2}(\lambda)}{V(\theta)V(\lambda)} \\
&= e^{-ct} \left[\sum_{\sigma_1, \sigma_2 \in S_m} \epsilon(\sigma_1) \sum_{n_1 \geq \dots \geq n_m \geq 0} e^{-2 \sum_{i=1}^m n_{\sigma_2(i)}(n_{\sigma_2(i)}+r+s+1)t} \prod_{i=1}^m P_{n_{\sigma_2(i)}}^{r,s}(\theta_i) P_{n_{\sigma_2(i)}}^{r,s}(\lambda_{\sigma_1(i)}) \right] \frac{W_m^{r,s,2}(\lambda)}{V(\theta)V(\lambda)}
\end{aligned}$$

Note that, for a given partition $(n_1 \geq \dots \geq n_m \geq 0)$ and a permutation $\sigma_2 \in S_m$, one has

$$\sum_{i=1}^m n_{\sigma_2(i)}(n_{\sigma_2(i)} + r + s + 1) = \sum_{i=1}^m n_i(n_i + r + s + 1)$$

Thus summing first over σ_1 with the change of variables $\sigma = \sigma_1 \sigma_2$, one gets:

$$\begin{aligned}
K_t^{r,s,2}(\theta, \lambda) &= e^{-ct} \sum_{n_1 \geq \dots \geq n_m \geq 0} e^{-2 \sum_{i=1}^m n_i(n_i+r+s+1)t} \frac{\det[P_{n_i}^{r,s}(\theta_j)]_{i,j}}{V(\theta)} \frac{\det[P_{n_i}^{r,s}(\lambda_j)]_{i,j}}{V(\lambda)} W_m^{r,s,2}(\lambda) \\
&= e^{-ct} \sum_{n_1 > \dots > n_m \geq 0} e^{-2 \sum_{i=1}^m n_i(n_i+r+s+1)t} \frac{\det[P_{n_i}^{r,s}(\theta_j)]_{i,j}}{V(\theta)} \frac{\det[P_{n_i}^{r,s}(\lambda_j)]_{i,j}}{V(\lambda)} W_m^{r,s,2}(\lambda)
\end{aligned}$$

Set $n_i = \tau_i + m - i$, then $\tau_1 \geq \dots \geq \tau_m \geq 0$. Moreover, with regard to (18), one easily checks that

$$d_{n,\tau}^2 = \sum_{i=1}^m \tau_i(\tau_i + r + s + 1 + 2(m - i))$$

so that

$$\sum_{i=1}^m n_i(n_i + r + s + 1) = d_{n,\tau}^2 - c/2$$

The final result writes

$$\begin{aligned} K_t^{r,s,2}(\theta, \lambda) &= \sum_{\tau_1 \geq \dots \geq \tau_m \geq 0} e^{-2r_{n,\tau}^2} \frac{\det[P_{\tau_i+m-i}^{r,s}(\theta_j)]_{i,j}}{V(\theta)} \frac{\det[P_{\tau_i+m-i}^{r,s}(\lambda_j)]_{i,j}}{V(\lambda)} W_m^{r,s,2}(\lambda) \\ &= \sum_{\tau_1 \geq \dots \geq \tau_m \geq 0} e^{-2r_{n,\tau}^2} P_\tau^{r,s,2}(\theta) P_\tau^{r,s,2}(\lambda) W_m^{r,s,2}(\lambda) \end{aligned}$$

where we used the determinantal representation of the Jacobi multivariate polynomials in the complex case⁸ (see [31]) :

$$P_\tau^{r,s,2}(\lambda) = \frac{\det[P_{\tau_i+m-i}^{r,s}(\lambda_j)]_{i,j}}{V(\lambda)}$$

From these observations, it is natural to claim that:

Proposition 9. *The semi group density of the β -Jacobi process is given by*

$$(19) \quad K_t^{r,s,\beta}(\theta, \lambda) := \sum_{n=0}^{\infty} \sum_{|\tau|=n} e^{-d_{n,\tau}^\beta} P_\tau^{r,s,\beta}(\theta) P_\tau^{r,s,\beta}(\lambda) W_m^{r,s}(\lambda) \mathbf{1}_{\{0 < \lambda_m < \dots < \lambda_1 < 1\}}$$

with respect to $d\lambda$. As a result, it is positive.

Proof: given a bounded symmetric function f on $[0, 1]^m$, define

$$T_t f(\theta) := \int_{0 < \lambda_m < \dots < \lambda_1 < 1} f(\lambda) \sum_{n=0}^{\infty} \sum_{|\tau|=n} e^{-2d_{n,\tau}^\beta} P_\tau^{r,s,\beta}(\theta) P_\tau^{r,s,\beta}(\lambda) W_m^{r,s}(\lambda) d\lambda$$

for $\theta = (0 < \theta_1 < \dots < \theta_m < 1)$ and $T_0 f = f$. The above expression makes sense: this uses the boundness of f , the exponential term with strictly positive t and Fubini Theorem. Besides, $T_t \mathbf{1} = 1$ and $\|T_t\|$ is bounded for all $t \geq 0$. The first claim follows easily from the orthogonality of the Jacobi polynomials and $P_0 = \mathbf{1}$ so that the only non zero term is that corresponding to $n = 0$. The second one is obvious for $t = 0$ and uses the exponential term when $t \geq \epsilon > 0$. One also easily checks that $T_t T_s = T_{t+s}$ and that $\mathcal{L} T_t f(\lambda) = \partial_t T_t f(\lambda)$ using the dominated convergence theorem. Now, let us consider the Cauchy problem associated to $\mathcal{L}_{p,q}^\beta$:

$$\begin{cases} \frac{\partial u_f}{\partial t}(t, \lambda) = \mathcal{L}_{p,q}^\beta u_f(t, \lambda) \\ u_f(0, \cdot) = f, \end{cases}$$

where $u_f \in C^{1,2}(\mathbb{R}_+^* \times S := \{0 < \lambda_m < \dots < \lambda_1 < 1\}) \cap C_b(\mathbb{R}^+ \cap S)$ with reflecting boundary condition :

$$\langle \nabla u(t, \lambda), n(\lambda) \rangle = 0, \quad (t, \lambda) \in \mathbb{R}_+^* \times \partial S$$

where $n(\lambda)$ is a unitary inward normal vector at λ . Define $u_t(f)(\lambda) := u_f(t, \lambda)$. It is known ([39]) that the above Cauchy problem has a unique solution. Consequently,

⁸We adopt a different normalization since we consider orthonormal polynomials.

$(T_t)_{t \geq 0}$ is the semi group of the eigenvalues process $(\lambda(t))_{t \geq 0}$ with density given by $K_t^{r,s,\beta}$. ■

Remark. For the real Jacobi matrix ($\beta = 1, [16]$), it is known that for $p \wedge q \geq m - 1$ and if the eigenvalues are distinct at time $t = 0$, then they remain distinct forever. It is then natural to wonder if this remains valid when starting from non distinct eigenvalues (see [16] p. 138-139). The Markov property together with the previous result for distinct eigenvalues are sufficient to claim that this is true provided that the eigenvalues semi group has a density which is absolutely continuous with respect to Lebesgue measure on \mathbb{R}^m . By virtue of $K_t^{r,s,1}(\theta, \lambda)$, for $p \wedge q > m$,

$$\mathbb{P}_{\lambda(0)}(\forall t \geq 0, \forall i \neq j, \lambda_i(t) \neq \lambda_j(t)) = 1, \quad \lambda_1(0) \geq \dots \geq \lambda_m(0).$$

We argue in the same way to claim that for $p \wedge q \geq m + 1$, the process will never hit 0 or 1 even if it did at time $t = 0$.

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