DUNKL OPERATORS : PROBABILISTIC OVERVIEW

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Dedicated to Professor Charles F. Dunkl

ABSTRACT. This survey may be considered as a continuation of the one written by M. Rösler on the theory of Dunkl operators. However, we focus on the probabilistic progress on this topic during approximately the last five years. We start with the study of the non symmetric case, that is, the right-continuous with left-hand limits Dunkl processes valued in a finite dimensional euclidean space V: time inversion property, two skew product decompositions, the analysis of jumps, martingale representation and Itô's formula, the Wiener Chaotic decomposition and an invariance principle. Some interesting consequences in the rank-one case are given, among which we cite the characterization of the Dunkl process using Gilat's Theorem. Then, we move to the symmetric or the so-called radial part: explicit expressions of the symmetrized Dunkl kernels are written down, a SDE with a singular drift is satisfied, absolute-continuity relations are set and precise results on the first hitting time of the boundary of the positive Weyl chamber are derived. The survey ends with some related processes such as Heckman-Opdam processes and eigenvalues of some matrix-valued processes as well as open questions.

1. MOTIVATION

During approximately the last decade, the study of Dunkl processes, after their introduction by Rösler in [49], has known a considerable growth and revealed interesting properties which allow to include these processes among various well-known families of homogeneous Markov processes:

- Starting with the Brownian scaling property, $\mathbb{R}^\star$-valued Dunkl processes are particular cases of the family of self-similar processes valued in a locally compact group which extends the famous class of Lamperti ([40]) and this aspect has been studied in [9].
- Their invariance under the action of a certain group of reflections provides them with a skew-product decomposition into radial and spherical parts ([8]), similar to the treatment in [47]. Moreover, a so-called k-invariance property satisfied by the semi-group density implies both the above mentioned invariance and the Feller property. The latter guarantees the existence of a right-continuous with left-hand limits (rcll) version.
- Dunkl processes enjoy the time-inversion property and are to our best knowledge the only multidimensional processes with jumps satisfying this property ([28]).
- They share with the multidimensional Brownian motion and Azéma martingales both martingale and Wiener chaotic representation properties ([29]).

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- Dunkl processes can be projected into a convex domain of their finite-dimensional state space yielding a diffusion that generalizes well-known diffusions obtained from eigenvalues of matrix-valued processes ([12]), in particular Brownian motions in Weyl chambers ([34], [4], [2]), and the reflected Brownian motion for which a Tanaka-type stochastic differential equation and a multidimensional Lévy-type Theorem are derived ([8]).

Apart from the above properties, the following holds:

- $\mathbb{R}$-valued Dunkl processes are the unique martingales such that their absolute values are Bessel processes ([30]).
- A Donsker-type invariance principle shows that Dunkl processes are a limit in distribution of rescaled generalized random walks ([31]).
- They can be constructed starting from their diffusion parts using independent Poisson processes and time changes. The latter are, in some particular cases, inverses of exponential functionals of Brownian motions with drifts ([8]).

The main motivation for writing this paper has been to gather this long but incomplete list of properties and to offer the opportunity to exhibit important however unpublished results on this topic obtained by O. Chybiryakov in his Ph.D.thesis. Though the manuscript is of expository nature, some improvements and new ways worth being explored are included.

2. Preliminaries: Probability and Harmonic Analysis

Some background on Dunkl operators and motivations leading to their study is provided in [52] and references therein and in [37] concerning facts on root systems. Let $(V, <>)$ be a finite dimensional euclidean space, R a reduced root system in V and let R_+, S denote its corresponding positive and simple sub-systems. Let k be a positive multiplicity function and $n := \dim V$. Then the Dunkl process X is a V-valued Feller process ([53]) with extended generator given for $u \in C^2(V)$ by

(1)

$$\mathscr{L}_k u(x) := \frac{1}{2}\Delta_k u(x) = \frac{1}{2}\Delta u(x) + \sum_{\alpha \in R_+} k(\alpha)\frac{< \nabla u(x), \alpha >}{< x, \alpha >} + \sum_{\alpha \in R_+} \frac{k(\alpha)|\alpha|^2}{2}\frac{u(\sigma_\alpha x) - u(x)}{< x, \alpha >^2}$$

where $|\alpha|^2 :=< \alpha, \alpha >$ and

$$\sigma_\alpha(x) = x - 2\frac{< \alpha, x >}{< \alpha, \alpha >}\alpha$$

denotes the reflection with respect to the hyperplane H_α. Δ_k is known as the *Dunkl Laplacian* and Δ is the usual Euclidean one. A first glance at $\mathscr{L}_k$ shows that it splits into two parts: a differential part and a difference part. Hence, one may think of the first part corresponding to a diffusion while, as we will see, the second one "hides" Poisson processes, therefore jumps.

Let $O(n)$ be the orthogonal group and let $W \in O(n)$ denote the reflection group spanned by $\{\sigma_\alpha, \alpha \in R\}$. Then, the difference part vanishes when restricted to W-invariant functions ($f(\sigma_\alpha x) = f(x), x \in V$) and the differential operator, known as the radial part of $\mathscr{L}_k$, is denoted by $\mathscr{L}_k^W$ (in fact, this superscript always refers to the W-invariant setting). Geometrically, this restriction amounts to bringing back each $x \in V$ to its unique conjugated representative in the positive Weyl chamber defined by:

$$C := \{x \in V, (< \alpha, x >) > 0, \alpha \in S\},$$

that is using the projection $\pi : V \leftarrow V/W$, where V/W denotes the space of conjugacy classes of V under the left action of W. The corresponding process has continuous paths and is called *radial Dunkl* process denoted by X^W. When $n = 1$, $\mathscr{L}_k^W, k > 0$ is the generator of a Bessel process of dimension $2k + 1$. The value $k = 0$ corresponds to a reflected Brownian motion. Notice that $\mathscr{L}_k^W$ has a singular drift as soon as k takes at least one strictly positive value, which may prevent X^W to exist for all times $t \geq 0$. Hopefully and similarly to the Bessel processes case, it is proved that the boundary ∂C is instantaneously reflecting when $k(\alpha) > 0$ for all $\alpha \in R$ and the existence of X^W for all $t \geq 0$ will follow provided that $k(\alpha) > 0$ for all $\alpha \in R$.
The semi-group density of X is written as ([52])

$$p_t^k(x, y) = \frac{1}{c_k t^{\gamma+n/2}} \exp - \left\{ \frac{|x|^2 + |y|^2}{2t} \right\} D_k \left(\frac{x}{\sqrt{t}}, \frac{y}{\sqrt{t}} \right) \omega_k(y)$$

with respect to $dy = dy_1 \dots dy_n$, where

$$\omega_k(y) := \prod_{\alpha \in R^+} | < \alpha, y > |^{2k(\alpha)}, \quad \gamma := \sum_{\alpha \in R^+} k(\alpha).$$

c_k is a normalizing constant and D_k is the so-called *Dunkl kernel* ([18]). Using $(X_t^W = y) = \cup_{w \in W}(X_t = wy), y \in C$ which is a disjoint union, one immediately deduces the semi-group density of X^W:

$$p_t^{k,W}(x, y) = \frac{1}{c_k t^{\gamma+n/2}} \exp - \left\{ \frac{|x|^2 + |y|^2}{2t} \right\} D_k^W \left(\frac{x}{\sqrt{t}}, \frac{wy}{\sqrt{t}} \right) \omega_k(y) \tag{2}$$

for $x, y \in C$, where

$$D_k^W(x, y) = \sum_{w \in W} D_k(x, wy)$$

is the *generalized Bessel function*, since it reduces to a Bessel function in the rank-one case.
An interesting subclass of W-invariant functions consists of those of the form $F(|x|)$ ($O(n)$-invariant). It is then easy to see that $\mathscr{L}_k$ then reduces to the infinitesimal generator of a Bessel process of dimension $\delta = 2\gamma + n$. This elementary property is of great relevance since on the one hand, Bessel processes are continuous and were intensively studied (see e.g. [55], Chap. XI). Another elegant proof uses a stretching of the uniform measure on the sphere, say μ_S, on its diameter ([53] p. 586), more precisely

$$\int_{S^{n-1}} D_k(ix, y)\omega_k(y)\mu_S(dy) = c(k) \int_{-1}^{1} e^{ir|x|}(1 - r^2)^{\gamma+(N-3)/2} dr = c(k) j_{\gamma+N/2-1}(|x|)$$

where j_s denotes the Bessel function of the first kind of index s ([42]). When acting on projections $u_i : x \mapsto x_i, 1 \leq i \leq n$, one gets

$$\mathscr{L}_k(u_i)(x) = 0$$

whence we deduce that each component X^i is a local martingale (with respect to the natural filtration of X). Since,

$$\sup_{s\leq t}\mathbb{E}(|X_s^i|) \leq \sup_{s\leq t}\mathbb{E}(|X_s|) < \infty$$

the X^i's are martingales. This property will be strongly needed to derive the martingale decomposition of X. In addition, when $n = 1$ (rank-one case), X is the unique martingale whose absolute value (positive submartingale) is a Bessel process, thereby solving Gilat's problem ([33]).
A distinguished feature of the Dunkl theory is the existence, for suitable values of k, of an *intertwining operator* for which we collect some needed facts.

2.1. **The intertwining operator.** It was shown ([18], [19]) that there is a subset $K \in \mathbb{C}$ containing the positive real half-line such that, for each $k \in K$, there exists an algebraic linear isomorphism V_k from the space of n-variables polynomials into itself characterized by the following:

- If $\mathscr{P}_d^n$ denotes the space of n-variables homogeneous polynomials of degree d, then $V_k\mathscr{P}_d^n = \mathscr{P}_d^n$ and $V_k\mathbf{1} = \mathbf{1}$ ($\mathbf{1}$ is the constant polynomial). V_k is said to be homogeneous of degree 0 or degree-preserving.
- If $T_\xi, \xi \in V \setminus \{0\}$ denotes the Dunkl directional derivative ([52]), then $T_\xi V_k = V_k\partial_\xi$.

V_k was even extended to a linear bounded operator from the algebra of absolutely convergent homogeneous series into itself ([18]) then to a homeomorphism from $C^\infty(V)$ onto itself ([57]). It is known that $\{T_\xi, \xi \in \setminus\{0\}\}$ form a commutative algebra ([18]). Thus, V_k interlaces between this algebra and its usual counterpart ($k = 0$) spanned by $\{\partial_\xi, \xi \in V \setminus \{0\}\}$. In particular, $\Delta_k V_k = V_k\Delta$ which is equivalent to $P_t^k V_k = V_k P_t^0$ where P_t^0, P_t^k denote the Brownian motion and Dunkl process semi-groups respectively. Such a relation was already of great interest in probability theory and we refer the interested reader to [6] and references therein for a list of interlaced Markov processes. Even more, authors there provides a fairly general framework for interlacing: Let $(X_t)_{t\geq 0}$ and $(Y_t)_{t\geq 0}$ be two measurable processes defined on the same probability space and taking values in two measurable spaces E and F respectively. Suppose that X and Y saistfy the following properties:

- there exist two filtrations $(\mathscr{G}_t)_{t\geq 0}$ and $(\mathscr{F}_t)_{t\geq 0}$ such that
  (1) $\mathscr{G}_t \subset \mathscr{F}_t$ for all t and
  (2) X is $(\mathscr{F}_t)_{t\geq 0}$-adapted and Y is $(\mathscr{G}_t)_{t\geq 0}$-adapted.
- X is Markovian w.r.t. $(\mathscr{F}_t)_{t\geq 0}$ with semi group $(P_t)_{t\geq 0}$ and Y is Markovian w.r.t. $(\mathscr{G}_t)_{t\geq 0}$ with semi group $(Q_t)_{t\geq 0}$.
- there exists a Markov kernel $\Lambda : E \to F$ such that $\forall t \geq 0$

$$\mathbb{E}[f(X_t)|\mathscr{G}_t] = \Lambda f(Y_t), \quad f : E \to \mathbb{R}.$$

Then, for every $s, t \geq 0$, one has

$$Q_t\Lambda f(Y_s) = \Lambda P_t f(Y_s), \quad a.s.$$

so that, under mild continuity assumptions, one obtains the identity

$$Q_t\Lambda = \Lambda P_t, \quad t \geq 0.$$

The proof of this quite interesting result is easy and sums up in writing the projection $\mathbb{E}(f(X_{t+s})|\mathscr{G}_s)$ of the $\mathscr{F}_{t+s}$-measurable variable X_{t+s} on the σ-algebra $\mathscr{G}_s \subset \mathscr{G}_{t+s} \subset \mathscr{F}_{t+s}$ in two different ways.
While V_k opened the way to set a parallel theory to Harish-Chandra's Spherical Harmonics (see [17] Ch.V), it gives an easy way to extend some properties of Brownian motions to Dunkl processes, such as the Wiener chaotic decomposition. More generally, when two Markov processes interlace, the intertwining operator allows to carry over well known properties enjoyed by one of them to the other, for instance we cite absolute continuity relations, limit theorems and time-reversal ([6]). The reader will notice from the examples given in [6] that the interlacing usually occurs as a multiplication operator by a random variable Z:

$$f \mapsto \mathbb{E}(f(\cdot Z))$$

which, as we will see a few lines later, may hold for V_k too. More precisely, V_k may admit the following integral representation

$$V_k f(x) = \int_{\Theta} f(\theta x)\phi(\theta)\mu_k(d\theta) \tag{3}$$

where Θ is a closed subset of the unit ball of $\mathbb{M}_n(\mathbb{R})$ if

- Θ is left and right W-invariant ($\omega\Theta = \Theta\omega = \Theta$).
- $\mu_k(d\theta)$ is a left and right W-invariant probablity measure ($\mu(\omega d\theta) = \mu(d\theta\omega) = \mu(d\theta)$).
- $\phi(\omega\theta\omega^{-1}) = \phi(\theta), \omega \in W$ and $\sum_{\omega\in W}\phi(\theta\omega) = |W|$.

Moreover, (3) simplifies to

$$V_k f(x) = \int_{\Theta} f(\theta x)\mu_k(d\theta)$$

for W-invariant functions. Letting $\psi := \phi - 1$, then $\psi(\omega\theta\omega^{-1}) = \psi(\theta)$, $\sum_{\omega\in W}\psi(\theta\omega) = 0$ and

$$\int_{\Theta} f(\theta x)\psi(\theta)\mu_k(d\theta) = 0$$

for W-invariant functions. It is not hard to see that if ψ is such that $\psi(\omega\theta) = \psi(\theta\omega) = \det(\omega)\psi(\theta)$ for all $\omega \in W$, then it satisfies the above properties. Indeed, $\psi(\omega\theta\omega^{-1}) = \psi(\theta)$ is straightforward,

$$\sum_{\omega\in W}\psi(\theta\omega) = \psi(\theta)\sum_{\omega\in W}\det(\omega) = 0$$

where the last equality follows from the change of variable $\omega \to s\omega$ for any reflection $s \in W$. Finally, for a W-invariant function f and using the W-invariance of Θ and μ, one has

$$\int_{\Theta} f(\theta x)\psi(\theta)\mu_k(d\theta) = \int_{\Theta} f(s\theta x)\psi(\theta)\mu_k(d\theta) = -\int_{\Theta} f(\theta x)\psi(\theta)\mu_k(d\theta)$$

for any reflection $s \in W$ and the integral vanishes. So far, Z is only known for few reflection groups: $W = (\mathbb{Z}/2\mathbb{Z})^n$ ([17] p. 201), $W = S_3$ ([19]), $W = S_2 \propto (\mathbb{Z}/2\mathbb{Z})^2$ corresponding to $R = B_2$ (but with a one parameter multiplicity function, [22]) and the dihedral group $I(2s), s \geq 1$ ([20]). In order to help the reader's motivation and interest after this long wave of information, we give an illustrating example: the rank-one case.

Here, $V = \mathbb{R}$ and $R = B_1 = \{\pm 1\}$ so that $W = \mathbb{Z}/2\mathbb{Z}$ acting by the sign change $x \mapsto -x$ which implies that $\Theta \in [-1, 1]$. We start with

$$\mathscr{L}_k u(x) = \frac{1}{2}u''(x) + \frac{k}{x}u'(x) - k\frac{u(x) - u(-x)}{x^2}$$

and

$$V_k(f)(x) = \frac{1}{B(k,1/2)}\int_{-1}^{1} f(\theta x)(1+\theta)(1-\theta^2)^{k-1}d\theta := \mathbb{E}(f(x\beta_k)).$$

where B stands for the Beta function, so that

$$\Theta =]-1.1[, \quad \phi(\theta) = (1+\theta), \quad \mu_k(d\theta) = (1-\theta^2)^{k-1},\ k > 0,$$

and β_k has the non symmetric Beta distribution of parameters $(k, k-1)$. From the definition of D_k ([18]), it follows that

$$D_k(x,y) = \frac{1}{B(k,1/2)}\int_{-1}^{1} e^{x\theta y}(1-\theta)^{k-1}(1+\theta)^k d\theta = e^{-xy}{}_1\mathscr{F}_1(k, 2k+1, 2xy) \tag{4}$$

where ${}_1\mathscr{F}_1$ is the confluent hypergeometric function ([42]). X^W is then a Bessel process of dimension $\delta = 2k+1 \geq 1$ (W-invariant functions are exactly even ones) and V_k interlaces between a Bessel process and a reflected Brownian motion. In that case, β_k reduces to the symmetric Beta distribution of parameter $k-1$ and this is a well known probabilistic fact (see [30]). More precisely, this follows from the intertwining property for squared Bessel processes which is a time-dependent analog of the Beta-Gamma algebra.
Similar results hold for $W = (\mathbb{Z}/2\mathbb{Z})^n$ (the n-th fold product of $\mathbb{Z}/2\mathbb{Z}$ acting by sign changes: $V = \mathbb{R}^n, R = \{\pm e_i, 1 \leq i \leq n\}$ so that $\mathscr{L}_k$ splits into a sum of n-copies of

$$\mathscr{L}_k^i u(x) = \frac{1}{2}\partial_i^2 u(x) + \frac{k}{x_i}\partial_i u(x) - k\frac{u(x_1,\ldots,x_i,\ldots,x_n) - u(x_1,\ldots,-x_i,\ldots,x_n)}{x_i^2}.$$

Thus, X^W is realized by n-independent Bessel processes. The intertwining operator is represented as ([17])

$$V_k(f)(x) = \left[\frac{1}{B(k,1/2)}\right]^n \int_{]-1,1[^n} f(x_1\theta_1,\ldots,x_n\theta_n)\prod_{i=1}^n (1+\theta_i)(1-\theta_i^2)^{k-1}d\theta_1\ldots d\theta_n$$

which simplifies in the W-invariant setting (that is for functions invariant under sign changes) to

$$V_k(f)(x) = \left[\frac{1}{B(k,1/2)}\right]^n \int_{]-1,1[^n} f(x_1\theta_1,\ldots,x_n\theta_n)\prod_{i=1}^n (1-\theta_i^2)^{k-1}d\theta_1\ldots d\theta_n$$

for $x = (x_1,\ldots,x_n) \in V$. Thus, β_k is a multivariate Beta distribution and the corresponding Dunkl kernel takes a product form, namely

$$D_k(x,y) = \left[\frac{1}{B(k,1/2)}\right]^n \int_{]-1,1[^n} \prod_{i=1}^n e^{x_i\theta_i y_i}(1+\theta_i)(1-\theta_i^2)^{k-1}d\theta_1\ldots d\theta_n \tag{5}$$

$$= \prod_{i=1}^n e^{-x_i y_i}{}_1\mathscr{F}_1(k, 2k+1, 2x_i y_i). \tag{6}$$

The paper now divides into two major parts devoted to a detailed discussion of the properties of both processes. The non invariant setting is of course much more technical than the W-invariant one for which explicit expressions of the semi-group densities for all irreducible infinite families of root systems by means of special functions are derived.

Remark. 1/If we identify $\theta = (\theta_1, \ldots, \theta_n)$ with a diagonal matrix, then one sees that V_k acts via random diagonal matrices.
2/ The explicit expressions of the Dunkl kernel given by (4) and (5) ($W = \mathbb{Z}/2\mathbb{Z})^n$) can be derived from probabilistic considerations. This was performed in [30] for $n = 1$ and its easy extension will be detailed later as well as further investigations ($R = B_2, D_2$).

3. Dunkl processes

Throughout this section, R will be normalized by $|\alpha|^2 = 2$ for all $\alpha \in R_+$.

3.1. **Self-similarity.** It is immediate to see from $\mathscr{L}_k$ or from p_t^k that the Dunkl process starting at $x \in V$ enjoys the Brownian scaling property, that is, it is semi-stable of index 1/2 :

$$(\sqrt{c}X_t^{x/\sqrt{c}})_{t\geq 0} \overset{d}{=} (X_{ct}^x)_{t\geq 0}, \quad c > 0.$$

In the rank-one case and when the space state is $\mathbb{R}^\star$, such a self-similar process was studied in [9] and belongs to a more wider class than the Lamperti class ([40]). Results there give a representation as an exponential of the sum of an independent Lévy process and a compound Poisson process N^U, with a random sign:

$$X_t = (-1)^{N_{A_t}} e^{\xi_{A_t} + N^U_{A_t}}, \quad N_t^U := \sum_{k=0}^{N_t} U_k \tag{7}$$

where $(A_t)_{t\geq 0}$ is the clock defined by

$$A_t := \int_0^t \frac{ds}{X_s^2}$$

and N is a Poisson process. This representation will be improved later when identifying ξ and N_t^U.

3.2. **Time-inversion property.** The above scaling property implies that $p_t(x, y) = p_1(x/\sqrt{t}, y/\sqrt{t})$. Together with $D_k(x, \lambda y) = D_k(\lambda x, y)$ for $\lambda \in \mathbb{C}$ ([52]) shows that the Dunkl process enjoys the time-inversion property, that is

$$(tX_{1/t})_{t>0}$$

is a *homogeneous* Markov process under the probability law P_x of X ([28]) which we shall denote by X^x. Indeed, the non-homogeneous semi-group density $q_{s,u}^{k,x}, s < u$ of X^x is given by ([28])

$$q_{s,u}^{k,x}(a, b) = \frac{1}{t^n} \frac{p_{1/u}^k(x, b/u)}{p_{1/s}^k(x, a/s)} p_{1/s-1/u}^k(b/u, a/s) = \exp\left\{-\frac{t|x|^2}{2}\right\} \frac{D_k(x, b)}{D_k(x, a)} p_t^k(a, b)$$

where $t = u - s$. Immediate consequences of this computation are:

- Trivially, the same holds for the radial part with D_k replaced by D_k^W.

- X^x is the h-transform of X in Doob's sense, h being given by
$$h(b) = h^{(x)}(b) = D_k(x,b) \quad \text{and} \quad \mathscr{L}_k[h^{(x)}](b) = \frac{|x|^2}{2}h^{(x)}(b)$$
where the last equality agrees with Proposition 2.31 in [52].
- The generator of X^x is given by
$$\mathscr{L}_k^x f = \frac{1}{D_k(x,\cdot)}\mathscr{L}_k(D_k(x,\cdot)f) - \frac{|x|^2}{2}f.$$
X^x is known as the Dunkl process with drift referring to both W-invariant rank-one and $k = 0$ cases for which it is respectively a Bessel process and a Brownian motion with drift (see [54]). For $k = 1$, X is a Bessel process of dimension 3 and X^x is the so-called hyperbolic Bessel process ([28]). Actually, Dunkl processes provide the first known example of multidimensional Markov processes with jumps enjoying this property (see [41] for further developments processes enjoying the time inversion property).

3.3. **Spherical Dunkl processes: a first Skew product decomposition.** It is well known that Brownian motion as well as rotationally invariant diffusions can be decomposed as the skew-product of a Bessel process and a time changed spherical Brownian motion ([32], see also [47] for diffusions in $\mathbb{R}^3$). The following Lemma shows the action of the orthogonal group $O(n)$ on $\mathscr{L}$:

Lemma 3.1. *Let X be a Dunkl process with root system R, mutiplicity function k and starting at $x \in V$. Then, for any $O \in O(n)$, θX is a Dunkl process with root system OR, multiplicity function $k_O : \alpha \in OR \mapsto k(O^t\alpha)$, starting at Ox.*

Proofs: One first proof, using the semi-group density and based on the injectivity of the Dunkl transform, was given in [53] and is valid for a more wider class of so-called *k-invariant* Markov kernels. This notion will be defined and used later when dealing with the Dunkl-Donsker principle.
A second one follows from easy computations of the action of $\mathscr{L}_k$ on $C^2(V)$-functions of the form $x \mapsto u(\sigma_\alpha x)$ for some $\alpha \in R$ and $u \in C^2(V)$, and the fact that $\theta \in O(n)$ is also a reduced root system. ∎
Let $x \in V \setminus \{0\}$. Using polar coordinates: $x = r\theta$ where
$$r = |x|, \quad \theta = \frac{x}{r} = (\theta_1, \ldots, \theta_n) \in S^{n-1}$$
The infinitesimal generator $\mathscr{L}_k$ splits into a sum of
$$(8) \quad \mathscr{L}_k^r = \frac{1}{2}\partial_r^2 + \frac{n+2\gamma-1}{2r}\partial_r$$
$$(9) \quad \mathscr{L}_k^\theta = \frac{1}{r^2}\left[\frac{1}{2}\Delta_{S^{n-1}} + \frac{1}{2\omega_k(\theta)} < P(\theta)\nabla_\theta, \nabla_\theta\omega_k(\theta) > + \sum_{\alpha\in R_+} k(\alpha)\frac{\sigma_\alpha - \mathbf{1}}{<\theta,\alpha>^2}\right]$$
where $\mathbf{1}$ is the identity operator, $\sigma_\alpha(u)(x) := u(\sigma_\alpha x)$ and $P = I_n - \theta\theta^T$ is a $n \times n$ matrix, I_n being the $n \times n$ identity matrix. (8) is obviously the generator of a Bessel process. (9) led O. Chybiryakov to the following definition ([8]):

Definition 3.1. The Markov process $(\Theta_t)_{t\geq 0}$ in S^{n-1} with extended generator given on $C^2(S^{n-1})$ by (9) will be called a spherical Dunkl process.

Both the existence and Markovianity of such a process follow from the skew-product presented in the next Theorem (see [8] p. 53):

Theorem 3.1. *Let X be a Dunkl process in V such that $k(\alpha) > 0$ for at least one α, then X can be decomposed as a skew-product:*

$$X_t = |X_t|\Theta_{A_t}, \quad A_t = \int_0^t \frac{ds}{|X_s|^2}$$

where Θ is a spherical Dunkl process independent of $|X|$.

Remark. Since $|X|$ is a Bessel process of dimension $\delta = 2\gamma + n > 2$, then A_t is a continuous, strictly increasing and additive functional and tends to ∞ a.s. so that its inverse $(\tau_t)_{t\geq 0}$ satisfies

$$\tau_t := \inf\{s \geq 0, A_s = t\} < \infty \quad \text{a.s.}$$

Thus Θ is equivalently defined by

$$\Theta_t := \frac{X_{\tau_t}}{|X|_{\tau_t}}, \quad t \geq 0$$

Corollary 3.1. *X^W admits a similar skew product decomposition with the radial part Θ^W of Θ. Moreover, the Dunkl process with drift $x \neq 0$ satisfies:*

$$X_t^x = |X_t^x|\Theta_{C_t}, \quad C_t = \int_t^{\infty} \frac{ds}{|X_s^{k,s}|^2}$$

where Θ is a spherical Dunkl process starting at $\Theta_0 = x/|x|$ and is independent of $(|X_t^x|)_{t\geq 0}$, the Bessel process with drift $|x|$.

Another interesting corollary concerns the hitting angle υ_t of X_t^x naturally defined by

$$\upsilon_t := \frac{X_t^x}{|X_t^x|} = \Theta_{C_t}$$

Let $T_a = \inf\{t \geq 0, X_t^{k,x} = a\}$, $a > 0$. Then the following extend results in [48]:

Corollary 3.2. *υ_{T_a} is independent of T_a.*

Proof: It follows from the strong Markov property of $|X^{k,x}|$ and the fact that C_{T_a} involves the motion of $|X^x|$ posterior to T_a. ■

3.4. **Analysis of Jumps.** Let us recall that the Lévy kernel $N(x, dy)$ of a homogeneous Markov process with semi-group $(P_t)_{t\geq 0}$ is determined by ([45] p.151):

$$\int f(y)N(x, dy) = \lim_{t\to 0} \frac{P_t f(x)}{t}$$

where f is a function in the domain of the infinitesimal generator vanishing in a neighborhood of x. Direct computations using $P_t f(x)/t = \mathscr{L}_k f(x) + \epsilon(t)$ for small t, with $\epsilon(t) \to 0$ when $t \to 0$, gives:

Proposition 3.1. *The Lévy kernel has the following form:*

$$N(x,dy)=\begin{cases}\displaystyle\sum_{\alpha\in R_+}\frac{k(\alpha)}{<\alpha,x>^2}\delta_{\sigma_\alpha x}(dy) & if \quad x\in V\setminus\cup_{\alpha\in R_+}H_\alpha\\ \displaystyle\sum_{\alpha\in R_+\setminus K}\frac{k(\alpha)}{<\alpha,x>^2}\delta_{\sigma_\alpha x}(dy) & if \qquad x\in\cup_{\alpha\in K}H_\alpha\\ 0 & if \qquad\qquad x=0\end{cases}$$

for any subset K of R_+. This shows that supp$N(x,dy)=\overline{\{y\in V/y=\sigma_\alpha x\neq x,\alpha\in R_+\}}$ which in turn implies that :

- The process can only jump from x to $\sigma_\alpha x\neq x$ for some $\alpha\in R_+$.
- There are no simultaneous jumps, that is, for each $x\in V\setminus\{0\}$ and each pair $(\alpha,\beta)\in R_+^2,\alpha\neq\beta$

  $$\mathbb{P}_x(\cup_{s\geq 0},\,\sigma_\alpha(X_{s-})=\sigma_\beta(X_{s-}),\text{ for some}\,\alpha\neq\beta\in R_+)=0$$

  and X cannot jump from 0. This is illustrated by the following ([29]):

Proposition 3.2. *Let*

$$f(x,y)=\mathbf{1}_{\{suppN(x,dz)\}^c}(y)$$

for $x\in V$. Then for all $t>0$,

$$\sum_{s\leq t}f(X_{s-},X_s)=0\quad P_x\,a.s.$$

Proof: This follows from the compensator of the positive discontinuous functional on the LHS of (10) ([45] p. 154)

$$\int_0^t ds\int_V f(X_{s-},y)N(X_{s-},dy) \tag{10}$$

and the definition of f. ■

Hence, when a jump occurs in the direction $\alpha\in R_+$, then

$$\Delta X_s:=X_s-X_{s-}=\sigma_\alpha X_{s-}-X_{s-}=-<\alpha,X_{s-}>\alpha.$$

Now, we will give two precise results about jumps occuring in a finite time interval. First, the magnitude of jumps in a finite interval is finite a.s. ([29]):

Proposition 3.3. *For all $t>0$ and $x\neq 0$, $\sum_{s\leq t}|\Delta X_s|<\infty\,P_x\,a.s.$*

Proof: Substituting the size of the jumps in the above sum and using the compensator of the resulting one, this amounts to showing that

$$H_t^\alpha:=\int_0^t\frac{ds}{|<\alpha,X_s>|}<\infty\quad P_x\,a.s.$$

for all $\alpha\in R_+$ such that $k(\alpha)>0$. It is sufficient to prove that $\mathbb{E}_x(H_t^\alpha)<\infty$ and this uses tedious computations that we detail below:
using the semi group density and Tonelli Theorem, it is equivalent to prove that

$$\mathbb{E}_x(H_t^\alpha)=\int_0^t ds\frac{e^{-|x|^2/(2s)}}{c_ks^{\gamma+n/2}}\int_V dye^{-|y|^2/(2s)}D_k\left(\frac{x}{\sqrt{s}},\frac{y}{\sqrt{s}}\right)\frac{\omega_k(y)}{|<\alpha,y>|}<\infty.$$

Using the fact that $D_k(\lambda x, y) = D_k(x, \lambda y)$ for $\lambda \in \mathbb{C}$ and that ω_k is homogeneous of degree 2γ, then the change of variable $y = sz$ in the second integral yields

$$\mathbb{E}_x(H_t^\alpha) = \int_0^t ds \frac{e^{-|x|^2/(2s)}}{c_k} s^{\gamma+n/2-1} \int_V dz\, e^{-s|z|^2/(2)} D_k(x,z) \frac{\omega_k(z)}{|<\alpha, z>|}.$$

Using the estimate $D_k(x,z) \leq e^{|x||z|}$ ([52]) and the Cauchy-Schwartz inequality, the integral w.r.t. the variable z is bounded by

$$\mathbb{E}_x(H_t^\alpha) \leq C \int_0^t ds \frac{e^{-|x|^2/(2s)}}{c_k} s^{\gamma+n/2-1} \int_V dz\, e^{-s|z|^2/(2)} e^{|x||z|} |z|^{2\gamma-2k(\alpha)} |<\alpha, z>|^{2k(\alpha)-1},$$

where C is a positive constant. Now, a suitable rotation $z \mapsto Oz$ allows us to assume that $\alpha = \sqrt{2}e_n$ where e_n is the n-th vector of the canonical basis of V. Using the generalized polar coordinates involving the radius ρ and the angles $\theta_1 \in [0, \pi/2]$ and $\theta_i \in [-\pi/2, \pi/2]$ for $1 \leq i \leq n-1$, one has

$$\int_V dz\, e^{-s|z|^2/(2)} e^{|x||z|} |<\alpha, z>|^{2k(\alpha)-1} = \int_0^\infty d\rho\, e^{-s\rho^2/(2)} e^{|x|\rho} \rho^{2\gamma+n-2} \int_{S^{n-1}} d\sigma \, \sin\theta_{n-1}^{2k(\alpha)-1}$$

where $d\sigma$ is the uniform probability measure on S^{n-1}. The last integral in the RHS converges since $k(\alpha) > 0$. Thus,

$$\mathbb{E}_x(H_t^\alpha) \leq \frac{C}{c_k} \int_0^\infty d\rho K(\rho) e^{|x|\rho} \rho^{2\gamma+n-2}$$

by Fubini-Tonelli Theorem, where

$$K(\rho) := \int_0^t ds\, e^{-s\rho^2/(2)} e^{-|x|^2/(2s)} s^{\gamma+n/2-1}.$$

Performing the change of variable $y = \rho^2/2$, then $K(\sqrt{2y})$ is the Laplace transform of the measure concentrated on the interval $[0, t]$ whose distribution function is given by

$$U(v) = \int_0^{v \wedge t} ds e^{-|x|^2/(2s)} s^{\gamma+n/2-1}.$$

By a Tauberian Theorem ([26]), $K(\sqrt{2y})$ is equivalent to $U(1/y)$ for large y and the result follows from easy computations. ■

The second one is proved in the same way and states that the Dunkl process has a finite number of jumps in a finite time interval. However, this requires a restriction on k:

Proposition 3.4. *Let $x \in V \setminus \{0\}$ and $k(\alpha) > 1/2, \alpha \in R_+$. Then, for all $t > 0$,*

$$\sum_{s \leq t} \sum_{\alpha \in R_+} \mathbf{1}_{\{X_s = \sigma_\alpha X_{s-} \neq X_{s-}\}} < \infty \quad P_x \, a.s.$$

3.5. **Martingale decomposition and Itô formula.** In this paragraph, we give the decomposition of X into its continuous and purely discontinuous parts. This relies on Itô's formula and the Lévy kernel $N(x, dy)$ used to compensate some discontinuous functionals. Recall that the general Itô's formula is written as for every $f \in C^2(V)$:

$$f(X_t) - f(X_0) - \int_0^t \partial_i f(X_{s-})dX_s^i = \frac{1}{2}\int_0^t \partial_{ij}^2 f(X_s)d\langle X^i, X^j\rangle_s^c + \sum_{s\leq t}[f(X_s) - f(X_{s-}) - \partial_i f(X_{s-})\Delta X_s^i] \tag{11}$$

where $\langle X^i, X^j\rangle^c$ denotes the continuous part of the bracket $[X^i, X^j]$ and, for sake of clarity, we used Einstein convention for summation. subtracting

$$\frac{1}{2}\int_0^t \mathscr{L}_k(f)(X_s)ds$$

from the LHS gives a local martingale since X^i is a martingale for all i. Compared with the RHS, this shows that the discontinuous functional there is compensated by

$$\frac{1}{2}\int_0^t \mathscr{L}_k(f)(X_s)ds - \frac{1}{2}\int_0^t \partial_{ij} f(X_s)d\langle X^i, X^j\rangle_s^c$$

But, using the expression of the Lévy kernel, this compensator is equal to

$$\begin{aligned}\int_0^t ds\int_V N(X_{s-}, dy)[f(y) - f(X_{s-}) - \partial_i f(X_{s-})(y - X_{s-})^i] &= \frac{1}{2}\int_0^t [\mathscr{L}_k - \Delta](f)(X_{s-})ds \\ &= \frac{1}{2}\int_0^t [\mathscr{L}_k - \Delta](f)(X_s)ds\end{aligned}$$

since $\{s, X_s \neq X_{s-}\}$ has zero Lebesgue measure. Finally,

$$\frac{1}{2}\int_0^t \Delta(f)(X_s)ds = \frac{1}{2}\int_0^t \partial_{ij} f(X_s)d\langle X^i, X^j\rangle_s^c$$

Specializing to $f_{ij}(x) = x_i x_j, 1 \leq i, j \leq n$ and using $\langle X^i, X^j\rangle^c = \langle X^{i,c}, X^{j,c}\rangle$ together with Lévy criterion, one concludes that X^c is a n-dimensional Brownian motion. For the purely discontinuous part, it only suffices to write $X_t = X_t - \sum_{s\leq t}\Delta X_s + \sum_{s\leq t}\Delta X_s$, then use $N(x, dy)$ to see that

$$\sum_{s\leq t}\Delta X_s = -\sum_{\alpha\in R_+}\sum_{s\leq t} < \alpha, X_{s-} > \mathbf{1}_{\{X_s = \sigma_\alpha X_{s-} \neq X_{s-}\}}\alpha$$

is compensated by

$$F_t := -\sum_{\alpha\in R_+} k(\alpha)\int_0^t \frac{ds}{< \alpha, X_{s-} >}\alpha = -\sum_{\alpha\in R_+} k(\alpha)\int_0^t \frac{ds}{< \alpha, X_s >}\alpha$$

Since $\sum_{s\geq t}\Delta X_s - F_t$ is a finite variation part martingale, it is purely discontinuous. This proves that ([29]):

Theorem 3.2. *Under P_x, the Dunkl martingale has the following decomposition :*

$$X_t = x + B_t + \sum_{\alpha\in R_+} M_t^\alpha \alpha$$

where B is a n-dimensional Brownian motion and $(M^\alpha)_{\alpha\in R+}$ is a family of purely discontinuous martingales given by :

$$M_t^\alpha = -\sum_{s\leq t} < \alpha, X_{s-} > \mathbf{1}_{\{X_s=\sigma_\alpha X_{s-}\neq X_{s-}\}} + k(\alpha)\int_0^t \frac{ds}{< \alpha, X_{s-} >}$$

Remark. Precise statements can be derived on $(M^\alpha)_{\alpha\in R_+}$: these martingales are normal in the sense of Meyer ([46]) :

$$< M^\alpha, M^\alpha >_t = \sum_{s\leq t}(\Delta X_s)^2 = t, \quad < M^\alpha, M^\beta >_t = 0\, a.s.$$

for $\alpha \neq \beta \in R_+$. The last assertion reflects the fact that simultaneous jumps are forbidden.

Corollary 3.3. *Let $f \in C^{2,1}(V \times \mathbb{R})$, then*

$$\begin{aligned} f(X_t,t) - f(X_0,0) &= \int_0^t < \nabla_x f(X_s-,s), dB_s > + \int_0^t \left[\partial_s f + \frac{1}{2}\mathscr{L}_k f\right](X_s,s)ds \\ &- \sum_{\alpha\in R_+}\int_0^t \frac{f(\sigma_\alpha X_{s-},s) - f(X_{s-},s)}{< \alpha, X_{s-} >} dM_s^\alpha. \end{aligned}$$

Proof: write the general Itô formula for $(X,t) \mapsto f(X_t,t)$ as done before and subtract needed terms to get a local martingale. It only remains to use the above decomposition in order to compute the bracket $(= \delta_{ij}t)$ and the discontinuous sum (in terms of M^α). ■

Corollary 3.4. *(Predictable representation property) Let $\mathscr{F}_\infty = \vee_{t\geq 0}\mathscr{F}_t$ denote the σ-field generated by the Dunkl process X. Then, every $(\mathscr{F}_t)_t$-local martingale Z under $P_x, x \in V$ may be written as*

$$Z_t = \mathbb{E}(Z_0) + \int_0^t < Z_s^c, dB_s > + \sum_{\alpha\in R_+}\int_0^t Z_s^\alpha dM_s^\alpha$$

where B is a n-dimensional Brownian motion and $Z^c, (Z^\alpha)_{\alpha\in R_+}$ are predictable processes satisfying

$$\int_0^t (Z_s^c)^2 ds < \infty, \quad \int_0^t (Z_s^\alpha)^2 ds < \infty, \quad P_x\, a.s.$$

Proof: It follows from the fact that local martingales of the form

$$M_t^f := f(X_t,t) - f(X_0,0) - \int_0^t [\partial_s f + \frac{1}{2}\mathscr{L}_k f](X_s,s)ds, \quad f \in C^{2,1}(V,\mathbb{R})$$

generate the space of locally square integrable martingales, in the sense of Kunita-Watanabe ([10], p. 246). ■

3.6. **Wiener Chaotic decomposition.** The Wiener chaos $C_m(X)$ of order $m \geq 1$ is defined as the subspace of $L^2(\mathscr{F}_\infty)$ spanned by integrals of the form:

$$\int_0^\infty dZ_{t_1}^{\epsilon_1} \int_0^{t_1-} dZ_{t_2}^{\epsilon_2} \dots \int_0^{t_{m-1}-} dZ_{t_m}^{\epsilon_m} f(t_1, \dots, t_m)$$

called *mixed* integrals, where $\{\epsilon_1, \dots, \epsilon_m\} \in [1..n + |R_+|]^m$, $Z^{\epsilon_i}, 1 \leq i \leq m$ is defined by

$$Z_t^{\epsilon_i} = \begin{cases} B_t^{\epsilon_i} & if \qquad \epsilon_i \in [1..n] \\ M_t^{\alpha_{\epsilon_i}} & if \quad \epsilon \in [n+1..n+|R_+|] \end{cases}$$

and f is a function defined on the simplex $\Delta_m = \{0 \leq t_m \leq \dots \leq t_1\}$ such that

$$\int_{\Delta_m} dt_1 \dots dt_m f^2(t_1, \dots, t_m) < \infty$$

We will show that X has the Wiener chaotic decomposition, that is

$$L^2(\mathscr{F}_\infty) = \bigoplus_{m \geq 0} C_m(X),$$

$C_0(X)$ is the space of constants. One possible proof is similar to the one given for the Brownian motion, mutatis mutandis, and uses the intertwining operator. Since V_k is degree-preserving, then the generalized monomials are naturally defined by:

$$m_\nu(x_1, \dots, x_n) = V_k(x \mapsto x_1^{\nu_1} \dots x_n^{\nu_n})(x)$$

and homogeneous monomials of degree $|\nu|$ form a basis of $\mathscr{P}_{|\nu|}$. Both properties allow to define the Dunkl-Hermite polynomials by

$$Q_\nu(x) = V_k(\prod_{i=1}^n H_{\nu_i}(\cdot))(x_1, \dots, x_n) := V_k(H_\nu)(x)$$

where H_{ν_i} is the classical Hermite polynomial orthogonal with respect to the standard Gaussian measure, and to see that Q_ν is homogeneous of degree $|\nu|$. H_ν is the classical multivariate Hermite polynomial and fits the generalized Hermite polynomial in the sense of Rösler in the case $k = 0$ ([52]). In order to set the link with stochastic processes, we introduce the space-time Dunkl-Hermite polynomials ([29]) as

$$Q_\nu(x, t) = V_k(\prod_{i=1}^n H_{\nu_i}(\cdot, t))(x) := V_k(H_\nu(\cdot, t))(x)$$

where $H_{\nu_i}(x_i, t) = t^{\nu_i/2} H_{\nu_i}(x_i/\sqrt{t})$ is the classical space-time Hermite polynomial and $H_\nu(\cdot, t)$ shall be called the classical multivariate space-time Hermite polynomial. Since

$$\left(\partial_t + \frac{1}{2}\Delta\right) H_\nu(x, t) = 0$$

(which follows from the fact that $(t^{\nu_i/2} H_{\nu_i}(B_t/\sqrt{t}))_{t \geq 0}$ is a martingale for a standard Brownian motion B) and

$$\mathbb{E}(B_t^\nu | \mathscr{F}_s^B) = H_\nu(B_s, s - t), \quad 0 \leq s \leq t$$

where W is a n-dimensional Brownian motion ([62]), the following is immediate from the intertwining relation and the Markovianity of X:

Proposition 3.5. *For all multi-index $\nu \in \mathbb{N}^n$ and $0 \leq s \leq t$,*

$$\mathbb{E}(m_\nu(X_t)|\mathscr{F}_s) = Q_\nu(X_s, s-t) \tag{12}$$

and $(Q_\nu(X_t,t))_{t\leq 0}$ is a $(\mathscr{F}_t)_{t\geq 0}$-martingale ($Q_\nu(x,t)$ is space-harmonic with respect to $\mathscr{L}_k$).

Now, we proceed to the proof of the chaotic decomposition. To reduce the problem, it suffices to find a total subset $E \subset L^2(\mathscr{F}_\infty)$ and to prove that every element in E can be expressed as a sum of mixed stochastic integrals. But, since any collection $(X_{t_1},\dots,X_{t_l})$ has an exponential moment, then the space of random polynomials $P(X_{t_1},\dots,X_{t_l}), l \geq 0, 0 \leq t_1 \leq \dots t_l$, is dense in $L^2(\mathscr{F}_\infty)$. Thus, the set of products of generalized monomials

$$\Lambda_l := \prod_{i=1}^{l} m_{\nu^i}(X_{t_i}),\ l \geq 0$$

where $(\nu^i)_i$ are multi-indices, forms a total subset of $L^2(\mathscr{F}_\infty)$. $|\nu^1|+\dots+|\nu^l|$ is the *total degree* of Λ_l. Putting $s = t = t_l$ in (12), one gets $m_{\nu^l}(X_{t_l}) = Q_{\nu^l}(X_{t_l}, 0)$. Besides, an application of Itô's formula to $Q_{\nu^l}(X_t, t-t_l), t \leq t_l$ together with the space-harmonicity of Q_{ν^l} gives

$$m_{\nu_l}(X_{t_l}) = Q_{\nu^l}(X_{t_{l-1}}, t_{l-1}-t_l) + \int_{t_{l-1}}^{t_l} < \nabla_x Q_{\nu^l}(X_{s-}, s-t_l), dW_u >$$
$$- \sum_{\alpha\in R_+} \int_{t_{l-1}}^{t_l} \frac{Q_{\nu^l}(\sigma_\alpha X_{s-}, s) - Q_{\nu^l}(X_{s-}, s)}{< \alpha, X_{s-} >} dM_s^\alpha$$

Note that $\nabla_x Q_{\nu^l}$ is a polynomial of degree $\leq |\nu^l| - 1$ and that the same holds for

$$\frac{Q_{\nu^l}(\sigma_\alpha x, t) - Q_{\nu^l}(x,t)}{< \alpha, x >}.$$

since the numerator vanishes when $x \in H_\alpha$. Since

$$\Lambda_l = m_{\nu^l}(X_{t_l})\Lambda_{l-1}$$

then, Λ_l is the sum of three terms: the first depends only on $X_{t_1},\dots,X_{t_{l-1}}$ and has degree equal to the one of (Λ_l), the remaining however depends moreover on $(X_u, t_{l-1} \leq u \leq t_l)$ and are of total degrees $\leq$ degree$(\Lambda_l) - 1$. The result follows then by a decreasing induction on l and the total degree of Λ_l similarly as for the Azéma-Emery martingales. ([62]).

3.7. **The rank-one case and Gilat's Theorem.** Recall that the rank-one case corresponds to $R = \pm\sqrt{2}, V = \mathbb{R}$ $(n = 1)$. There is only one conjugacy class so that $k(\alpha) := k \geq 0$ and $W = \{Id, \sigma\}$ where $\sigma(x) = -x$. Gilat's Theorem states that every non-negative submartingale is the absolute value of a martingale ([33], the converse is obvious). It was shown in [30] that the one dimensional Dunkl martingale solve "Gilat's problem" when the submartingale is a Bessel process of dimension $\delta = 2k_0 + 1 \geq 1$. It suffices to show that a martingale whose absolute value is a Bessel process is a Markov process with generator

$$\mathscr{L}_k(f)(x) = \frac{1}{2}f''(x) + \frac{k}{x}f'(x) - k\frac{f(x) - f(-x)}{2x^2}, \quad x \in \mathbb{R}, f \in C^2(\mathbb{R}).$$

Since X can only jump at a given time s from X_{s-} to $\sigma(X_{s-}) = -X_{s-} = X_s$, then $\Delta X_s = -2X_{s-}$. The Lévy kernel is given by:

$$N(x, dy) = \frac{k}{2x^2}\delta_{(-x)}(y), \quad x \neq 0.$$

It follows that

$$\sum_{s \leq t}[f(X_s) - f(X_{s-}) - f'(X_{s-})\Delta X_s]$$

is compensated by

$$k\int_0^t \frac{f(-X_{s-}) - f(X_{s-}) + 2X_{s-}f(X_{s-})}{2X_{s-}^2}ds = k\int_0^t \frac{f(-X_s) - f(X_s) + 2X_s f'(X_s)}{2X_s^2}ds.$$

$$= \int_0^t [\mathscr{L}_k f - (1/2)\partial^2 f](X_s)ds$$

Finally, we write Itô's formula and use the fact that X is a martingale to see that

$$f(X_t) - f(X_0) - \int_0^t \mathscr{L}_k f(X_s)ds$$

defines a local martingale. ■

3.8. **Dunkl process and martingale problems.** For the sake of completeness, we recall some definitions from [24] concerning martingale problems. Let (S, d) be a metric space equipped with the topology induced by the metric d and its Borel σ-field. Let $D(\mathbb{R}_+, S)$ be the space of rcll functions valued in S. Denote by $\mathcal{P}$ the set of Borel probability measures. Given a linear operator $\mathscr{A}$, let $\mathscr{D}(A)$ denote its domain. Let X be a measurable process defined on some probability space $(\Omega, \mathscr{F}, \mathbb{P})$ taking values in S. For $\mu \in \mathscr{P}(S)$, we say that X solves the $D(\mathbb{R}_+, S)$ *martingale problem* $(\mathscr{A}, \mu)$ if X has sample paths in $D(\mathbb{R}_+, S)$, $\mathbb{P}(X_0 \in \cdot) = \mu(\cdot)$ and for $f \in \mathscr{D}(A)$,

$$M_t := f(X_t) - f(X_0) - \int_0^t \mathscr{A}f(X_s)ds$$

defines a $(\mathscr{F}_t^X)$-martingale. If a martingale problem admits a unique solution, then we say that it is well-posed. The following was proved in [53] and states that the Dunkl process solves the $D(\mathbb{R}_+, V)$ martingale problem associated with the restriction of $\mathscr{L}_k$ on $C_c^2(V)$, the set of compactly-supported and twice-differentiable functions defined on V. Moreover, the local martingale problem is well-posed under the assumption that $|X|$ is a positive real-valued diffusion.

Theorem 3.3. *Let X be a process with sample paths in $D(\mathbb{R}_+, V)$ such that $|X|$ has continuous paths. Then the following are equivalent:*

- *X is the Dunkl process.*
- *For any $f \in C^2(V)$, M^f is a $(\mathscr{F}_t^X)$-local martingale.*
- *For any $f \in C_c^2(V)$, M^f is a $(\mathscr{F}_t^X)$-martingale.*

Proof: It is based on the two ingredients below:

(1) for each $y \in V$,

$$\left(\frac{D_k(X_t, iy)}{\widehat{P_t^k(0,\cdot)}(y)} \right)_{t \geq 0}$$

is a $\mathscr{F}_t^X$-martingale (the result is valid for a more general class of processes). The denominator in (1) is the Dunkl transform of the Markov kernel $P_t^k(0, dz) := p_t^k(0, z)dz$ defined by:

$$\widehat{P_t^k(0,\cdot)}(y) := \int_V p_t^k(0, z) D_k(iy, z) dz.$$

This is the analogue of

$$\left(\frac{e^{iyX_t}}{\mathbb{E}(e^{iyX_t})} \right)$$

for Lévy processes.

(2) One has

$$\widehat{P_t^k(0,\cdot)}(y) = e^{-t|y|^2}$$

which, together with the above martingale characterizes, as in the Brownian motion case, the Dunkl process. ■

Remark. Further extensions of well-known results for Brownian motions were derived in [53], we cite both the recurrence and transience properties and a Dunkl version of the Ornstein-Ulhenbeck process.

3.9. **Dunkl-Donsker invariance principle.** As is well-known, Donsker's invariance principle is a functional version of the central limit Theorem. What makes this result crucial is that it allows to translate some properties from Brownian motions to random walks and vice versa.

The statement of this principle is as follows ([55]): let $(S_j^x = Z_1 + \cdots + Z_j)_{j \geq 1}$ be a classical random walk in $\mathbb{R}^n$ started from $x \in \mathbb{R}^n$, where $(Z_j)_{j \geq 1}$ are independent random vectors of mean zero and covariance matrix given by $\text{cov}(Z_j, Z_j) = \sigma_j^2 I_n, \sigma_k > 0$ for all $r \geq 1$. Write $S_j^x = (S_j^{x,r})_{1 \leq r \leq n}$, then the sequence of processes defined by :

$$\frac{1}{\sqrt{j}} \left(\frac{S_{[jt]}^{x,r}}{\sigma_r}, \, t \geq 0 \right)_{1 \leq r \leq n}, \quad j \geq 1$$

converges as $j \to \infty$ to a n-dimensional Brownian motion in the Skorohod space $D(\mathbb{R}_+, \mathbb{R}^n)$. Since $D_k(x, y), x, y \in V$ reduces to $e^{<x,y>}$ for $k = 0$ ([31]) and for a simple random walk with transition kernel $P(x, \cdot), x \in \mathbb{R}^n$ satisfies

$$(\widehat{P(x,\cdot)})(\xi) = e^{<ix,\xi>} \widehat{(\mu(\cdot))}(\xi)$$

where $\mu(\cdot) := P(0, \cdot)$ and $\widehat{\cdot}$ stands for the usual Fourier transform, then authors in [31] extended the above principle to the Dunkl setting. We shall call it the *Dunkl-Donsker invariance principle.*

Its proof is a Dunkl version of the classsical one since all the needed ingredients, such as Fourier-transform and Lévy continuity Theorem, have their Dunkl counterparts. The first one is given by the Dunkl-transform, replacing in the defining integral $e^{i<x,\cdot>}$ by

$D_k(ix, \cdot)$ ([52]). The second was derived in ([32]). Of course, this suggests to introduce *Dunkl-moments* by successive derivatives with respect to the parameter x. For instance, first and second ones are respectively $-i\nabla$ (the gradient) and $-\nabla\nabla^T$ (The Hessian matrix) evaluated at $x = 0$.
Next, one requires the splitting of V into a direct sum of orthogonal subspaces $\oplus_{j=0}^{l} V_j$, which in turn splits $D_k(x, y)$ into a product of $D_k(x^j, y^j), 0 \leq j \leq l$ where $x = x_0 + \cdots + x_l, y = y_0 + \cdots + y_l$. This is equivalent to the fact that the coordinates of the Dunkl process in this new decomposition are independent. Then, the following extends the simple random walk:

Definition 3.2. A generalized random walk $(S_k)_{k\geq 0}$ in $\mathbb{R}^m$ is a Markov chain with transition kernel $P(x, \cdot), x \in \mathbb{R}^n$ such that

$$\mathscr{F}_k(P(x, \cdot))(\xi) = D_k(ix, \xi)\mathscr{F}_k(\mu(\cdot))(\xi) \tag{13}$$

where $\mathscr{F}_k$ denotes the Dunkl transform and $\mu := P(0, \cdot)$. More generally, a Markov kernel satisfying (13) is called k-invariant ([53]) and we refer the reader to [53] for some examples.

It only remains to use the Markovianity and an additional criterion which, combined with the convergence of finite-dimensional distributions, yield the functional one.

Proposition 3.6.
- *There exists a unique decomposition of V as a direct orthogonal sum of W-irreducible subspaces*

$$V = \overset{l}{\underset{r=0}{\oplus}} V_r$$

where $V_0 = \cap_{\alpha \in R} H_\alpha$.
- *Let $(S_j)_{j\geq 0}$ be a generalized random walk with $\mu(\cdot) = P(0, \cdot)$ having zero first Dunkl moment and second Dunkl moment equal to $\sigma_r^2 I_r$ on V_r (I_r being the identity matrix of size dim(V_r). Let $S_j^r, 0 \leq r \leq l$ be the coordinates of S_k with respect to the above decomposition. Then the sequence of processes defined by :*

$$\frac{1}{\sqrt{j}} \left(\frac{S_{[jt]}^{x,r}}{\sigma_r}, t \geq 0 \right)_{0 \leq r \leq l}, \quad j \geq 1$$

converges in distribution as $j \to \infty$ to a $l + 1$-dimensional Dunkl process.

Remarks. 1/The symbol V was used in [31] to denote the space spanned by the root vectors which may be different from the euclidean space that contains the root system. This is for instance the case of the A_m-type root system which spans a hyperplane of $\mathbb{R}^{m+1} = V$ (in fact, $\cap_{\alpha \in \mathbb{R}^+} H_\alpha$ is spanned by the vector $(1, \ldots, 1) \in \mathbb{R}^{m+1}$).
2/When $k = 0$, one has $P(x, \cdot) = \delta_x \star P(0, \cdot)$ which is coherent since the usual convolution is probability-preserving. This is no longer true in the Dunkl setting for which the convolution is given by the generalized translation $\star_k$ and $P(x, \cdot)$ may be a signed measure. To get a probability measure, one should restrict this convolution to a smaller class of measures (see [51] for details).

4. Radial Dunkl processes

Hereafter, R is a reduced root system with arbitrary root lengths. Let X be a Dunkl process associated with R and a multiplicity function $R_+ \ni \alpha \mapsto k(\alpha) \in \mathbb{R}_+$. Since $V = \sum_{\omega \in W} w\overline{C}$ and each $x \in V$ is conjugated with one and only one $y \in \overline{C}$, then one constructs a *continuous-paths* $\overline{C}$-valued process by identifying each X_t with its unique representative in $\overline{C}$. This is equivalent to use the projection π from the Euclidean space V into the space V/W of orbits of the action of W on V and one defines the resulting process by

$$X_t^W := \pi(X_t) \quad t \geq 0,$$

which is called the *radial Dunkl process*. It is a diffusion with infinitesimal generator $\mathscr{L}_k^W$ acting on $C^2(\overline{C})$-functions u as follows:

$$\mathscr{L}_k^W u(x) := \frac{1}{2}\Delta_k^W u(x) = \frac{1}{2}\Delta u(x) + \sum_{\alpha \in R_+} k(\alpha) \frac{< \nabla u(x), \alpha >}{< x, \alpha >} \tag{14}$$

with the boundary condition $< \nabla u(x), \alpha >= 0$ whenever $< \alpha, x >= 0$. The operator $\mathscr{L}^W$ is the radial part of $\mathscr{L}$, that is, $\mathscr{L}_k^W u = \mathscr{L}_k f$ when the latter acts on W-invariant $C^2(V)$-functions f and u is then the restriction of f to $\overline{C}$. The basic example is provided by the rank-one case B_1 for which

$$\mathscr{L}_k^W u(x) = \frac{1}{2}u''(x) + \frac{k}{x}u'(x)$$

so that when $k > 0$, X^W is a Bessel process of dimension $\delta = 2k + 1 > 1$ (or index $\nu = k - 1/2$. When $k = 0$, X is a Brownian motion and X^W is a reflected Brownian motion, that is, the absolute value of a Brownian motion. For higher ranks and when k is not identically zero, then X^W is closely related to eigenvalues processes of matrix-valued processes. An interesting case is given by the so-called the Brownian motion in the Weyl chamber which fits X^W when $k \equiv 1$. When $k \equiv 0$, X^W is a reflected n-dimensional Brownian motion. We postpone the description of these processes after providing the reader with the following two results:

Theorem 4.1. *Let $(B_t)_{t \geq 0}$ be a n-dimensional Brownian motion and k a strictly positive multiplicity function. Then, the SDE*

$$dY_t = dB_t + \sum_{\alpha \in R_+} k(\alpha) \frac{dt}{< \alpha, Y_t >}\alpha, \quad Y_0 \in \overline{C} \tag{15}$$

has a unique strong solution for all time $t \geq 0$. Moreover, this solution is a radial Dunkl process X^W starting at $X_0^W = Y_0$.

Proof: Three proofs of this result are known: the first proof is in [56] as the limiting (in law) process of a similar existence and uniqueness result on radial Heckman-Opdam processes, then in ([8]) as the limiting (in law) process of a sequence involving the Dunkl process X using the fact that the latter is the unique solution of the martingale problem associated with $(\mathscr{L}_k, D(\mathbb{R}_+, V)$ (however, the author supposed that the simple system is a basis of V which is not true when $R = A_{n-1}$), and finally in [12] using a result of Cépa and Lépingle ([7]) on the existence and uniqueness of a solution of SDEs with a singular drift given by a convex function. This third proof is inspired from results on

Bessel processes of dimensions > 1 which is a semimartingale with a.s. zero local time at 0 ([55]). ■

Remark. When $k(\alpha) = 0$ for at least one α and $X_0 \in C$, then there exists a unique strong solution up to time T_0 ([8]).

Theorem 4.2. *Let T_0 denote the first hitting time of the boundary of the Weyl chamber:*

$$T_0 := \inf\{t > 0,\, X_t^W \in \partial C\} \tag{16}$$

where X^W is a radial Dunkl process starting at $X_0 = x \in C$. Then

- *If $k(\alpha) \geq 1/2$ for all $\alpha \in S$, then $T_0 = \infty$ a.s.*
- *If $0 \leq k(\alpha) < 1/2$ for at least $\alpha \in S$, then $< s, X^W >$ hits a.s. zero so that $T_0 < \infty$ a.s.*

Proof: note first that

$$T_0 = \inf_{\alpha \in S} \inf\{t > 0, < \alpha, X_t >= 0\} := \inf_{\alpha \in S} T_\alpha.$$

Both statements about the finiteness of T_0 were first proved in [8] using local martingales arguments, that is, for $x \in C$,

$$\prod_{\substack{\alpha \in R_+ \\ k(\alpha) > 1/2}} < \alpha, x >^{1-2k(\alpha)}, \qquad \prod_{\substack{\alpha \in R_+ \\ k(\alpha) \neq 1/2}} < \alpha, x >^{1-2k(\alpha)} \log \prod_{\substack{\alpha \in R_+ \\ k(\alpha) = 1/2}} < \alpha, x >$$

define harmonic functions with respect to $\mathscr{L}_k^W$ which vanish on the boundary, and the fact that a continuous local martingale is a time-changed Brownian motion which can not reach infinity in a finite time. Another proof of the second statement and the precision that $< s, X^W >$ hits zero a.s. for small multiplicities is given in [12]. This proof relies on stochastic calculus techniques and a comparison Theorem and is an adaptation to arbitrary reduced root systems of the one used in [7] for the A-type root system. ■

Let us now discuss some examples showing some realizations of X^W as the process of eigenvalues for some matrix-valued processes.

4.1. **The A-type root system and symmetric and Hermitian BMs.** Let $(B^i)_i$, $(B^{ij,1})_{i,j}, (B^{ij,2})_{i,j}$ be independent families of independent real BMs. The $n \times n$ symmetric BM is a $Sym(n, \mathbb{R})$-valued process G whose entries $(G^{ij})_{i,j=1}^n$ are given by

$$G^{ij} = \begin{cases} B^i & \text{if} \quad i = j \\ B^{ij,1} & \text{if} \quad i < j. \end{cases}$$

Its Hermitian version, known as the Dyson-BM ([23]), is defined by

$$G^{ij} = \begin{cases} B^i & \text{if} \quad i = j \\ \dfrac{B^{ij,1} + \sqrt{-1}B^{ij,2}}{\sqrt{2}} & \text{if} \quad i < j. \end{cases}$$

The eigenvalues processes $\lambda = (\lambda^1, \lambda^2, \ldots, \lambda^m)$ of these matrix-valued processes satisfy the SD systems

$$d\lambda_t^i = dZ_t^i + k \sum_{j \neq i} \frac{dt}{\lambda_t^i - \lambda_t^j}, \quad 1 \leq i \leq n,\, t < \tau,$$

for $k = 1/2, 1$ respectively, where $(\lambda_0^1 > \cdots > \lambda_0^n)$, $(Z^1, \dots, Z^n)$ is a n-dimensional BM and

$$\tau = \inf\{t > 0,\ \lambda_t^i = \lambda_t^j \text{ for some } (i,j)\}$$

is the first collision time. The above SDE is obviously a particular case of (16) when $R = A_{n-1}$ and $T_0 = \tau$. As mentioned above, the process λ with an arbitrary $k > 0$ was deeply studied in [7] where Theorem (4.1) was proved for this special case. However, to the best of our knowledge, we do not know of any underlying matrix-valued process except the so called geometric cases for which $k = 1/2, 1, 2$. Nevertheless, in the independent-time regime $t = 1$ and for $\lambda_0 = 0_{\mathbb{R}^n}$, the geometric cases are known as the eigenvalues of random matrices from the GOE, GUE respectively and tridiagonal matrix models were derived in [16] for arbitrary $k > 0$.

4.2. **The B-type root system and Wishart and Laguerre processes.** Let G be a $m \times n$ real Brownian matrix, that is, a matrix whose entries are nm independent real BMs. The Wishart process is a $n \times n$ matrix-valued process defined by $G^T G$ ([5]). Its Hermitian version, called the Laguerre process ([39], [11]), is defined in the same way starting with a complex Brownian matrix and substituting the transpose by the complex adjoint. Their eigenvalue processes satisfy

$$d\lambda_t^i = 2\sqrt{\lambda_t^i} dZ_t^i + \beta \sum_{j \neq i} \frac{\lambda_t^i + \lambda_t^j}{\lambda_t^i - \lambda_t^j} dt, \quad 1 \leq i \leq n,\ t < R_0 \wedge \tau,$$

for $\beta = 1, 2$ respectively, where $(\lambda_0^1 > \cdots > \lambda_0^n > 0)$, $(Z^1, \dots, Z^n)$ is a n-dimensional BM and

$$R_0 = \inf\{t > 0,\ \lambda_t^n = 0\}$$

is the first hitting time of 0. Setting $r_i = \sqrt{\lambda_i}$, it easily follows that

$$dr_t^i = dZ_t^i + \frac{k_0}{r_t^i} dt + k_1 \sum_{j \neq i} \left[\frac{1}{r_t^i - r_t^j} + \frac{1}{r_t^i + r_t^j} \right] dt$$

where $k_1 = \beta/2$ and k_0 depending on β, n, m. For arbitrary $k_0, k_1 > 0$, r fits the radial Dunkl process associated with a B_n-type root system and $T_0 = R_0 \wedge \tau$. Thus, there exists unique strong solution for all time t and in the geometric cases $k_1 = 1/2, 1$, this improves well known results from matrix theory (see [12] for more details).

4.3. **BMs in Weyl chambers.** For a given root system,the BM in the corresponding Weyl chamber satisfies (16) with $k \equiv 1$. The main feature of such a process is its representation as the Doob h-transform of a n-dimensional process whose coordinates are one dimensional BMs killed when they hit the boundary of the Weyl chamber ([34]). h being the product over the positive roots:

$$h(x) = \prod_{\alpha \in R_+} < \alpha, x >$$

which vanishes on the hyperplanes, is positive on the Weyl chamber and is a harmonic function ([14],[34]). The process is then interpreted as n BMs conditioned never to hit this boundary. Note also that this is coherent with the fact that $T_0 = \infty$ a.s. More details will be given after exhibiting the semi-group densities.

4.4. **Generalized Bessel functions.** Unlike the non-symmetric case, the symmetric Dunkl kernel, known as the generalized Bessel function since it reduces to a Bessel function in the rank-one case, is explicitely written for root systems of types A, B, C, D by means of multivariate hypergeometric functions of two arguments. Recall first that the latter are defined by

$$
{}_pF_q^{(2/\beta)}((a_i)_{1\leq i\leq p}, (b_i)_{1\leq i\leq q}, x, y) := \sum_{m=0}^{\infty}\sum_{|\tau|=m} \frac{(a_1)_\tau \dots (a_p)_\tau}{(b_1)_\tau \dots (b_q)_\tau} \frac{J_\tau^{(2/\beta)}(x) J_\tau^{(2/\beta)}(y)}{J_\tau^{(2/\beta)}(1)|\tau|!}
$$

where $1 = (1, \dots, 1), (a)_\tau$ is the generalized Pochhammer symbol, $2/\beta, \beta > 0$ is the Jack parameter, $x = (x_1, \dots, x_n), y = (y_1, \dots, y_n), \tau = (\tau_1 \geq \tau_2 \dots \geq \tau_n)$ is a partition of length n and weight $|\tau| = \tau_1 + \dots + \tau_n, J_\tau^{(2/\beta)}$ is the Jack polynomial and $(b_j)_\tau$ is not zero for all $1 \leq j \leq q$. The reader is referred to [3] and [44] for all the definitions. The radius of convergence of the infinite series depends on p, q ([3]) and the series reduces to a polynomial when at least one $(a_i)_\tau$ vanishes. For both A and B-types, the generalized Bessel function is given by ([1]):

$$
{}_0F_0^{(1/k)}(x, y), \qquad {}_0F_1^{(1/k_1)}\left(k_0 + (m-1)k_1 + \frac{1}{2}, \frac{x^2}{2}, \frac{y^2}{2}\right)
$$

respectively, where $x^2 := (x_1^2, \dots, x_n^2)$, and $k, (k_0, k_1)$ denote the multiplicity functions in both cases respectively. For the D-type, it is given by ([12]):

$$
{}_0F_1^{(1/k_1)}\left((m-1)k_1 + \frac{1}{2}, \frac{x^2}{2}, \frac{y^2}{2}\right) + \prod_{i=1}^{m} \frac{x_i y_i}{2} {}_0F_1^{(1/k_1)}\left(k_0 + (m-1)k_1 + \frac{3}{2}, \frac{x^2}{2}, \frac{y^2}{2}\right)
$$

and the reader may see that it is different from the function corresponding to the B-type specialized to $k_0 = 0$. Nevertheless, the latter, that is

$$
{}_0F_1^{(1/k_1)}\left((m-1)k_1 + \frac{1}{2}, \frac{x^2}{2}, \frac{y^2}{2}\right)
$$

is the symmetrical of the generalized Bessel function of type D w.r.t. the reflection that acts on the last coordinate y_m by a sign change. This is in agreement with the fact that the Weyl chamber of type D is the union of the Weyl chamber of type B and its symmetrical w.r.t. the above mentioned reflection ([37]). Notice also that the generalized Bessel function of type D is not invariant under the action of the B-type Weyl group acting by permutations and sign changes. However, it is invariant under the action of the D-type Weyl group acting by permutations and even sign changes.

4.5. **BMs in Weyl chambers and determinantal representations.** An interesting feature of the multivariate hypergeometric functions is given by the determinantal representations they admit when the Jack parameter equals 1 (see [35] and references therein). For instance,

$$
\begin{aligned}
{}_pF_q(1)((m+\mu_i)_{1\leq i\leq p},&(m+\phi_j)_{1\leq j\leq q};x,y)\\
&= \pi^{\frac{m(m-1)}{2}(p-q-1)}\Gamma_m(m)\prod_{i=1}^{p}\frac{\Gamma(\mu_i+1)}{\Gamma_m(m+\mu_i)}\prod_{j=1}^{q}\frac{\Gamma_m(m+\phi_j)}{\Gamma(\phi_j+1)}\\
&\quad \frac{\det\left({}_p\mathscr{F}_q((\mu_i+1)_{1\leq i\leq p},(1+\phi_j)_{1\leq j\leq q};x_ly_f)\right)_{l,f}}{V(x)V(y)}
\end{aligned}
$$

where $\mu_i, \phi_i > -1$, $V(x)$ stands for the Vandermonde determinant, Γ_m is the multivariate Gamma function ([44]) and ${}_p\mathscr{F}_q$ is the univariate hypergeometric function ($m = 1$, [42]). When $p = q = 0$, this formula is understood as:

$$
{}_0F_0^{(1)}(x,y) = \pi^{\frac{-m(m-1)}{2}}\Gamma_m(m)\frac{\det\left({}_0\mathscr{F}_0(x_ly_f)\right)_{l,f}}{V(x)V(y)} = C(m)\frac{\det\left(e^{x_ly_f}\right)_{l,f}}{V(x)V(y)}.
$$

With regard to (2), the semi group density of the radial Dunkl process of A_{n-1}-type is written as:

$$
p_t^{1,W}(x,y) = \frac{V(y)}{V(x)}\det(N_t^B(y_j-x_i))_{i,j=1}^n, \tag{17}
$$

where $N_t(v) = (1/\sqrt{2\pi t})e^{-v^2/2t}$ is the heat kernel. Similarly, since $D_k^W = |W|{}_0F_1^{(1/k_1)}$ for $R = B_n$, then:

$$
p_t^{(k_0,1),W}(x,y) = C_m\frac{h(y)}{h(x)}\frac{e^{-(|x|^2+|y|^2)/2t}}{t^{m/2}}\prod_{i,j=1}^{m}\left(\frac{x_iy_j}{t}\right)\det\left[{}_0\mathscr{F}_1\left(k_0+\frac{1}{2},\frac{(x_iy_j)^2}{4t^2}\right)\right]_{i,j=1}^n,
$$

where $h(y) = V(y^2)\prod_{i=1}^n y_i$. Taking $k_0 = 1$ and using the identity ([42])

$$
{}_0\mathscr{F}_1(\frac{3}{2},z) = \frac{C}{2\sqrt{z}}\sinh(2\sqrt{z}).
$$

for some constant C, one gets

$$
p_t^{(1,1),W}(x,y) = \frac{h(y)}{h(x)}\det\left[N_t(y_j-x_i)-N_t(y_j+x_i)\right]_{i,j=1}^n. \tag{18}
$$

Finally, one has for $R = D_n$:

$$
p_t^{1,W}(x,y) = \frac{V(y^2)}{V(x^2)}\frac{\det[N_t^-(x_j,y_i)]_{i,j=1}^n + \det[N_t^+(x_j,y_i)]_{i,j=1}^n}{2} \tag{19}
$$

where

$$
\begin{aligned}
N_t^-(x_j,y_i) &= N_t(y_i-x_j)-N_t(y_i+x_j)\\
N_t^+(x_j,y_i) &= N_t(y_i-x_j)-N_t(y_i+x_j).
\end{aligned}
$$

Setting $h(y) := \prod_{\alpha\in R_+} <\alpha,y>$, then h is a $\mathscr{L}_k^W$-harmonic function ([34]) and (17) and (18) share the form

$$
\frac{h(y)}{h(x)}\det(q_t(x_i,y_j))_{i,j=1}^n \tag{20}
$$

where q_t is respectively the semi group of a real BM and a real BM killed when it first hits zero ([55]). We say then that the radial Dunkl process is the h-transform in Doob's sense of a process consisting of m-independent BMs killed when they first hit ∂C of type A, B respectively. The h-process is then known as *the BM in the Weyl chamber* m-independent BMs conditioned never to hit ∂C which agrees with the fact that $T_0 = \infty$ a.s. For the D_n-type root system, a similar result holds however we preferred the above expression (19) since it involves the semi-group density of a real BM killed when it first hits 0 and the one of a reflected BM. This actually reflects the geometrical structure of the Weyl chamber of type D.
All these results were first derived in [34] using the reflection principle and then reproved in [12] using determinantal representations as our analytic tools.

4.6. **Absolute Continuity Relations.** Let X^W be the radial Dunkl process associated with a reduced root system R and a positive multiplicity function k. Imitating the rank-one case for which X^W is a Bessel process of dimension $\delta = 2k+1$ or index $\nu = k - 1/2$ ([55]), one similarly defines the index function associated with k as the W-invariant function given by $\nu(\alpha) := k(\alpha) - 1/2$ for all α. Denote the law of X^W starting at $x \in C$ by $\mathbb{P}_x^\nu$. Then, under $\mathbb{P}_x^\nu$, the coordinate process is a radial Dunkl process that a.s. hits ∂C when $-1/2 \leq \nu(\alpha) < 0$ for at least one α and will never do that a.s. when $\nu(\alpha) \geq 0$ for all $\alpha \in R$. The absolute continuity relations between $\mathbb{P}_x^\nu$ for different index functions ν were derived in [8] and the results may be as follows:

Proposition 4.1. (1) *If $\nu(\alpha) \geq 0$ for all $\alpha \in R$, then*

$$\mathbb{P}^0_{x|\mathscr{F}_t} = \prod_{\alpha \in R_+} \left(\frac{<\alpha, X_t>}{<\alpha, x>}\right)^{-\nu(\alpha)} \exp\left(\frac{1}{2}\sum_{\alpha,\zeta \in R_+}\int_0^t \frac{<\alpha,\zeta>\nu(\alpha)\nu(\zeta)}{<\alpha, X_s^W><\zeta, X_s^W>}ds\right)\mathbb{P}^\nu_{x|\mathscr{F}_t}.$$

(2) *If $-1/2 \leq \nu(\alpha) < 0$ for at least one $\alpha \in R$, then*

$$\mathbb{P}^0_{x|\mathscr{F}_t} = \prod_{\alpha \in R_+} \left(\frac{<\alpha, X_t>}{<\alpha, x>}\right)^{-\nu(\alpha)} \exp\left(\frac{1}{2}\sum_{\alpha,\zeta \in R_+}\int_0^t \frac{<\alpha,\zeta>\nu(\alpha)\nu(\zeta)}{<\alpha, X_s^W><\zeta, X_s^W>}ds\right)\mathbb{P}^\nu_{x|\mathscr{F}_{t\wedge T_0}}.$$

(3) *If $0 \leq \nu(\alpha) \leq 1/2$ for all $\alpha \in R$, then*

$$\mathbb{P}_x^{-\nu}{}_{|\mathscr{F}_{t\wedge T_0}} = \prod_{\alpha \in R_+} \left(\frac{<\alpha, X_t>}{<\alpha, x>}\right)^{-\nu(\alpha)} \mathbb{P}^\nu_{x|\mathscr{F}_t}.$$

Proof: It uses the SDE (15) together with Girsanov Theorem applied with the martingale

$$\prod_{\alpha \in R_+} \left(\frac{<\alpha, X_\cdot>}{<\alpha, x>}\right)^{-\nu(\alpha)} \exp\left(\frac{1}{2}\sum_{\alpha,\zeta \in R_+}\int_0^\cdot \frac{<\alpha,\zeta>\nu(\alpha)\nu(\zeta)}{<\alpha, X_s^W><\zeta, X_s^W>}ds\right)$$

and with the previous results on the finiteness of T_0. ■
Three straightforward consequences follow:

Corollary 4.1. *Let μ, ν be two W-invariant positive functions, then for every $\mathscr{F}_t$-measurable random variable $Z \geq 0$*

$$\mathbb{E}_x^{\mu}\left[Z \prod_{\alpha \in R_+}\left(\frac{<\alpha, X_t>}{<\alpha, x>}\right)^{-\mu(\alpha)} \exp\left(-\frac{1}{2}\sum_{\alpha,\zeta \in R_+}\int_0^t \frac{<\alpha,\zeta> \nu(\alpha)\nu(\zeta)}{<\alpha, X_s^W><\zeta, X_s^W>}ds\right)\right]$$
$$= \mathbb{E}_x^{\nu}\left[Z \prod_{\alpha \in R_+}\left(\frac{<\alpha, X_t>}{<\alpha, x>}\right)^{-\nu(\alpha)} \exp\left(-\frac{1}{2}\sum_{\alpha,\zeta \in R_+}\int_0^t \frac{<\alpha,\zeta> \mu(\alpha)\mu(\zeta)}{<\alpha, X_s^W><\zeta, X_s^W>}ds\right)\right].$$

Using the semi group density (2), one gets

Corollary 4.2. *Let $k_\mu : \mu + 1/2, k_\nu = \nu + 1/2$ and $\kappa(\alpha,\zeta) := \nu(\alpha)\nu(\zeta) - \mu(\alpha)\mu(\zeta)$. Then*

$$\mathbb{E}_x^{\mu}\left[\exp\left(-\frac{1}{2}\sum_{\alpha,\zeta \in R_+}\int_0^t \frac{<\alpha,\zeta> \kappa(\alpha,\zeta)}{<\alpha, X_s^W><\zeta, X_s^W>}ds\right)|X_t = y\right] \tag{21}$$

$$= \prod_{\alpha \in R_+}\left(\frac{<\alpha, y>}{<\alpha, x>}\right)^{\mu(\alpha)-\nu(\alpha)} \frac{p_t^{k_\nu,W}(x,y)}{p_t^{k_\mu,W}(x,y)}. \tag{22}$$

Taking $\mu \equiv 0$, one easily derives

Corollary 4.3 (Generalized Hartman-Watson law)**.** *Let*

$$H(x) := \sum_{\alpha \in R_+} \nu(\alpha)\frac{\alpha}{<\alpha, x>}, \quad x \in C,$$

then

$$\mathbb{E}_x^{0}\left[\exp\left(-\frac{1}{2}\int_0^t <H(X_s^W), H(X_s^W)> ds\right)|X_t^W = y\right] = \tag{23}$$

$$C(\nu)t^r \frac{D^W_{\nu+1/2}(x/\sqrt{t}, y/\sqrt{t})}{D_{1/2}(x/\sqrt{t}, y/\sqrt{t})}\prod_{\alpha \in R_+}(<\alpha, y><\alpha, x>)^{\nu(\alpha)} \tag{24}$$

where $r := \sum_{\alpha \in R_+}[(1/2) - \lambda(\alpha)]$.

Remarks. 1/In the rank one case, the D_k^W reduces to the hypergeometric function of Bessel-type ${}_0F_1$ ([42]) and X^W is a Bessel process of index 0 or dimension 2. The above equality simplifies to

$$\mathbb{E}_x^{0}\left[\exp -\frac{\nu^2}{2}\int_0^t \frac{ds}{(X_s^W)^2}|X_t^W = y\right] = \frac{I_\nu(xy/t)}{I_0(xy/t)}$$

where I_ν is the modified Bessel function of index ν. The RHS is the Laplace transform in $\nu^2/2$ of the so-called Hartman-Watson law ([61]) of parameter $r = xy/t$ and similar multivariate results corresponding to the B-type root system with $k_1 = 1, 2$ (X^W is the square root of the eigenvalues process of Wishart and Laguerre processes respectively) were already derived in [11],[13].

2/For $R = B_2 = \{\pm e_1, \pm e_2, \pm e_1 \pm e_2\}$, (21) is given by

$$\mathbb{E}_x^\mu\left[\exp\left(-\frac{1}{2}\sum_{\alpha,\zeta\in R_+}\int_0^t \frac{<\alpha,\zeta>\kappa(\alpha,\zeta)}{<\alpha,X_s^W><\zeta,X_s^W>}ds\right)|X_t=y\right] \tag{25}$$

$$= \mathbb{E}_x^\mu\left[\exp\left(-\frac{1}{2}\sum_{\alpha\in R_+}\int_0^t \frac{|\alpha|^2\kappa(\alpha)}{<\alpha,X_s^W>^2}ds\right)|X_t=y\right] \tag{26}$$

$$= \prod_{\alpha\in R_+}\left(\frac{<\alpha,y>}{<\alpha,x>}\right)^{\mu(\alpha)-\nu(\alpha)}\frac{p_t^{k_\nu,W}(x,y)}{p_t^{k_\mu,W}(x,y)}. \tag{27}$$

where $\kappa(\alpha) := \kappa(\alpha,\alpha) = \nu(\alpha)^2 - \mu(\alpha)^2$ for two index functions ν,μ. In fact, both orbits consists of orthogonal roots and the sum over roots belonging to different orbits is zero. For fixed $\mu = (\mu_0,\mu_1)$ corresponding to (k_0,k_1), one may choose $\nu_1 = \mu_1$ to get

$$\mathbb{E}_x^\mu\left[\exp\left(-\frac{(\nu_0^2-\mu_0^2)}{2}\sum_{i=1}^2\int_0^t\frac{ds}{<e_i,X_s^W>^2}ds\right)|X_t=y\right] = \prod_{i=1}^2\left(\frac{y_i}{x_i}\right)^{\mu_0-\nu_0}\frac{p_t^{k_\nu,W}(x,y)}{p_t^{k_\mu,W}(x,y)},$$

and similarly take $\mu_0 = \nu_0$ to get

$$\mathbb{E}_x^\mu\left[\exp\left(-\frac{(\nu_1^2-\mu_1^2)}{2}\sum_{i=1}^2\int_0^t\frac{ds}{<e_1+\epsilon_i e_2,X_s^W>^2}ds\right)|X_t=y\right] = \left(\frac{y_1^2-y_2^2}{x_1^2-x_2^2}\right)^{\mu_1-\nu_1}\frac{p_t^{k_\nu,W}(x,y)}{p_t^{k_\mu,W}(x,y)}$$

where $\epsilon_1 = 1, \epsilon_2 = -1$.

4.7. **The law of T_0.** The above absolute continuity relations allow to derive the tail distribution of T_0. We shall distinguish two cases:

- $0 \leq \nu(\alpha) \leq 1/2$ with at least one α such that $\nu(\alpha) > 0$, for which we use the third part of Proposition 4.1 to write

$$P_x^{-\nu}(T_0 > t) = E_x^\nu\left[\left(\prod_{\alpha\in R_+}\frac{<\alpha,X_t>}{<\alpha,x>}\right)^{-2\nu(\alpha)}\right].$$

- $-1/2 \leq \nu(\alpha) < 0$ for at least one $\alpha \in R$ for which we use the second part of the same proposition to write

$$P_x^\nu(T_0 > t) = E_x^0\left[\prod_{\alpha\in R_+}\left(\frac{<\alpha,X_t>}{<\alpha,x>}\right)^{\nu(\alpha)}\exp\left(-\frac{1}{2}\sum_{\alpha,\zeta\in R_+}\int_0^t\frac{<\alpha,\zeta>\nu(\alpha)\nu(\zeta)}{<\alpha,X_s><\zeta,X_s>}ds\right)\right].$$

One may simplify the last expectation using the first part of the same Proposition

$$P_x^\nu(T_0 > t) = E_x^r\left[\prod_{\alpha\in R_+}\left(\frac{<\alpha,X_t>}{<\alpha,x>}\right)^{\nu(\alpha)-r(\alpha)}\exp\left(-\frac{1}{2}\sum_{\alpha,\zeta\in R_+}\int_0^t\frac{<\alpha,\zeta>\kappa(\alpha,\zeta)}{<\alpha,X_s><\zeta,X_s>}ds\right)\right]$$

where

$$r(\alpha) = \begin{cases}\nu(\alpha) & \text{if} \quad \nu(\alpha)\geq 0\\ -\nu(\alpha) & \text{if} \quad \nu(\alpha) < 0.\end{cases}$$

It follows that $\kappa(\alpha,\zeta) = 0$ if $\nu(\alpha)\nu(\zeta) \geq 0$ and $\kappa(\alpha,\zeta) = -2r(\alpha)r(\zeta)$ otherwise. Thus,

$$P_x^\nu(T_0 > t) = E_x^r\left[\prod_{\substack{\alpha\in R_+\\ \nu(\alpha)<0}}\left(\frac{<\alpha,x>}{<\alpha,X_t>}\right)^{2r(\alpha)}\exp\left(\sum_{\substack{\alpha,\zeta\in R_+\\ \nu(\alpha)\nu(\zeta)<0}}\int_0^t \frac{<\alpha,\zeta>r(\alpha)r(\zeta)}{<\alpha,X_s><\zeta,X_s>}ds\right)\right].$$

Note that the exponential functional in the RHS equals 1 for the irreducible root systems of types A, B, C, D. For both types A and D, it is obvious since R consists of one orbit so that $\{\alpha,\,\zeta\in R_+, \nu(\alpha)\nu(\zeta) < 0\}$ is empty and one fits into the first case. For the type B, note that $\nu(\alpha)\nu(\zeta) < 0$ may not be satisfied unless α and ζ belong to different orbits. Thus, writing $R_+ = \{e_i, 1 \leq i \leq m\} \cup \{e_j \pm e_k, 1 \leq j < k \leq m\}$ so that $< e_i, e_j \pm e_k >= \delta_{ij} \pm \delta_{ik}$ gives

$$\begin{aligned} S &= \sum_{i=1}^m\sum_{i<k}\frac{1}{X_t^i}\left[\frac{1}{X_t^i - X_t^k} + \frac{1}{X_t^i + X_t^k}\right] + \sum_{i=1}^m\sum_{k<i}\frac{1}{X_t^i}\left[\frac{-1}{X_t^k - X_t^i} + \frac{1}{X_t^k + X_t^i}\right] \\ &= \sum_{i=1}^m\sum_{i<k}\frac{2}{(X_t^i)^2 - (X_t^k)^2} - \sum_{i=1}^m\sum_{k<i}\frac{2}{(X_t^k)^2 - (X_t^i)^2} = \sum_{1\leq i\neq k\leq m}\frac{2}{(X_t^i)^2 - (X_t^k)^2} = 0, \end{aligned}$$

where S stands for the sum between brackets (up to a constant). The same obviously holds for the type C since it is the coroot system of B:

$$C_m = \{2e_i, 1 \leq i \leq m, \pm e_i \pm e_j, 1 \leq i < j \leq m\}.$$

Therefore, for the above mentioned irreducible root systems, one has:

$$P_x^\nu(T_0 > t) = E_x^r\left[\prod_{\substack{\alpha\in R_+\\ \nu(\alpha)<0}}\left(\frac{<\alpha,x>}{<\alpha,X_t>}\right)^{-2\nu(\alpha)}\right].$$

We will deal with the first case with an arbitrary root system then adapt our reasoning to the second case for the root system of type B.

4.7.1. *first formulae.* Easy computation using (2) yield

$$\begin{aligned} P_x^{-\nu}(T_0 > t) &= \prod_{\alpha\in R_+} <\alpha,x>^{2\nu(\alpha)}\frac{e^{-|x|^2/2t}}{c_k t^{\gamma'}}\int_C e^{-|y|^2/2}D_k^W(\frac{x}{\sqrt{t}},y)\prod_{\alpha\in R_+}<\alpha,y>dy \\ &:= \prod_{\alpha\in R_+}<\alpha,\frac{x}{\sqrt{t}}>^{2\nu(\alpha)}\frac{e^{-|x|^2/2t}}{c_k t^{\gamma'}}g\left(\frac{x}{\sqrt{t}}\right). \end{aligned}$$

g is a W-invariant function with initial value $g(0)$ given by the Selberg-Mehta integral

$$g(0) = \int_C e^{-|y|^2/2}\prod_{\alpha\in R_+}<\alpha,y>dy = \int_V e^{-|y|^2/2}\prod_{\alpha\in R_+}|<\alpha,y>|dy$$

since $\cup_{w\in W} w\overline{C}$. g is characterized as follows:

Theorem 4.3. *g solves the spectral problem*

$$[\Delta_k - \sum_{i=1}^{n} x_i\partial_i]g = [\Delta_k^W - \sum_{i=1}^{n} x_i\partial_i]g = (m + |R_+|)g$$

with the initial value given by $g(0)$.

Proof: It uses the action of the eigenoperator $\mathscr{L}_k - \sum_{i=1}^{n} x_i\partial_i$ on the Dunkl kernel and an integration by parts (see [12] for details). ■

Once this is done, we specialize this Theorem to the irreducible root systems A, B, C, D and identify g with multivariate hypergeometric series of one argument. Our results summarize into the Propositions below:

Proposition 4.2. *Let* $R = A_{n-1}$*, then for* $1/2 \leq k_1 \leq 1$,

$$g(x) = g(0)C(n, k_1) \lim_{b\to\infty} {}_2F_1^{(1/k_1)}\left[\frac{n+1}{2}, b, \frac{b}{2} + \frac{k_1}{2}(m-1) + \frac{m+3}{4}, \frac{1}{2}\left(1 - \frac{x}{\sqrt{b}}\right)\right]$$

where

$$C(n, k_1)^{-1} = \lim_{b\to\infty} {}_2F_1^{(1/k_1)}\left(\frac{n+1}{2}, b, \frac{b}{2} + \frac{k_1}{2}(m-1) + \frac{m+3}{4}, \frac{1}{2}\right).$$

${}_2F_1^{(1/k_1)}$ *is the multivariate Gauss hypergeometric function.*

Proposition 4.3. *Let* $R = B_n$*, then one has for* $1/2 \leq k_0, k_1 \leq 1$,

$$P_x^{-\nu}(T_0 > t) = C_k \prod_{i=1}^{n}\left(\frac{x_i^2}{2t}\right)^{\nu_0}\left(V\left(\frac{x^2}{2t}\right)\right)^{2\nu_1} e^{-|x|^2/2t}{}_1F_1^{(1/k_1)}(\frac{n+1}{2}, k_0+(n-1)k_1+\frac{1}{2}, \frac{x^2}{2t})$$

where $\nu_0 := k_0 - 1/2, \nu_1 := k_1 - 1/2$. ${}_1F_1^{(1/k_1)}$ *is the multivariate confluent hypergeometric function.*

Proposition 4.4. *Let* $R = D_n$*, then for* $1/2 \leq k_1 \leq 1$*, the tail distribution is given by:*

$$\mathbb{P}_x^{-\nu}(T_0 > t) = C_k\left[V\left(\frac{x^2}{2t}\right)\right]^{2\nu} e^{-|x|^2/2t}{}_1F_1\left(\frac{n}{2}, (n-1)k_1 + \frac{1}{2}, \frac{x^2}{2}\right).$$

Remark. For the A-type root system, one cannot exchange the infinite sum and the limit operation. Indeed, expand the generalized Pochhammer symbol as ([38])

$$(a)_\tau = \prod_{i=1}^{n}(a - k(i-1))_{\tau_i} = \prod_{i=1}^{n}\frac{\Gamma(a - k(i-1) + \tau_i)}{\Gamma(a - k(i-1))}$$

and use Stirling formula to see that each term in the above product is equivalent to $(a + k(m-1) + \tau_i)^{\tau_i}$ for large enough positive a. Moreover since $J_\tau^{(1/k)}$ is homogeneous, one has

$$J_\tau^{(1/k)}\left[\frac{1}{2}\left(1 - \frac{x}{\sqrt{b}}\right)\right] = 2^{-p}J_\tau^{(1/k)}\left(1 - \frac{x}{\sqrt{b}}\right), \quad |\tau| = p.$$

It follows that

$$\frac{(b)_\tau}{(b/2 + (n-1)k/2 + (n+3)/4)_\tau}J_\tau^{(1/k)}\left[\frac{1}{2}\left(1 - \frac{x}{\sqrt{b}}\right)\right] \approx J_\tau^{(1/k)}(1)$$

for positive large b. Thus, the above Gauss hypergeometric function reduces to

$$ {}_1F_0^{(1/k)}\left(\frac{n+1}{2},1\right) $$

which diverges for instance for $k = 1/2, 2$ since one has in that cases ([25])

$$ {}_1F_0^{(1/k)}(a,x) = \prod_{i=1}^{n}(1-x_i)^{-a} = \det(I_n - x)^{-a} $$

where x is identified with a $n \times n$ matrix whose eigenvalues are $(x_i)_i$ and I_n stands for the identity matrix.

4.7.2. *Second formulae.* These formulae give the tail distribution of T_0 for $R = B_n$ with two indices of opposite signs. Theorem 4.3 still applies and the following holds

Proposition 4.5. *If $k_0 < 1/2$, $k_1 \geq 1/2$, then*

$$ P_x^\nu(T_0 > t) = C_k \prod_{i=1}^{n}\left(\frac{x_i^2}{2t}\right)^{\nu_0} e^{-|x|^2/2t}{}_1F_1^{(1/k_1)}((1/2) + k_1(n-1), k_0 + (n-1)k_1 + \frac{1}{2}, \frac{x^2}{2t}). $$

Finally

Proposition 4.6. *If $k_0 \geq 1/2, k_1 < 1/2$, then*

$$ P_x^\nu(T_0 > t) = C_k V\left(\frac{x^2}{2t}\right)^{2\nu_1} e^{-|x|^2/2t}{}_1F_1^{(1/k_1)}(\nu_0 + \frac{n}{2}, k_0 + (n-1)k_1 + \frac{1}{2}, \frac{x^2}{2t}). $$

Remark. Since D_k^W is given by hypergeometric series for the types A, B, C, D, possible computations may be performed using the McDonald conjecture proved in [38] (Corollary 2 p.1107). The latter gives the value of the integral of a Jack polynomial against the Selberg weight which is a particular case of the weight given by powers of the product over the positive roots.

5. Brownian motions reflected on the walls of Weyl chambers

In the probability scope, the reflected BM is the absolute value of a real BM. It is easy to see that its semi group density is given for $x \geq 0$ by:

$$ q_t(x,y) = [N_t(y-x) + N_t(y+x)]\mathbf{1}_{\{y\geq 0\}} \tag{28} $$

where as before $N_t(x)$ is the heat kernel. The reflected BM is not a Itô's process, yet it is a semimartingale which satisfies the SDE

$$ |B_t - a| = |B_0 - a| + \int_0^t \operatorname{sgn}(B_s - a)dB_s + L_t^a, \quad a \in \mathbb{R} \tag{29} $$

where L_t^a is the local time up to time t at the level a of the BM $(B_t)_{t\geq 0}$. For all a, the process $t \mapsto L_t^a$ is of bounded variation, $(a,t) \mapsto L_t^a$ has a bi-continuous version and dL_t^a is supported by the set $\{t, B_t = a\}$ (see [55] for more facts on local times). Another famous result related to the reflected BM is known as Lévy's representation ([55]) which states that

$$ (S_t - B_t, S_t)_{t\geq 0} \overset{\mathscr{L}}{=} (|B_t|, L_t^0)_{t\geq 0} $$

where $S_t := \sup_{s\leq t} B_s$. Regarding $\mathbb{R}_+$ as a cone, it was natural to define the reflected BM in higher dimensions: this was first done for a cone in the complex plane of the form

$$C_\phi := \{re^{i\theta},\, r \geq 0,\, |\theta| \leq \phi\}, \quad 0 < \phi < \pi/2$$

using a sub-martingale problem ([58]). Another way to define this process is to define it as a process U valued in C_ϕ which solves:

$$U_t = x + B_t + A_t^1 u_1 + A_t^2 u_2, \quad x \in C_\phi$$

where $(B_t)_{t\geq 0}$ is a planar BM ([55]), u_1, u_2 are two vectors in $\mathbb{R}^2$ along which the reflections on both sides of the cone Δ_1, Δ_2 is done, and A^1, A^2 are two adapted (w.r.t the filtration of B), continuous and increasing processes that only increase when U hits the sides Δ_1, Δ_2 respectively ([60]). The last definition allows to define the reflected BM in a cone in a similar way as soon as one characterizes the sides of the cone. This is for instance the case of simplexes in $\mathbb{R}^n$ defined by

$$SIM := \{\sum_{i=1}^n x_i v_i,\, x_i \geq 0\}.$$

where $(v_i)_{1\leq i\leq n}$ is a basis of $\mathbb{R}^n$. The side $\Delta_i, 1 \leq i \leq n$, is then characterized by $x_i = 0$ (see the end of [43]). In particular, C_ϕ is a simplex of $\mathbb{R}^2$ with basis $e^{i\phi}, e^{-i\phi}$ and so is $\overline{C}$ for span(R) with the basis that consists of the *fundamental weights* or the dual basis of S in V defined by

$$\omega_i \text{ is a fundamental weight} \Leftrightarrow <\omega_i, \alpha_j> = \delta_{ij}.$$

For instance, $\omega_i = e_1 + \cdots + e_i$ for $R = B$. For $R = A_1$, $C = \{x \in \mathbb{R}^2, x_1 > x_2\}$ is a simplex for the hyperplane consisting of vectors whose coordinates sum to zero. This Weyl chamber can be mapped to the half plane $\{x \in \mathbb{R}^2,\, x_1 > 0, x_2 \in \mathbb{R}\}$ which may be viewed as $C_{\pi/2}$.

5.1. **Definition and semi group density.** The BM reflected on the walls of a given Weyl chamber C is defined in [8] as the projection of a n-dimensional BM (Dunkl process of zero multiplicity function) in $\overline{C}$, that is, as a radial Dunkl process of zero multiplicity function. Its semi group density is easily deduced from (2): recall that the Dunkl kernel D_k reduces when $k \equiv 0$ to the exponential function ([49]):

$$D_0(x, y) = \exp(<x, y>)$$

so that

$$p_t^{0,W}(x, y) = \frac{1}{c_0 t^{n/2}} \exp - \left\{\frac{|x|^2 + |y|^2}{2t}\right\} \sum_{w\in W} \exp\left(\frac{1}{t} <x, wy>\right).$$

For $R = B_1$, $W = \{Id, \sigma\}$ where $\sigma(x) = -x, x \in \mathbb{R}$ and one recovers (28).We shall see that it solves a SDE similar to (29) and that a representation of Lévy-type holds.

5.2. **Reflected BMs on the walls as solutions of SDEs.** The multidimensional extension of (29) is given in the following Theorem ([8]):

Theorem 5.1. *Let $(\pi(B_t))_{t\geq 0}$ be the reflected BM in $\overline{C}$ associated with a reduced root system R with a simple system S. Suppose that S is a basis of V, then there exists a $\overline{C}$-valued continuous process $(L_t^0)_{t\geq 0}$ such that $(<\alpha, L_t^0>)_{t\geq 0}$ is an increasing process for all simple root $\alpha \in S$ and*

$$d\pi(B_t) = dY_t + dL_t^0$$

where Y is a BM in V. Furthermore, one has

$$\int_0^t \mathbf{1}_{\{<\alpha,\pi(B_s)>\neq 0\}} d(<\alpha, L_s^0>) = 0, \quad \alpha \in S.$$

and

$$L_t^0 = \lim_{\epsilon\to 0} \Lambda^{-1} \int_0^t b_\epsilon(B_s) ds$$

where $b_\epsilon = (b_\epsilon^1, \dots, b_\epsilon^n)$ with

$$b_\epsilon^i(x) = \frac{1}{\epsilon} \sum_{w\in W} \mathbf{1}_{\{0\leq <s_i,\omega^\star x>\leq \epsilon\}}, \quad S = \{s_1, \dots, s_n\}.$$

An illustrative example is given by $R = B_2$ for which $C = \{x \in \mathbb{R}^2,\ x_1 > x_2 > 0\}$. Let $S = (s_1 = e_1 - e_2, s_2 = e_2)$, then (see [8] p. 70)

$$\Lambda = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \qquad \Lambda^{-1} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

and

$$L_t^0 = \Lambda^{-1} \begin{pmatrix} L_t^0(B_1 - B_2) + L_t^0(B_1 + B_2) \\ L_t^0(B_2) + L_t^0(B_1) \end{pmatrix}.$$

Remark. One may define the process $(L_t^x)_{t\geq 0}$ for any $x \in V$ as the unique $\overline{C}$-valued and bounded variation process in the semimartingale decomposition of $(\pi(B_t - x))_{t\geq 0}$.

5.3. **Lévy's representation.** The extension of the Lévy's representation to simplexes was given in [43]: let $(B_t)_{t\geq 0}$ be a n-dimensional BM and for every t, let S_t be the vertex of the smallest cone of the form $x - SIM$ and containing the trajectory $B([0,t])$ up to time t. More precisely, the i-th coordinate of $(B_t)_{t\geq 0}$ in the basis $(v_i)_i$ is (up to a constant) a real BM, say $(K_t^i)_{t\geq 0}$ and the i-th coordinate of $(S_t)_{t\geq 0}$ in the basis $(v_i)_i$ is $(\sup_{s\leq t} K_t^i)_{t\geq 0}$ which has the same law as the local time at 0 of $(K_t^i)_{t\geq 0}$ by the Lévy's representation. Then, the process $(S_t - B_t)_{t\geq 0}$ is a reflected BM in SIM and the direction of the reflection on the side $x_i = 0$ is given by the vector v_i. We still suppose that S is a basis of V, then

Theorem 5.2. *The processes $(\pi(B_t), L_t^0)_{t\geq 0}$ and $(\Lambda^{-1} \sup_{s\leq t} \Lambda X_s - X_t, \Lambda^{-1} \sup_{s\leq t} \Lambda X_s)_{t\geq 0}$ have the same law, where the supremum procedure is applied to each coordinate.*

5.4. **Hitting time.** The formulae below are particular cases of the ones given in subsection 4.7.1 for the particular multiplicity function $k \equiv 1$. Indeed, by the definition of the BM reflected on ∂C, its index function equals $-1/2$. The involved hypergeometric functions take determinantal forms ([35]) and we get:

- A-type root system. In this case, one has

$$g(x) = C \lim_{b\to\infty} \frac{\det\left[(x_i^b)^{m-j}{}_2\mathscr{F}_1\left[(m+1)/2 - j + 1, b - j + 1, b/2 + m - j + 5/4, x_i^b\right]\right]_{i,j=1}^m}{V(x^b)}$$

where $x^b := (1/2)(1 - x/\sqrt{b})$. Note that

$$V(x^b) = \prod_{i<j}(x_i^b - x_j^b) = (2\sqrt{b})^{-m(m-1)/2}V(-x).$$

Thus,

$$g(x) = \frac{C}{V(x)} \lim_{b\to\infty} \det\left[b^{(m-1)/4}(x_i^b)^{m-j}{}_2\mathscr{F}_1\left[\frac{m+1}{2} - j + 1, b - j + 1, \frac{b}{2} + m - j + 5/4, x_i^b\right]\right]_{i,j=1}^m$$

so that the tail distribution is written as:

$$P_x^{-\nu}(T_0 > t) = Ce^{-|x|^2/2t} \lim_{b\to\infty}$$
$$\det\left[b^{(m-1)/4}\left(1 - \frac{x_i}{\sqrt{bt}}\right)^{m-j}{}_2\mathscr{F}_1\left[\frac{m+1}{2} - j + 1, b - j + 1, \frac{b}{2} + m - j + 5/4, \frac{1}{2}\left(1 - \frac{x_i}{\sqrt{tb}}\right)\right]\right]_{i,j=1}^m$$

- B-type root system.

$$\begin{aligned} P_x^{-1/2}(T_0 > t) &= C \det\left[\left(\frac{x_i^2}{2t}\right)^{m-j+1/2} e^{-x_i^2/2t}{}_1\mathscr{F}_1\left(\frac{m+1}{2} - j + 1, m + \frac{1}{2} - j + 1, \frac{x_i^2}{2t}\right)\right]_{i,j=1}^m \\ &= c_k \det\left[\left(\frac{x_i^2}{2t}\right)^{m-j+1/2}{}_1\mathscr{F}_1\left(\frac{m}{2}, m - j + \frac{3}{2}, -\frac{x_i^2}{2t}\right)\right]_{i,j=1}^m \end{aligned}$$

where the last line follows from Kummer's transformation ([42]).

- D-type root system. Similarly, one gets

$$\mathbb{P}_x^{-1/2}(T_0 > t) = C \det\left[{}_1\mathscr{F}_1\left(\frac{m-1}{2}, m - j + \frac{1}{2}, -\frac{x_i^2}{2t}\right)\right]_{i,j=1}^m.$$

Remark. Let B be a n-dimensional BM starting at $x \in C$ and let g be the last hitting time before time $t = 1$ of ∂C. By time inversion property and scaling, one may relate g to T_0 by the following identity:

$$\mathbb{P}_x\left(g < \frac{1}{1+u}\right) = \mathbb{E}_y(e^{-u/2T_0}), \quad y = x/|x|,\ u \geq 0.$$

5.5. **The heat equation.** Recall that $(x,t) \mapsto u(x,t) = \mathbb{P}_x^{-1/2}(T_0 > t)$ is the unique solution of the heat equation

$$\frac{1}{2}\Delta u(x,t) = \partial_t u(x,t),$$

with boundary values $u(x,t) = 0$ if $x \in \partial C$ and $u(x,0) = 1$ if $x \in C$ ([15]). This may be checked using the fact that

$$x \mapsto g(x) = \int_C e^{-|y|^2/2} D_1^W(x,y) \prod_{\alpha \in R_+} < \alpha, y > dy$$

satisfies

$$[2\mathscr{L}_1^W - \sum_{i=1}^m x_i \partial_i] g = (m + |R_+|) g. \tag{30}$$

Indeed, the tail distribution is given in this case by

$$\mathbb{P}_x^{-1/2}(T_0 > t) = C(vhg)\left(\frac{x}{\sqrt{t}}\right), \, v(x) := e^{-|x|^2/2}, h(x) = \prod_{\alpha \in R_+} < \alpha, x > .$$

It is easy to see from

$$\nabla h = h \nabla \log h = h \sum_{\alpha \in R_+} \frac{\alpha}{< \alpha, \cdot >}$$

that

$$\sum_{i=1}^m x_i \partial_i h(x) =< x, \nabla \log h(x) > h(x) = |R_+| h(x).$$

Therefore, using

$$\Delta(gh) = h\Delta g + g\Delta h + 2 < \nabla g, \nabla h >,$$

then $2\mathscr{L}_1^W$ takes the form (h-transform property, [55])

$$2\mathscr{L}_1^W(\cdot) = \frac{1}{h}\Delta(h\cdot)$$

so that (30) is equivalent to

$$\Delta(gh)(x) = m(gh)(x) + < x, \nabla(gh)(x) > .$$

Now, since $\nabla v(x) = -xv(x)$ and using (5.5), then

$$\begin{aligned}
\Delta(vgh)(x) &= (gh)(x)\Delta(v)(x) + v(x)\Delta(gh)(x) + 2 < \nabla v(x), \nabla(gh)(x) > \\
&= (gh)(x) \sum_{i=1}^m \partial_i[-x_i v(x)] + v(x)[\Delta(gh)(x) - 2 < x, \nabla(gh)(x) >] \\
&= (vgh)(x) \sum_{i=1}^m [x_i^2 - 1] + v(x)[m(gh)(x) - < x, \nabla(gh)(x) >] \\
&= |x|^2 (vgh)(x) - < x, \nabla(gh)(x) > v(x) = - < x, \nabla(vgh)(x) > .
\end{aligned}$$

Keeping in mind that $u(x,t) = C(vgh)(x/\sqrt{t})$, then

$$\frac{1}{2}\Delta[u(x,t)] = \frac{C}{2t}\Delta(vgh)\left(\frac{x}{\sqrt{t}}\right) = \frac{C}{2t^{3/2}} < x, \nabla(vgh)\left(\frac{x}{\sqrt{t}}\right) > .$$

Finally, the derivative w.r.t. the time variable t is easily computed once one writes $t \mapsto u(x,t)$ as a compostion of the functions $t \mapsto x/\sqrt{t}$ for fixed $x \in E$ and $y \mapsto (vgh)(y)$.

Remark. The fact that the tail distribution is the unique solution of the heat equation with the corresponding boundary values is valid for all homogeneous W-invariant Markov processes. Particularly, it is true for the radial Dunkl processes and we will not do here it since the computations are tedious. Nevertheless, since $\mathbb{P}_x^{-\nu}(T_0 > t) = (vgh^{2\nu})(x/\sqrt{t})$ and v, g are W-invariant, then we hint the interested reader at the fact that the Dunkl partial derivative T_i ([49]) acts on the product of two functions as a derivation when at least one of them is W-invariant.

6. From radial Dunkl processes to Dunkl processes

We have already seen that the projection of the V-valued Dunkl process X on $\overline{C}$ gave the previously-called radial Dunkl process X^W. Conversely, as it was proved in [8], the V-valued Dunkl process may be constructed from its radial part using a skew-product decomposition involving independent processes and time changes. This is done by viewing $\mathscr{L}_k$ as a perturbation of $\mathscr{L}_k^W$ ([24]). This perturbation encodes the jumps of X and one adds them carefully with suitable time-changes in order to get the desired generator $\mathscr{L}_k$. Before giving the general construction, let us describe how this works in the rank-one case, a construction that was already provided in [30]:

Proposition 6.1. *A $\mathbb{R}^\star$-valued Dunkl process ($\nu = k - 1/2 \geq 0$) starting from $x \neq 0$ admits the representation below:*

$$X_t = (-1)^{N_{A_t}} \exp(\beta_{A_t} + \nu A_t) = (-1)^{N_{A_t}}|X_t|, \quad A_t = \int_0^t \frac{ds}{X_s^2},$$

where $(N_t)_{t\geq 0}$ is a Poisson process of parameter $\lambda = k/2$ independent of X.

Proof: the second equality follows from Lamperti's representation of semi-stable processes taking values in $\mathbb{R}_+^\star$ and from the fact that $|X|$ is a Bessel process of index ν. The first equality may be checked as follows: define a process Y by

$$Y_t := X_{\tau_t}, \quad \tau_t := \inf\{s, A_s > t\} = \int_0^t Y_s^2 ds.$$

It is known that the time change τ_t preserves the Markovianity ([59]) thereby Y is a Markov process and its generator, say $\mathscr{L}_k^Y$ acts on smmoth functions as follows:

$$\begin{aligned}\mathscr{L}_k^Y f(x) &= x^2 \mathscr{L}_k f(x)\\ &= \frac{x^2}{2} f''(x) + kxf'(x) + \frac{k}{2}(f(-x) - f(x)).\end{aligned}$$

Now, an application of the Itô's formula to the process defined by

$$Z_t := \exp(\beta_t + \nu t + i\pi N_t)$$

together with the fact that $Z_{t-} = -Z_t$ shows that its generator coincides with $\mathscr{L}_k^Y$ on smooth functions. ■

Remark. Notice that the proof remains valid for a wider class of two parameters Markov processes whose generators acts as

$$\frac{1}{2}\mathscr{L}_{k,\lambda} f(x) = \frac{1}{2} f''(x) + kxf'(x) + \lambda(f(-x) - f(x)), \quad \lambda > 0, k \geq 1/2.$$

The absolute value of each element in this class is a Bessel process of dimension $2k+1$ and the above representation holds with a Poisson process $(N_t)_{t\geq 0}$ of parameter λ independent of the BM β. This class were first introduced in [30] and its multidimensional extension was studied in [8]. It is known as the class of *generalized Dunkl processes.*

The general construction is not obvious and it can be hardly deduced from the following observation concerning the rank-one case:

$$X_t = \sigma^{N_{A_t}} |X_t| = \sigma^{N_{A_t}} X_t^W,$$

where σ is the reflection acting by sign change: $x \mapsto -x$. It is stated as follows:

Theorem 6.1. *Let X be the Dunkl process such that $X_0 \in C$ and suppose that $k(\alpha) \geq 1/2$ for all $\alpha \in R_+$, that is $T_0 = \infty$ a.s..*

- *For $1 \leq i \leq |R_+|$, there exist Poisson processes N^i with intensity $k(\alpha_i)$ respectively and processes Y^i defined recursively by:*
$$Y^0 = X^W, \quad Y^i_{\tau_i} = Y^{i-1}_{\tau_{i-1}} \star_{\alpha_i} N^i$$
where
$$A_t^{i,i} := A_t^i := \int_0^t \frac{ds}{< Y_s^i, \alpha_i >^2}, \quad \tau_t^{i,i} := \tau_t^i := \inf\{s, A_s^i > t\}$$
such that $A_t^i < \infty$ a.s., $\tau_t^i < \infty$ a.s. and Y^i is a Markov process with extended generator acting on $u \in C_c^\infty$ as follows:
$$\mathscr{G}_i u(x) = \mathscr{L}_k^W u(x) + \sum_{\alpha \in R_+^i} k(\alpha) \frac{u(\sigma_\alpha x) - u(x)}{< x, \alpha >^2}.$$
- $Y^{|R_+|} \overset{d}{=} X$.
- *If for some numbering of the positive roots $R_+ = \{\alpha_1, \ldots, \alpha_{|R_+|}\}$, one has*

(31) $$\sigma_{\alpha_i}(R^{i-1}) = R^{i-1},$$

then

(32) $$Y_t^i = \sigma_{\alpha_i}^{N^i_{A_t^{i-1,i}}} Y_t^{i-1}, \quad A_t^{i-1,i} = \int_0^t \frac{ds}{< Y_s^{i-1}, \alpha_i >^2}.$$

Remarks. 1/The condition (31) is satisfied for instance by orthogonal root systems, that is, those such that $< \alpha, \beta >= 0$ for all $\alpha \neq \beta \in R$. In that case, the roots subsets $R^i, 1 \leq i \leq |R_+|$ are pointwise fixed. Examples include $\{\pm e_i, 1 \leq i \leq n\}$ corresponding to the product group $(\mathbb{Z}/2\mathbb{Z})^n$, and $R = D_2 := \{\pm e_1 \pm e_2\}$. Nevertheless, $R = B_2 = \{\pm e_1, \pm e_2, \pm e_1 \pm e_2\}$ satisfies (31) with the numbering $e_1, e_2, e_1 - e_2, e_1 + e_2$ while it is not an orthogonal root system. For this root system, a root subset R^i is either pointwisely or globally fixed. However, if a root system satisfies (31), then it contains at least two orthogonal roots, for otherwise, R^1, which contains only one root, will not be stable. Examples include $R = A_{n-1}, n \geq 3$ and the root system $I_2(m), m \neq 4$ corresponding to the dihedral group (see next paragraph). In fact, when m is odd, there is no orthogonal subset of roots. Otherwise, for an even integer $m \neq 4$, the roots come by orthogonal pairs however the global invariance breaks down (the value $m = 4$ corresponds to $R = B_2$). 2/ Since X^W is a C-valued process, then, given (31), Y^1 is valued in $C^1 := C \cup \sigma_1 C$ where σ_1 denotes the reflection w.r.t. $\alpha_1 \in R_+$, Y^2 is valued in $C^2 := C_1 \cup \sigma_2 C_1$ and so on. Moreover, (31) depends on the way one enumerates the positive roots so that different enumerations lead to different skew-product decompositions.

6.1. **Application to the computation of the semi-group density.** It is possible to recover the semi-group density of the Dunkl process using the skew-product decomposition (32). As always, we start with the rank-one case for which one easily writes for a nice function f:

$$\mathbb{E}_x(f(X_t)) = \mathbb{E}_x\left[f(|X_t|)\mathbf{1}_{\{N_{A_t} \text{ is even}\}}\right] + \mathbb{E}_x\left[f(-|X_t|)\mathbf{1}_{\{N_{A_t} \text{ is odd}\}}\right]$$

where $x \neq 0$. Without loss of generality, take $x > 0$. Since N is a Poisson process of parameter $k/2$, then one gets

$$\mathbb{P}(N_{A_t} \text{ is even} | \sigma(X_s, 0 \leq s \leq t)) = \frac{1}{2}(1 + e^{-kA_t})$$

and therefore:

$$\begin{aligned}\mathbb{E}_x(f(X_t)) &= \frac{1}{2}\mathbb{E}_x\left[f(|X_t|)(1 + e^{-kA_t})\right] + \frac{1}{2}\mathbb{E}_x\left[f(-|X_t|)(1 - e^{-kA_t})\right] \\ &= \frac{1}{2}\mathbb{E}_x\left[f(|X_t|) + f(-|X_t|)\right] + \frac{1}{2}\mathbb{E}_x\left[[f(|X_t|) - f(-|X_t|)]e^{-kA_t})\right].\end{aligned}$$

Now, recall that $|X|$ is a Bessel process of index $\nu = k - 1/2$ and that the Laplace transform of the Bessel clock A_t under the Bridge $|X_0| = x, |X_t| = y$ is given by ([61]):

$$\mathbb{E}_x(e^{-kA_t} || X_t| = y) = \frac{I_{\nu^2+2k}}{I_\nu}\left(\frac{xy}{t}\right)$$

where I_ν denotes for the modified Bessel function of index ν ([42]). Thus, the semi group density follows after some computations left to the reader. For the product group $(\mathbb{Z}/2\mathbb{Z})^n$, X^W consists of n independent Bessel processes and

$$A_t^{i-1,i} = \int_0^t \frac{ds}{(X_s^{W,i})^2}$$

so that the result follows by tensorization (as expected!!). The orthogonal root system $R = D_2 = \{\pm e_1 \pm e_2$ is isomorphic to $\{\pm e_1, \pm e_2\}$ correponding to $(\mathbb{Z}/2\mathbb{Z})^2$ and there is nothing to prove. One may also check using (15) that $X^{W,1} - X^{W,2}$ and $X^{W,1} + X^{W,2}$ define two (up to a a deterministic time change) independent Bessel processes. More precisely:

$$\begin{aligned} dX_t^{W,1} &= dB_t^1 + k_1\left[\frac{dt}{X_t^{W,1}-X_t^{W,2}} + \frac{dt}{X_t^{W,1}+X_t^{W,2}}\right] \\ dX_t^{W,2} &= dB_t^2 + k_1\left[\frac{dt}{X_t^{W,2}-X_t^{W,1}} + \frac{dt}{X_t^{W,2}+X_t^{W,1}}\right] \end{aligned}$$

thereby

$$\begin{aligned} d[X_t^{W,1}-X_t^{W,2}] &= d[B_t^1 - B_t^2] + 2k_1\left[\frac{dt}{X_t^{W,1}-X_t^{W,2}}\right] \\ d[X_t^{W,1}+X_t^{W,2}] &= d[B_t^1 + B_t^2] + 2k_1\left[\frac{dt}{X_t^{W,1}+X_t^{W,2}}\right]. \end{aligned}$$

Since $B^1 - B^2$ and $B^1 + B^2$ are orthogonal continuous martingales, then they are independent by Knight's Theorem which implies the independence of $X^{W,1} - X^{W,2}$ and $X^{W,1} + X^{W,2}$ since the Bessel process of dimension δ is the unique **strong** solution of ([55]):

$$dR_t = dW_t + \frac{\delta - 1}{2R_t}dt.$$

An interesting (non orthogonal) example is provided by $R = B_2 = \{\pm e_1, \pm e_2, \pm e_1, \pm e_2\}$ for which, it exist independent Poisson processes $N^i, 1 \leq i \leq 4$ of parameters k_0, k_0, k_1, k_1 ($k_i \geq 1/2, i = 0, 1$) respectively (recall that there are two orbits $\{\pm e_1, \pm e_2\}$ and $\{\pm e_1 - e_2, \pm e_1 + e_2\}$ so that $k = (k_0, k_1)$. Then, using the numbering $e_1, e_2, e_1 + e_2, e_1 - e_2$ of positive roots, (32) transforms to

$$X_t = \sigma_{e_1-e_2}^{N^4_{A^4_t}} \sigma_{e_1+e_2}^{N^3_{A^3_t}} \sigma_{e_2}^{N^2_{A^2_t}} \sigma_{e_1}^{N^1_{A^1_t}} X_t^W,$$

where

$$\begin{aligned} dX_t^{W,1} &= dB_t^1 + \frac{k_0}{X_t^{W,1}}dt + k_1\left[\frac{1}{X_t^{W,1}-X_t^{W,2}} + \frac{1}{X_t^{W,1}+X_t^{W,2}}\right]dt \\ dX_t^{W,2} &= dB_t^2 + \frac{k_0}{X_t^{W,2}}dt + k_1\left[\frac{1}{X_t^{W,2}-X_t^{W,1}} + \frac{1}{X_t^{W,2}+X_t^{W,1}}\right]dt \end{aligned}$$

and

$$A_t^i = \int_0^t \frac{ds}{(X_s^{W,i})^2}, \ i = 1, 2.$$

Remark. the Dunkl kernel of type B was already computed in [21] in the special case $k_1 = k_2$.

7. Trigonometric and hyperbolic Extensions

A parallel theory to the rational Dunkl operators gave rise to the so-called Cherednik operators which may be used to define trigonometric and hyperbolic analogues of Dunkl processes. In the scope of symmetric spaces, these analogues correspond to symmetric spaces with negative and positive curvatures such that the Dunkl setting is a rescaled limit as the curvature tends to zero. The underlying Laplacian is known as the Cherednik Laplacian and involves $u \mapsto \cot u$ or $u \mapsto \coth u$ instead of $u \mapsto 1/u$. The hyperbolic analogue is known as the radial Heckman-Opdam process and is valued in the Weyl chamber of a possibly **non reduced** root system. It was introduced and studied in [56]. The radial part of the trigonometric analogue was considered in [12]. The process is valued in a bounded convex domain known as *the principal Weyl alcove* defined by means of an *affine root system* ([12]). Moreover, it involves the non reduced root system of type BC ([37]). The process evolves like particles in an interval. It is related for some particular values of the multplicity function to the eigenvalues process of the real and the complex matrix Jacobi process ([14]). In the complex case, the process is interpreted as a Doob-transform of independent real BMs killed when they first collide or exit the pre-mentioned interval. Note also that the eigenvalues processes of both the orthogonal and the unitary BMs ([36]) may be seen as multidimensional processes valued in the principal alcove of type A ([37]). The eigenvalues evolve then like particles on the unit circle.

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References

[1] *T. H. Baker, P. J. Forrester.* The Calogero-Sutherland model and generalized classical polynomials. *Comm. Math. Phys.* **188**. 1997, 175-216.

[2] *P. Biane, P. Bougerol, N. O'Connell.* Littelmann paths and Brownian paths. *Duke Math. J.* **130**, no. 1. 2005, 127-167.

[3] *R. J. Beerends, E. M. Opdam.* Certain hypergeometric series related to the root system BC. *Trans. Amer. Math. Soc.* **339**, no. 2. 1993, 581-607.

[4] *P. Bougerol, T. Jeulin.* Paths in Weyl chambers and random matrices. *P. T. R. F.* **124**, no. 4. 2002, 517–543.

[5] *M. F. Bru.*Wishart Processes. *J. Theoretical Probability*, **4** , no. 4. 1991, 725 -751.
[6] *P. Carmona, F. Petit, M. Yor.* Beta-gamma random variables and intertwining relations between certain Markov processes. *Rev. Math. Ibero.* **14**, no. 2, 1998, 311-367.
[7] *E. Cépa.* Equations différentielles stochastiques multivoques. *Sém. Proba.* **XXIX**. 1995, 86-107.
[8] *O. Chybiryakov.* Processus de Dunkl et Relation de Lamperti. Ph. D. Thesis, Paris VI University, 2006.
[9] *O. Chybiryakov.* The Lamperti correspondence extended to Lévy processes and semi-stable Markov processes in locally compact groups. *Stoch. Proc. Appl.* **116**, 2006, 857-872.
[10] *C. Dellacherie, P. A. Meyer.* Probabilités et potentiel. Ch. XII-XVI. Théorie du potentiel associée à une résolvante. Théorie des processus de Markov. *Hermann, Paris.* 1987.
[11] *N. Demni.* Laguerre process and generalized Hartman-Watson law. *Bernoulli.* **13**, no. 2. 2007, 556-580.
[12] *N. Demni.* Topics on radial Dunkl processes. Submitted to *SIGMA Journal.*
[13] *Donati-Martin C, Doumerc Y, Matsumoto H, Yor M.* (2004) Some properties of Wishart process and a matrix extension of the Hartman-Watson law. *P. R. I. M. S. Kyoto University*, **40**, No 4, 2004, 1385-1412.
[14] *Y. Doumerc.* Matrix Jacobi Process. *Ph. D. Thesis.* Paul Sabatier Univ. May 2005.
[15] *Y. Doumerc, N. O'Connell.* Exit problems associated with finite reflection groups. *P. T. R. F.* **132**, no. 4. 2005, 501–538.
[16] *I. Dumitriu, A, Edelman.* Eigenvalue statistics for Beta-ensembles. *J. Math. Phy.* **43**, (11). 2002, 5830-5847.
[17] *C. F. Dunkl, Y. Xu.* Orthogonal Polynomials of Several Variables. *Encyclopedia of Mathematics and Its Applications. Cambridge University Press.* 2001.
[18] *C. F. Dunkl.* Integral kernels with reflection group invariance. *Canad. J. Math.* **43**, no. 6. 1991, 1213-1227.
[19] *C. F. Dunkl, M. F. E. De Jeu, E. M. Opdam.* Singular polynomials for finite reflection groups. *Trans. Amer. Math. Soc.* **346**, no. 1, 1994, 237-256.
[20] *C. F. Dunkl.* Intertwining operators associated to the group S_3. *Trans. Amer. Math. Soc.* **347**. no. 9. 1995, 3347-3374.
[21] *C. F. Dunkl.* Polynomials associated with dihedral groups. *Symmetry, Integrability and Geometry: Methods and Applications, SIGMA.* **3**, 2007.
[22] *C. F. Dunkl.* An intertwining operator for the group B_2. *Glasg. Math. J.* **49**, no.2. 2007, 291–319.
[23] *F. J. Dyson.* A Brownian motion model for the eigenvalues of a random matrix. *J. Math. Phys.* **3**. 1962, 1191-1198.
[24] *S. N. Ethier, T. G. Kurtz.* Markov Processes. Characterization and Convergence. *Wiley Series.* New York, 1986.
[25] *J. Faraut, A. Koryani.* Analysis on symmetric Cones. *Clarendon Press, Oxford*, 1994.
[26] *W. Feller.* An introduction to Probability Theory and Its Applications. **2**, 2nd ed. 1971. Wiley, New York.
[27] *L. Gallardo.* Paul Lévy's continuity Theorem on a product space. *C. R. A. S. Paris. Ser I.* **315**, no. 1. 1992, 73-77.
[28] *L. Gallardo, M. Yor.* Some new examples of Markov processes which enjoy the time-inversion property. *P. T. R. F.* **132**, 2005, 150-162.
[29] *L. Gallardo, M. Yor.* A chaotic representation property of the multidimensional Dunkl processes. *Ann. Probab.* **34**, no. 4. 2006, 1530-1549.
[30] *L. Gallardo, M. Yor.* Some remarkable properties of the Dunkl martingales. *In memoriam Paul-André Meyer: Séminaire de Probabilités XXXIX.* Lecture Notes in Math. **1874**, Springer, Berlin, 2006, 337–356.
[31] *L. Gallardo, L. Godefroy.* An invariance principle related to a process which generalizes N-dimensional Brownian motion. *C. R. A. S. Paris. Sér I.* **338**, no.6. 2004, 487–492.
[32] *A. R. Galmarino.* Representation of an isotropic diffusion as a skew product. *Zeit. Wahr. Verw. Geb.* **1**, 1962, 359-378.

[33] *D. Gilat*. Every non-negative submartingale is the absolute value of a martingale. *Ann. Probab.* **5**, 1977, 475-481.

[34] *D. J. Grabiner*. Brownian motion in a Weyl chamber, non-colliding particles and random matrices. *Ann. IHP.* **35**, 1999, no. 2, 177-204.

[35] *K. I. Gross, St. P. Richards*. Special functions of matrix argument, *Bull. Amer. Math. Soc.* **24**, no 2, 1991, 349-355.

[36] *D. G. Hobson, W. Werner*. Non-colliding Brownian motions on the circle. *Bull. London Math. Soc.* **28**, no. 6. 1996, 643-650.

[37] *J. E. Humphreys*. Reflections Groups and Coxeter Groups. *Cambridge University Press.* **29**. 2000.

[38] *J. Kaneko*. Selberg integrals and hypergeometric functions with Jack polynomials. *SIAM J. Math. Anal.* **24**, 1993, 1086-1110.

[39] *W. König, N. O' Connell*. Eigenvalues of the Laguerre process as non-colliding squared Bessel processes. *Elec. Comm. in Proba.* **6**, 2001, 107-114.

[40] *J. Lamperti*. Semi-stable Markov processes. *Zeit. Wahr. Verw. Gebiete.* **22**, 1972, 205-225.

[41] *S. Lawi*. Towards a characterization of Markov processes enjoying the time-inversion property. *J. Theo. Probab.* **21**. 2008, 144-168.

[42] *N. N. Lebedev*. Special Functions And Their Applications. *Dover Publications, INC.* 1972.

[43] *J. F. Legall*. Mouvement Brownien, cônes et processus stables. *P. T. R. F.* **76**. 1987, 587-627.

[44] *L. G. MacDonald*. Symmetric Functions and Hall Polynomials, 2^{nd} ed. *Clarendon Press, Oxford*, 1995.

[45] *P. A. Meyer*. Intégrales stochastiques. *Sém. Proba. I, Lecture Notes in Math.* **39**, 1967, 72-162.

[46] *P. A. Meyer*. Constructions de solutions d'équations de structure. *Sém. Proba. XXIII, Lecture Notes in Math.* **1372**, 1989, 142-145.

[47] *E. J. Pauwels, L. C. G. Rogers*. Skew-product decompositions of Brownian motions. *Geometry of random motion (Ithaca, N.Y., 1987), Contemp. Math.* **73**, Amer. Math. Soc., Providence, RI, 1988, 237-262.

[48] *J. Pitman, M. Yor*. Bessel processes and infinitely divisible laws. *In: Stochastic integrals. Ed. D. Williams. Proc. Sympos., Univ. Durham, Durham.* 1980, 285-370, Lecture Notes in Math., **851**, Springer, Berlin, 1981.

[49] *M. Rösler*. Generalized Hermite polynomials and the Heat equation for Dunkl operators. *Comm. Math. Phy.* **192**, 1998, 519-542.

[50] *M. Rösler*. Positivity of Dunkl's intertwining operator. *Duke Math. J.* **98**, 1999, 445-463.

[51] *M. Rösler*. A positive radial product formula for the Dunkl kernel. *Trans. Amer. Math. Soc.* **355**, no. 6. 2003, 2413-2438.

[52] *M. Rösler*. Dunkl operator : theory and applications, Orthogonal polynomials and special functions (Leuven, 2002). *Lecture Notes in Math.* Vol. 1817, Springer, Berlin, 2003, 93-135.

[53] *M. Rösler, M. Voit*. Markov processes related with Dunkl operators. *Adv. App. Math.* **21**, n0. 4. 1998, 575-643.

[54] *S. Watanabe*. On time-inversion of one dimensional diffusion processes. *Zeit. Wahr. Verw. Geb.* **31**, 1975, 115-124.

[55] *D. Revuz, M. Yor*. Continuous Martingales And Brownian Motion, 3^{rd} ed, Springer, 1999.

[56] *B. Schapira*. The Heckman-Opdam Markov processes. *Probab. Theory Related Fields.* **138**. no. 3-4. 2007, 495–519.

[57] *K. Trimèche*. Paley-Wiener Theorems for the Dunkl transform and Dunkl translation operators. *Int. Trans. Spec. Funct.* **13**. 2002, 17-38.

[58] *S. R. S. Varadhan, R. J. Williams*. Brownian motion in a wedge with oblique reflection. Comm. Pure Appl. Math. **38**. 1985, 405-443.

[59] *V. A. Volkonski*. Random time changes in strong Markov processes. *Teor. Vero. Prim.* **3**. 1958, 310-326.

[60] *R. J. Williams*. Reflected Brownian motion in a wedge: semi-martingale property. *Zeit. Wahr.* **69**. 1985, 161-176.

[61] *M.Yor*. Loi de l'indice du lacet brownien et distribution de Hartman-Watson, *Zeit. Wahr. Verw. Geb.* **53**, no.1, 1980, 71-95.

[62] *M. Yor*. Some Aspects of Brownian Motion. Part II: Some Recent Martingale Problems. *Birkhäuser, Basel.* 1997.