On the approximation of integrated semigroups

Miao Li and Sergey Piskarev

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Abstract

This paper is devoted to the approximation of integrated semigroups with respect to space and time variables. The presentation is given in the abstract framework of discrete approximation schemes, which includes finite element, finite difference, and projection methods.

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1 Introduction

Let $B(E)$ denote the Banach algebra of linear bounded operators on a complex Banach space E. The set of all linear closed densely defined operators in E will be denoted by $\mathcal{C}(E)$.

Let A be the generator of a C_0-semigroup $\exp(tA)$, $t \geq 0$, and consider in the Banach space E the Cauchy problem

$$u'(t) = Au(t) + f(t),\ t \geq 0, \tag{1.1}$$

$$u(0) = u^0, \tag{1.2}$$

with some function $f(\cdot) \in C([0,T];E)$. Usually, one assumes $u^0 \in D(A)$ in order to obtain well-posedness.

Definition 1.1 *A function $u(\cdot)$ is called a solution of* (1.1) *in the classical sense if* $u(\cdot) \in C^1([0,T];E) \cap C([0,T];D(A))$ *and $u(\cdot)$ satisfies* (1.1) – (1.2).

It is known that the mild solution of (1.1)-(1.2) is defined by the function

$$u(t) = \exp(tA)u^0 + \int_0^t \exp((t-s)A)f(s)\,ds. \tag{1.3}$$

If $u(\cdot)$ in (1.3) is continuously differentiable, then such $u(\cdot)$ is a classical solution of (1.1)-(1.2). If one puts $f(s) = \exp(sA)x$, $s \geq 0$, then boundedness of $Au(t) = \exp(tA)Au^0 + tA\exp(tA)x$ as $t \to 0$ implies that A generates an analytic C_0-semigroup. But even if A generates an analytic C_0-semigroup and if $f(\cdot) \in C([0,T];E)$, then $u(\cdot)$ from (1.3) is generally not a classical solution (see [30]). Therefore, in the numerical analysis of these equations one can only expect a maximal regularity inequality with a logarithm. Actually, if (1.1)-(1.2) is coercive well-posed in $C([0,T];E)$, then either A is bounded or E contains a subspace isomorphic to c_0 (see [11]).

Therefore, in order to have well-posedness of $(1.1)-(1.2)$ one needs to impose some smoothness assumption on $f(\cdot)$ in case of a C_0-semigroup. For instance, one may assume that $f(\cdot) \in C^1([0,T];E)$. The situation changes dramatically if the operator A satisfies weaker conditions than those for a generator of C_0-semigroup.

In the literature [2, 14, 17, 18] there has been quite some interest in solving problem (1.1)-(1.2) under conditions weaker than those for a generator A of a C_0-semigroup, namely under the condition that A generates an integrated semigroup e_1^{tA}, $t \geq 0$, or a C-semigroup $S(t)$, $t \geq 0$. For example, the Schrödinger operator $i\Delta$ generates a C_0-semigroup on $L^p(I\!R^n)$ iff $p = 2$. Moreover, if $\alpha > n|\frac{1}{2} - \frac{1}{p}|$, then $i\Delta$ generates an α-times integrated semigroup. The starting point for this paper is the observation that there seems to be no systematically developed approximation theory for integrated semigroups, not even for the homogeneous case.

In recent years a rather general approach has been developed for studying the approximation of solutions of C_0-semigroups. We give here a short historical overview of some simple general results from this approach. Let us consider a well-posed Cauchy problem in a Banach space E with some operator $A \in \mathcal{C}(E)$

$$(1.4) \qquad \begin{aligned} &u'(t) = Au(t), \ t \in [0,\infty), \\ &u(0) = u^0 \in E. \end{aligned}$$

If A generates a C_0-semigroup $\exp(\cdot A)$ then, as is well known, the generalized solution of (1.4) is given by $u(t) = \exp(tA)u^0$ for $t \geq 0$. The theory of well-posed problems and the numerical analysis of these problems have been developed extensively, see for instance the papers [13, 15, 23, 25]. Let us consider a general discretization scheme obtained from the a semidiscrete approximation of (1.4) in some Banach spaces E_n :

$$(1.5) \qquad \begin{aligned} &u_n'(t) = A_n u_n(t), \ t \in [0,\infty), \\ &u_n(0) = u_n^0 \in E_n, \end{aligned}$$

Here we assume $u_n^0 \xrightarrow{\mathcal{P}} u^0$ and operators $A_n \in \mathcal{C}(E_n)$ that generate C_0-semigroups, consistent with the operator $A \in \mathcal{C}(E)$. For a precise definition of discrete convergence of elements and operators see Section 3.

First, we state the following version of Trotter-Kato's Theorem for general approximation schemes:

Theorem 1.1 [*28*] (*Theorem ABC*) *Assume that $A \in \mathcal{C}(E), A_n \in \mathcal{C}(E_n)$ generate C_0-semigroups. Then the following conditions (A) and (B) are equivalent to condition (C).*

(A) Consistency. There exists $\lambda \in \rho(A) \cap \bigcap_n \rho(A_n)$ such that the resolvents converge $(\lambda I_n - A_n)^{-1} \xrightarrow{\mathcal{PP}} (\lambda I - A)^{-1}$;

(B) Stability. There are some constants $M \geq 1$ and ω, independent of n such that $\|\exp(tA_n)\| \leq M\exp(\omega t)$ for $t \geq 0$ and any $n \in I\!N$;

(C) Convergence. For any finite $T > 0$ one has $\max_{t\in[0,T]} \|\exp(tA_n)u_n^0 - p_n \exp(tA)u^0\| \to 0$ as $n \to \infty$, whenever $u_n^0 \xrightarrow{\mathcal{P}} u^0$ for any $u_n^0 \in E_n, u^0 \in E$.

Remark 1.1 *Condition (A) of this Theorem is equivalent to compatibility of the operators (A_n, A).*

Theorem 1.2 [*23*] *Let operators A and A_n generate analytic C_0-semigroups. The following conditions (A) and (B_1) are equivalent to condition (C_1).*

(A) Consistency. There exists $\lambda \in \rho(A) \cap \bigcap_n \rho(A_n)$ such that the resolvents converge $(\lambda I_n - A_n)^{-1} \xrightarrow{\mathcal{PP}} (\lambda I - A)^{-1}$;

(B_1) Stability. There exist constants $M_2 \geq 1$ and ω_2 independent of n such that for any $Re\lambda > \omega_2$

$$\|(\lambda I_n - A_n)^{-1}\| \leq \frac{M_2}{|\lambda - \omega_2|} \quad \textit{for all } n \in I\!N;$$

(C_1) *Convergence. For any finite* $\mu > 0$ *and some* $0 < \theta < \frac{\pi}{2}$ *we have*

$$\max_{\eta \in \Sigma(\theta,\mu)} \| \exp(\eta A_n) u_n^0 - p_n \exp(\eta A) u^0 \| \to 0 \quad as \quad n \to \infty \quad whenever \quad u_n^0 \xrightarrow{\mathcal{P}} u^0.$$

Here $\Sigma(\theta, \mu) = \{z \in \Sigma(\theta) : |z| \leq \mu\}$, *and* $\Sigma(\theta) = \{z \in \mathbb{C} \ : |arg\, z| \leq \theta\}$.

Normally one assumes that conditions (A) and (B) for the corresponding C_0-semigroup are satisfied without any restriction of generality if discretization processes in time are considered. We denote by $T_n(\cdot)$ a family of discrete semigroups as in [15], i.e. $T_n(t) = T_n(\tau_n)^{k_n}$, where $k_n = [\frac{t}{\tau_n}]$, as $\tau_n \to 0, n \to \infty$. The generator of the discrete semigroup is defined by $\breve{A}_n = \frac{1}{\tau_n}(T_n(\tau_n) - I_n) \in B(E_n)$ so that $T_n(t) = (I_n + \tau_n \breve{A}_n)^{k_n}$, for $t = k_n \tau_n$.

Theorem 1.3 *[28] (Theorem* $ABC - discr$*) The following conditions* (A) *and* (B') *are equivalent to condition* (C').

(A) *Consistency. There exists* $\lambda \in \rho(A) \cap \cap_n \rho(\breve{A}_n)$ *such that the resolvents converge* $(\lambda I_n - \breve{A}_n)^{-1} \xrightarrow{\mathcal{PP}} (\lambda I - A)^{-1}$;

(B') *Stability. There exist constants* $M_1 \geq 1$ *and* $\omega_1 \in \mathbb{R}$ *such that*

$$\|T_n(t)\| \leq M_1 \exp(\omega_1 t) \ for\ t \in \overline{\mathbb{R}}_+ = [0, \infty), n \in \mathbb{N};$$

(C') *Convergence. For any finite* $T > 0$ *one has* $\max_{t \in [0,T]} \|T_n(t) u_n^0 - p_n \exp(tA) u^0\| \to 0$ *as* $n \to \infty$, *whenever* $u_n^0 \xrightarrow{\mathcal{P}} u^0$ *for any* $u^0 \in E, u_n^0 \in E_n$.

Theorem 1.4 *[28] Assume that operators* $A \in \mathcal{C}(E), A_n \in \mathcal{C}(E_n)$ *are given that generate* C_0*-semigroups. Assume also that conditions* (A) *and* (B) *of Theorem* 1.1 *hold. Then the implicit difference scheme*

$$\frac{\overline{U}_n(t + \tau_n) - \overline{U}_n(t)}{\tau_n} = A_n \overline{U}_n(t + \tau), \ \overline{U}_n(0) = u_n^0, \tag{1.6}$$

is stable, i.e. $\|(I_n - \tau_n A_n)^{-k_n}\| \leq M_1 e^{\omega_1 t}, t = k_n \tau_n \in \overline{\mathbb{R}}_+$, *and gives an approximation to the solution of the problem* (1.4), *i.e.* $\overline{U}_n(t) \equiv (I_n - \tau_n A_n)^{-k_n} u_n^0 \xrightarrow{\mathcal{P}} \exp(tA) u_n^0$ *uniformly with respect to* $t = k_n \tau_n \in [0, T]$ *as* $u_n^0 \xrightarrow{\mathcal{P}} u^0, n \to \infty, k_n \to \infty, \tau_n \to 0$.

Note that in this case $T_n(\tau_n) = (I_n - \tau_n A_n)^{-1}$ and $\breve{A}_n = ((I_n - \tau_n A_n)^{-1} - I_n)/\tau_n = A_n(I_n - \tau_n A_n)^{-1}$.

Theorem 1.5 *[28] Assume that conditions* (A) *and* (B) *of Theorem* 1.1 *hold and condition*

$$\tau_n \|A_n^2\| \leq C, \ n \in \mathbb{N}, \tag{1.7}$$

is fulfilled. Then the difference scheme

$$\frac{U_n(t + \tau_n) - U_n(t)}{\tau_n} = A_n U_n(t), \ \ U_n(0) = u_n^0, \tag{1.8}$$

is stable and gives an approximation to the solution of the problem (1.4), *i.e.* $U_n(t) \equiv (I_n + \tau_n A_n)^{k_n} u_n^0 \xrightarrow{\mathcal{P}} u(t)$ *uniformly with respect to* $t = k_n \tau_n \in [0, T]$ *as* $u_n^0 \xrightarrow{\mathcal{P}} u^0, n \to \infty, k_n \to \infty, \tau_n \to 0$.

Theorem 1.6 *[12, 23] Assume that conditions* (A) *and* (B_1) *of Theorem* 1.2 *hold and condition*

$$\tau_n\|A_n\| \le 1/(M+2),\ n \in I\!N, \tag{1.9}$$

is fulfilled. Then the difference scheme (1.8) *is stable and gives an approximation to the solution of the problem* (1.4)*, i.e.* $U_n(t) \equiv (I_n + \tau_n A_n)^{k_n} u_n^0 \xrightarrow{\mathcal{P}} u(t)$ *uniformly with respect to* $t = k_n\tau_n \in [0,T]$ *as* $u_n^0 \xrightarrow{\mathcal{P}} u^0, n \to \infty, k_n \to \infty, \tau_n \to 0$.

In this case we have $T_n(\tau_n) = I_n + \tau_n A_n$ and $\breve{A}_n = A_n$.

Let us introduce the following equivalent conditions:

(B_1') Stability. There are constants M', ω' such that

$$\|\exp(tA_n)\| \le M' e^{\omega' t},\ \|A_n \exp(tA_n)\| \le \frac{M'}{t} e^{\omega' t},\ t \in I\!R_+.$$

(B_1'') Stability. There are constants M'', ω'' and $\tau^* > 0$ such that

$$\|(I_n - \tau_n A_n)^{-k}\| \le M'' e^{\omega'' k\tau_n},\ \|k\tau_n A_n (I_n - \tau_n A_n)^{-k}\| \le M'' e^{\omega'' k\tau_n},\ 0 < \tau_n < \tau^*, n, k \in I\!N.$$

Theorem 1.7 *Conditions* (A) *and* (B_1') *are equivalent to the condition* (C_1).

Remark 1.2 *Note that conditions* $(B_1), (B_1')$ *and* (B_1'') *are equivalent, see [26].*

Let us recall that the constant M_2 in condition (B_1), which defines $\alpha \in (0, \frac{\pi}{2})$ by $M_2 \sin\alpha < 1$ [16], satisfies

$$\|(\lambda I_n - A_n)^{-1}\| \le \frac{M_2}{|\lambda - \omega|} \text{ for all } \lambda \in \Sigma(\pi/2 + \alpha). \tag{1.10}$$

Recall that there exists a unique Padé approximation of degree $(p.q)$ for e^{-z} given by the formula $R_{p,q}(z) = P_{p,q}(z)/Q_{p,q}(z)$, where

$$P_{p,q}(z) = \sum_{j=0}^{p} \frac{(p+q-j)!p!(-z)^j}{(p+q)!j!(p-j)!},\ Q_{p,q}(z) = \sum_{j=0}^{q} \frac{(p+q-j)!q!z^j}{(p+q)!j!(q-j)!}.$$

Definition 1.2 *A rational approximation* $r_{p,q}(\cdot) \in \pi_{p,q}$ *for* e^{-z} *is called*

(a) A-acceptable if $|r_{p,q}(z)| < 1$ *for* $Re(z) > 0$;

(b) A(θ)-acceptable if $|r_{p,q}(z)| < 1$ *for* $z \in \Sigma(\theta) = \{z : -\theta < arg(z) < \theta, z \ne 0\}$.

It is well known that $R_{q,q}(z), R_{q-1,q}(z)$, and $R_{q-2,q}(z)$ are A-acceptable. But for $q \ge 3$ and $p = q-3$, the Padé functions are not A-acceptable.

Since $r(\cdot) \in \pi_{p,q}$ is an approximation of e^{-z}, it is natural to construct the operator-function $r(\tau_n A_n)^k$ which can be considered as an approximation of $\exp(tA_n)$ for $t = k\tau_n$. For simplicity, we assume in the following Theorems of this section that $\|\exp(tA_n)\| \le M, t \in \overline{I\!R}_+$.

Theorem 1.8 *[6] Let condition* (B) *be satisfied. Then there exists a constant* C*, depending on* $r(\cdot)$*, such that if* $r(\cdot)$ *is A-acceptable, then*

$$\|r(\tau_n A_n)^k\| \le CM\sqrt{k} \text{ for } \tau_n > 0, k \in I\!N.$$

Remark 1.3 *The term* $\sqrt{k}$ *in Theorem* 1.8 *cannot be removed in general; in fact, there are examples (see [9]), for which the inequality* $\|r(\tau_n A_n)^k\| \ge c\sqrt{k}, k \in I\!N$*, holds.*

We say that $r(\cdot) \in \pi_{p,q}$ is accurate of order $1 \le d \le p+q$ if $|e^{-z} - r(z)| = O(|z|^{d+1})$ as $|z| \to 0$.

Theorem 1.9 [9] *Let condition* (B_1) *be satisfied. Then there is a constant* C *depending on* r, *such that if* r *is* $A(\theta)$*-acceptable, accurate of order* d, *and* $\theta \in (\pi/2 - \alpha, \pi/2]$ *with* α *from condition* (1.10), *then*

$$\|r(\tau_n A_n)^k\| \le CM \text{ for } \tau_n > 0, k \in I\!N.$$

The difference scheme corresponding to the rational function $r(\cdot)$ of Pade type $R_{1,1}(z)$ (also called the Crank-Nicolson scheme) is given by the formula

$$\frac{U_n(t+\tau_n) - U_n(t)}{\tau_n} = A_n \frac{U_n(t+\tau) + U_n(t)}{2},\ U_n(0) = u_n^0, \tag{1.11}$$

It is easy to see that in this case $T_n(\tau_n) = \frac{I_n + \frac{\tau_n}{2} A_n}{I_n - \frac{\tau_n}{2} A_n}$ and $\breve{A}_n = (\frac{I_n + \frac{\tau_n}{2} A_n}{I_n - \frac{\tau_n}{2} A_n} - I_n)/\tau_n = A_n(I_n - \frac{\tau_n}{2} A_n)^{-1}$.

2 Preliminaries

Let us consider the Cauchy problem in the Banach space E

$$\begin{aligned} u'(t) &= Au(t) + f(t),\ t \in [0,T], \\ u(0) &= u^0 \in E, \end{aligned} \tag{2.1}$$

where the operator A generates k-times integrated semigroup and $f(\cdot) \in L^1([0,T];E)$. A function $u(\cdot)$ is called a classical solution of (2.1) if it belongs to $C^1([0,T];E) \cap C([0,T];D(A))$ and satisfies both equations in (2.1).

The k-times integrated semigroup is a family of bounded linear operators e_k^{tA}, that is strongly continuous in $t \in [0,\infty)$ and satisfies the equation

$$e_k^{tA} = A \int_0^t e_k^{sA} ds + \frac{t^k}{k!}, \quad t \ge 0. \tag{2.2}$$

Let us define the function

$$v(t) = e_k^{tA} u^0 + \int_0^t e_k^{(t-s)A} f(s) ds, \quad t \in [0,T].$$

If there is a classical solution of (2.1), then $v(\cdot) \in C^{k+1}([0,T];E)$ and $v^{(k)}(\cdot) = u(\cdot)$. If the k-times integrated semigroup is exponentially bounded, i.e. $\|e_k^{tA}\| \le Me^{\omega t}, t \in \overline{I\!R}_+$, then resolvents exist and satisfy

$$(\lambda I - A)^{-1} = \lambda^k \int_0^\infty e^{-\lambda t} e_k^{tA} dt \quad \text{for} \quad \lambda > \omega.$$

Let us consider the problem (2.1) when $k = 1$.

If we have a once integrated semigroup and $f(\cdot) \equiv 0$, $u^0 \in D(A^2)$ then the solution of (2.1) is given by $u(t) = (e_1^{tA} u^0)'_t$. As an example, let us approximate (2.2) in this case as follows $e_1^{\tau A} \approx \tau A e_1^{\tau A} + \tau I$. We can write the approximation to e_1^{tA} in the form

$$W(\tau) = \tau (I - \tau A)^{-1}. \tag{2.3}$$

It is also well-known that a once integrated semigroup satisfies the equation

$$e_1^{sA} e_1^{tA} = \int_0^s (e_1^{(r+t)A} - e_1^{rA}) dr \quad \text{for any} \quad s,t \ge 0. \tag{2.4}$$

Setting $s = \tau, t = k\tau$, one obtains a discrete 1-times integrated semigroup using the approximation

$$e_1^{k\tau A} e_1^{\tau A} \approx \tau e_1^{(k+1)\tau A} - \tau e_1^{\tau A}.$$

This leads to the difference scheme

$$W((k+1)\tau) = W(k\tau)W(\tau)/\tau + W(\tau), \quad W(\tau) = \tau(I - \tau A)^{-1}, \tag{2.5}$$

from which the discrete approximation function of e_1^{tA} can be calculated. The expression (2.5) is the analog of the scheme (1.6) for C_0-semigroups (see Proposition 4.2 for details). In this paper we are going to construct a general approximation theory for exponentially bounded once integrated semigroups.

3 Discretisation of integrated semigroups in space

The general approximation scheme, due to [29] can be described in the following way. Let E_n and E be Banach spaces and $\{p_n\}$ be a sequence of linear bounded operators $p_n : E \to E_n, p_n \in B(E, E_n), n \in I\!N = \{1, 2, \cdots\}$, with the property:

$$\|p_n x\|_{E_n} \to \|x\|_E \text{ as } n \to \infty \text{ for any } x \in E. \tag{3.1}$$

From (3.1) it follows (see [29]) that $\|p_n\| \le C, n \in I\!N$.

Definition 3.1 *The sequence of elements $\{x_n\}, x_n \in E_n, n \in I\!N$, is called $\mathcal{P}$-convergent to $x \in E$ iff $\|x_n - p_n x\|_{E_n} \to 0$ as $n \to \infty$. We write this as $x_n \xrightarrow{\mathcal{P}} x$.*

Definition 3.2 *The sequence of bounded linear operators $B_n \in B(E_n), n \in I\!N$, is called $\mathcal{PP}$-convergent to the bounded linear operator $B \in B(E)$ if for every $x \in E$ and for every sequence $\{x_n\}, x_n \in E_n, n \in I\!N$, such that $x_n \xrightarrow{\mathcal{P}} x$ one has $B_n x_n \xrightarrow{\mathcal{P}} Bx$. We then write $B_n \xrightarrow{\mathcal{PP}} B$.*

Remark 3.1 *If we set $E_n = E$ and $p_n = I$ for each $n \in I\!N$, where I is the identity operator on E, then Definition* 3.1 *leads to the traditional pointwise convergence of bounded linear operators which we denote by $B_n \to B$.*

Since our infinitesimal generators are generally unbounded we consider the notion of *compatibility*.

Definition 3.3 *The sequence of closed linear operators $\{A_n\}, A_n \in \mathcal{C}(E_n), n \in I\!N$, is called compatible with a closed linear operator $A \in \mathcal{C}(E)$ iff for each $x \in D(A)$ there is a sequence $\{x_n\}, x_n \in D(A_n) \subseteq E_n, n \in I\!N$, such that $x_n \xrightarrow{\mathcal{P}} x$ and $A_n x_n \xrightarrow{\mathcal{P}} Ax$. We write (A_n, A) are compatible.*

Note, that (A_n, A) are compatible if resolvents converge $(\lambda I_n - A_n)^{-1} \xrightarrow{\mathcal{PP}} (\lambda I - A)^{-1}$.

Let us define the space

$$c(E_n, E) = \{\{x_n\}, x_n \in E_n : \text{ there exists } x \in E \text{ such that } x_n \xrightarrow{\mathcal{P}} x\},$$

where the norm of an element $\bar{x} = \{x_n\} \in c(E_n, E)$ is defined as

$$\|\{x_n\}\|_{c(E_n,E)} = \sup_n \|x_n\|_{E_n}.$$

We also introduce the space

$$l^\infty(E_n) = \{\{x_n\}, x_n \in E_n : \text{ with a norm } \sup_n \|x_n\|_{E_n} < \infty\}.$$

Next we show that these two normed spaces are closed.

Lemma 3.1 *The spaces $c(E_n, E)$ and $l^\infty(E_n)$ are closed. Consequently, $c(E_n, E)$ is a closed subspace of $l^\infty(E_n)$.*

Proof. It is clear that $l^\infty(E_n)$ is closed. We now show that $c(E_n, E)$ is closed as well.

Let $\bar{y}^m = \{x_n^m\} \in c(E_n, E)$ be a Cauchy sequence in $c(E_n, E)$, i.e.

$$\|\bar{y}^m - \bar{y}^k\|_{c(E_n,E)} = \sup_n \|x_n^m - x_n^k\| \to 0, \quad m, k \to \infty. \tag{3.2}$$

For every $\epsilon > 0$ there exists $K \in I\!N$ such that

$$\|x_n^m - x_n^k\| < \epsilon/4, \quad m, k \geq K, \forall n \in I\!N. \tag{3.3}$$

Since E_n are Banach spaces, one obtains $x_n^m \to x_n$ as $m \to \infty$ in E_n. Define $\bar{y} = \{x_n\}$. We have to show that $\bar{y} \in c(E_n, E)$ and $\bar{y}^m \to \bar{y}$ in $c(E_n, E)$ as $m \to \infty$.

By definition, for every $\bar{y}^m \in c(E_n, E)$ there exists $x^m \in E$ such that

$$\|x_n^m - p_n x^m\| \to 0 \text{ as } \quad n \to \infty. \tag{3.4}$$

We show that $\{x^m\}$ is a Cauchy sequence in E. Using (3.1), for every fixed $m, k \geq K$, there exists some $N_0(m, k) \in I\!N$ such that

$$\|x^m - x^k\| \leq \epsilon/4 + \|p_n(x^m - x^k)\|, \quad n \geq N_0(m, k).$$

Since (3.4) holds we can choose $N_1 \geq N_0(m, k)$ such that

$$\|x_{N_1}^m - p_{N_1} x^m\|_{E_{N_1}}, \|x_{N_1}^k - p_{N_1} x^k\|_{E_{N_1}} < \epsilon/4.$$

Hence

$$\begin{aligned} \|x^m - x^k\| &\leq \epsilon/4 + \|p_{N_1}(x^m - x^k)\|_{N_1} \\ &\leq \epsilon/4 + \|x_{N_1}^m - p_{N_1} x^m\|_{N_1} + \|x_{N_1}^k - p_{N_1} x^k\|_{N_1} + \|x_{N_1}^m - x_{N_1}^k\|_{N_1} \\ &< \epsilon \end{aligned}$$

for any $m, k \geq K$. This proves that $\{x^m\}$ is a Cauchy sequence in E. Let $x \in E$ be an element such that $x^m \to x$ in E. Then $x_n \xrightarrow{\mathcal{P}} x$, where $\{x_n\} = \bar{y}$, i.e. $\bar{y} \in c(E_n, E)$. In fact, for given $\epsilon > 0$, take $m = K$. Then there exists $N_2 \in I\!N$ such that for $n > N_2$,

$$\|x_n^K - p_n x^K\| < \epsilon/4,$$

and

$$\|p_n x^K - p_n x\| < C\epsilon$$

since $\|x^K - x\| \leq \epsilon$. We have for $n > N_2$,

$$\begin{aligned} \|x_n - p_n x\| &\leq \|x_n - x_n^K\| + \|x_n^K - p_n x^K\| + \|p_n x^K - p_n x\| \\ &\leq \epsilon/4 + \epsilon/4 + C\epsilon < (1 + C)\epsilon. \end{aligned}$$

The $\mathcal{P}$-convergence of x_n to x implies that $\bar{y} \in c(E_n, E)$. Finally, by (3.2) and (3.3) it follows

$$\|\bar{y}^m - \bar{y}\|_{c(E_n,E)} = \sup_n \|x_n^m - x_n\| \to 0, \quad m \to \infty.$$

This proves that $\bar{y}^m \to \bar{y}$ in $c(E_n, E)$.

Now we can derive a discrete version of Theorem 1.7.5 from [2].

Theorem 3.1 *Let $f_n(\cdot) \in C(I\!R_+; E_n)$ with $\|f_n(t)\|_{E_n} \le Me^{\omega t}$ for some $M > 0$, $\omega \in I\!R$ and all $n \in I\!N$. Let $\lambda_0 \ge \omega$. The following are equivalent:*

(i) The Laplace transforms $\hat{f}_n(\cdot)$ $\mathcal{P}$-converge pointwise on (λ_0, ∞) to $\hat{f}(\cdot)$ and the sequence $\{f_n(\cdot)\}, n \in I\!N$, is equicontinuous on compact subsets of $I\!R_+$;

(ii) The functions $f_n(\cdot)$ $\mathcal{P}$-converge uniformly on compact subsets of $I\!R_+$ to $f(\cdot)$.

Moreover, if (ii) holds, then $\hat{f}(\lambda) = \mathcal{P}\text{-}\lim_{n\to\infty} \hat{f}_n(\lambda)$ for all $\lambda > \lambda_0$, where $f(t) := \mathcal{P}\text{-}\lim_{n\to\infty} f_n(t)$.

Proof. By Lemma 3.1, $c(E_n, E)$ is a closed subspace of $l^\infty(E_n)$. Define $w(\cdot) : I\!R_+ \to l^\infty(E_n)$ by $w(t) = \{f_n(t)\}$. Assume that (i) holds, from the equicontinuity of $\{f_n(\cdot)\}$ we know that $w(t)$ is continuous. The $\mathcal{P}$-convergence of $\hat{f}_n(\cdot)$ implies $\hat{w}(\lambda) = \{\hat{f}_n(\lambda)\} \in c(E_n, E)$ for all $\lambda > \lambda_0$. Consider the quotient mapping $q : l^\infty(E_n) \to l^\infty(E_n)/c(E_n, E)$, then $(\widehat{q \circ w})(\lambda) = q(\hat{w}(\lambda)) = 0$ for all $\lambda > \lambda_0$. Since $q \circ w : I\!R_+ \to l^\infty(E_n)/c(E_n, E)$ is continuous, by the uniqueness theorem we have $q \circ w(t) = 0$ for all $t \ge 0$, that is, $w(t) \in c(E_n, E)$ for all $t \ge 0$. This implies that the $f_n(t)$ $\mathcal{P}$-converge pointwise. Define $f(t) = \mathcal{P}\text{-}\lim_{n\to\infty} f_n(t)$. We show that this convergence is uniformly on compact intervals of $I\!R_+$. Let $[0, T]$ be such an interval. For every $\epsilon > 0$, since $f(t)$ is uniformly continuous on $[0, T]$, an δ_1 can be chosen such that $\|f(t) - f(s)\| \le \epsilon$ when $t, s \in [0, T]$ satisfying $|t - s| < \delta_1$. Since $\{f_n(\cdot)\}$ are equicontinuous on $[0, T]$ one has $\|f_n(t) - f_n(s)\| \le \epsilon$ for $|t - s| < \delta_1$ and $t, s \in [0, T]$, where δ_1 is some positive constant. Next we choose m large enough such that $T/m \le \delta$, and let $t_j = Tj/m$ for $j = 0, 1, \cdots, m$. Since $f_n(t_j) \xrightarrow{\mathcal{P}} f(t_j)$, we find an $n(m) \in I\!N$ such that $\|f_n(t_j) - p_n f(t_j)\| \le 2\epsilon$ for all $n \ge n(m)$ and $j = 0, 1, \cdots, m$. For any $t \in [0, T]$, there is a t_j such that $|t - t_j| < \delta_2 := \min(\delta, \delta_1)$. Thus for $n > n(m)$ we have

$$\|f_n(t) - p_n f(t)\| \le \|f_n(t) - f_n(t_j)\| + \|f_n(t_j) - p_n f(t_j)\| + \|p_n f(t_j) - p_n f(t)\| < (2 + C)\epsilon,$$

where $C = \sup \|p_n\|$. This prove that the $f_n(\cdot)$ $\mathcal{P}$-converge to $f(\cdot)$ uniformly on $[0, T]$.

The implication $(ii) \to (i)$ follows from the dominated convergence theorem.

Let us first consider a general discrete version of the ABC Theorem for integrated semigroups.

Theorem 3.2 *($Theorem\, ABC - int$) Assume that closed operators A, A_n on E and E_n respectively generate exponentially bounded k-times integrated semigroups. The following conditions (A) and (B_{int}) are equivalent to condition (C_{int}).*

(A) Consistency. There exists $\lambda \in \rho(A) \cap \bigcap_n \rho(A_n)$ such that the resolvents converge $(\lambda I_n - A_n)^{-1} \xrightarrow{\mathcal{PP}} (\lambda I - A)^{-1}$;

(B_{int}) Stability. There exist constants $M \ge 1$ and ω_1, which are independent of n and such that $\|e_k^{tA_n}\| \le M \exp(\omega_1 t)$ for $t \ge 0$ and any $n \in I\!N$, and the sequence $\{e_k^{tA_n} p_n x\}, n \in I\!N$, is equicontinuous on compact subsets of $I\!R_+$ for every $x \in E$.

(C_{int}) Convergence. For some finite $\omega > 0$ one has $\max_{t\in[0,\infty)} e^{-\omega t}\|e_k^{tA_n} u_n^0 - p_n e_k^{tA} u^0\| \to 0$ as $n \to \infty$, whenever $u_n^0 \xrightarrow{\mathcal{P}} u^0$ for any $u_n^0 \in E_n, u^0 \in E$.

Proof. Assume that conditions (A) and (B_{int}) hold. In the same way as in Theorem 3.1 we introduce the subspace $c_0(E_n, E)$, i.e. the space of the sequences which converge to zero, and put $f_n(t) = e_1^{tA_n} x_n - p_n e_1^{tA} x$, where $x_n \xrightarrow{\mathcal{P}} x$. From the convergence of resolvents (condition (A)), one has that the Laplace transforms of $f_n(t)$ converge to zero. Then for any $x \in E$ there is a sequence $\{x_n\}$ such that $x_n \xrightarrow{\mathcal{P}} x$ and by Theorem 3.1 $\max_{t\in[0,T]} \|e_k^{tA_n} x_n - p_n e_k^{tA} x\| \to 0$ as $n \to \infty$.

Conversely, assume that (C_{int}) holds. To prove (A) and (B_{int}), by Theorem 3.1 we only need to show (B_{int}). From (C_{int}) it follows that $\max_{t\in[0,\infty)} e^{-\omega t}\|e_k^{tA_n}\|_{B(E_n)} \le C$. If this is not true, then one finds sequences $\|x_n\| = 1$ and $t_n \in [0, \infty)$ such that $e^{-\omega t_n}\|e_k^{t_n A_n} x_n\| \to \infty$. Then the sequence

$y_n = \frac{x_n}{e^{-\omega t_n}\|e_k^{t_n A_n} x_n\|}$ converges to zero and satisfies $e^{-\omega t_n} e_k^{t_n A_n} y_n \xrightarrow{\mathcal{P}} e^{-\omega t_n} e_k^{t_n A} 0 = 0$ uniformly in t_n, but this contradicts $e^{-\omega t_n}\|e_k^{t_n A_n} y_n\| = 1$. The equicontinuity of $\{e_k^{tA_n} p_n x\}$ follows as in Theorem 3.1.

When $D(A)$ is dense, we have the following result:

Theorem 3.3 $(Theorem\, ABC - int - dense)$ *Assume that a closed densely defined operator A on E generates an exponentially bounded k-times integrated semigroup, and closed operators A_n generate k-times integrated semigroups on E_n respectively. The following conditions (A) and (B'_{int}) are equivalent to condition (C'_{int}).*

(A) Consistency. There exists $\lambda \in \rho(A) \cap \bigcap_n \rho(A_n)$ such that the resolvents converge $(\lambda I_n - A_n)^{-1} \xrightarrow{\mathcal{PP}} (\lambda I - A)^{-1}$;

(B'_{int}) Stability. There are some constants $M \geq 1$ and ω_1, which are not depending on n and such that $\|e_k^{tA_n}\| \leq M \exp(\omega_1 t)$ for $t \geq 0$ and any $n \in I\!N$;

(C'_{int}) Convergence. For some finite $\omega > 0$ one has $\max_{t\in[0,\infty)} e^{-\omega t}\|e_k^{tA_n} u_n^0 - p_n e_k^{tA} u^0\| \to 0$ as $n \to \infty$, whenever $u_n^0 \xrightarrow{\mathcal{P}} u^0$ for any $u_n^0 \in E_n, u^0 \in E$.

Proof. For $u^0 \in D(A)$, $u_n^0 \in D(A_n)$ such that $u_n^0 \xrightarrow{\mathcal{P}} u^0$ and $A_n u_n^0 \xrightarrow{\mathcal{P}} A u^0$ we have

$$\begin{aligned}\|e_k^{tA_n} u_n^0 - e_k^{sA_n} u_n^0\|_{E_n} &= \|\int_s^t e_k^{\tau A_n} A_n u_n^0 d\tau + \frac{t^k - s^k}{k!} u_n^0\|_{E_n} \\ &\leq |t-s| C_k(T)(\|A_n u_n^0\|_{E_n} + \|u_n^0\|_{E_n}),\end{aligned}$$

where $C_k(T)$ is a constant depending only on M, ω, T and k. This implies the equicontinuity of $\{e_k^{tA_n} u_n^0\}$ on $[0,T]$ since $\|A_n u_n^0\|_{E_n}$ and $\|u_n^0\|_{E_n}$ are uniformly bounded. By Theorem 3.1 we have $\max_{t\in[0,T]} \|e_k^{tA_n} u_n^0 - p_n e_k^{tA} u^0\| \to 0$ as $n \to \infty$. This yields (C'_{int}), since $D(A)$ is dense in E.

Corollary 3.1 *Assume that A, A_n generate locally Lipschitz continuous k-times integrated semigroups on E, E_n respectively. Then conditions (A) and (B''_{int}) imply (C''_{int}).*

(A) Compatibility. There exists $\lambda \in \rho(A) \cap \bigcap_n \rho(A_n)$ such that the resolvents converge:

$$(\lambda I_n - A_n)^{-1} \xrightarrow{\mathcal{PP}} (\lambda I - A)^{-1};$$

(B''_{int}) Stability-Uniformity. There are some constants $M \geq 0$ and ω such that

$$\|e_k^{(t+h)A_n} - e_k^{tA_n}\| \leq M e^{\omega(t+h)} h, \text{ for } t, h \geq 0, n \in I\!N;$$

(C''_{int}) Convergence. For any finite $T > 0$ one has $\max_{t\in[0,T]} \|e_k^{tA_n} u_n^0 - p_n e_k^{tA} u^0\| \to 0$ as $n \to \infty$, whenever $u_n^0 \xrightarrow{\mathcal{P}} u^0$ for any $u_n^0 \in E_n, u^0 \in E$.

Proof. From condition (B''_{int}) the condition (B_{int}) of Theorem 3.2 follows.

At last we give an ABC Theorem for analytic integrated semigroups.

Theorem 3.4 *Assume that closed operators A, A_n on E and E_n respectively generate exponentially bounded analytic k-times integrated semigroups. The following conditions (A) and (B'''_{int}) are equivalent to condition (C'''_{int}).*

(A) Consistency. There exists $\lambda \in \rho(A) \cap \bigcap_n \rho(A_n)$ such that the resolvents converge $(\lambda I_n - A_n)^{-1} \xrightarrow{\mathcal{PP}} (\lambda I - A)^{-1}$;

(B'''_{int}) *Stability. There are some constants* $M \geq 1$, $0 < \theta \leq \pi/2$ *and* ω_1, *which are independent of* n *and such that the sector* $\omega_1 + \Sigma_{\theta+\pi/2}$ *is included in* $\rho(A_n)$ *and*

$$\sup_{\lambda \in \omega_1 + \Sigma_{\theta+\pi/2}} \|(\lambda - \omega_1) R(\lambda, A_n)/\lambda^k\| \leq M$$

for any $n \in I\!N$, *and the sequence* $\{e_k^{zA_n} p_n x\}, n \in I\!N$, *is equicontinuous on compact subsets of* Σ_θ *for every* $x \in E$.

(C'''_{int}) *Convergence. For some finite* $\omega > 0$ *one has* $\max_{z \in \Sigma(\theta)} e^{-\omega Re z} \|e_k^{zA_n} u_n^0 - p_n e_k^{zA} u^0\| \to 0$ *as* $n \to \infty$, *whenever* $u_n^0 \xrightarrow{\mathcal{P}} u^0$ *for any* $u_n^0 \in E_n, u^0 \in E$.

Proof. By Theorem 4.3 in [8], the first part of (B'''_{int}) is equivalent to the stability of the integrated semigroups.

Remark 3.2 *Trotter-Kato's Theorem involving integrated semigroups was considered in* [3] − [5]. *This subject was discussed also in* [7, 20], [31] − [33].

4 Discretisation of integrated semigroups in time

In case when the operator A does not generate s C_0-semigroup the approach described in the Introduction fails in the following sense. There is no strongly continuous bounded linear family of operators which represents the solutions and which could be successively approximated, i.e. the semigroup family of operators, even if it exists like $\exp(tA)$, is not a C_0-semigroup anymore. But one can assume that some other family of bounded linear operators exists, for instance e_1^{tA}, $t \in I\!R_+$. There is a way to construct approximations of such a family of operators by analogy to the approach from the Introduction. The solution of the original problem

$$\begin{aligned} &u'(t) = Au(t),\ t \in [0, \infty), \\ &u(0) = u^0 \in E, \end{aligned} \tag{4.1}$$

where A generates a once integrated semigroup e_1^{tA}, can be approximated by taking the discrete derivative of the discrete once integrated semigroup. This approach will be the subject of a forthcoming paper. In this section we consider discrete once integrated semigroups.

4.1 Explicit scheme

Let $T_n(\tau_n) \in B(E_n)$, $\{\tau_n\}$ and $\tau_n > 0$, be a sequence converging to 0 as $n \to \infty$ and $\breve{A}_n = (T_n(\tau_n) - I_n)/\tau_n \in B(E_n)$. The once integrated discrete semigroup can be defined as $W_n^e(t) := \int_0^t T_n^{[s/\tau_n]} ds$, where $[s/\tau_n]$ is entire part of number s/τ_n, i.e. it is $\tau_n \Sigma_{j=0}^{[t/\tau_n]-1} (I_n + \tau_n \breve{A}_n)^j$. By definition we assume that $\tau_n \Sigma_{j=1}^{[t/\tau_n]-1} (I_n + \tau_n \breve{A}_n)^j = 0$ as $t = 0$. We can also give a formal definition

Definition 4.1 *The discrete family of operators* $\{W_n^e(k\tau_n)\}, k = 0, 1, 2, \ldots,$ *is called discrete 1-times explicit integrated semigroup if* $W_n^e(0) = 0, W_n^e(\tau_n) = \tau_n I_n, W_n^e(2\tau_n) = \breve{A}_n W_n^e(\tau_n)\tau_n + 2\tau_n I_n$ *and*

$$W_n^e(k\tau_n) W_n^e(2\tau_n) = (W_n^e((k+1)\tau_n) + W_n^e(k\tau_n))\tau_n - \tau_n^2 I_n.$$

Proposition 4.1 *The discrete 1-times explicit integrated semigroup in case* $\breve{A}_n^{-1}$ *exists is given by the formulas*

$$W_n^e((k+1)\tau_n) = W_n^e(k\tau_n)(I_n + \tau_n \breve{A}_n) + \tau_n I_n,\ k = 1, 2, \ldots,$$

$$W_n^e(k\tau_n) = \tau_n \sum_{j=0}^{k-1} (I_n + \tau_n \breve{A}_n)^j = ((I_n + \tau_n \breve{A}_n)^k - I_n)\breve{A}_n^{-1},\ k = 0, 1, 2, \ldots.$$

Proof. From Definition 4.1 it follows that

$$W_n^e((k+1)\tau_n) = W_n^e(k\tau_n)(W_n^e(2\tau_n)/\tau_n - I_n) + \tau_n I_n = W_n^e((k-1)\tau_n)(I_n + \tau_n \breve{A}_n)^2 + \tau_n(I_n + \tau_n \breve{A}_n) + \tau_n I_n.$$

One gets $W_n^e((k+1)\tau_n) = W_n^e(k\tau)(I_n + \tau_n \breve{A}_n) + \tau I_n$. Therefore

$$W_n^e((k+1)\tau_n) = \frac{(I_n + \tau_n \breve{A}_n)^{k+1} - I_n}{(I_n + \tau_n \breve{A}_n) - I_n}\tau_n = ((I_n + \tau_n \breve{A}_n)^{k+1} - I_n)\breve{A}_n^{-1}. \tag{4.2}$$

Theorem 4.1 (*Theorem ABC−discr−int*) *Suppose that* $A, \breve{A}_n$ *generate exponentially bounded once integrated semigroups respectively. The following conditions* (A) *and* $(\tilde{B}_{int})$ *are equivalent to condition* $(\tilde{C}_{int})$.

(A) *Consistency. There exists* $\lambda \in \rho(A) \cap \bigcap_n \rho(\breve{A}_n)$ *such that the resolvents converge* $(\lambda I_n - \breve{A}_n)^{-1} \xrightarrow{\mathcal{PP}} (\lambda I - A)^{-1}$;

$(\tilde{B}_{int})$ *Stability. There are some constants* $M_1 \geq 1$ *and* $\omega_1 \in I\!R$ *such that the explicit discrete once integrated semigroup* $W_n^e(\cdot)$ *is stable, i.e.*

$$\| \int_0^t T_n^{[s/\tau_n]} ds \| \leq M_1 \exp(\omega_1 t) \ \text{for}\ t \in \overline{I\!R}_+ = [0, \infty), n \in I\!N,$$

and $\{\int_0^t T_n^{[s/\tau_n]} p_n x ds\}$ *is equicontinuous on bounded intervals of* $I\!R^+$ *for every* $x \in E$;

$(\tilde{C}_{int})$ *Convergence. For some finite* $\omega > 0$ *one has* $\max_{t \in [0,\infty)} e^{-\omega t} \| \int_0^t T_n^{[s/\tau_n]} u_n^0 ds - p_n e_1^{tA} u^0 \| \to 0$ *as* $n \to \infty$, *whenever* $u_n^0 \xrightarrow{\mathcal{P}} u^0$ *for any* $u^0 \in E, u_n^0 \in E_n$.

Proof. Since

$$\begin{aligned}
\int_0^\infty e^{-\lambda t} T_n^{[t/\tau_n]} dt &= \sum_{k=0}^\infty T_n^k \int_{k\tau_n}^{(k+1)\tau_n} e^{-\lambda t} dt \\
&= \frac{e^{\lambda \tau_n} - 1}{\lambda} (e^{\lambda \tau_n} I_n - T_n)^{-1} \\
&= \frac{e^{\lambda \tau_n} - 1}{\lambda \tau_n} (\frac{e^{\lambda \tau_n} - 1}{\tau_n} I_n - \frac{T_n - I_n}{\tau_n})^{-1} \\
&= \frac{e^{\lambda \tau_n} - 1}{\lambda \tau_n} (\frac{e^{\lambda \tau_n} - 1}{\tau_n} I_n - \breve{A}_n)^{-1},
\end{aligned}$$

we get by integration by parts and by the stability condition $(\tilde{B}_{int})$ that

$$\int_0^\infty e^{-\lambda t} \int_0^t T_n^{[s/\tau_n]} ds dt = \frac{e^{\lambda \tau_n} - 1}{\lambda^2 \tau_n} (\frac{e^{\lambda \tau_n} - 1}{\tau_n} I_n - \breve{A}_n)^{-1},$$

which $\mathcal{PP}$-converge to $\frac{1}{\lambda}(\lambda I - A)^{-1}$ by (A). Since $\{\int_0^t T_n^{[s/\tau_n]} ds\}$ are uniformly exponentially bounded and equicontinuous on bounded intervals, the rest of the proof is similar to the proof of Theorem 3.2.

Remark 4.1 (1) *Note that* $\int_0^{k\tau_n} T_n^{[s/\tau_n]} ds := W_n(k\tau_n) = \tau_n \sum_{j=0}^{k-1} (I_n + \tau_n \breve{A}_n)^j$ *is a discrete once explicit integrated semigroup with* $W_n(0) = 0$ *by definition. So* $\int_0^t T_n^{[s/\tau_n]} ds$ *can be considered as the continuous realization of discrete once explicit integrated semigroup, which means* $\int_0^t T_n^{[s/\tau_n]} ds := \tau_n \sum_{j=0}^{k-1} (I_n + \tau_n \breve{A}_n)^j$ *for* $(k-1)\tau_n < t \leq k\tau_n$.

(2) *Let us compare condition* $(\tilde{B}_{int})$ *with condition* $(\tilde{B}'_{int})$, *which is defined in the followin way: there are some constants* $M \geq 1$ *and* $\omega \in I\!R$ *such that*

$$\|T_n^k\| \leq Me^{\omega k\tau_n} \text{ for all } n, k \in I\!N.$$

Condition $(\tilde{B}'_{int})$ *is the main hypothesis in [27]. Condition* $(\tilde{B}'_{int})$ *implies condition* $(\tilde{B}_{int})$. *Indeed we have for* $t = k\tau_n + r$ *and* $0 \leq r < \tau_n$,

$$\| \int_0^t T_n^{[s/\tau_n]} ds \| = \| \sum_{j=1}^{k-1} \int_{j\tau_n}^{(j+1)\tau_n} T_n^j ds + \int_0^r T_n^k ds \| \leq \sum_{j=1}^{k-1} \tau_n M e^{\omega j\tau_n} + rMe^{\omega k\tau_n} \leq$$

$$\leq tMe^{\omega k\tau_n} \leq Me^{(\omega+1)t}.$$

Similarly, one can show that in this case

$$\| \int_0^t T_n^{[\xi/\tau_n]} d\xi - \int_0^s T_n^{[\xi/\tau_n]} d\xi \| \leq Me^{\omega \max(t,s)} |t-s|. \tag{4.3}$$

Moreover, it is easy to prove that $(\tilde{B}'_{int})$ *is equivalent to* (4.3).

(3) *We assume in this Section 4.1 that the discrete generators* $\breve{A}_n$ *are bounded, i.e.* $\breve{A}_n \in B(E_n)$. *We will continue to use* A_n, *but keep in mind that in general* $\|A_n\| \to \infty$ *as* $n \to \infty$.

Theorem 4.2 *Suppose that conditions* (*A*) *and* (B_{int}) *of Theorem* 3.2 *hold and*

$$\tau_n \|A_n^2\|, \|A_n^{-1}\| \leq C, n \in I\!N.$$

Then discrete explicit once integrated semigroup $\int_0^t (I_n + \tau_n A_n)^{[s/\tau_n]} ds$ *is exponentially stable, i.e.*

$$\| \sum_{j=0}^{k_n} \tau_n (I_n + \tau_n A_n)^j \| \leq M_1 \, e^{\omega_1 \tau_n k_n}, \tag{4.4}$$

and it gives an approximation of once integrated semigroup, i.e. $\tau_n \sum_{j=0}^{k_n-1} (I_n + \tau_n A_n)^j u_n^0 \xrightarrow{\mathcal{P}} e_1^{tA} u^0$ *uniformly with respect to* $t = k_n \tau_n \in [0,T]$ *as* $u_n^0 \xrightarrow{\mathcal{P}} u^0, n \to \infty$.

Proof. Since

$$\begin{aligned}
\sum_{k=0}^{m-1} \tau_n (I_n + \tau_n A_n)^k &= ((I_n + \tau_n A_n)^m - I_n) A_n^{-1} \\
&= \Big((I_n - \tau_n^2 A_n^2)^m (I_n - \tau_n A_n)^{-m} - I_n \Big) A_n^{-1} \\
&= (I_n - \tau_n^2 A_n^2)^m \Big((I_n - \tau_n A_n)^{-m} A_n^{-1} - A_n^{-1} \Big) + (I_n - \tau_n^2 A_n^2)^m A_n^{-1} - A_n^{-1} \\
&= (I_n - \tau_n^2 A_n^2)^m \sum_{k=1}^{m} \tau_n (I_n - \tau_n A_n)^{-k} + (I_n - \tau_n^2 A_n^2)^m A_n^{-1} - A_n^{-1},
\end{aligned}$$

then stability follows from estimates $(1 + \tau_n \|\tau_n A_n^2\|)^m \leq Me^{\omega m\tau_n}$ and Theorem 4.5. To prove convergence one can apply Theorem 4.1.

Remark 4.2 *We can give a different proof of Theorem* 4.2. *Let us define an operator* $Q_n = -A_n(I_n + \tau_n A_n)^{-1}$. *It is clear that operators* $A_n + Q_n = A_n(I_n + \tau_n A_n)^{-1} \tau_n A_n$ *are uniformly in n bounded if* $\|\tau_n A_n^2\| \leq Constant$. *It follows by [19] or [22] that the operators* Q_n *generate integrated semigroups and by Theorem 4.5 one gets* $\|\tau_n \Sigma_{j=0}^{k-1} (I_n - \tau_n Q_n)^{-j}\| = \|\tau_n \Sigma_{j=0}^{k-1} (I_n + \tau_n A_n)^j\| \leq constant.$

Theorem 4.3 *Suppose that conditions* (A) *and* (B'''_{int}) *of Theorem* 3.4 *hold with* $\omega_1 = 0$ *and*

$$\sup_n \tau_n \|A_n\| < \mu < 2\sin\theta, \quad 0 \in \rho(A_n), \quad n \in I\!N.$$

Then the statements of Theorem 4.2 *hold.*

Proof. Since A_n are bounded operators, they generate C_0-semigroups and the once integrated semigroups are given by $e_1^{tA_n} = A_n^{-1}(e^{tA_n} - I)$. We know that $e_1^{tA_n}$ are uniformly bounded, i.e. $\|e_1^{tA_n}\| \leq M$. If we show that $\|(e^{tA_n} - I_n)A_n^{-1} - \sum_{k=0}^{m-1} \tau_n (I_n + \tau_n A_n)^k\| \leq Ct$, then stability, i.e. condition $(\tilde{B}_{int})$, is proved. One can write

$$(e^{tA_n} - I_n)A_n^{-1} - ((I_n + \tau_n A_n)^m - I_n)A_n^{-1} = -\int_0^{\tau_n} \frac{d}{ds}\Big(e^{m(\tau_n - s)A_n}(I_n + sA_n)^m\Big) ds A_n^{-1} =$$

$$= -\int_0^{\tau_n} \Big(- mA_n e^{m(\tau_n - s)A_n}(I_n + sA_n)^m + e^{m(\tau_n - s)A_n}(I_n + sA_n)^{m-1} mA_n\Big) ds A_n^{-1} =$$

$$= \int_0^{\tau_n} sme^{m(\tau_n - s)A_n}(I_n + sA_n)^{m-1} ds\, A_n = \frac{1}{2\pi i}\int_0^{\tau_n} sm\Big(\int_\Gamma e^{m(\tau_n - s)\lambda}(1 + s\lambda)^{m-1}(\lambda I_n - A_n)^{-1} d\lambda\Big) ds\, A_n,$$

where the positively oriented contour Γ is composed of $\Gamma_1 = \{re^{\pm i(\theta + \pi/2)} : 0 \leq r \leq R_n\}$ and $\Gamma_2 = \{R_n e^{i\varphi} : \theta + \pi/2 \leq \varphi \leq -\theta + 3\pi/2)\}$ with R_n satisfying $\tau_n R_n = \mu$. First we can choose a positive γ, which depends only on θ and μ such that

$$|1 + s\lambda|^2 = 1 + s^2 r^2 - 2sr\sin\theta = 1 - sr(2\sin\theta - sr) \leq 1 - \gamma sr,$$

for any $0 \leq s \leq \tau_n, 0 \leq r \leq R_n$. Thus for $2\sin\theta - \mu = \gamma > 0$

$$|1 + sz| \leq 1 - \gamma sr, \quad 0 \leq sr < \mu,\ z \in \Gamma_1.$$

For $z = re^{\pm i(\theta + \pi/2)} \in \Gamma_1$, the integral over Γ_1 can be estimated by

$$\int_0^{R_n} e^{-m(\tau_n - s)r\sin\theta}(1 - \gamma sr)^{m-1} dr\|A_n\| \leq \int_0^{R_n} e^{-m(\tau_n - s)r\sin\theta} e^{-\gamma sr(m-1)} dr\|A_n\| \leq$$

$$\leq e^{\gamma\tau_n R_n}\int_0^{R_n} e^{-[m(\tau_n - s)\sin\theta + \gamma sm]r} dr\|A_n\| \leq \frac{e^{\gamma\mu}}{m(\tau_n - s)\sin\theta + \gamma sm}\|A_n\|$$

since $(\tau_n - s)\sin\theta > 0$ and γ is positive. Moreover, since the function $[0, \tau_n] \ni s \mapsto m(\tau_n - s)\sin\theta + \gamma sm$ is linear, it reaches its minimum at 0 or τ_n. Thus one has

$$e^{\gamma\mu}\int_0^{\tau_n} \frac{sm}{m(\tau_n - s)\sin\theta + \gamma sm} ds\|A_n\| \leq \frac{e^{\gamma\mu}\tau_n^2 m}{\min\{m\tau_n\sin\theta, \gamma\tau_n m\}}\|A_n\| \leq \frac{e^{\gamma\mu}\mu}{\min\{\sin\theta, \gamma\}},$$

for any $m \geq 1$. For $\lambda = R_n e^{i\varphi} \in \Gamma_2$, choose $\beta < 1$ such that $\tau_n\|A_n\| \leq \beta\mu$, then

$$\|(\lambda I_n - A_n)^{-1}\| \leq \frac{1}{|\lambda| \cdot (1 - \|A_n\|/|\lambda|)} = \frac{1}{R_n \cdot (1 - \|A_n\|/R_n)} \leq \frac{1}{R_n(1 - \beta)}.$$

Therefore we can bound the integral over Γ_2 by

$$C\int_{\theta + \pi/2}^{3\pi/2 - \theta} e^{-m\tau_n R_n\sin\theta}\frac{R_n d\varphi}{R_n(1 - \beta)}\|A_n\| \leq C' e^{-m\tau_n R_n\sin\theta}\|A_n\|.$$

Also for $\lambda = \mu e^{i\varphi}/\tau_n \in \Gamma_2$, we have

$$|1+\tau_n\lambda| \leq |1+\mu e^{i(\theta+\pi/2)}| = (1+\mu^2-2\mu\sin\theta)^{1/2} = [1-\mu(2\sin\theta-\mu)]^{1/2} < 1;$$

combining this with the inequality $|1+s\lambda| \leq 1$ for $\lambda \in \Gamma_1$ and $0 \leq s \leq \tau_n$, the Maximal Modulus Principle yields $|1+s\lambda| \leq 1$ for all $\lambda \in \Gamma$, $0 \leq s \leq \tau_n$. Thus we have

$$\begin{aligned} &\| \int_0^{\tau_n} sm\Big(\int_{\Gamma_2} e^{m(\tau_n-s)\lambda}(1+s\lambda)^{m-1}(\lambda I_n - A_n)^{-1}d\lambda\Big)dsA_n\| \\ \leq\ & C'\int_0^{\tau_n} sme^{-m\tau_n R_n\sin\theta}ds\|A_n\| \leq C\frac{\tau_n^2 m}{2}\|A_n\| \leq Ct\mu. \end{aligned}$$

Since $\tau_n\|A_n\| < \mu < \infty$ the Theorem is proved.

Remark 4.3 *In Theorems 4.2-4.3 we assumed that exist A_n^{-1}. However, if A generates an exponentially bounded once integrated semigroup e_1^{tA} satisfying $\|e_1^{tA}\| \leq Me^{\omega t}$, then by [14] the once integrated semigroup generated by $A-\omega I$ is related to e_1^{tA} by formula*

$$e_1^{tA} = e^{\omega t}e_1^{t(A-\omega I)} - \omega\int_0^t e^{\omega s}e_1^{s(A-\omega I)}ds.$$

One can show that $e_1^{t(A-\omega I)}$ is still exponentially bounded, but because of the choice of $\omega > 0$ one can achieve $0 \in \rho(A-\omega I)$. Because of Theorem 3.2 we can find $\omega_3 > 0$ such that $0 \in \rho(A_n - \omega_3 I_n)$ for any $n \geq n_0$. Now one can construct an approximation of e_1^{tA} in the following way

$$e^{\omega_3 t}W_n^e(k_n\tau_n) - \omega_3\tau_n\Sigma_{j=0}^{k_n}e^{\omega_3 j\tau_n}W_n^e(j\tau_n)\xrightarrow{\mathcal{PP}}e_1^{tA},\ t = k_n\tau_n, \tag{4.5}$$

where $W_n^e(k_n\tau_n)$ is constructed by operators $A_n - \omega_3 I_n$. Of course, in (4.5) we can use different quadrature formulas for approximating the integral $\int_0^t e^{\omega s}e_1^{s(A-\omega I)}ds$.

4.2 Implicit scheme

Let us now set $T_n(\tau_n) = (I_n - \tau_n A_n)^{-1}$. Then in this subsection the discrete once integrated semigroup can be defined as $\int_0^t T_n^{[s/\tau_n]}ds$, i.e. as $\tau_n\Sigma_{j=1}^{[t/\tau_n]}(I_n - \tau_n A_n)^{-j}$ where we put by definition $\tau_n\Sigma_{j=1}^{[t/\tau_n]}(I_n - \tau_n A_n)^{-j} = 0$ for $t = 0$. Therefore, in this subsection we have a special choice of $\breve{A}_n = A_n(I_n - \tau_n A_n)^{-1}$, with the operator A_n taken from Theorem 3.2.

Definition 4.2 *The discrete family of operators $\{W_n^i(k\tau_n)\}, k = 0,1,2,\ldots,$ is called a discrete 1-times implicit integrated semigroup if $W_n^i(0) = 0, W_n^i(\tau_n) = \tau_n(I_n - \tau_n A_n)^{-1}$ and*

$$W_n^i(k\tau_n)W_n^i(\tau_n) = \tau_n W_n^i((k+1)\tau_n) - \tau_n W_n^i(\tau_n).$$

Proposition 4.2 *The discrete 1-times implicit integrated semigroup in case of existence of A_n^{-1} is given by the formulas*

$$W_n^i((k+1)\tau_n) = W_n^i(k\tau_n)(I_n - \tau_n A_n)^{-1} + W_n^i(\tau_n),\ k = 1,2,\ldots, \tag{4.6}$$

$$W_n^i(k\tau_n) = \sum_{j=1}^k \tau_n(I_n - \tau_n A_n)^{-j} = ((I_n - \tau_n A_n)^{-k} - I_n)A_n^{-1},\ k = 0,1,2,\ldots. \tag{4.7}$$

Proof. From Definition 4.2 it follows that

$$W_n^i((k+1)\tau_n) = W_n^i(k\tau_n)W_n^i(\tau_n)/\tau_n + W_n^i(\tau_n).$$

One gets $W_n^i((k+1)\tau_n) = W_n^i(k\tau_n)(I_n - \tau_n A_n)^{-1} + W_n^i(\tau_n)$. Therefore

$$W_n^i((k+1)\tau_n) = \frac{(I_n - \tau_n A_n)^{-(k+1)} - I_n}{(I_n - \tau_n A_n)^{-1} - I_n}\tau_n(I_n - \tau_n A_n)^{-1} = ((I_n - \tau_n A_n)^{-(k+1)} - I_n)A_n^{-1}.$$

Theorem 4.4 *Suppose that A, A_n generate exponentially bounded integrated semigroups respectively. The following conditions (A) and $(\check{B}_{int})$ are equivalent to condition $(\check{C}_{int})$.*

(A) Consistency. There exists $\lambda \in \rho(A) \cap \cap_n \rho(A_n)$ such that the resolvents converge $(\lambda I_n - A_n)^{-1} \xrightarrow{\mathcal{PP}} (\lambda I - A)^{-1}$;

$(\check{B}_{int})$ Stability. There are some constants $M_1 \geq 1$ and $\omega_1 \in I\!R$ such that

$$\|\int_0^t T_n^{[s/\tau_n]} ds\| \leq M_1 \exp(\omega_1 t) \text{ for } t \in \overline{I\!R}_+ = [0, \infty), \quad n \in I\!N,$$

and $\{\int_0^t T_n^{[s/\tau_n]} p_n x ds\}$ is equicontinuous on bounded intervals of $I\!R^+$ for every $x \in E$;

$(\check{C}_{int})$ Convergence. For some finite $\omega > 0$ one has $\max_{t \in [0,\infty)} e^{-\omega t}\| \int_0^t T_n^{[s/\tau_n]} u_n^0 ds - p_n e_1^{tA} u^0\| \to 0$ as $n \to \infty$, whenever $u_n^0 \xrightarrow{\mathcal{P}} u^0$ for any $u^0 \in E, u_n^0 \in E_n$.

Proof. Taking Laplace transform of $\int_0^t T_n^{[s/\tau_n]} ds$, as in the proof of Theorem 4.1 one gets

$$\begin{aligned}
\int_0^\infty e^{-\lambda t} \int_0^t T_n^{[s/\tau_n]} ds\, dt &= \frac{e^{\lambda\tau_n} - 1}{\lambda}(e^{\lambda\tau_n} I_n - T_n)^{-1} \\
&= \frac{e^{\lambda\tau_n} - 1}{\lambda}(e^{\lambda\tau_n} I_n - (I_n - \tau_n A_n)^{-1})^{-1} \\
&= \frac{1 - e^{-\lambda\tau_n}}{\lambda\tau_n}(I_n - \tau_n A_n)(\frac{1 - e^{-\lambda\tau_n}}{\tau_n} I_n - A_n)^{-1} \\
&\xrightarrow{\mathcal{PP}} \frac{1}{\lambda}(\lambda I - A)^{-1}.
\end{aligned}$$

Similarly, as in Subsection 4.1, we have $\int_0^t T_n^{[s/\tau_n]} ds \xrightarrow{\mathcal{PP}} e_1^{tA}$ uniformly on bounded intervals. Now the result follows from Theorem 3.1.

Theorem 4.5 *Suppose that conditions (A) and (B_{int}) of Theorem 3.2 hold. Then discrete 1-times integrated semigroup is exponentially stable, i.e. $\| \sum_{j=0}^k \tau_n (I_n - \tau_n A_n)^{-j}\| \leq M_1 e^{\omega_1 \tau_n k}$ and gives an approximation to once integrated semigroup, i.e. $\sum_{j=0}^{k_n} \tau_n (I_n - \tau_n A_n)^{-j} \xrightarrow{\mathcal{P}} e_1^{tA} u_n^0$ uniformly with respect to $t = k_n \tau_n \in [0, T]$ as $u_n^0 \xrightarrow{\mathcal{P}} u^0, n \to \infty$.*

Proof. We only need to show that conditions (A) and (B_{int}) of Theorem 3.2 imply $(\check{B}_{int})$. By the stability condition (B_{int}) of Theorem 3.2, we know that

$$\|\Big(\frac{R(\lambda, A_n)}{\lambda}\Big)_\lambda^{(m)}\| \leq \frac{Mm!}{|\lambda - \omega|^{m+1}}, \ \lambda > \omega,$$

where $(\cdot)^{(m)}_\lambda$ denotes the m-th derivative with respect to λ. By the formulas for derivatives of resolvents $(R(\lambda,A_n))^{(m)}_\lambda = (-1)^m m! R(\lambda,A_n)^{m+1}$ and $(1/\lambda)^{(m)}_\lambda = (-1)^m m!(1/\lambda)^{m+1}$, one obtains

$$\begin{aligned}(R(\lambda,A_n)/\lambda)^{(m)}_\lambda &= \sum_{j=0}^{m} C^j_m (R(\lambda,A_n))^{(j)}_\lambda (1/\lambda)^{(m-j)}_\lambda \\ &= \sum_{j=0}^{m} C^j_m (-1)^j j! R(\lambda,A_n)^{j+1}(-1)^{m-j}(m-j)!(1/\lambda)^{m-j+1} \\ &= (-1)^m m! \sum_{j=1}^{m+1} R(\lambda,A_n)^j/\lambda^{m+2-j},\end{aligned}$$

and therefore

$$\|\frac{1}{\lambda}\sum_{j=1}^{m+1}\lambda^j R(\lambda,A_n)^j\| \le \frac{M}{|1-\frac{\omega}{\lambda}|^{m+1}}.$$

Now choosing $\lambda = 1/\tau_n$, we have

$$\|\tau_n \sum_{j=0}^{m}(I_n-\tau_n A_n)^{-j}\| \le \frac{M}{|1-\tau_n\omega|^m} \le M' e^{\omega'\tau_n m}.$$

Remark 4.4 *In the uniformly Lipschitz case considered by Tanaka [27] one can even show $\|T^k_n\| \le$ constant, $k \in I\!N$. In general one assumes just (B_{int}) from Theorem 3.2, then we have just last inequality in the proof of Theorem* 4.5.

4.3 Crank-Nicholson scheme

In this subsection let us consider $T_n(\tau_n) = (I_n + \frac{\tau_n}{2}A_n)(I_n - \frac{\tau_n}{2}A_n)^{-1}$, so that $\breve{A}_n = A_n(I_n - \frac{\tau_n}{2}A_n)^{-1}$.

Definition 4.3 *The discrete family of operators $\{W^{cd}_n(k\tau_n)\}, k = 0,1,2,\ldots,$ is called a central difference discrete 1-times integrated semigroup if $W^{cd}_n(0) = 0, W^{cd}_n(\tau_n) = \tau_n(I_n - \frac{\tau_n}{2}A_n)^{-1}$, and*

$$W^{cd}_n(k\tau_n)W^{cd}_n(\tau_n) = \tau_n\frac{W^{cd}_n((k+1)\tau_n) + W^{cd}_n(k\tau_n)}{2} - \frac{\tau_n}{2}W^{cd}_n(\tau_n).$$

Proposition 4.3 *The discrete central difference 1-times integrated semigroup is given by the formulas*

$$W^{cd}_n((k+1)\tau_n) = W^{cd}_n(k\tau_n)\frac{I_n + \frac{\tau_n}{2}A_n}{I_n - \frac{\tau_n}{2}A_n} + W^{cd}_n(\tau_n),\ k = 1,2,\ldots,$$

(4.8) $$W^{cd}_n(k\tau_n) = \Big((\frac{I_n + \frac{\tau_n}{2}A_n}{I_n - \frac{\tau_n}{2}A_n})^k - I_n\Big)A^{-1}_n,\ k = 0,1,2,\ldots.$$

Proof. From Definition 4.3 it follows that

$$W^{cd}_n((k+1)\tau_n) = W^{cd}_n(k\tau_n)\Big(2W^{cd}_n(\tau_n)/\tau_n - I_n\Big) + W^{cd}_n(\tau_n).$$

One gets $W^{cd}_n((k+1)\tau_n) = W^{cd}(k\tau_n)\Big(2(I_n - \frac{\tau_n}{2}A_n)^{-1} - I_n\Big) + W^{cd}(\tau_n)$. Therefore

$$W^{cd}_n((k+1)\tau_n) = W^{cd}_n(k\tau_n)\frac{I_n + \frac{\tau}{2}A_n}{I_n - \frac{\tau}{2}A_n} + W^{cd}_n(\tau_n).$$

Using the identity $\frac{I_n+\frac{\tau_n}{2}A_n}{I_n-\frac{\tau_n}{2}A_n} - I_n = \Big(I_n + \frac{\tau_n}{2}A_n - (I_n - \frac{\tau_n}{2}A_n)\Big)(I_n - \frac{\tau_n}{2}A_n)^{-1}$ we get the following.

Theorem 4.6 *Suppose that A and A_n generate exponentially bounded integrated semigroups. The following conditions (A) and $(\hat{B}_{int})$ are equivalent to condition $(\hat{C}_{int})$.*

(A) Consistency. There exists $\lambda \in \rho(A) \cap \cap_n \rho(A_n)$ such that the resolvents converge $(\lambda I_n - A_n)^{-1} \xrightarrow{\mathcal{PP}} (\lambda I - A)^{-1}$;

$(\hat{B}_{int})$ Stability. There exist constants $M_1 \geq 1$ and $\omega_1 \in I\!R$ such that

$$\| \int_0^t T_n^{[s/\tau_n]} ds \| \leq M_1 \exp(\omega_1 t) \text{ for } t \in \overline{I\!R}_+ = [0,\infty), n \in I\!N,$$

and $\{\int_0^t T_n^{[s/\tau_n]} p_n x ds\}$ is equicontinuous on bounded intervals of $I\!R^+$ for every $x \in E$;

$(\hat{C}_{int})$ Convergence. For some finite $\omega > 0$ one has $\max_{t\in[0,\infty)} e^{-\omega t} \| \int_0^t T_n^{[s/\tau_n]} u_n^0 ds - p_n e_1^{tA} u^0 \| \to 0$ as $n \to \infty$, whenever $u_n^0 \xrightarrow{\mathcal{P}} u^0$ for any $u^0 \in E, u_n^0 \in E_n$.

Proof. As in Theorem 4.1, using relation $\int_0^\infty e^{-\lambda t} \int_0^t T_n^{[s/\tau_n]} ds dt = \frac{e^{\lambda\tau_n} - 1}{\lambda} (e^{\lambda\tau_n} I_n - T_n)^{-1}$, one gets

$$\begin{aligned} \int_0^\infty e^{-\lambda t} \int_0^t T_n^{[s/\tau_n]} ds\, dt &= \frac{e^{\lambda\tau_n} - 1}{\lambda} (e^{\lambda\tau_n} - (I + \tau_n A_n/2)(I - \tau_n A_n/2)^{-1})^{-1} \\ &= \frac{1 - e^{-\lambda\tau_n}}{\lambda\tau_n} (I - \tau_n A_n/2) (\frac{1 - e^{-\lambda\tau_n}}{\tau_n} - (1 + e^{-\lambda\tau_n}) A_n/2)^{-1} \\ &\xrightarrow{\mathcal{PP}} \frac{1}{\lambda} (\lambda I - A)^{-1}. \end{aligned}$$

Thus we obtain the convergence of $\int_0^t T_n^{[s/\tau_n]} ds$ to e_1^{tA} from Theorem 3.1.

Theorem 4.7 *Suppose that conditions (A) and (B_{int}) of Theorem 3.2 hold and*

$$\tau_n \|A_n^2\|, \|A_n^{-1}\| \leq C, \, n \in I\!N.$$

Then the discrete central difference once integrated semigroup $\int_0^t (\frac{I_n + \tau_n A_n/2}{I_n - \tau_n A_n/2})^{[s/\tau_n]} ds$ is exponentially stable, i.e.

$$\|\tau_n \Sigma_{j=1}^{k_n} (\frac{I_n + \tau_n A_n/2}{I_n - \tau_n A_n/2})^j \| \leq M_1 \, e^{\omega_1 \tau_n k_n}, \; 0 \leq \tau_n k_n \leq T, \tag{4.9}$$

and it provides an approximation of the once integrated semigroup, i.e.

$$\tau_n \Sigma_{j=0}^{k_n} (\frac{I_n + \tau_n A_n/2}{I_n - \tau_n A_n/2})^j u_n^0 \xrightarrow{\mathcal{P}} e_1^{tA} u^0,$$

uniformly with respect to $t = k_n\tau_n \in [0,T]$ as $u_n^0 \xrightarrow{\mathcal{P}} u^0, k_n \to \infty, n \to \infty$.

Proof. To prove stability of (4.8) in the form $\tau_n \Sigma_{j=1}^k (\frac{I_n + \frac{\tau_n}{2} A_n}{I_n - \frac{\tau_n}{2} A_n})^j = \tau_n \Sigma_{j=1}^k (I_n - \tau_n Q_n)^{-j}$, where $Q_n = A_n(I_n + \tau_n A_n/2)^{-1}$, we apply Theorem 4.5. Let us consider the difference $(Q_n - A_n)x_n = A_n(I_n + \tau_n A_n/2)^{-1} \tau_n A_n x_n/2$. The operators $(Q_n - A_n)$ are uniformly bounded if $\|\tau_n A_n^2\| \leq constant$ and they commute with A_n. By Theorem in [19] or [22] we find that the operators Q_n generate exponentially bounded once integrated semigroups and under condition $\|\tau_n A_n^2\| \leq constant$ we obtain stability by Theorem 4.5. So one has

$$\tau_n \Sigma_{j=0}^{k_n - 1} (\frac{I_n + \tau_n A_n/2}{I_n - \tau_n A_n/2})^j (I_n - \tau_n A_n/2)^{-1} = \tau_n \Sigma_{j=1}^{k_n} (\frac{I_n + \tau_n A_n/2}{I_n - \tau_n A_n/2})^j (I_n + \tau_n A_n/2)^{-1},$$

and the estimate $\|\frac{1}{I_n + \tau_n A_n/2}\| = \|\frac{I_n - \tau_n A_n/2}{I_n + \tau_n^2 A_n^2/4}\| \leq \|I_n - \tau_n A_n^2 A_n^{-1}/2\| \frac{1}{1 - \tau_n c} < \infty$, since $\tau_n \|A_n^2\| \leq constant$.

Theorem 4.8 *Suppose that conditions* (A) *and* (B'''_{int}) *of Theorem* 3.4 *hold with* $\omega_1 = 0$ *and*

$$\sup_n \tau_n \|A_n\| < \mu < 2\sin\theta, \quad 0 \in \rho(A_n), n \in I\!N. \tag{4.10}$$

Then the discrete central difference once integrated semigroup $\int_0^t T_n(\tau_n)^{[s/\tau_n]}ds$ *is exponentially stable, i.e.* (4.9) *holds. Moreover, it gives an approximation of the once integrated semigroup in the sense that*

$$\tau_n \sum_{k=0}^{k_n} (\frac{I_n + \tau_n A_n/2}{I_n - \tau_n A_n/2})^j u_n^0 \xrightarrow{\mathcal{P}} e_1^{tA} u^0 \ \text{ as } n \to \infty,$$

uniformly with respect to $t = k_n\tau_n \in [0,T]$ *as* $u_n^0 \xrightarrow{\mathcal{P}} u^0, n \to \infty$.

Proof. One can write

$$(e^{tA_n} - I_n)A_n^{-1} - ((\frac{I_n + \tau_n A_n/2}{I_n - \tau_n A_n/2})^m - I_n)A_n^{-1} = -\int_0^{\tau_n} \frac{d}{ds}\Big(e^{m(\tau_n - s)A_n}(\frac{I_n + sA_n/2}{I_n - sA_n/2})^m\Big)ds\, A_n^{-1} =$$

$$= \int_0^{\tau_n} \frac{ms^2}{4} e^{m(\tau_n - s)A_n}\Big(\frac{I_n + sA_n/2}{I_n - sA_n/2}\Big)^{m-1} \frac{A_n^3}{(I_n - sA_n/2)^2} ds A_n^{-1} =$$

$$= \frac{1}{2\pi i}\int_0^{\tau_n} \frac{ms^2}{4}\Big(\int_\Gamma e^{m(\tau_n - s)\lambda}\frac{(1+s\lambda/2)^{m-1}}{(1-s\lambda/2)^{m+1}}(\lambda I_n - A_n)^{-1}d\lambda\Big)ds\, A_n^2,$$

where the positively oriented contour Γ is composed of $\Gamma_1 = \{re^{\pm i(\theta+\pi/2)} : 0 \le r \le R_n\}$ and $\Gamma_2 = \{R_n e^{i\varphi} : \theta + \pi/2 \le \varphi \le -\theta + 3\pi/2)\}$. Since for $\lambda \in \Gamma, 0 \le s \le \tau_n$, $|1 - s\lambda/2| \ge 1$, we have $\frac{|1+s\lambda/2|^{m-1}}{|1-s\lambda/2|^{m+1}} \le |1 + s\lambda/2|^{m-1}$. The rest of the proof is similar to that of Theorem 4.3.

Remark 4.5 *Note that the analog of Theorem* 4.8 *for the case of analytic* C_0*-semigroups, namely Theorem* 1.9, *does not involve stability conditions like* (4.10). *Here we follow the idea of* [24]. *The proof of Theorem* 1.9 *is based on the fact that if* A *generates a bounded analytic* C_0*-semigroup and* A^{-1} *exists, then this inverse also generates a bounded analytic* C_0*-semigroup. Unfortunately, for analytic integrated semigroups such a statement does not make sense. As was shown in* [10] *if one assumes that both* A *and* A^{-1} *generate bounded analytic integrated semigroups, then* A *generates in fact a bounded analytic* C_0*-semigroup.*

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Miao Li
Department of Mathematics,
Sichuan University,
Chengdu, Sichuan 610064,
P.R. China
limiao1973@hotmail.com

Sergey Piskarev
Scientific Research Computer Center,

Moscow State University
Vorobjevy Gory, Moscow 119991,
Russia
piskarev@gmail.com