ON THE OCCURRENCE OF THE SINE KERNEL IN CONNECTION WITH THE SHIFTED MOMENTS OF THE RIEMANN ZETA FUNCTION

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ABSTRACT. We point out an interesting occurrence of the sine kernel in connection with the shifted moments of the Riemann zeta function along the critical line. We discuss rigorous results in this direction for the shifted second moment and for the shifted fourth moment. Furthermore, we conjecture that the sine kernel also occurs in connection with the higher (even) shifted moments and show that this conjecture is closely related to a recent conjecture by CONREY, FARMER, KEATING, RUBINSTEIN, and SNAITH [CFKRS1, CFKRS2].

1. INTRODUCTION

Since the discovery by Montgomery and Dyson that the pair correlation function of the non-trivial zeros of the Riemann zeta function seems to be asymptotically the same as that of the eigenvalues of a random matrix from the Gaussian Unitary Ensemble (GUE), the relationship between the theory of the Riemann zeta function and the theory of random matrices has attracted considerable interest. This interest intensified in the last few years after KEATING and SNAITH [KS1] compared the moments of the characteristic polynomial of a random matrix from the Circular Unitary Ensemble (CUE) with the – partly conjectural – moments of the value distribution of the Riemann zeta function along the critical line, and also found some striking similarities. These findings have sparked intensive further research. On the one hand, there are now a number of new conjectures, derived from random matrix theory, about the moments of the value distribution of the Riemann zeta function and more general L-functions (see the papers by KEATING and SNAITH [KS1, KS2] as well as CONREY, FARMER, KEATING, RUBINSTEIN, and SNAITH [CFKRS1, CFKRS2, CFKRS3] and the references contained therein). On the other hand, various authors have investigated the moments and the correlation functions of the characteristic polynomial also for other random matrix ensembles (see e.g. BRÉZIN and HIKAMI [BH1, BH2], MEHTA and NORMAND [MN], STRAHOV and FYODOROV [SF], GÖTZE and KÖSTERS [GK]).

A recurring phenomenon on the random matrix side is the emergence of the sine kernel in the asymptotics of the correlation functions (or shifted moments) of the characteristic polynomial. For instance, for the Circular Unitary Ensemble (CUE) (see FORRESTER [Fo] or MEHTA [Me]), the second-order correlation function

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of the characteristic polynomial

$$f_{\mathrm{CUE}}(N,\mu,\nu) := \int_{\mathcal{U}_N} \det(U-\mu I)\,\overline{\det(U-\nu I)}\;dU$$

(where I denotes the $N\times N$ identity matrix and integration is with respect to the normalized Haar measure on the group $\mathcal{U}_N$ of $N\times N$ unitary matrices) satisfies

$$\lim_{N\to\infty} \frac{1}{N}\cdot f_{\mathrm{CUE}}\left(N; e^{2\pi i\mu/N}, e^{2\pi i\nu/N}\right) = e^{\pi i(\mu-\nu)}\cdot \frac{\sin\pi(\mu-\nu)}{\pi(\mu-\nu)} \tag{1.1}$$

for any $\mu,\nu\in\mathbb{R}$. This can be deduced using standard arguments from random matrix theory (see e.g. Chapter 4 in Forrester [Fo]). More generally, using similar arguments, it can be shown that for any $M\geq 1$, the correlation function of order $2M$ of the characteristic polynomial

$$f_{\mathrm{CUE}}(N,\mu_1,\ldots,\mu_M,\nu_1,\ldots,\nu_M) := \int_{\mathcal{U}_N}\prod_{j=1}^{M}\det(U-\mu_j I)\,\overline{\det(U-\nu_j I)}\;dU$$

satisfies

$$\begin{aligned}&\lim_{N\to\infty}\frac{1}{N^{M^2}}\cdot f_{\mathrm{CUE}}\left(N; e^{2\pi i\mu_1/N},\ldots,e^{2\pi i\mu_M/N},e^{2\pi i\nu_1/N},\ldots,e^{2\pi i\nu_M/N}\right)\\ &= \frac{\exp(\sum_{j=1}^{M}\pi i(\mu_j-\nu_j))}{\Delta(2\pi\mu_1,\ldots,2\pi\mu_M)\cdot\Delta(2\pi\nu_1,\ldots,2\pi\nu_M)}\cdot\det\left(\frac{\sin\pi(\mu_j-\nu_k)}{\pi(\mu_j-\nu_k)}\right)_{j,k=1,\ldots,M}\end{aligned} \tag{1.2}$$

for any pairwise different $\mu_1,\ldots,\mu_M,\nu_1,\ldots,\nu_M\in\mathbb{R}$, where $\Delta(x_1,\ldots,x_M) := \prod_{1\leq j<k\leq M}(x_k-x_j)$ denotes the Vandermonde determinant. For completeness, the proof of (1.2) is sketched in Appendix B of this paper.

Similarly, for the Gaussian Unitary Ensemble (GUE) (see Forrester [Fo] or Mehta [Me]), the second-order correlation function of the characteristic polynomial

$$f_{\mathrm{GUE}}(N,\mu,\nu) := \int_{\mathcal{H}_N}\det(X-\mu I)\,\det(X-\nu I)\;\mathbb{Q}(dX)$$

(where I denotes the $N\times N$ identity matrix and $\mathbb{Q}$ denotes the Gaussian Unitary Ensemble on the space $\mathcal{H}_N$ of $N\times N$ Hermitian matrices) satisfies

$$\lim_{N\to\infty}\sqrt{\frac{\pi}{2N}}\cdot\frac{2^N}{N!}\cdot f_{\mathrm{GUE}}\left(N;\frac{\pi\mu}{\sqrt{2N}},\frac{\pi\nu}{\sqrt{2N}}\right) = \frac{\sin\pi(\mu-\nu)}{\pi(\mu-\nu)} \tag{1.3}$$

for any $\mu,\nu\in\mathbb{R}$ (see e.g. Chapter 4 in Forrester [Fo]). Also, an analogue of (1.2) holds as well. Even more, these results can be generalized both to the class of unitary-invariant matrix ensembles (Brézin and Hikami [BH1], Mehta and Normand [MN], Strahov and Fyodorov [SF]) and – at least for the second-order correlation function – to the class of Hermitian Wigner ensembles (Götze and Kösters [GK]). In particular, it is noteworthy that the emergence of the sine kernel is universal, as it occurs in all the cases previously mentioned, irrespective of the particular details of the definition of the random matrix ensemble. (More precisely, the emergence of the sine kernel depends on the symmetry class of the random matrix ensemble. For instance, for the Gaussian Orthogonal Ensemble (GOE) on the space of real symmetric matrices, the asymptotics are different; see Brézin and Hikami [BH2].)

In view of the above-mentioned similarities between random matrices and the Riemann zeta function, it seems natural to ask whether there is an analogue of (1.1) and (1.2) for the shifted moments of the Riemann zeta function along the critical line. Although we have the feeling that such analogues should be well-known, we have not been able to find an explicit statement of such analogues in the (extensive!) recent literature on the relationship between the theory of random matrices and the theory of the Riemann zeta function. This seems somewhat surprising, particularly since there exist some more general results (or conjectures) from which an analogue of (1.1) or (1.2) could be deduced rather easily, at least on a formal level. The main aim of this note is to fill this gap.

More precisely, by an analogue of (1.1) and (1.3), we mean a result of the form

$$\lim_{T\to\infty} \frac{1}{C(T)} \int_{T_0}^{T} \zeta\left(\tfrac{1}{2}+i\left(t+\tfrac{2\pi\mu}{\log t}\right)\right) \zeta\left(\tfrac{1}{2}-i\left(t+\tfrac{2\pi\nu}{\log t}\right)\right) dt = e^{\pm i\pi(\mu-\nu)}\cdot\frac{\sin\pi(\mu-\nu)}{\pi(\mu-\nu)},$$

where μ and ν are arbitrary real numbers, $T_0 > 1$ is a constant, and $C(T)$ is some normalizing factor depending on T. To account for our choice of scaling for the shift parameters μ and ν, note that both in (1.1) and in (1.3), the scaling factor is equal to the mean spacing of eigenvalues. For instance, for a random $N \times N$ matrix from the CUE, there are N eigenvalues distributed over the unit circle of length 2π, which gives rise to a mean spacing of $2\pi/N$. Similarly, for a random $N \times N$ matrix from the GUE, it is well-known that the mean spacing at the origin is $\pi/\sqrt{2N}$ (see e.g. Chapter 6 in Mehta [Me]). Now recall that, if $N(T)$ denotes the number of zeros of $\zeta(\sigma + it)$ in the region $0 \le \sigma \le 1$, $0 \le t \le T$, it is known that $N(T) \sim (2\pi)^{-1} T \log T$ (see e.g. Chapter 9 in Titchmarsh [Ti]), so that the *empirical* mean spacing at location t is $\sim 2\pi/\log t$. Since this mean spacing depends on t, it seems natural to multiply the shift parameters μ and ν by the *location-dependent* scaling factor $2\pi/\log t$.

For the shifted second moment of the Riemann zeta function, this result was obtained (in a slightly different formulation) already by Atkinson [At] in 1948, and thus even before the sine kernel was "discovered" in random matrix theory. (Curiously, this paper seems not to get cited in the recent literature on the interplay between random matrix theory and number theory.) Atkinson's theorem can be restated as follows:

Theorem 1.1. *For any $T_0 > 1$ and any $\mu, \nu \in \mathbb{R}$,*

$$\lim_{T\to\infty} \frac{1}{T\log T} \int_{T_0}^{T} \zeta\left(\tfrac{1}{2}+i\left(t+\tfrac{2\pi\mu}{\log t}\right)\right) \zeta\left(\tfrac{1}{2}-i\left(t+\tfrac{2\pi\nu}{\log t}\right)\right) dt = e^{-i\pi(\mu-\nu)}\cdot\mathbb{S}(\pi(\mu-\nu)),$$

where $\mathbb{S}(x) := \sin x/x$ for $x \neq 0$ and $\mathbb{S}(x) := 1$ for $x = 0$.

In particular, for $\mu, \nu = 0$, this reduces to the classical result that

$$\lim_{T\to\infty} \frac{1}{T\log T} \int_{0}^{T} \left|\zeta\left(\tfrac{1}{2}+it\right)\right|^2 dt = 1$$

(see e.g. Theorem 7.3 in Titchmarsh [Ti]).

Actually, ATKINSON's theorem states that for any $\alpha \geq 0$,

$$\int_{T_0}^{T} \zeta\Big(\tfrac{1}{2} + iu(t)\Big)\, \zeta\Big(\tfrac{1}{2} - it\Big)\, dt \sim e^{-iu} \cdot \mathbb{S}(u) \cdot T \log T \qquad (T \to \infty)\,,$$

where $u(t)$ is defined by the relation $\vartheta(u(t)) - \vartheta(t) = \alpha$, with $\vartheta(t) := -\frac{1}{2} \arg \chi(\frac{1}{2} + it)$. However, as $u(t) - t \sim 2\alpha / \log t$ $(t \to \infty)$, it seems clear that Theorem 1.1 is virtually the same, and in fact this result can be established by the same proof as ATKINSON's theorem.

For the shifted fourth moment of the Riemann zeta function, we have the following result, which constitutes an analogue of (1.2) in the special case $M := 2$, $\mu_1 = \nu_1 =: \mu$, $\mu_2 = \nu_2 =: \nu$:

Theorem 1.2. *For any $T_0 > 1$ and any $\mu, \nu \in \mathbb{R}$,*

$$\lim_{T \to \infty} \frac{1}{T (\log T)^4} \int_{T_0}^{T} \left| \zeta\Big(\tfrac{1}{2} + i\Big(t + \tfrac{2\pi\mu}{\log t}\Big)\Big) \right|^2 \left| \zeta\Big(\tfrac{1}{2} + i\Big(t + \tfrac{2\pi\nu}{\log t}\Big)\Big) \right|^2 dt = \frac{3}{2\pi^2} \cdot \mathbb{T}(\pi(\mu - \nu)),$$

where $\mathbb{T}(x) := \frac{1}{x^2}\Big(1 - \big(\frac{\sin x}{x}\big)^2\Big)$ for $x \neq 0$ and $\mathbb{T}(0) := 1/3$ for $x = 0$.

In particular, for $\mu, \nu = 0$, this reduces to the classical result that

$$\lim_{T \to \infty} \frac{1}{T (\log T)^4} \int_{0}^{T} \left| \zeta\Big(\tfrac{1}{2} + it\Big) \right|^4 dt = \frac{1}{2\pi^2}$$

(see Theorem B in INGHAM [In]).

For the proof of Theorem 1.2, we will closely follow the proof of Theorem B in INGHAM [In]. In particular, our proof is also based on the approximate functional equation for $\zeta(s)^2$. (This is analogous to the proof of Theorem 1.1 indicated above, which closely follows the proof of the corresponding result for the non-shifted second moment, starting from the approximate functional equation for $\zeta(s)$.)

As pointed out by an anonymous referee, it should also be possible to deduce Theorems 1.1 and 1.2 (and even more precise versions involving information about the lower-order terms) from the existing (more general) mean value theorems for the second and fourth moment of the Riemann zeta function with *constant* shifts (see Theorem A in INGHAM [In] and Theorem 4.2 in MOTOHASHI [Mot]). However, we will not pursue this issue further here, since it is our main aim to point out that the highest-order terms of the appropriately shifted moments of the Riemann zeta function give rise to the sine kernel. Furthermore, weighing the shifts with the factor $2\pi / \log t$ seems to simplify the problem, and we therefore think that our independent (and comparatively simple) proof of Theorem 1.2 might be of its own interest.

As regards the higher (even) shifted moments of the Riemann zeta function along the critical line, we will show that a recent conjecture by CONREY, FARMER, KEATING, RUBINSTEIN, and SNAITH [CFKRS2], when combined with our choice of scaling, gives rise to the following analogue of (1.2):

Conjecture 1.3. *For any $M = 1, 2, 3, \ldots$, for any $T_0 > 1$ and for any $\mu_1, \ldots, \mu_M$, $\nu_1, \ldots, \nu_M \in \mathbb{R}$,*

$$\lim_{T\to\infty} \frac{1}{T(\log T)^{M^2}} \int_{T_0}^{T} \prod_{j=1}^{M} \zeta\left(\tfrac{1}{2} + it + \tfrac{2\pi i\mu_j}{\log t}\right) \prod_{j=1}^{M} \zeta\left(\tfrac{1}{2} - it - \tfrac{2\pi i\nu_j}{\log t}\right) \, dt$$
$$= a_M \cdot \frac{\exp(-\pi i \sum_{j=1}^{M}(\mu_j - \nu_j))}{\Delta(2\pi\mu_1, ..., 2\pi\mu_M) \cdot \Delta(2\pi\nu_1, ..., 2\pi\nu_M)} \cdot \det\left(\frac{\sin\pi(\mu_j - \nu_k)}{\pi(\mu_j - \nu_k)}\right)_{j,k=1,\ldots,M},$$

where $\Delta(x_1, \ldots, x_M) := \prod_{1\le j<k\le M}(x_k - x_j)$ is the Vandermonde determinant and

$$a_M := \prod_{p\in\mathcal{P}} \left(1 - \tfrac{1}{p}\right)^{M^2} \sum_{j=0}^{\infty} \left(\frac{\Gamma(j+M)}{j!\,\Gamma(M)}\right)^2 p^{-j},$$

the product being taken over the set $\mathcal{P}$ of prime numbers. (Naturally, in the case where two or more of the shift parameters are equal, the right-hand side should be regarded as defined by continuous extension, similarly as in the preceding theorems.)

It is easy to see that $a_1 = 1$ and $a_2 = 6/\pi^2$. Thus, Theorem 1.1 confirms Conjecture 1.3 in the special case $M = 1$, and Theorem 1.2 confirms Conjecture 1.3 in the special case $M = 2$, $\mu_1 = \nu_1$, $\mu_2 = \nu_2$.

Furthermore, Equation (1.2) and Conjecture 1.3 clearly have a similar structure. A notable difference is given by the factor a_M which occurs in Conjecture 1.3 for the Riemann zeta function but not in Equation (1.2) for the CUE. It is well-known (see e.g. Keating and Snaith [KS1, KS2] and Conrey, Farmer, Keating, Rubinstein, and Snaith [CFKRS1, CFKRS2]) that this "arithmetic factor" is not predicted by random matrix theory. Another difference is given by the sign in the phase factor $\exp(\pm\pi i \sum_{j=1}^{M}(\mu_j - \nu_j))$. This difference could have been avoided if we had defined the characteristic polynomial by $\det(I - \xi^{-1}U)$ instead of $\det(U - \xi I)$.

As already mentioned, apart from Atkinson's theorem, we are not aware of explicit statements of *continuous* mean value theorems involving the sine kernel. This seems somewhat surprising, given that the choice of the shift parameters on the scale $2\pi/\log t$ seems completely natural in view of the existing similarities to random matrix theory. In particular, this scaling also occurs in the pair correlation function of the zeros of the Riemann zeta function (see e.g. Montgomery [Mon]) as well as in a number of *discrete* mean value theorems related to the zeros of the Riemann zeta function (see e.g. Gonek [Go], Hughes [Hu], Mozer [Moz1, Moz2, Moz3]).

Throughout this paper, we use the following notation: Let $\zeta(s)$ denote the Riemann zeta function, which is defined by the Dirichlet series

$$\zeta(s) := \sum_{n=1}^{\infty} n^{-s}$$

for $\mathrm{Re}\,(s) > 1$ and by analytic continuation for $\mathrm{Re}\,(s) \le 1$, and let

$$\chi(s) := 2^s\, \pi^{s-1} \sin(\tfrac{1}{2}\pi s)\, \Gamma(1-s) = \pi^{s-\frac{1}{2}}\, \Gamma(\tfrac{1}{2} - \tfrac{1}{2}s) \,/\, \Gamma(\tfrac{1}{2}s)$$

for any $s \in \mathbb{C}$. We follow the convention of denoting the real and imaginary part of the argument s by σ and t, respectively. Furthermore, for any integer $n \geq 1$, we denote by $d(n)$ the number of divisors of n. Finally, we make the convention that, unless otherwise indicated, the $\mathcal{O}$-bounds occurring in the proofs may depend on μ and ν (which are regarded as fixed) but not on any other parameters.

This paper is structured as follows. Section 2 is devoted to the proof of Theorem 1.2. In Section 3 we discuss the relationship between Conjecture 1.3 for the higher (even) shifted moments of the Riemann zeta function and the conjecture by Conrey, Farmer, Keating, Rubinstein, and Snaith [CFKRS1, CFKRS2]. Finally, for the convenience of the reader, the appendices A and B contain some auxiliary results from analytic number theory and random matrix theory which have been used in the preceding sections.

2. The Mean Value of the Fourth Moment

The proof of Theorem 1.2 is modelled on the proof of Theorem B in Ingham [In]).

Proof of Theorem 1.2. We will show that for any $\mu, \nu \in \mathbb{R}$,

$$\lim_{T\to\infty} \frac{1}{T(\log T)^4} \int_T^{2T} \left|\zeta\Big(\tfrac12 + i\Big(t + \tfrac{2\pi\mu}{\log t}\Big)\Big)\right|^2 \left|\zeta\Big(\tfrac12 + i\Big(t + \tfrac{2\pi\nu}{\log t}\Big)\Big)\right|^2 dt = \frac{3}{2\pi^4(\mu-\nu)^2} \cdot \left(1 - \left(\frac{\sin\pi(\mu-\nu)}{\pi(\mu-\nu)}\right)^2\right). \quad (2.1)$$

The assertion of Theorem 1.2 then follows by using (2.1) for $T/2^1$, $T/2^2$, $T/2^3$, ... and taking the sum.

For the proof of (2.1), we start from the approximate functional equation for ζ^2 (see e.g. Theorem 4.2 in Ivić [Iv]) , which states that for any $h > 0$,

$$\zeta^2(s) = \sum_{n\leq x} d(n)n^{-s} + \chi^2(s) \sum_{n\leq y} d(n)n^{-1+s} + \mathcal{O}(x^{1/2-\sigma}\log t)$$

for $0 < \sigma < 1$, $4\pi^2 xy = t^2$, $x, y, t > h > 0$. Taking $\sigma = \frac12$, $t > 2$, $x(t) = y(t) = t/2\pi$, it follows that

$$\zeta^2(\tfrac12 + it) = \sum_{n\leq x(t)} d(n)\, n^{-\frac12 - it} + \chi^2(\tfrac12 + it) \sum_{n\leq x(t)} d(n)\, n^{-\frac12 + it} + \mathcal{O}(\log t)\,.$$

Using the functional equation

$$\zeta(\tfrac12 + it) = \chi(\tfrac12 + it)\,\zeta(\tfrac12 - it) = \chi(\tfrac12 + it)\,\overline{\zeta(\tfrac12 + it)}$$

(see e.g. Equation (1.23) in Ivić [Iv]) and multiplying by $\chi(\frac12 + it)^{-1} = \overline{\chi(\frac12 + it)}$, which is of order $\mathcal{O}(1)$, we therefore obtain

$$\left|\zeta(\tfrac12 + it)\right|^2 = 2\,\mathrm{Re}\left(\chi(\tfrac12 + it) \sum_{n\leq x(t)} d(n)\, n^{-\frac12 + it}\right) + \mathcal{O}(\log t)\,.$$

(This is Equation (4.11) in Ivić [Iv].)

In the following, we will repeatedly use the fact that for any fixed $\varepsilon > 0$,

$$d(n) = \mathcal{O}(n^{\varepsilon})$$

(see e.g. Equation (1.71) in Ivić [Iv]). In particular, this implies that

$$\sum_{n \le x(t)} d(n) n^{-\frac{1}{2}} = \mathcal{O}\Big(\sum_{n \le x(t)} n^{-\frac{1}{2}+\varepsilon} \Big) = \mathcal{O}\Big(\int_1^{x(t)} u^{-\frac{1}{2}+\varepsilon} \, du \Big) = \mathcal{O}(t^{\frac{1}{2}+\varepsilon}) .$$

Using the approximation

$$\chi(\tfrac{1}{2}+it) = e^{\pi i/4} \left(\frac{2\pi e}{t} \right)^{it} + \mathcal{O}(t^{-1}) \qquad (t > 2)$$

(see e.g. Equation (1.25) in Ivić [Iv]), it follows that

$$\left| \zeta(\tfrac{1}{2}+it) \right|^2 = 2 \operatorname{Re} \left(e^{\pi i/4} \left(\frac{2\pi e}{t} \right)^{it} \sum_{n \le x(t)} d(n) \, n^{-\frac{1}{2}+it} \right) + \mathcal{O}(\log t) .$$

Using this equation for $t + 2\pi\lambda / \log t$ instead of t, where λ is a fixed real number and t is sufficiently large (depending on λ), and using the straightforward estimate

$$\begin{aligned} \left(\frac{2\pi e}{t + \frac{2\pi\lambda}{\log t}} \right)^{+it+\frac{2\pi i\lambda}{\log t}} &= \left(\frac{2\pi e}{t} \right)^{+it+\frac{2\pi i\lambda}{\log t}} \left(\frac{1}{1 + \frac{2\pi\lambda}{t \log t}} \right)^{+it+\frac{2\pi i\lambda}{\log t}} \\ &= \left(\frac{2\pi e}{t} \right)^{+it+\frac{2\pi i\lambda}{\log t}} \exp\left(-\frac{2\pi i\lambda}{\log t} \right) + \mathcal{O}(t^{-1}) , \end{aligned}$$

it finally follows that

$$\left| \zeta(\tfrac{1}{2}+it+2\pi i\lambda/\log t) \right|^2 = 2 \operatorname{Re} \big(S(\lambda,t) \big) + \mathcal{O}(\log t) ,$$

where

$$S(\lambda,t) := e^{\pi i/4} \left(\frac{2\pi e}{t} \right)^{it} \left(\frac{2\pi}{t} \right)^{\frac{2\pi i\lambda}{\log t}} \sum_{n \le x(t)} d(n) \, n^{-\frac{1}{2}+it+\frac{2\pi i\lambda}{\log t}} . \tag{2.2}$$

Now suppose that we can show that

$$\begin{aligned} \lim_{T \to \infty} \frac{1}{T (\log T)^4} \int_T^{2T} & 2 \operatorname{Re} \left(S(\mu,t) \right) \cdot 2 \operatorname{Re} \left(S(\nu,t) \right) \, dt \\ &= \frac{3/2}{\pi^4 (\mu-\nu)^2} \left(1 - \left(\frac{\sin \pi(\mu-\nu)}{\pi(\mu-\nu)} \right)^2 \right) \end{aligned} \tag{2.3}$$

for any $\mu, \nu \in \mathbb{R}$.

It then follows by the Cauchy-Schwarz inequality that, for T sufficiently large,

$$\begin{aligned}
&\int_T^{2T} \left|\zeta(\tfrac{1}{2} + it + \tfrac{2\pi i\mu}{\log t})\right|^2 \left|\zeta(\tfrac{1}{2} - it - \tfrac{2\pi i\nu}{\log t})\right|^2 \; dt \\
&= \int_T^{2T} 2\,\mathrm{Re}\,(S(\mu,t)) \cdot 2\,\mathrm{Re}\,(S(\nu,t)) \; dt \\
&\quad + \mathcal{O}\left(\left(\int_T^{2T} \left|2\,\mathrm{Re}\,(S(\mu,t))\right|^2 \; dt\right)^{1/2} \left(\int_T^{2T} (\log t)^2 \; dt\right)^{1/2}\right) \\
&\quad + \mathcal{O}\left(\left(\int_T^{2T} (\log t)^2 \; dt\right)^{1/2} \left(\int_T^{2T} \left|2\,\mathrm{Re}\,(S(\nu,t))\right|^2 \; dt\right)^{1/2}\right) \\
&\quad + \mathcal{O}\left(\int_T^{2T} (\log t)^2 \; dt\right) \\
&= \int_T^{2T} 2\,\mathrm{Re}\,(S(\mu,t)) \cdot 2\,\mathrm{Re}\,(S(\nu,t)) \; dt + o\Big(T \log^4 T\Big) \,,
\end{aligned}$$

and the theorem is proved.

Thus, it remains to prove (2.3). To begin with, we have

$$2\,\mathrm{Re}\,(S(\mu,t)) \cdot 2\,\mathrm{Re}\,(S(\nu,t)) = 2\,\mathrm{Re}\,(S(\mu,t)\,\overline{S(\nu,t)}) + 2\,\mathrm{Re}\,(S(\mu,t)\,S(\nu,t))$$

and therefore

$$\begin{aligned}
&\int_T^{2T} 2\,\mathrm{Re}\,(S(\mu,t)) \cdot 2\,\mathrm{Re}\,(S(\nu,t)) \; dt \\
&\qquad = 2\,\mathrm{Re}\left(\int_T^{2T} S(\mu,t)\,\overline{S(\nu,t)} \; dt\right) + 2\,\mathrm{Re}\left(\int_T^{2T} S(\mu,t)\,S(\nu,t) \; dt\right) . \qquad (2.4)
\end{aligned}$$

An elaboration of the argument in Ingham [In] shows that the second integral on the right-hand side in (2.4) is of order $o(T \log^4 T)$ and therefore tends to zero after division by $T \log^4 T$ as in (2.3). Indeed, from (2.2), we have

$$\int_T^{2T} S(\mu,t)\,S(\nu,t) \; dt = i \sum_{m,n \le x(2T)} \frac{d(m)\,d(n)}{\sqrt{m}\,\sqrt{n}} \int_{T'}^{2T} \exp(iF(t)) \; dt \,,$$

for all $T \ge 2$, where $T' := T'(T,m,n) := \max\{T, 2\pi m, 2\pi n\}$ and

$$F(t) := t\log(mn) + 2t\log(2\pi e/t) + \tfrac{2\pi\mu}{\log t}\log(2\pi m/t) + \tfrac{2\pi\nu}{\log t}\log(2\pi n/t) \,.$$

The derivatives of $F(t)$ are given by

$$F'(t) = 2\log(2\pi\sqrt{mn}/t) - (2\pi\mu\log(2\pi m) + 2\pi\nu\log(2\pi n)) \; \frac{1}{t(\log t)^2}$$

and

$$F''(t) = -2/t + (2\pi\mu\log(2\pi m) + 2\pi\nu\log(2\pi n)) \; \frac{2 + \log t}{t^2(\log t)^3} \,.$$

Clearly, for T sufficiently large (depending on μ,ν), $F''(t) \le -1/t \le -1/2T$ for all $t \in [T', 2T]$. Hence, by Lemma 4.4 in Titchmarsh [Ti] (= Lemma A.2), we have the estimate

$$\left|\int_{T'}^{2T} \exp(iF(t))\ dt\right| = \mathcal{O}\big(\sqrt{T}\,\big)\,.$$

Moreover, for $m \neq n$, $T' \geq 2\pi \max\{m,n\} > 2\pi\sqrt{mn}$. It follows that, for T sufficiently large (depending on μ,ν), $F'(t) \leq \log(2\pi\sqrt{mn}/t) \leq \log(2\pi\sqrt{mn}/T')$ $\leq -\frac{1}{2}|\log(n/m)|$ for all $t \geq T'$. Hence, by Lemma 4.2 in Titchmarsh [Ti] (= Lemma A.1), we have the estimate

$$\left|\int_{T'}^{2T} \exp(iF(t))\ dt\right| = \mathcal{O}\bigg(\frac{1}{|\log(n/m)|}\bigg)\,.$$

Combining these two estimates and using Lemmas B.3 and B.1 in Ingham [In] (= Lemmas A.4 and A.6), it follows that

$$\begin{aligned}\int_{T}^{2T} S(\mu,t)\,S(\nu,t)\ dt &= i \sum_{m,n\leq x(2T)} \frac{d(m)\,d(n)}{\sqrt{m}\,\sqrt{n}} \cdot \int_{T'}^{2T} \exp(iF(t))\ dt \\ &= \mathcal{O}\left(\sum_{1\leq m<n\leq x(2T)} \frac{d(m)\,d(n)}{\sqrt{mn}\,\log(n/m)}\right) + \mathcal{O}\left(\sum_{n\leq x(2T)} \frac{d(n)^2}{n}\sqrt{T}\right) \\ &= \mathcal{O}\big(T\log^3 T\big) + \mathcal{O}\big(\sqrt{T}\log^4 T\big) = o\big(T\log^4 T\big)\,,\end{aligned}$$

as claimed.

Thus, it remains to examine the first integral on the right-hand side in (2.4). To begin with, from (2.2), we have

$$\begin{aligned}&\int_{T}^{2T} S(\mu,t)\,\overline{S(\nu,t)}\ dt \\ &\quad= \sum_{m,n\leq x(2T)} \frac{d(m)\,d(n)}{\sqrt{m}\,\sqrt{n}} \int_{T'}^{2T} \left(\frac{2\pi}{t}\right)^{\frac{2\pi i(\mu-\nu)}{\log t}} m^{+it+\frac{2\pi i\mu}{\log t}}\, n^{-it-\frac{2\pi i\nu}{\log t}}\ dt \\ &\quad= e^{-2\pi i(\mu-\nu)} \sum_{m,n\leq x(2T)} \frac{d(m)\,d(n)}{\sqrt{m}\,\sqrt{n}} \int_{T'}^{2T} (m/n)^{it}\,(2\pi m)^{+\frac{2\pi i\mu}{\log t}}\ (2\pi n)^{-\frac{2\pi i\nu}{\log t}}\ dt\end{aligned}$$

for all $T \geq 2$, where $T' := T'(T,m,n) := \max\{T, 2\pi m, 2\pi n\}$.

Now, for those pairs (m,n) with $m \neq n$, we find using integration by parts that

$$\begin{aligned}&\int_{T'}^{2T} (m/n)^{it}\,(2\pi m)^{+\frac{2\pi i\mu}{\log t}}\ (2\pi n)^{-\frac{2\pi i\nu}{\log t}}\ dt \\ &= \left[\frac{(m/n)^{it}}{i\log(m/n)}\,(2\pi m)^{+\frac{2\pi i\mu}{\log t}}\,(2\pi n)^{-\frac{2\pi i\nu}{\log t}}\right]_{t=T'}^{t=2T} + \int_{T'}^{2T} \frac{(m/n)^{it}}{i\log(m/n)} \\ &\qquad \cdot\ (2\pi m)^{+\frac{2\pi i\mu}{\log t}}\,(2\pi n)^{-\frac{2\pi i\nu}{\log t}} \left[\frac{2\pi i\mu\log(2\pi m)}{t(\log t)^2} - \frac{2\pi i\nu\log(2\pi n)}{t(\log t)^2}\right]\ dt \\ &= \mathcal{O}\left(\frac{1}{|\log(m/n)|}\right)\,,\end{aligned}$$

where the last step uses the inequalities $T \leq T' \leq 2T$ and $m,n \leq x(2T) \leq T$.

Hence, using Lemma B.3 in INGHAM [In] (= Lemma A.4), it follows that

$$\sum_{m \neq n} \frac{d(m)\,d(n)}{\sqrt{m}\,\sqrt{n}} \int_{T'}^{2T} (m/n)^{it}\,(2\pi m)^{+\frac{2\pi i\mu}{\log t}}\,(2\pi n)^{-\frac{2\pi i\nu}{\log t}}\,dt$$

$$= \mathcal{O}\left(\sum_{m \neq n} \frac{d(m)\,d(n)}{\sqrt{mn}\,|\log(m/n)|}\right) = \mathcal{O}\left(x(2T)\log^3 x(2T)\right) = \mathcal{O}\left(T\log^3 T\right),$$

so that the sum over the pairs (m,n) with $m \neq n$ tends to zero after division by $T\log^4 T$ as in (2.3).

Consequently, to determine the asymptotic behaviour of the first integral on the right-hand side in (2.4), it remains to consider the sum over the pairs (m,n) with $m = n$ and to show that

$$\lim_{T\to\infty} \frac{1}{T\log^4 T} \cdot 2\,\mathrm{Re}\left(e^{-2\pi i(\mu-\nu)} \sum_{n \leq x(2T)} \frac{d(n)^2}{n} \int_{T'}^{2T} (2\pi n)^{\frac{2\pi i(\mu-\nu)}{\log t}}\,dt\right)$$

$$= \frac{3/2}{\pi^4(\mu-\nu)^2}\left(1 - \left(\frac{\sin \pi(\mu-\nu)}{\pi(\mu-\nu)}\right)^2\right). \quad (2.5)$$

Clearly, in doing so, we may assume without loss of generality that $\nu = 0$.

To evaluate the integral on the left-hand side in (2.5), write

$$(2\pi n)^{2\pi i\mu/\log t} = (2\pi n)^{2\pi i\mu/\log T} - \int_T^t (2\pi n)^{2\pi i\mu/\log u}\,\log(2\pi n)\,\frac{2\pi i\mu}{u\,(\log u)^2}\,du$$

and note that for $T \leq t \leq 2T$ and $n \leq 2T$,

$$\int_T^t (2\pi n)^{2\pi i\mu/\log u}\,\log(2\pi n)\,\frac{2\pi i\mu}{u(\log u)^2}\,du = \mathcal{O}\left((t-T)\cdot\frac{\log(2\pi n)}{T(\log T)^2}\right) = \mathcal{O}\left(\frac{1}{\log T}\right).$$

Hence, since $T' \geq T$ and $x(2T) \leq 2T$, it follows that

$$\int_{T'}^{2T} (2\pi n)^{2\pi i\mu/\log t}\,dt = \left(2T - T'\right)(2\pi n)^{\frac{2\pi i\mu}{\log T}} + \mathcal{O}\left(\frac{T}{\log T}\right)$$

$$= T\,(2\pi n)^{\frac{2\pi i\mu}{\log T}} - \left(T' - T\right)(2\pi n)^{\frac{2\pi i\mu}{\log T}} + \mathcal{O}\left(\frac{T}{\log T}\right)$$

and therefore

$$e^{-2\pi i\mu} \sum_{n \leq x(2T)} \frac{d(n)^2}{n} \int_{T'}^{2T} (2\pi n)^{\frac{2\pi i\mu}{\log t}}\,dt = e^{-2\pi i\mu} \sum_{n \leq x(2T)} T\,\frac{d(n)^2}{n}\,(2\pi n)^{\frac{2\pi i\mu}{\log T}}$$

$$- e^{-2\pi i\mu} \sum_{n \leq x(2T)} \left(T' - T\right)\frac{d(n)^2}{n}\,(2\pi n)^{\frac{2\pi i\mu}{\log T}} + \mathcal{O}\left(\frac{T}{\log T}\sum_{n \leq x(2T)} \frac{d(n)^2}{n}\right). \quad (2.6)$$

By Lemma B.1 in INGHAM [In] (= Lemma A.6), we have

$$\sum_{n \leq T} \frac{d(n)^2}{n} = \frac{1}{4\pi^2}\log^4 T + \mathcal{O}(\log^3 T).$$

Thus, the $\mathcal{O}$-term in (2.6) is obviously of order $o(T\log^4 T)$. Also, since $T' = T$ for $n \le x(T)$,

$$\begin{aligned}
\sum_{n \le x(2T)} (T'-T)\,\frac{d(n)^2}{n} &= \sum_{x(T)<n\le x(2T)} (T'-T)\,\frac{d(n)^2}{n} \\
&\le T\left(\sum_{n\le x(2T)} \frac{d(n)^2}{n} - \sum_{n\le x(T)} \frac{d(n)^2}{n}\right) \\
&= T\left(\frac{1}{4\pi^2}\log^4(2T) - \frac{1}{4\pi^2}\log^4(T) + \mathcal{O}(\log^3 T)\right) \\
&= o(T\log^4 T)\,,
\end{aligned}$$

so that the second sum on the right-hand side in (2.6) is also of order $o(T\log^4 T)$.

Thus, it remains to consider the first sum on the right-hand side in (2.6). Similarly as in the proof of Lemma B.1 in Ingham [In], we can approximate this sum by an integral. Setting

$$D(t) := \sum_{n\le t} d(n)^2$$

and using Lemma A.5, we have, for $\lambda \in \mathbb{R}$ from a bounded set,

$$\begin{aligned}
&\sum_{n\le x(2T)} d(n)^2\, n^{-1+i\lambda} \\
&= \sum_{n\le x(2T)} (D(n) - D(n-1))\; n^{-1+i\lambda} \\
&= \sum_{n\le x(2T)-1} D(n)\left(n^{-1+i\lambda} - (n+1)^{-1+i\lambda}\right) + \mathcal{O}(\log^3 T) \\
&= (1-i\lambda)\int_1^{x(2T)} \frac{D(u)}{u^{2-i\lambda}}\, du + \mathcal{O}(\log^3 T) \\
&= (1-i\lambda)\,\frac{1}{\pi^2}\int_1^{x(2T)} \frac{\log^3 u}{u^{1-i\lambda}}\, du + \mathcal{O}\left(\int_1^{x(2T)} \frac{\log^2 u}{u}\, du\right) + \mathcal{O}(\log^3 T) \\
&= (1-i\lambda)\,\frac{1}{\pi^2}\int_1^{x(2T)} \frac{\log^3 u}{u^{1-i\lambda}}\, du + \mathcal{O}(\log^3 T)\,.
\end{aligned}$$

Substituting $v = \log u$ and $w = v/\log T$ yields

$$\begin{aligned}
\int_1^{x(2T)} \frac{\log^3 u}{u^{1-i\lambda}}\, du &= \int_0^{\log x(2T)} v^3\, e^{i\lambda v}\, dv \\
&= (\log T)^4 \int_0^{\log x(2T)/\log T} w^3\, e^{i\lambda w \log T}\, dw
\end{aligned}$$

and therefore

$$\sum_{n\le x(2T)} d(n)^2\, n^{-1+i\lambda} = (\log T)^4\,(1-i\lambda)\cdot\frac{1}{\pi^2}\int_0^1 w^3\, e^{i\lambda w\log T}\, dw + \mathcal{O}(\log^3 T)\,.$$

Thus, with λ replaced by $2\pi\mu/\log T$, it follows that

$$\begin{aligned} & T\, e^{-2\pi i\mu} \sum_{n \le x(2T)} \frac{d(n)^2}{n}\, (2\pi n)^{\frac{2\pi i\mu}{\log T}} \\ &= T\,(\log T)^4\,(2\pi)^{\frac{2\pi i\mu}{\log T}}\,(1 - \tfrac{2\pi i\mu}{\log T}) \cdot \frac{1}{\pi^2} \int_0^1 w^3\, e^{2\pi i\mu(w-1)}\; dw + \mathcal{O}(T \log^3 T) \\ &= T\,(\log T)^4 \cdot \frac{1}{\pi^2} \int_0^1 w^3\, e^{2\pi i\mu(w-1)}\; dw + \mathcal{O}(T \log^3 T)\,. \end{aligned}$$

Dividing by $T\,\log^4 T$ and taking real parts, we therefore obtain

$$\begin{aligned} \lim_{T\to\infty} \frac{1}{T \log^4 T} \cdot 2\,\mathrm{Re} \left(T\, e^{-2\pi i\mu} \sum_{n \le x(2T)} \frac{d(n)^2}{n}\, (2\pi n)^{\frac{2\pi i\mu}{\log T}} \right) \\ = \frac{2}{\pi^2} \int_0^1 w^3\; \cos(2\pi\mu(w-1))\; dw\,. \end{aligned}$$

A direct calculation using the trigonometric identity $1 - \cos(z) = 2\sin^2(z/2)$ yields

$$\begin{aligned} \int_0^1 w^3\; \cos(2\pi\mu(w-1))\; dw &= \frac{3}{(2\pi\mu)^2} \left(1 - \frac{2 - 2\cos(2\pi\mu)}{(2\pi\mu)^2} \right) \\ &= \frac{3}{(2\pi\mu)^2} \left(1 - \left(\frac{\sin \pi\mu}{\pi\mu} \right)^2 \right)\,. \end{aligned}$$

This is true also for $\mu = 0$, provided that we consider the continuous extension of the right-hand side, i.e. $1/4$. This concludes the proof of (2.5), and hence of Theorem 1.2. □

3. The Conjecture for the Higher Shifted Moments

In this section we comment on the relationship between Conjecture 1.3 for the higher (even) shifted moments of the Riemann zeta function and the conjecture by Conrey, Farmer, Keating, Rubinstein, and Snaith [CFKRS1, CFKRS2], which we will simply call the CFKRS-Conjecture from now on.

In the special case of the Riemann zeta function, this conjecture can be stated as follows:

Conjecture 3.1 (Conjecture 2.2 in [CFKRS1])**.** *For any $M = 1, 2, 3, \ldots,$ and any $\mu_1, \ldots, \mu_M, \nu_1, \ldots, \nu_M \in \mathbb{R}$,*

$$\begin{aligned} \int_0^T \prod_{j=1}^M \zeta\left(\tfrac{1}{2} + it + i\mu_j\right) \prod_{j=1}^M \zeta\left(\tfrac{1}{2} - it - i\nu_j\right)\; dt \\ = \int_0^T W_M(t; i\mu_1, \ldots, i\mu_M; i\nu_1, \ldots, i\nu_M) \left(1 + \mathcal{O}(t^{-(1/2)+\varepsilon}) \right)\; dt\,, \end{aligned}$$

where

$$\begin{aligned} &W_M(t;\xi_1,\dots,\xi_M,\xi_{M+1},\dots,\xi_{2M}) \\ &:= \exp(\tfrac{1}{2}\log\tfrac{t}{2\pi}\cdot\sum_{j=1}^{M}(-\xi_j+\xi_{M+j}))\cdot\sum_{\sigma\in\mathcal{S}'_{2M}}\exp(\tfrac{1}{2}\log\tfrac{t}{2\pi}\cdot\sum_{j=1}^{M}(\xi_{\sigma(j)}-\xi_{\sigma(M+j)})) \\ &\qquad\cdot A_M(\xi_{\sigma(1)},\dots,\xi_{\sigma(2M)})\cdot\prod_{j,k=1,\dots,M}\zeta(1+\xi_{\sigma(j)}-\xi_{\sigma(M+k)})\,. \end{aligned}$$

Here, $\mathcal{S}'_{2M}$ denotes the subset of permutations σ of the set $\{1,\dots,2M\}$ satisfying $\sigma(1)<\dots<\sigma(M)$ and $\sigma(M+1)<\dots<\sigma(2M)$, and $A_M(z_1,\dots,z_{2M})$ is a certain function which is analytic in a neighborhood of the origin and for which $A_M(0,\dots,0)=a_M$.

We will show that Conjecture 1.3 follows from the CFKRS-conjecture provided that one permits replacing $\mu_1,\dots,\mu_M,\nu_1,\dots,\nu_M$ with $2\pi\mu_1/\log t,\dots,2\pi\mu_M/\log t$, $2\pi\nu_1/\log t,\dots,2\pi\nu_M/\log t$. In this respect, Conjecture 1.3 may be regarded as a special case of the CFKRS-conjecture.

Similarly as in the proof of Theorem 1.2, we prefer working with the interval $[T,2T]$ instead of $[0,T]$. Besides that, we will only consider the leading-order terms. We then have the approximation

$$\begin{aligned} &\int_T^{2T}\prod_{j=1}^{M}\zeta\left(\tfrac{1}{2}+it+\tfrac{2\pi i\mu_j}{\log t}\right)\prod_{j=1}^{M}\zeta\left(\tfrac{1}{2}-it-\tfrac{2\pi i\nu_j}{\log t}\right)\,dt \\ \approx&\int_T^{2T}\exp(\tfrac{1}{2}\log\tfrac{t}{2\pi}\cdot\sum_{j=1}^{M}(-\tfrac{\xi_j}{\log t}+\tfrac{\xi_{M+j}}{\log t}))\cdot\sum_{\sigma\in\mathcal{S}'_{2M}}\exp(\tfrac{1}{2}\log\tfrac{t}{2\pi}\cdot\sum_{j=1}^{M}(\tfrac{\xi_{\sigma(j)}}{\log t}-\tfrac{\xi_{\sigma(M+j)}}{\log t})) \\ &\qquad\cdot A_M(\tfrac{\xi_{\sigma(1)}}{\log t},\dots,\tfrac{\xi_{\sigma(2M)}}{\log t})\cdot\prod_{j,k=1,\dots,M}\zeta(1+\tfrac{\xi_{\sigma(j)}}{\log t}-\tfrac{\xi_{\sigma(M+k)}}{\log t})\,dt\,, \end{aligned} \tag{3.1}$$

where we have put $\xi_j:=2\pi i\mu_j$ for $j=1,\dots,M$, $\xi_{M+j}:=2\pi i\nu_j$ for $j=1,\dots,M$, and $\mathcal{S}'_{2M}$ and A_M are the same as in the CFKRS-conjecture. Alternatively, the approximation (3.1) could be obtained by starting from the expression

$$\int_T^{2T}\prod_{j=1}^{M}\zeta(\tfrac{1}{2}+it+\tfrac{2\pi i\mu_j}{\log t})\prod_{j=1}^{M}\zeta(\tfrac{1}{2}-it-\tfrac{2\pi i\nu_j}{\log t})\,dt$$

and by following the (non-rigorous) "recipe" leading to the CFKRS-conjecture. (In fact, since the factor $\frac{1}{\log t}$ is essentially constant, it is irrelevant for the question which terms are rapidly oscillating and should therefore be discarded.)

To simplify (3.1) as $T\to\infty$, recall that A_M is regular at $(0,\dots,0)$ and ζ has a simple pole with residual 1 at $z=1$. Thus, concentrating on leading-order terms, we obtain

$$\int_T^{2T} \prod_{j=1}^M \zeta\left(\tfrac{1}{2} + it + \tfrac{2\pi i \mu_j}{\log t}\right) \prod_{j=1}^M \zeta\left(\tfrac{1}{2} - it - \tfrac{2\pi i \nu_j}{\log t}\right)\, dt$$

$$\approx \int_T^{2T} \exp(\tfrac{1}{2} \cdot \sum_{j=1}^M (-\xi_j + \xi_{M+j}) \cdot \sum_{\sigma \in \mathcal{S}'_{2M}} \exp(\tfrac{1}{2} \cdot \sum_{j=1}^M (\xi_{\sigma(j)} - \xi_{\sigma(M+j)}))$$

$$\cdot\, A_M(0, \ldots, 0) \cdot \frac{(\log t)^{M^2}}{\prod_{j,k=1,\ldots,M}(\xi_{\sigma(j)} - \xi_{\sigma(M+k)})}\, dt\,. \quad (3.2)$$

Therefore, since

$$\int_T^{2T} (\log t)^{M^2}\, dt = T(\log T)^{M^2} + \mathcal{O}\left(T(\log T)^{M^2-1}\right)\,,$$

we should expect that

$$\lim_{T\to\infty} \frac{1}{T(\log T)^{M^2}} \int_T^{2T} \prod_{j=1}^M \zeta(\tfrac{1}{2} + it + \tfrac{2\pi i \mu_j}{\log t}) \prod_{j=1}^M \zeta(\tfrac{1}{2} - it - \tfrac{2\pi i \nu_j}{\log t})\, dt$$

$$= \exp(\tfrac{1}{2} \cdot \textstyle\sum_{j=1}^M (-\xi_j + \xi_{M+j})) \cdot \displaystyle\sum_{\sigma \in \mathcal{S}'_{2M}} \exp(\tfrac{1}{2} \cdot \textstyle\sum_{j=1}^M (\xi_{\sigma(j)} - \xi_{\sigma(M+j)}))$$

$$\cdot\, A_M(0, \ldots, 0) \cdot \frac{1}{\prod_{j,k=1,\ldots,M}(\xi_{\sigma(j)} - \xi_{\sigma(M+k)})}\,.$$

Since $A_M(0, \ldots, 0) = a_M$ (see Equation (2.7.10) in [CFKRS2]) and

$$\sum_{\sigma \in \mathcal{S}'_{2M}} \exp(\tfrac{1}{2} \cdot \textstyle\sum_{j=1}^M (\xi_{\sigma(j)} - \xi_{\sigma(M+j)})) \cdot \displaystyle\frac{1}{\prod_{j,k=1,\ldots,M}(\xi_{\sigma(j)} - \xi_{\sigma(M+k)})}$$

$$= \frac{1}{\Delta(2\pi\mu_1, \ldots, 2\pi\mu_M) \cdot \Delta(2\pi\nu_1, \ldots, 2\pi\nu_M)} \cdot \det\left(\frac{\sin \pi(\mu_j - \nu_k)}{\pi(\mu_j - \nu_k)}\right)_{j,k=1,\ldots,M}$$

(see equation (B.3) in Appendix B), this yields Conjecture 1.3.

Appendix A. Some Estimates from the Literature

In this appendix, we state some estimates from the literature which have been used in the proofs of Theorems 1.1 and 1.2.

Lemma A.1 (Titchmarsh [Ti], Lemma 4.2)**.** *Let $F(x)$ be a real differentiable function such that $F'(x)$ is monotonic, and $F'(x) \geq \varepsilon > 0$ or $F'(x) \leq -\varepsilon < 0$, throughout the interval $[a, b]$. Then*

$$\left| \int_a^b \exp(iF(x))\, dx \right| \leq \frac{4}{\varepsilon}\,.$$

Lemma A.2 (Titchmarsh [Ti], Lemma 4.4)**.** *Let $F(x)$ be a real function, twice differentiable, such that $F''(x) \geq \varepsilon > 0$ or $F''(x) \leq -\varepsilon < 0$, throughout the interval $[a, b]$. Then*

$$\left| \int_a^b \exp(iF(x))\, dx \right| \leq \frac{8}{\sqrt{\varepsilon}}\,.$$

The $\mathcal{O}$-bounds in the following lemmas relate to the case that $T \to \infty$.

Lemma A.3 (Titchmarsh [Ti], Lemma 7.2)**.**

$$\sum_{1 \le m < n \le T} \frac{1}{\sqrt{mn}\,\log(n/m)} = \mathcal{O}\left(T\,\log T\right) .$$

Lemma A.4 (Ingham [In], Lemma B.3)**.**

$$\sum_{1 \le m < n \le T} \frac{d(m)\,d(n)}{\sqrt{mn}\,\log(n/m)} = \mathcal{O}\left(T\,\log^3 T\right) .$$

Lemma A.5 (see e. g. Ivić [Iv], Equation (5.24))**.**

$$\sum_{n \le T} d(n)^2 = \frac{1}{\pi^2}\,T\,\log^3 T + \mathcal{O}\left(T\,\log^2 T\right) .$$

Lemma A.6 (Ingham [In], Lemma B.1)**.**

$$\sum_{n \le T} \frac{d(n)^2}{n} = \frac{1}{4\pi^2}\log^4 T + \mathcal{O}(\log^3 T)\,.$$

Appendix B. On the Characteristic Polynomial of the CUE

The purpose of this appendix is to show that

$$\lim_{N\to\infty} \frac{1}{N^{M^2}} \cdot f_{\mathrm{CUE}}\left(N; e^{2\pi i \mu_1/N}, \ldots, e^{2\pi i \mu_M/N}, e^{2\pi i \nu_1/N}, \ldots, e^{2\pi i \nu_M/N}\right)$$
$$= \frac{\exp(\sum_{j=1}^M \pi i(\mu_j - \nu_j))}{\Delta(2\pi\mu_1, \ldots, 2\pi\mu_M) \cdot \Delta(2\pi\nu_1, \ldots, 2\pi\nu_M)} \cdot \det\left(\frac{\sin\pi(\mu_j - \nu_k)}{\pi(\mu_j - \nu_k)}\right) \quad \text{(B.1)}$$

and

$$\lim_{N\to\infty} \frac{1}{N^{M^2}} \cdot f_{\mathrm{CUE}}\left(N; e^{2\pi i \mu_1/N}, \ldots, e^{2\pi i \mu_M/N}, e^{2\pi i \nu_1/N}, \ldots, e^{2\pi i \nu_M/N}\right)$$
$$= \exp(\tfrac{1}{2}\sum_{j=1}^M (\xi_j - \xi_{M+j})) \cdot \sum_{\sigma \in \mathcal{S}'_{2M}} \frac{\exp(\frac{1}{2}\sum_{j=1}^M (\xi_{\sigma(j)} - \xi_{\sigma(M+j)}))}{\prod_{j,k=1,\ldots,M}(\xi_{\sigma(j)} - \xi_{\sigma(M+k)})}\,, \quad \text{(B.2)}$$

where $\Delta(x_1, \ldots, x_M) := \prod_{j<k}(x_k - x_j)$ denotes the Vandermonde determinant, $\mathcal{S}'_{2M}$ denotes the subset of permutations σ of the set $\{1, \ldots, 2M\}$ satisfying $\sigma(1) < \cdots < \sigma(M)$ and $\sigma(M+1) < \cdots < \sigma(2M)$, $\xi_j := 2\pi i \mu_j$ for $j = 1, \ldots, M$, and $\xi_{M+j} := 2\pi i \nu_j$ for $j = 1, \ldots, M$. In particular, by combining (B.1) and (B.2), it follows that

$$\sum_{\sigma \in \mathcal{S}'_{2M}} \frac{\exp(\frac{1}{2}\sum_{j=1}^M (\xi_{\sigma(j)} - \xi_{\sigma(M+j)}))}{\prod_{j,k=1,\ldots,M}(\xi_{\sigma(j)} - \xi_{\sigma(M+k)})}$$
$$= \frac{1}{\Delta(2\pi\mu_1, \ldots, 2\pi\mu_M) \cdot \Delta(2\pi\nu_1, \ldots, 2\pi\nu_M)} \cdot \det\left(\frac{\sin\pi(\mu_j - \nu_k)}{\pi(\mu_j - \nu_k)}\right)\,, \quad \text{(B.3)}$$

which was used at the end of Section 3.

The proofs of (B.1) and (B.2) use well-known arguments from random matrix theory, and are included here mainly for the sake of completeness.

To prove (B.1), we use an argument from Section 4.1 in Forrester [Fo]. Recall that the correlation function of order $2M$ of the characteristic polynomial of a random matrix from the Circular Unitary Ensemble is defined by

$$f(\mu_1,\ldots,\mu_M;\nu_1,\ldots,\nu_M) = \int_{\mathcal{U}_N} \prod_{j=1}^{M} \det(U-\mu_j I)\,\overline{\det(U-\nu_j I)}\ dU\,.$$

It is well-known that the probability measure on the space of eigenvalue angles induced by the CUE is given by

$$Z_N^{-1} \prod_{1\le j<k\le N} \left|e^{i\vartheta_k}-e^{i\vartheta_j}\right|^2\ d\lambda^N(\vartheta_1,\ldots,\vartheta_N)$$

(see Forrester [Fo] or Mehta [Me]), where $Z_N := (2\pi)^N\,N!$ and λ denotes the Lebesgue measure on the interval $[0,2\pi]$. We therefore obtain

$$\begin{aligned}
&f(e^{i\mu_1},\ldots,e^{i\mu_M};e^{i\nu_1},\ldots,e^{i\nu_M})\\
&= Z_N^{-1}\int \prod_{j=1}^{M}\prod_{k=1}^{N}(e^{i\vartheta_k}-e^{i\mu_j})\prod_{j=1}^{M}\prod_{k=1}^{N}\overline{(e^{i\vartheta_k}-e^{i\nu_j})}\\
&\qquad\qquad\cdot\prod_{1\le j<k\le N}\left|e^{i\vartheta_k}-e^{i\vartheta_j}\right|^2\ d\lambda^N(\vartheta_1,\ldots,\vartheta_N)\\
&= \frac{Z_N^{-1}}{C(\mu,\nu)}\int \Delta(e^{i\mu_1},\ldots,e^{i\mu_M},e^{i\vartheta_1},\ldots,e^{i\vartheta_N})\\
&\qquad\qquad\cdot\Delta(e^{-i\nu_1},\ldots,e^{-i\nu_M},e^{-i\vartheta_1},\ldots,e^{-i\vartheta_N})\ d\lambda^N(\vartheta_1,\ldots,\vartheta_N)\\
&= \frac{Z_N^{-1}}{C(\mu,\nu)}\int \det\left(\frac{e^{ik\mu_j}}{e^{ik\vartheta_j}}\right)_{jk}\cdot\det\left(\ e^{-ik\nu_l}\ \middle|\ e^{-ik\vartheta_l}\ \right)_{kl}\ d\lambda^N(\vartheta_1,\ldots,\vartheta_N)\\
&= \frac{Z_N^{-1}}{C(\mu,\nu)}\int \det\left(\begin{array}{c|c} S_{N+M}(\mu_j,\nu_l) & S_{N+M}(\mu_j,\vartheta_l)\\ \hline S_{N+M}(\vartheta_j,\nu_l) & S_{N+M}(\vartheta_j,\vartheta_l)\end{array}\right)_{jl}\ d\lambda^N(\vartheta_1,\ldots,\vartheta_N)\,,
\end{aligned}$$

where $\Delta(x_1,\ldots,x_n) := \prod_{1\le j<k\le n}(x_k-x_j)$ denotes the Vandermonde determinant,

$$C(\mu,\nu) := \Delta(e^{i\mu_1},\ldots,e^{i\mu_M})\ \cdot\ \Delta(e^{-i\nu_1},\ldots,e^{-i\nu_M})\,,$$

and

$$S_n(\mu,\nu) := \sum_{k=0}^{n-1} e^{ik(\mu-\nu)} = \frac{e^{in(\mu-\nu)}-1}{e^{i(\mu-\nu)}-1} = e^{i(n-1)(\mu-\nu)/2}\cdot\frac{\sin(n(\mu-\nu)/2)}{\sin((\mu-\nu)/2)}\,.$$

Carrying out the integration with respect to $\vartheta_N,\ldots,\vartheta_1$ as in the proof of Proposition 4.2 in Forrester [Fo], it follows that

$$f(e^{i\mu_1},\ldots,e^{i\mu_M};e^{i\nu_1},\ldots,e^{i\nu_M}) = \frac{1}{C(\mu,\nu)}\cdot\det\Big(S_{N+M}(\mu_j,\nu_l)\Big)_{jl}\,.$$

Replacing $e^{i\mu_j}$, $e^{i\nu_j}$ with $e^{2\pi i\mu_j/N}$, $e^{2\pi i\nu_j/N}$, multiplying by N^{-M^2} and letting $N\to\infty$, we therefore obtain

$$\begin{aligned}
&\lim_{N\to\infty}\Big(N^{-M^2}f(e^{2\pi i\mu_1/N},\dots,e^{2\pi i\mu_M/N};e^{2\pi i\nu_1/N},\dots,e^{2\pi i\nu_M/N})\Big)\\
&=\lim_{N\to\infty}\frac{\exp(\sum_{j=1}^M \pi i(N+M-1)(\mu_j-\nu_j)/N)}{\Delta(Ne^{2\pi i\mu_1/N},\dots,Ne^{2\pi i\mu_M/N})\,\Delta(Ne^{-2\pi i\nu_1/N},\dots,Ne^{-2\pi i\nu_M/N})}\\
&\qquad\qquad\cdot\det\left(\frac{\sin(\pi(N+M)(\mu_j-\nu_l)/N)}{N\sin(\pi(\mu_j-\nu_l)/N)}\right)\\
&=\frac{\exp(\sum_{j=1}^M \pi i(\mu_j-\nu_j))}{\Delta(2\pi\mu_1,\dots,2\pi\mu_M)\,\Delta(2\pi\nu_1,\dots,2\pi\nu_M)}\cdot\det\left(\frac{\sin\pi(\mu_j-\nu_l)}{\pi(\mu_j-\nu_l)}\right),
\end{aligned}$$

and (B.1) is proved.

To prove (B.2), we use the representation

$$\begin{aligned}
&f(e^{2\pi i\mu_1},\dots,e^{2\pi i\mu_M};e^{2\pi i\nu_1},\dots,e^{2\pi i\nu_M})\\
&=\exp\big(\tfrac12 N\sum_{j=1}^M(\xi_j-\xi_{M+j})\big)\cdot\sum_{\sigma\in\mathcal{S}'_{2M}}\frac{\exp\big(\frac12 N\sum_{j=1}^M(\xi_{\sigma(j)}-\xi_{\sigma(M+j)})\big)}{\prod_{j,k=1,\dots,M}\big(1-e^{\xi_{\sigma(M+k)}-\xi_{\sigma(j)}}\big)},
\end{aligned}$$

where $\mathcal{S}'_{2M}$ and ξ_j are defined as below (B.2). See Equation (2.21) in Conrey, Farmer, Keating, Rubinstein, and Snaith [CFKRS1], but note that we use a slightly different definition of the characteristic polynomial, which explains why some signs have changed.

Replacing $e^{2\pi i\mu_j}$, $e^{2\pi i\nu_j}$ with $e^{2\pi i\mu_j/N}$, $e^{2\pi i\nu_j/N}$, multiplying by N^{-M^2} and letting $N\to\infty$, it follows that

$$\begin{aligned}
&\lim_{N\to\infty}\Big(N^{-M^2}f(e^{2\pi i\mu_1/N},\dots,e^{2\pi i\mu_M/N};e^{2\pi i\nu_1/N},\dots,e^{2\pi i\nu_M/N})\Big)\\
&=\exp\big(\tfrac12\sum_{j=1}^M(\xi_j-\xi_{M+j})\big)\cdot\sum_{\sigma\in\mathcal{S}'_{2M}}\frac{\exp\big(\frac12\sum_{j=1}^M(\xi_{\sigma(j)}-\xi_{\sigma(M+j)})\big)}{\prod_{j,k=1,\dots,M}\lim_{N\to\infty}\big(N\cdot\big(1-e^{(\xi_{\sigma(M+k)}-\xi_{\sigma(j)})/N}\big)\big)}\\
&=\exp\big(\tfrac12\sum_{j=1}^M(\xi_j-\xi_{M+j})\big)\cdot\sum_{\sigma\in\mathcal{S}'_{2M}}\frac{\exp\big(\frac12\sum_{j=1}^M(\xi_{\sigma(j)}-\xi_{\sigma(M+j)})\big)}{\prod_{j,k=1,\dots,M}\big(\xi_{\sigma(j)}-\xi_{\sigma(M+k)}\big)},
\end{aligned}$$

and (B.2) is proved.

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