ULTRASPHERICAL TYPE GENERATING FUNCTIONS FOR ORTHOGONAL POLYNOMIALS

Nizar Demni [1]

Abstract. We characterize the probability distributions of finite all order moments having generating functions for orthogonal polynomials of an ultraspherical type.

1. Motivation: Meixner families

There is a one to one correspondance between probability distributions on the real line and polynomials of a one variable satisfying a three-terms recurrence relation subject to some positive conditions ([9]). That is why in most of the cases, if not all, one tries to characterize probability distributions using generating functions for orthogonal polynomials. Among the famous generating functions are the ones of *exponential type*, that is if μ is a probability distribution with a finite exponential moment in a neighborhood of zero

$$\int_{\mathbb{R}} e^{zx}\mu(dx) \quad < \quad \infty,$$

then

$$\psi(z,x) := \sum_{n\geq 0} P_n(x)z^n = \frac{e^{xH(z)}}{\mathbb{E}(e^{XH(z)})}, \tag{1}$$

where H is analytic around $z=0$ such that $H(0)=0, H'(0)=1$, X is a random variable in some probability space $(\Omega, \mathscr{F}, \mathbb{P})$ with law $\mu = \mathbb{P}\circ X^{-1}$ and $(P_n)_{n\geq 0}$ is the set of orthogonal polynomials with respect to μ. Up to translations and dilations, there are six probability distributions which form the so-called Meixner family referring to its first appearance with J. Meixner ([15]). It consists of Gaussian, Poisson, Gamma, negative binomial, Meixner and binomial distributions. This family appeared many times under differents guises ([17], [14], [1], [16], [11]).
Another well known example was first suggested and studied in [2] and is given by a *Cauchy-Stieltjes type* kernel. Namely, if μ is a probability distribution of finite all order moments, then

$$\psi(z,x) := \sum_{n\geq 0} P_n(x)z^n = \frac{1}{u(z)[f(z)-x]} \tag{2}$$

where u and $z \mapsto zf(z)$ are analytic functions around zero such that

$$\lim_{z\to 0}\frac{u(z)}{z} = \lim_{z\to 0} zf(z) = 1.$$

Date: September 2, 2008.

[1] supported by CRC 701, Faculty of Mathematics, Bielefeld University, Germany. email: demni@math.uni-bielefeld.de

This family, known as the free Meixner family due to its intimate relation to free probability theory, covers six compactly-supported probability measures too. We refer the reader to [4], [8], [13], [5] for more characterizations and more interpretations. The natural q-deformation that interpolates the forementioned families for arbitrary $|q| \leq 1$ was defined and studied in [3] and is up to affine transformations the so-called Al-Salam and Chihara family of orthogonal polynomials ([1]). Their generating functions is given by an infinite product and is similar to the q-exponential function. Another characterization of the last family was recently given in [7].

After this sketchy overview, we suggest another type of generating functions which may be viewed as a generalization of the free Meixner family. It is inspired from the case of Gegenbauer or ultraspherical polynomials which satisfy ([9])

$$\sum_{n\geq 0} 2^n \frac{(\lambda)_n}{n!} C_n^{\lambda}(x) z^n = \frac{1}{(1-2zx+z^2)^{\lambda}}, \quad \lambda > 0. \tag{3}$$

We adapted here the monic normalization for $(C_n^{\lambda})_n$ and henceforth all the polynomials are monic so that they satisfy the normalized recurrence relation

$$xP_n(x) = P_{n+1}(x) + \alpha_n P_n(x) + \omega_n P_{n-1}(x),\, n \geq 0, P_{-1} := 0, \omega_0 = 1.$$

The sequences $(\alpha_n)_{n\geq 0}, (\omega_n)_{n\geq 0}$ are known as the Jacobi-Szegö parameters and $\omega_n > 0$ for all n unless μ is has finite support ([9]).

It is then natural to adress the problem of characterizing probability measures with finite all-order moments, say μ_λ, such that

$$\psi_\lambda(z,x) := \sum_{n\geq 0} a_n^{\lambda} P_n^{\lambda}(x) z^n = \frac{1}{u_\lambda(z)(f_\lambda(z)-x)^{\lambda}}, \quad \lambda > 0 \tag{4}$$

for some sequence $(a_n^{\lambda})_n$ and some functions u_λ, f_λ such that u_λ and $z \mapsto zf_\lambda(z)$ are analytic in a neighborhood of zero (possibly depending on λ with

$$u_\lambda(0) = 0, \quad \lim_{z\to 0} zf_\lambda(z) = 1, \quad \lim_{z\to 0} \frac{u_\lambda(z)}{z^{\lambda}} = 1.$$

We shall say that ψ_λ is an OP generating function of ultraspherical-type referring to ultraspherical polynomials. Without loss of generality, we may assume that μ_λ is standard, that is, has mean zero and variance one. Equivalently, if $(\alpha_n^{\lambda})_{n\geq 0}, (\omega_n^{\lambda})_{n\geq 0}$ denote the Jacobi-Szegö parameters of μ_λ, then one has $\alpha_0^{\lambda} = 0, \omega_1^{\lambda} = 1$. Our strategy is based on the following general remark that was stated in [6]:

Claim: to a given generating function of orthogonal polynomials $(z,x) \mapsto \psi(z,x)$ associated with a probability measure satisfying a moments condition, there corresponds a one parameter family of probability measures such that the variance is a polynomial of degree 2 in the mean. Namely, if ψ is of the form

$$\psi(z,x) = \sum_{n\geq 0} a_n P_n(x) z^n$$

such that

$$\sum_{n\geq 0} |a_n m_n z^n| < \infty, \quad m_n := \int x^n \mu(dx)$$

in a neighborhood of zero, then

$$\mathbb{P}_z(dx) := \psi(z,x)\mu(dx)$$

defines a one parameter family of probability measures such that the first and the second moments of $\mathbb{P}_z$ are at most linear and at most quadratic polynomials in z. Indeed, this follows immediately from exchanging the order of integration and one uses the recurrence relation once and twice when computing the first and the second moments respectively. The family $\mathbb{P}_z$ is then referred to as a ψ-family of an at most quadratic variance referring to exponential and Cauchy-Stieltjes families ([16], [8]).

When ψ is handable so that one can perform computations of the first and of the second moments of $\mathbb{P}_z$, one recovers two equations that may be used to solve the problem of characterization of probability measures whose generating function for orthogonal polynomials is of ψ-type. In the case of the Meixner and the free Meixner families, this was noticed in [6]. In the present case, both equations allow to derive a nonlinear one order differential equation for f_λ and our main result is

Proposition 1.1.

(1) *The function f_λ satisfies in some neighborhood of zero*
$$Q_2(z)f'_\lambda(z) = f^2_\lambda(z) - Q_1(z)f_\lambda(z) + R_1(z)$$
where Q_2, R_1 are polynomials of degree 2 while Q_1 is a polynomial of degree 1. Moreover the coefficients of these polynomials depend only on $\lambda, \alpha_1^\lambda, \omega_2^\lambda$.

(2) *The function u_λ is related to f_λ by*
$$\frac{u'_\lambda(z)}{u_\lambda(z)} = \lambda\frac{1 - f'_\lambda(z)}{f_\lambda(z) - \lambda z}.$$

Once we did, we investigate the case of symmetric measures and we show that if
$$f_\lambda(z) := \frac{P_\lambda(z)}{z} \tag{5}$$
where P_λ is a polynomial of degree 2, then there exists only two families of probability measures for which
$$\text{either} \quad \omega_2^\lambda = \frac{2\lambda+1}{\lambda+2} \quad \text{or} \quad \omega_2^\lambda = \frac{2\lambda-1}{\lambda+1}, \lambda > 1/2.$$
A similar result holds for non symmetric measures with f_λ as above and there are two families corresponding to
$$\omega_2^\lambda = \frac{4\lambda^3}{2\lambda^3 + 3\lambda^2 - 1}, \alpha_1^\lambda = \pm\sqrt{\frac{2}{(\lambda+1)^2(\lambda - 1/2)}},$$
however the range of λ is restricted to $\lambda > 1/2, \lambda \neq 1$. In both cases, the orthogonal polynomials may be expressed through the ultraspherical polynomials. The reader may wonder about the fact that P_λ is a polynomial of degree 2. in fact, we show that if f_λ is given by (5) where P_λ is a polynomial, then the degree of P_λ can not exceed 2. Unfortunately, we do not know if there exists a solution f_λ with P_λ an entire function. Nevertheless, we conjecture that this alternative is not possible.

2. Proofs

2.1. **First and second moments.** First of all, it is easily seen from (4) together with the limiting conditions on u_λ and f_λ that
$$a_n = \frac{(\lambda)_n}{n!}.$$

In fact, it suffices to substitute x with x/z in (4) then let z go to zero and use the binomial theorem

$$\lim_{z\to 0}\psi(z, x/z) = \frac{1}{(1-x)^{\lambda}} = \sum_{n\geq 0}\frac{(\lambda)_n}{n!}x^n.$$

Second, let us proceed to the compuations of the first and the second moments. On the one hand, the three-terms recurrence relation (applied twice for the second moment), one gets

$$m_1^{\lambda}(z) = \lambda z, \quad m_2^{\lambda}(z) = \frac{\lambda(\lambda+1)}{2}\omega_2^{\lambda}z^2 + \lambda\alpha_1^{\lambda}z + 1.$$

On the other hand, the explicit form of $\psi_{\lambda}(z,x)$ enables to derive the following: first since the expectation of $\psi_{\lambda}(z,\cdot)$ equals one then

$$u_{\lambda}(z) = \int_{\mathbb{R}}\frac{1}{(f_{\lambda}(z)-x)^{\lambda}}\mu_{\lambda}(dx)$$

Second, using the elematary operations $x = (x - f(z)) + f(z)$ and $x^2 = x(x - f(z)) + xf(z)$, it follows then

$$m_1^{\lambda}(z) = f(z) - \frac{u_{\lambda,1}(z)}{u_{\lambda}(z)}, \quad u_{\lambda,1}(z) := \int_{\mathbb{R}}\frac{1}{(f(z)-x)^{\lambda-1}}\mu_{\lambda}(dx).$$

Using the relation $(1-\lambda)f'(z)u_{\lambda}(z) = (u_{\lambda,1})'(z)$ ina neighborhood of 0, one gets:

$$\frac{u_{\lambda}'(z)}{u_{\lambda}(z)} = \lambda\frac{1-f_{\lambda}'(z)}{f_{\lambda}(z)-\lambda z} \tag{6}$$

which can be written as

$$(u_{\lambda}(z)[f_{\lambda}(z)-\lambda z])' = (1-\lambda)u_{\lambda}(z)f_{\lambda}'(z). \tag{7}$$

Finally,

$$m_2^{\lambda}(z) = \lambda z f_{\lambda}(z) - \frac{1}{u_{\lambda}(z)}\int_{\mathbb{R}}\frac{x}{(f_{\lambda}(z)-x)^{\lambda-1}}\mu_{\lambda}(dx). \tag{8}$$

Note that

$$\left(\int_{\mathbb{R}}\frac{x}{(f_{\lambda}(z)-x)^{\lambda-1}}\mu_{\lambda}(dx)\right)' = (1-\lambda)f_{\lambda}'(z)\int_{\mathbb{R}}\frac{x}{(f_{\lambda}(z)-x)^{\lambda}}\mu_{\lambda}(dx) = \lambda(1-\lambda)zu_{\lambda}(z)f_{\lambda}'(z)$$

so that (8) implies that

$$\big([\lambda z f_{\lambda}(z) - m_2^{\lambda}(z)]u_{\lambda}(z)\big)' = \lambda(1-\lambda)zu_{\lambda}(z)f_{\lambda}'(z). \tag{9}$$

2.2. **A non linear differential equation.** By the virtue of (7), (9) implies that

$$\big([\lambda z f_{\lambda}(z) - m_2^{\lambda}(z)]u_{\lambda}(z)\big)' = \lambda z(u_{\lambda}(z)[f_{\lambda}(z)-\lambda z])'$$

which gives

$$\begin{aligned}&[\lambda z f_{\lambda}(z) - m_2^{\lambda}(z)]u_{\lambda}'(z) + [\lambda f_{\lambda}(z) + \lambda z f_{\lambda}'(z) - (m_2^{\lambda})'(z)]u_{\lambda}(z)\\ &= \lambda z[f_{\lambda}(z)-\lambda z]u_{\lambda}'(z) + \lambda z[f_{\lambda}'(z)-\lambda)]u_{\lambda}(z),\end{aligned}$$

therefore

$$[\lambda^2 z^2 - m_2^{\lambda}(z)]u_{\lambda}'(z) = [(m_2^{\lambda})'(z) - \lambda f_{\lambda}(z) - \lambda^2 z]u_{\lambda}(z).$$

If $\lambda z - m_2^{\lambda}(z) \neq 0$, one gets after the comparison of the last equality to (6)

$$\frac{(m_2^{\lambda})'(z) - \lambda f_{\lambda}(z) - \lambda^2 z}{\lambda^2 z^2 - m_2^{\lambda}(z)} = \lambda\frac{1-f_{\lambda}'(z)}{f_{\lambda}(z)-\lambda z}$$

which shows after elemantary computations that f_λ satisfies the following nonlinear one order differential equation:

$$Q_2(z)f'_\lambda(z) = f^2_\lambda(z) - Q_1(z)f_\lambda(z) + R_1(z) \tag{10}$$

where

$$\begin{aligned}
Q_2(z) &= \lambda\left[\lambda - \frac{\lambda+1}{2}\omega_2^\lambda\right]z^2 - \lambda\alpha_1^\lambda z - 1,\\
Q_1(z) &= (\lambda+1)\omega_2^\lambda z + \alpha_1^\lambda,\\
R_1(z) &= \frac{\lambda(\lambda+1)}{2}\omega_2^\lambda z^2 - 1.
\end{aligned}$$

Setting $g_\lambda(z) := f_\lambda(z) - [Q_1(z)/2]$, (10) transforms to

$$Q_2(z)g'_\lambda(z) = g^2_\lambda(z) + \tilde{Q}_2(z) \tag{11}$$

where

$$\begin{aligned}
\tilde{Q}_2(z) &= R_1(z) - \frac{1}{4}[Q_1(z)]^2 - \frac{\lambda+1}{2}\omega_2^\lambda Q_2(z)\\
&= [(\lambda+1)\omega_2^\lambda - 2\lambda]\frac{\lambda^2-1}{4}\omega_2^\lambda z^2 + \frac{\lambda^2-1}{2}\alpha_1^\lambda\omega_2^\lambda z + \frac{(\lambda+1)\omega_2^\lambda}{2} - 1 - \frac{(\alpha_1^\lambda)^2}{4}.
\end{aligned}$$

Finally, once g_λ is given, one deduces f_λ by adding $Q_1/2$ then use (6) to derive u_λ. Now, we shall look for a solution of the form

$$g_\lambda(z) := \frac{E_\lambda(z)}{z}, \quad E_\lambda(0) = 1$$

for a second degree polynomial P_λ. In fact, since $z \mapsto zg_\lambda(z)$ is analytic around zero, one may always assume that $g_\lambda(z)$ has the above form for an entire function E_λ. But if E_λ is a polynomial of degree ≥ 3, then all the terms of degree ≥ 3 will vanish only by equating both sides of (11). For instance, let

$$E_\lambda(z) = a_0z^3 + a_1z^2 + a_2z + a_3$$

and write (11) as

$$Q_2(z)[zE'_\lambda(z) - E_\lambda(z)] - E^2_\lambda(z) = z^2\tilde{Q}_2(z). \tag{12}$$

Then by equating terms of degree 6 is this equation, one easily gets $a_0 = 0$ so that P has degree 2. For P a polynomial of degree 4, start with equating terms of degree 4 and so on. However, this way of thinking fails or rather become cumbersome when P is an entire function.

Remark 2.1. *Note that $\tilde{Q}_2$ is a constant polynomial for $\lambda = 1$ so that* (11) *transforms for this parameter value to*

$$[(1-\omega_2^\lambda)z^2 - \alpha_1^\lambda z - 1]g'_\lambda(z) = g^2(z) + (\omega_2^\lambda - 1) - (\alpha_1^\lambda)^2/4.$$

But it is well known that if μ belongs to the free Meixner family, than its Jacobi-Szegö parameters are given by ([4])

$$\alpha_n^1 = a, a \in \mathbb{R}, n \geq 1, \quad \omega_n^1 = (1+b), b \geq -1, n \geq 2,$$

where we used the fact that μ has mean zero ($\alpha_0^1 = 0$) and variance one ($\omega_1^1 = 1$). Moreover, one has ([5])

$$f_1(z) = \frac{1 + az + (1+b)z^2}{z} \quad \Rightarrow \quad g_1(z) = \frac{(a/2)z + 1}{z} = \frac{a}{2} + \frac{1}{z}.$$

It is then an easy exercice to check that g *satisfies* (11) *which reads in this case*

$$-[bz^2 + az + 1]g_1'(z) = g_1^2(z) + b - a^2/4. \tag{13}$$

3. Symmetric measures: Gegenbauer polynomials

In the sequel, we shall focus on the case $\alpha_n^\lambda = 0$ for all n. This is equivalent to the fact that μ_λ is symmetric, that is the image of μ_λ by the map $x \mapsto -x$ is still μ_λ. In this case, one gets by taking $\alpha_1^\lambda = 0$

$$\begin{aligned} Q_2(z) &= \frac{\lambda}{2}[2\lambda - (\lambda+1)\omega_2^\lambda]z^2 - 1 \\ \tilde{Q}_2(z) &= [(\lambda+1)\omega_2^\lambda - 2\lambda]\frac{\lambda^2-1}{4}\omega_2^\lambda z^2 + \frac{(\lambda+1)\omega_2^\lambda}{2} - 1. \end{aligned}$$

Writing $P_\lambda(z) = a_0 z^2 + a_1 z + a_2$ and equating both sides in (11), one gets:

$$\begin{aligned} a_2 &= 1, \\ a_1 &= 0, \\ -3a_0 - \frac{\lambda}{2}[2\lambda - (\lambda+1)\omega_2^\lambda] &= \frac{(\lambda+1)\omega_2^\lambda}{2} - 1, \\ -a_0^2 + a_0\frac{\lambda}{2}[2\lambda - (\lambda+1)\omega_2^\lambda] &= [(\lambda+1)\omega_2^\lambda - 2\lambda]\frac{\lambda^2-1}{4}\omega_2^\lambda. \end{aligned}$$

The third equation gives

$$a_0 = \frac{(1-\lambda^2)(2-\omega_2^\lambda)}{6}.$$

Hence, it remains to check when the above a_0 satisfies the fourth equation. For $\lambda = 1$, this gives $a = 0 = 0$ and one has a unique solution. For $\lambda \neq 1$ and after substituting a_0 in the fourth equation, one can remove the term $(1-\lambda^2)$ and see that ω_2^λ satisfies

$$-(\lambda+1)(\lambda+2)(\omega_2^\lambda)^2 + (4\lambda^2 + 6\lambda - 1)\omega_2^\lambda + (1 - 4\lambda^2) = 0.$$

What is quite interesting, that though this polynomial looks complicated, its descriminant is equal 9 so that there are two solutions given by

$$\omega_2^\lambda = \frac{2\lambda+1}{\lambda+2} \qquad \omega_2^\lambda = \frac{2\lambda-1}{\lambda+1}, \lambda > 1/2.$$

This gives the values

$$a_0 = \frac{1-\lambda^2}{2(\lambda+2)}, \qquad a_0 = \frac{1-\lambda^2}{2(\lambda+1)} = \frac{1-\lambda}{2}, \lambda > 1/2.$$

Thus

$$f_\lambda(z) = \frac{1+\lambda}{2}z + \frac{1}{z}, \quad f_\lambda(z) = \frac{\lambda}{2}z + \frac{1}{z}, \lambda > 1/2$$

and from (6)

$$\frac{u_\lambda'(z)}{u_\lambda(z)} = \frac{\lambda}{z}, \quad \frac{u_\lambda'(z)}{u_\lambda(z)} = \lambda\frac{z^2 + 1 - (\lambda/2)z^2}{z(1-(\lambda/2)z^2)}, \lambda > 1/2.$$

Finally

$$u_\lambda(z) = z^\lambda, \quad u_\lambda(z) = \frac{z^\lambda}{1-(\lambda/2)z^2}, \lambda > 1/2$$

for sufficiently small z.

3.1. **Orthogonal polynomials.** The first value of ω_2^λ corresponds to the Gegenbauer polynomials as we shall see and the Jacobi-Szegö parameters are well known ([9]). However in order to fit into our setting, one has to consider the monic Gegenbauer polynomials which are orthogonal with respect to the standard ultraspherical distribution

$$(1 - x^2/[2(1+\lambda)])^{\lambda-1/2}dx, \quad x \in [\pm\sqrt{2(1+\lambda)}].$$

Easy computations show that these polynomials are given by

$$\tilde{C_n^\lambda}(x) := [2(1+\lambda)]^{n/2} C_n^\lambda\left(\frac{x}{\sqrt{2(1+\lambda)}}\right) \tag{14}$$

and that their Jacobi-Szegö parameters are given by (see [9] for the Jacobi-Szegö parameters of C_n^λ):

$$\alpha_n^\lambda = 0, n \geq 0, \quad \omega_n^\lambda = \frac{n(\lambda+1)(n+2\lambda-1)}{2(n+\lambda-1)(n+\lambda)}, n \geq 1.$$

Hence, on the one hand, it follows that

$$\omega_2^\lambda = \frac{2\lambda+1}{\lambda+2}.$$

On the other hand, it is easy to see from (3) that the generating function for the polynomials in (14) is given by

$$\begin{aligned}\sum_{n\geq 0} \frac{(\lambda)_n}{n!}\tilde{C_n^\lambda}(x) z^n &= \sum_{n\geq 0} 2^n \frac{(\lambda)_n}{n!} C_n^\lambda\left(\frac{x}{\sqrt{2(1+\lambda)}}\right)\left(\frac{\sqrt{1+\lambda}z}{\sqrt{2}}\right)^n \\ &= \frac{1}{(1 - zx + (1+\lambda)z^2/2)^\lambda} \\ &= z^{-\lambda}\left[\frac{1 + (1+\lambda)z^2/2}{z} - x\right]^{-\lambda} = \frac{1}{u_\lambda(z)(f_\lambda(z) - x)^\lambda}.\end{aligned}$$

For the second value, let us first write down ψ_λ:

$$\psi_\lambda(z,x) = \frac{1 - (\lambda/2)z^2}{z^\lambda(\lambda z/2 + 1/z - x)^\lambda} = \frac{1 - (\lambda/2)z^2}{(\lambda z^2/2 + 1 - zx)^\lambda}$$

then use the above generating function for $\tilde{C_n^\lambda}$ to expand

$$\frac{1}{(\lambda z^2/2 + 1 - zx)^\lambda} = \sum_{n\geq 0} \frac{(\lambda)_n}{n!}\tilde{C_n^\lambda}\left(\sqrt{\frac{\lambda+1}{\lambda}}x\right)\left(\sqrt{\frac{\lambda}{1+\lambda}}z\right)^n.$$

Set

$$B_n^\lambda(x) := \left(\sqrt{\frac{\lambda}{1+\lambda}}\right)^n \tilde{C_n^\lambda}\left(\sqrt{\frac{\lambda+1}{\lambda}}x\right)\left(\sqrt{\frac{\lambda}{1+\lambda}}\right)^n = (\sqrt{2\lambda})^n C_n^\lambda\left(\frac{x}{\sqrt{2\lambda}}\right),$$

then B_n^λ are monic and one easily sees that the family of orthogonal polynomials corresponding to the second value of ω_2^λ are given by

$$P_n^\lambda(x) = B_n^\lambda(x) - \frac{\lambda}{2}\frac{n(n-1)}{(\lambda+n-1)(\lambda+n-2)}B_{n-2}^\lambda(x), n \geq 2$$

with $S_0^\lambda = 1$ and $S_1^\lambda(x) = B_1^\lambda(x) = x$.

Remark 3.1. *Both above families of symmetric measures extend to the case of free Meixner family. In fact, when $\lambda = 1$, the first family reduces to the Tchebycheff polynomials of the second kind while the second family reduces to*

$$P_n^1(x) := B_n^1(x) - \frac{\lambda}{2}B_{n-2}^1(x) = (\sqrt{2})^n U_n\left(\frac{x}{\sqrt{2}}\right) - \frac{1}{2}(\sqrt{2})^{n-2}U_{n-2}\left(\frac{x}{\sqrt{2}}\right)$$

for $n \geq 2$, where U_n is the n-th Tchebycheff polynomial with second kind. The latter polynomials are orthogonal with respect to the Kesten measure (the latter is a particular case of the free binomial measure derived in [4], *see also* [10], [12] *).*

4. NON-SYMMETRIC PROBABILITY MEASURES

Henceforth, we suppose that $\alpha_1^\lambda \neq 0$ and we will show that there are two families of probability measures correponding to

$$g_\lambda(z) = \frac{a_0 z^2 + a_1 z + a_2}{z}.$$

Then, we get the following equations

$$\begin{aligned}
a_2 &= 1, \\
a_1 &= \frac{\lambda\alpha_1^\lambda}{2} \neq 0, \\
-3a_0 - \frac{\lambda}{2}[2\lambda - (\lambda+1)\omega_2^\lambda] - a_1^2 &= \frac{(\lambda+1)\omega_2^\lambda}{2} - 1 - \frac{(\alpha_1^\lambda)^2}{4}, \\
-a_0\alpha_1\lambda - 2a_0a_1 &= \frac{\lambda^2-1}{2}\alpha_1^\lambda\omega_2^\lambda, \\
-a_0^2 + a_0\frac{\lambda}{2}[2\lambda - (\lambda+1)\omega_2^\lambda] &= [(\lambda+1)\omega_2^\lambda - 2\lambda]\frac{\lambda^2-1}{4}\omega_2^\lambda.
\end{aligned}$$

From the second, third and fourth equations, it follows that

$$a_0 = \frac{1-\lambda^2}{6}\left[\frac{(\alpha_1^\lambda)^2}{2} + 2 - \omega_2^\lambda\right] = \frac{1-\lambda^2}{4\lambda}\omega_2^\lambda.$$

Actually, this gives a constraint on $\lambda, \alpha_1^\lambda, \omega_2^\lambda$:

$$\left(\frac{(\alpha_1^\lambda)^2}{2} + 2\right)\lambda = \left(\lambda + \frac{3}{2}\right)\omega_2^\lambda. \tag{15}$$

Substituting a_0 by $(1-\lambda^2)\omega_2^\lambda/(4\lambda)$ and assuming $\lambda \neq 1$, the fifth equation becomes

$$-\frac{1-\lambda^2}{16\lambda^2}(\omega_2^\lambda)^2 + \frac{\omega_2^\lambda}{8}[2\lambda - (\lambda+1)\omega_2^\lambda] = [2\lambda - (\lambda+1)\omega_2^\lambda]\frac{\omega_2^\lambda}{4}.$$

In the non degenerate case $\omega_2^\lambda \neq 0$,

$$\omega_2^\lambda = \frac{4\lambda^3}{2\lambda^3 + 3\lambda^2 - 1}.$$

But -1 is a double root of the polynomial in the denominator so that

$$\omega_2^\lambda = \frac{2\lambda^3}{(\lambda+1)^2(\lambda-1/2)},$$

which is positive for $\lambda > 1/2$. Finally, one deduces from (15) that

$$\begin{aligned}
(\alpha_1^\lambda)^2 &= 2\left[\frac{(2\lambda+3)\lambda^2}{(\lambda+1)^2(\lambda-1/2)} - 2\right] = \frac{2}{(\lambda+1)^2(\lambda-1/2)} > 0, \\
a_0 &= = \frac{(1-\lambda^2)\lambda^2}{2(\lambda+1)^2(\lambda-1/2)} = \frac{(1-\lambda)\lambda^2}{(\lambda+1)(2\lambda-1)}.
\end{aligned}$$

It follows that

$$f_\lambda(z) = \frac{a_0 z^2 + a_1 z + a_2}{z} + \frac{(1+\lambda)\omega_2^\lambda z + \alpha_1^\lambda}{2} = \frac{1}{z}\left[\frac{\lambda^2}{2\lambda-1}z^2 \pm \frac{1}{\sqrt{2\lambda-1}}z + 1\right]$$

and

$$\frac{u'_\lambda(z)}{u_\lambda(z)} = \frac{\lambda}{z}\left[1 - \frac{(\lambda-1)^2}{2\lambda-1}z^2\right]\left[\frac{\lambda(1-\lambda)}{2\lambda-1}z^2 \pm \pm\frac{1}{\sqrt{2\lambda-1}}z + 1\right]^{-1}.$$

The descriminant of the polynomial

$$\frac{\lambda(\lambda-1)}{2\lambda-1}z^2 \pm \pm\frac{1}{\sqrt{2\lambda-1}}z + 1$$

is easily seen to be:

$$\frac{1}{2\lambda-1} - \frac{4\lambda(1-\lambda)}{2\lambda-1} = 2\lambda - 1 > 0.$$

It follows that, when $\alpha_1^\lambda > 0$, the roots are given by

$$z_1 = -\frac{\sqrt{2\lambda-1}}{\lambda}, \quad z_2 = -\frac{\sqrt{2\lambda-1}}{1-\lambda}.$$

Writing

$$1 - \frac{(\lambda-1)^2}{2\lambda-1}z^2 = -\frac{(\lambda-1)^2}{2\lambda-1}\left[z + \frac{\sqrt{2\lambda-1}}{1-\lambda}\right]\left[z - \frac{\sqrt{2\lambda-1}}{1-\lambda}\right],$$

one gets

$$\frac{u'_\lambda(z)}{u_\lambda(z)} = \frac{\lambda-1}{z}\left[z + \frac{\sqrt{2\lambda-1}}{\lambda-1}\right]\left[z + \frac{\sqrt{2\lambda-1}}{\lambda}\right]^{-1} = \frac{\lambda}{z} - \frac{1}{z + \sqrt{2\lambda-1}/\lambda}.$$

As a result

$$u_\lambda(z) = \frac{\sqrt{2\lambda-1}}{\lambda}\frac{z^\lambda}{z + \sqrt{2\lambda-1}/\lambda}$$

and the generating function is written as

$$\psi_\lambda(z,x) = \frac{\lambda}{\sqrt{2\lambda-1}}\left[z + \frac{\sqrt{2\lambda-1}}{\lambda}\right]\left[1 - z\left(x - \frac{1}{\sqrt{2\lambda-1}}\right) + \frac{\lambda^2}{2\lambda-1}z^2\right]^{-\lambda}.$$

Using the generating function for the monic Gegenbauer polynomials $\tilde{C_n^\lambda}$, one gets

$$\begin{aligned}
\left[1 - z\left(x - \frac{1}{\sqrt{2\lambda-1}}\right) + \frac{\lambda^2}{2\lambda-1}z^2\right]^{-\lambda} &= \left[1 - z\left(x - \frac{1}{\sqrt{2\lambda-1}}\right) + \frac{\lambda^2}{(\lambda+1)(\lambda-1/2)}\frac{1+\lambda}{2}z^2\right]^{-\lambda}. \\
&= \sum_{n\geq 0}\frac{(\lambda)_n}{n!}K_n^\lambda(x)z^n
\end{aligned}$$

where K_n^λ is the shifted Gegenbauer polynomial given by

$$K_n^\lambda(x) := \left[\frac{\lambda}{\sqrt{(\lambda+1)(\lambda-1/2)}}\right]^n \tilde{C_n^\lambda}\left[\frac{\sqrt{(\lambda+1)(\lambda-1/2)}}{\lambda}\left(x - \frac{1}{\sqrt{2\lambda-1}}\right)\right].$$

As a result,

$$P_n^\lambda(x) = K_n^\lambda(x) + \frac{\lambda}{\sqrt{2\lambda-1}}\frac{n}{n+\lambda-1}K_{n-1}^\lambda(x),\, n \geq 1,$$

with $P_0^\lambda = 1$. In the case $\alpha_1^\lambda < 0$, similar computations yield

$$u_\lambda(z) = -\frac{\sqrt{2\lambda-1}}{\lambda}\frac{z^\lambda}{z - \sqrt{2\lambda-1}/\lambda}$$

and

$$\psi_\lambda(z,x) = -\frac{\lambda}{\sqrt{2\lambda-1}}\left[z - \frac{\sqrt{2\lambda-1}}{\lambda}\right]\left[1 - z\left(x + \frac{1}{\sqrt{2\lambda-1}}\right) + \frac{\lambda^2}{2\lambda-1}z^2\right]^{-\lambda}.$$

Similar computations as above yield

$$P_n^\lambda(x) = S_n^\lambda(x) - \frac{\lambda}{\sqrt{2\lambda-1}}\frac{n}{n+\lambda-1}S_{n-1}^\lambda(x),\, n \geq 1,$$

with $P_0^\lambda = 1$, where

$$S_n^\lambda(x) := \left[\frac{\lambda}{\sqrt{(\lambda+1)(\lambda-1/2)}}\right]^n \tilde{C_n^\lambda}\left[\frac{\sqrt{(\lambda+1)(\lambda-1/2)}}{\lambda}\left(x + \frac{1}{\sqrt{2\lambda-1}}\right)\right].$$

Remark 4.1. *Unlike the case of symmetric case, the results derived in the non symmetric case do not extend to the parameter value $\lambda = 1$. In fact, there are more than two families of non symmetric probability distributions with Cauchy-Stieltjes type generating functions* ([4]).

Acknowledgments: the author thanks Professor Marek Bozejko for stimulating discussions in the Wroclaw Institute and Patrycja Polińska for her help in checking tedious computations.

References

[1] *W. Al-Salam, T. S. Chihara.* Convolutions of orthogonal polynomials. *SIAM J. Funct. Anal.* **7**, 1976, 16-28.

[2] *M. Anshelevich.* Free martingale polynomials. *J. Funct. Anal.* **201**, 2003, 228-261.

[3] *M. Anshelevich.* Appell polynomials and their relatives. *Int. Math. Res. Not.* **65**, 2004, 3469–3531.

[4] *M Bozejko, W. Bryc.* On a class of free Lévy laws related to a regression problem. *J. Funct. Anal.* **236**, no. 1, 2006, 59-77.

[5] *M. Bozejko, N. Demni.* Generating functions of Cauchy-Stieltjes type for orthogonal polynomials. *Submitted to Infinite. Dimen. Anal. Quantum Probab. Rel. Top.*

[6] *M. Bozejko, N. Demni.* Topics on Meixner families. *Submitted to The Proceedings of the 11-th Workshop on Noncommutative Harmonic Analysis with Applications to Probability. Poland, 2008.*

[7] *W. Bryc, M. E. H. Ismail.* Approximation operators, q-exponential and free exponential families. *Available on arXiv.*

[8] *W. Bryc, M. E. H. Ismail.* Cauchy-Stieltjes kernel families. *Available on arXiv.*

[9] *M. E. H. Ismail.* Classical and Quantum orthogonal polynomials in one variable. *Cambridge University Press.* 2005.

[10] *H. Kesten.* Symmetric random walks on groups. *Trans. Amer. Math. Soc.* **92**, 1959, 336-354.

[11] *I. Kubo.* Generating functions of exponential-type for orthogonal polynomials. *Inf. Dimens. Anal, Quantum Probab. Related Topics.* **7**, no 1, 2004, 155-159.

[12] *I. Kubo, H. H. Kuo, S. Namli.* Interpolation of Chebyshev polynomials and interacting Fock spaces. *Infin. Dimens. Anal. Quantum Probab. Relat. Top. 9.* **3**. 2006, 361-371.

[13] *I. Kubo, H. H. Kuo, S. Namli.* The characterization of a class of probability measures by multiplicative renormalization. *Communications on Stochastic Analysis.* **1**, no. 3. 2007, 455–472.

[14] *R. G. Laha, E. Lukacs.* On a problem connected with quadratic regression. *Biometrika.* **47**, 1960, 335-343.

[15] *J. Meixner.* Orthogonale polynomsysteme mit einer besonderern der erzeugenden funktion. *J. London. Math. Soc.* **9**, 1934, 6-13.

[16] *C. N. Morris.* Natural exponential families with quadratic variance function. *Ann. Stat.* **10**, no. 1. 1982, 65-80.

[17] *I. M. Sheffer.* Some properties of polynomial set of type zero. *Duke Math. J.* **5**, 1939, 590-622.