RADIAL DUNKL PROCESSES ASSOCIATED WITH DIHEDRAL SYSTEMS

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Abstract. We interest in radial Dunkl processes associated with dihedral systems. We write down the semi group density and as a by-product the generalized Bessel function and the Dunkl-Hermite polynomials. Then, a skew product decomposition, involving only independent Bessel processes, is given and the tail distribution of the first hitting time of the Weyl chamber is written down.

1. A quick reminder

We refer the reader to [8] and [13] for facts on root systems and to [4], [17] for facts on radial Dunkl processes. Let R be a reduced root system in a finite euclidean space $(V, <,>)$ with positive system R_+ and simple system S. Let W be its reflection group and C be its positive Weyl chamber. The radial Dunkl process X associated with R is a continuous paths Markov process valued in $\overline{C}$ whose generator acts on $C^2(\overline{C})$-functions as

$$\mathscr{L}_k u(x) = \frac{1}{2}\Delta u(x) + \sum_{\alpha \in R_+} k(\alpha) \frac{< \nabla u(x), \alpha >}{< x, \alpha >}$$

where Δ, ∇ denote the euclidean Laplacian and the gradient respectively and k is a positive multiplicity function. Its semi group density with respect to the Lebesgue measure in V is given by

$$p_t^k(x,y) = \frac{1}{c_k t^{\gamma+m/2}} e^{-(|x|^2+|y|^2)/2t} D_k^W\left(\frac{x}{\sqrt{t}}, \frac{y}{\sqrt{t}}\right) \omega_k^2(y), \quad x, y \in \overline{C} \tag{1}$$

where $\gamma = \sum_{\alpha \in R_+} k(\alpha)$ and $m = \dim V$ is the rank of R. The weight function ω_k is given by

$$\omega_k(y) = \prod_{\alpha \in R_+} < \alpha, y >^{k(\alpha)}$$

and D_k^W is the generalized Bessel function. Thus, $\mathscr{L}_k$ may be written as

$$\mathscr{L}_k u(x) = \frac{1}{2}\Delta u(x) + < \nabla u(x), \nabla \omega_k(x) > . \tag{2}$$

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2. Motivation

Several reasons motivated us to investigate radial Dunkl processes associated with dihedral root systems. First, the dihedral group is a Coxeter, yet non Weyl in general, reflections group and covers an exceptional Weyl group known in the literature as G_2 which is of a particular interest ([1]). Second, the study of the Dunkl operators associated with dihedral root systems revealed a close relation with Gegenbauer and Jacobi polynomials which have interesting geometrical interpretations such as spherical harmonics and the radial part of the Laplacian on the sphere ([8], [11]). The latter is a particular case of the Jacobi operator which generates a diffusion known as the Jacobi process that may be represented, up to a random time change, by means of two independent Bessel processes ([18]). Since the norm of the radial Dunkl process is a Bessel process, we wanted to gather all these materials in the present work and to see how do they interact. The last reason is that [6] emphasizes the irreducible root systems of types A, B, C, D which, together with dihedral root systems, exhaust the infinite families of irreducible root systems associated with finite Coxeter groups.

2.1. **Dihedral groups and Dihedral systems.** The dihedral group, denoted by $\mathcal{D}_2(n)$ for $n \geq 3$, is defined as the group of orthogonal transformations that preserve a regular n-sided polygon in $V = \mathbb{R}^2$ centered at the origin. Without loss of generality, one may assume that the y-axis is a mirror for the polygon. It contains n rotations through multiples of $2\pi/n$ and n reflections about the diagonals of the polygon. By a diagonal, we mean a line joining two opposite vertices or two midpoints of opposite sides if n is even, or a vertex to the midpoint of the opposite side if n is odd. The corresponding dihedral root system, $I_2(n)$, is characterized by its positive and simple systems given by:

$$R_+ = \{-ie^{i\pi l/n} := -ie^{i\theta_l}, \, 1 \leq l \leq n\}, \quad S = \{e^{i\pi/n}e^{-i\pi/2}, e^{i\pi/2}\}$$

so that the Weyl chamber is a wedge of angle π/m. The reader can check that, for instance, $I_2(3)$ (equilateral triangle-preserving) is isomorphic to $R = A_2$ and $I_2(4)$ (square-preserving) is nothing but $R = B_2$. However, it is a bit more delicate to see that $I_2(6)$ (hexagon-preserving) corresponds to the exceptional Weyl group G_2 ([1]). When $n = 2p, p \geq 2$, there are two orbits so that $k = (k_0, k_1)$, otherwise, there is only one orbit and k takes only one value. The paper is organized as follows: we first write down the semi group density. To proceed, we write down the semi group density of the spherical motion (see below). As a by-product, we deduce the generalized Bessel function and the W-invariant counterparts of the generalized Hermite polynomials ([17]). Then, we give a skew product decomposition of the radial Dunkl process using only independent Bessel processes. This mainly follows from the skew-product decomposition of the Jacobi process derived in [18]. Finally, we compute the tail distribution of the first hitting time of ∂C.

2.2. **On the Spherical motion.** Let us recall the skew-product decomposition of the radial Dunkl process into a radial and a spherical parts ([4]):

$$(X_t)_{t\geq 0} = (|X_t|\Theta_{A_t})_{t\geq 0} \quad A_t = \int_0^t \frac{ds}{|X_s|^2},$$

where $|X|$ is a Bessel process of index γ ([15]) and Θ is the spherical motion of X valued on the sphere and is independent of $|X|$. For dihedral systems, Θ is valued in the unit circle and may be written as $(\cos\theta, \sin\theta)$.

2.3. **Semi group density.** In order to write down p_t^k, we mainly focus on the process θ. Let us first split $\mathscr{L}_k$ of X into a radial and a spherical parts which easily follows from (2) and from ([8])

$$\begin{aligned}\omega_k(r,\theta) &= r^{nk}(\sin n\theta)^k \\ \omega_k(r,\theta) &= r^{p(k_0+k_1)}[\sin(p\theta)]^{k_0}[\cos(p\theta)]^{k_1}\end{aligned}$$

for n being odd, even respectively and $y = (r,\theta) \in \mathbb{R}^2$. Thus

$$\mathscr{L}_k = \frac{1}{2}\left[\partial_r^2 + \frac{2\gamma+1}{r}\partial_r\right] + \frac{1}{r^2}\left[\frac{\partial_\theta^2}{2} + nk\cot(n\theta)\partial_\theta\right]$$

when m is odd, where $\gamma = nk$, and

$$\mathscr{L}_k = \frac{1}{2}\left[\partial_r^2 + \frac{2\gamma+1}{r}\partial_r\right] + \frac{1}{r^2}\left[\frac{\partial_\theta^2}{2} + p(k_0\cot(p\theta) - k_1\tan(p\theta))\partial_\theta\right]$$

when $n = 2p$, where $\gamma = p(k_0 + k_1)$. A first glance at $\mathscr{L}_k^W$ shows that a radial Dunkl process associated with $\mathcal{D}_2(2p), p$ is odd, and with equal multiplicities $k_0 = k_1 = k$ is a radial Dunkl process associated with $\mathcal{D}_2(p)$ and multiplicity function k. Besides, the generator of θ, say $\mathscr{L}_k^\theta$ acts on smooth functions as

$$\begin{aligned}\mathscr{L}_k^\theta &= \frac{\partial_\theta^2}{2} + nk\cot(n\theta)\partial_\theta \\ \mathscr{L}_k^\theta &= \frac{\partial_\theta^2}{2} + p(k_0\cot(p\theta) - k_1\tan(p\theta))\partial_\theta\end{aligned}$$

when n is odd, even respectively. Now, it is easy to see that the process N defined by $N_t := n\theta^W_{t/n^2}$ satisfies

$$dN_t = dB_t + k\cot(N_t)dt$$

when m is odd, while $(M_t := p\theta_{t/p^2})_{t\geq 0}$ satisfies

$$dM_t = dB_t + [k_0\cot(M_t) - k_1\tan(M_t)]dt$$

when n is even, B being a real Brownian motion. This shows that M has the same law as N (as $\pi/2 - N$) when $k_0 = k > 0, k_1 = 0, p = n$ (when $k_0 = 0, k_1 = k > 0, p = n$) so that N may be seen as a particular case of M. Thus, we shall mainly focus on the even case and deduce results for the odd case by substitutuing $k_1 = 0, k_0 = k$ and $p = n$.
The generator of M has a discrete spectrum given by $\lambda_j = -2j(j + k_0 + k_1), j \geq 0$ corresponding to the Jacobi polynomials $P_j^{k_1-1/2,k_0-1/2}(\cos(2\theta))$ (see [8], p. 201). It is known ([16]) that this set of orthogonal eigenpolynomials is complete for the Hilbert space $L^2([0,\pi/2], \mu_k(\theta)d\theta)$ where

$$\mu_k(\theta) := C(k_0,k_1)\sin(\theta)^{2k_0}\cos(\theta)^{2k_1}$$

for some normalizing constant $C(k_0, k_1)$. Accordingly, M has a semi group density, say $m_t^k(\phi,\theta)$, given by (we use orthonormal polynomials, [16])

$$(3) \qquad m_t^k(\phi,\theta) = \sum_{j\geq 0} e^{\lambda_j t} P_j^{k_1-1/2,k_0-1/2}(\cos(2\phi)) P_j^{k_1-1/2,k_0-1/2}(\cos(2\theta))\mu_k(\theta)$$

for $\phi,\theta \in [0,\pi/2]$. It follows that the semi group density of θ, say K_t^k, is given by

$$K_t^k(\phi,\theta) = m_{p^2 t}(p\phi, p\theta), \quad \phi,\theta \in [0,\pi/(2p)].$$

Now, let $(r,\theta) \mapsto f(r,\theta)$ be a nice function and let $\mathbb{P}_{\rho,\phi}$ denote the law of X starting at $x = (\rho,\phi) \in C$. Then, using the independence of θ and $|X|$ together with Fubini's Theorem, one has

$$\begin{aligned}\mathbb{E}_{\rho,\phi}[f(|X_t|,\theta_{A_t})] &= \mathbb{E}_{\rho,\phi}[\mathbb{E}_{\rho,\phi}[f(|X_t|,\theta_{A_t})|\sigma(|X_s|, s\leq t)]] \\ &= \int_0^{\pi/(2p)} \sum_{j\geq 0} \mathbb{E}_\rho^\gamma[f(|X_t|,\theta)e^{\lambda_j p^2 A_t}] P_j^{l_1,l_0}(\cos(2p\phi)) P_j^{l_1,l_0}(\cos(2p\theta))\mu_k(p\theta)d\theta\end{aligned}$$

where $\mathbb{P}_\rho^\gamma$ is the law of the Bessel process $|X|$ starting at ρ and of index γ. Next, for every $\theta \in [0,\pi/(2p)]$

$$\mathbb{E}_\rho^\gamma[f(|X_t|,\theta)e^{\lambda_j p^2 A_t}] = \mathbb{E}_\rho^\gamma[\mathbb{E}_\rho^\gamma[f(|X_t|,\theta)e^{\lambda_j p^2 A_t}||X_t|]] = \int_0^\infty \mathbb{E}_\rho^\gamma[e^{\lambda_j p^2 A_t}||X_t| = r]f(r,\theta)q_t(\rho,r)dr$$

where $q_t(\rho,r)$ is the semi group density of the Bessel process $|X|$ of index γ ([15]):

$$q_t(\rho,r) = \frac{1}{t}\left(\frac{r}{\rho}\right)^\gamma re^{-(\rho^2+r^2)/2t} I_\gamma\left(\frac{\rho r}{t}\right)$$

where I_γ is the modified Bessel function of index γ ([14]). Moreover (see [19] p. 80)

$$\mathbb{E}_\rho^\gamma[e^{\lambda_j p^2 A_t}||X_t| = r] = \frac{I_{\sqrt{\gamma^2 - 2\lambda_j p^2}}(\rho r/t)}{I_\gamma(\rho r/t)}, \ \lambda_j = -2j(j+k_0+k_1).$$

Thus, we proved that

Proposition 2.1. *The semi group of the radial Dunkl process associated with even Dihedral groups $\mathcal{D}_2(2p)$ is given by*

$$\begin{aligned}p_t^k(\rho,\phi,r,\theta) = \frac{1}{c_k t}\left(\frac{r}{\rho}\right)^\gamma & e^{-(\rho^2+r^2)/2t}\sin^{2k_0}(p\theta)\cos^{2k_1}(p\theta) \\ & \sum_{j\geq 0} I_{2jp+\gamma}\left(\frac{\rho r}{t}\right) P_j^{l_1,l_0}(\cos(2p\phi))P_j^{l_1,l_0}(\cos(2p\theta))\end{aligned}$$

where c_k is a normalizing constant, $l_0 = k_0 - 1/2, l_1 = k_1 - 1/2$ and $\rho, r \geq 0, 0 \leq \phi,\theta \leq \pi/2p$. For odd Dihedral groups $\mathcal{D}_2(n)$, take $k_1 = 0, k_0 = k$ and $p = n$ in the above expression.

Remarks. 1/The j-th Jacobi polynomial $P_j^{k_1-1/2,k_0-1/2}(\cos(2p\theta))$ can be replaced by the generalized Gegenbauer polynomial $C_{2j}^{k_1,k_0}(\cos(p\theta))$ (see [8], p. 27). For $k_1 = 0$,

$C_{2j}^{k_1,k_0}(\cos(p\theta))$ reduces to the Gegenbauer polynomial $C_{2j}^{k_0}(\cos(p\theta))$.
2/The heat kernel is easily deduced by taking $k \equiv 0$ in the above formula:

$$p_t^0(\rho,\phi,r,\theta) = \frac{1}{c_0 t} e^{-(\rho^2+r^2)/2t} \sum_{j\geq 0} I_{2jp}\left(\frac{\rho r}{t}\right) T_j(\cos(2p\phi)) T_j(\cos(2p\theta))$$

where T_j is the orthonormal j-th Tchebycheff polynomial defined by

$$T_j(\cos\theta) = f_j \cos(j\theta), \quad j \geq 0$$

for some constant f_j. Thus

$$p_t^0(\rho,\phi,r,\theta) = \frac{1}{c_0 t} e^{-(\rho^2+r^2)/2t} \sum_{j\geq 0} f_j^2 I_{2jp}\left(\frac{\rho r}{t}\right) \cos(2p\phi)\cos(2p\theta).$$

For $k \equiv 1$, one recovers the Brownian motion conditioned to stay in a wedge of angle π/n which is the $h = \omega_1$-transform in Doob's sense of a planar Brownian motion killed when it first hits the edge ([12]). For $n = 2p$, one gets

$$p_t^1(\rho,\phi,r,\theta) = \frac{1}{c_1 t}\left(\frac{1}{r\rho}\right)^{2p} e^{-(\rho^2+r^2)/2t}\omega_1^2(r,\theta) \sum_{j\geq 0} d_j^2 I_{2(j+1)p}\left(\frac{\rho r}{t}\right) U_j(\cos(2p\phi)) U_j(\cos(2p\theta)).$$

where U_j is the j-th Tchebycheff polynomial of the second kind defined by

$$U_j(\cos\theta) = \frac{\sin(j+1)\theta}{\sin\theta}, j \geq 0$$

and d_j is a normalizing constant so that $d_j U_j$ has unit norm. Having in mind that

$$\omega_1(r,\theta) = (1/2) r^{2p} \sin(2p\theta)$$

short computations yield

$$p_t^1(\rho,\phi,r,\theta) = \frac{\omega_1(r,\theta)}{\omega_1(\rho,\phi)} \frac{e^{-(r^2+\rho^2)/2t}}{c_k t} \sum_{j\geq 0} d_j^2 I_{2(j+1)p}\left(\frac{\rho r}{t}\right) \sin[2p(j+1)\phi)] \sin[2(j+1)p\theta]$$

which agrees with Grabiner's result . Besides, the semi group density of a planar Brownian motion killed when it first hits the edge corresponds is the heat kernel with Dirichlet boundary conditions ([15]). From the last equality, it is given by

$$p_t^C(\rho,\phi,r,\theta) = \frac{e^{-(r^2+\rho^2)/2t}}{t} \sum_{j\geq 0} d_j^2 I_{2(j+1)p}\left(\frac{\rho r}{t}\right) \sin[2p(j+1)\phi)] \sin[2(j+1)p\theta]$$

where t runs till the first hitting time of the edge. The last equality should be compared with Lemma 1 in [3]. For odd values of n, a similar result holds and the interested reader may use the Tchebycheff polynomials of the third kind $P_j^{-1/2,1/2}$ denoted in the literature by V_j and defined by

$$V_j(\cos 2\theta) = \frac{\cos[(2j+1)\theta]}{\cos\theta}.$$

3/There is a factor r that was ommited since computations are performed using polar coordinates.

The above density is written as

$$p_t^k(\rho,\phi,r,\theta) = \frac{1}{c_k t^{\gamma+1}}\left(\frac{t}{r\rho}\right)^{\gamma} e^{-(\rho^2+r^2)/2t}\omega_k^2(r,\theta)$$
$$\sum_{j\geq 0} I_{(2j+k_0+k_1)p}\left(\frac{\rho r}{t}\right) P_j^{l_1,l_0}(\cos(2p\phi))P_j^{l_1,l_0}(\cos(2p\theta)).$$

Compared with (1), it leads to the by-product

Corollary 2.1 (Generalized Bessel function)**.** *For even dihedral groups, the generalized Bessel function is given by*

$$D_k^W(\rho,\phi,r,\theta) = c_{p,k}\left(\frac{2}{r\rho}\right)^{\gamma}\sum_{j\geq 0} I_{2jp+\gamma}(\rho r)P_j^{l_1,l_0}(\cos(2p\phi))P_j^{l_1,l_0}(\cos(2p\theta))$$

where $\gamma = p(k_0+k_1)$. For odd dihedral groups, one has

$$D_k^W(\rho,\phi,r,\theta) = c_{n,k}\left(\frac{2}{r\rho}\right)^{\gamma}\sum_{j\geq 0} I_{2jn+\gamma}(\rho r)P_j^{-1/2,l_0}(\cos(2m\phi))P_j^{-1/2,l_0}(\cos(2m\theta))$$

where $\gamma = nk$. The constant $c_{p,k}$ and $c_{n,k}$ are such that $D_k^W(0,0,r,\theta) = |W|$.

2.4. **A Mehler-type formula and Dunkl-Hermite polynomials.** Recall from [17] that the generalized Hermite polynomials $(H_\tau)_{\tau\in\mathbb{N}^m}$ are defined by

$$H_\tau(x) = [e^{-\Delta_k/2}\phi_\tau](x).$$

where $(\phi_\tau)_{\tau\in\mathbb{N}^m}$ is a basis of homogeneous polynomials orthogonal with respect to the pairing inner product defined in [10]. $(H_\tau)_\tau$ is then said to be associated to the basis $(\phi_\tau)_\tau$. Their W-invariant counterparts are defined by

$$H_\tau^W(x) := \sum_{w\in W} H_\tau(wx).$$

By the W-invariance of Δ_k ([8] p. 169), one has

$$H_\tau^W(x) := [e^{-\Delta_k/2}\phi_\tau^W](x), \quad \phi_\tau^W(x) := \sum_{w\in W}\phi_\tau(wx).$$

$(H_\tau)_\tau$ satisfies a Mehler-type formula ([8] p. 246[2])

$$\sum_{\tau\in\mathbb{N}^m} H_\tau(x)H_\tau(y)r^{|\tau|} = \frac{1}{(1-r^2)^{\gamma+m/2}}\exp -\frac{r^2(|x|^2+|y|^2)}{2(1-r^2)} D_k\left(x,\frac{r}{1-r^2}y\right)$$

for $0<r<1$, $x,y\in V$, which after summing twice over W transforms to

$$\sum_{\tau\in\mathbb{N}^m} H_\tau^W(x)H_\tau^W(y)r^{|\tau|} = \frac{|W|}{(1-r^2)^{\gamma+m/2}}\exp -\frac{r^2(|x|^2+|y|^2)}{2(1-r^2)} D_k^W\left(x,\frac{r}{1-r^2}y\right). \tag{4}$$

[2]we use a different normalization than the one used there.

To derive (4) for even dihedral systems, let us first express D_k^W through the hypergeometric function ${}_0\mathscr{F}_1$ ([14])

$$D_k^W(\rho,\phi,r,\theta) = c_{p,k}\sum_{j\geq 0}\frac{(\rho r/2)^{2jp}}{\Gamma(2jp+\gamma+1)}{}_0\mathscr{F}_1\left(2jp+\gamma+1, \frac{\rho^2 r^2}{4}\right)P_j^{l_1,l_0}(\cos(2p\phi))P_j^{l_1,l_0}(\cos(2p\theta)),$$

then use the Mehler-type formula for univariate Laguerre polynomials ([2] p. 200):

$$\sum_{q\geq 0}\frac{q!}{(2jp+\gamma+1)_q}L_q^{2jp+\gamma}(\rho^2/2)L_q^{2jp+\gamma}(r^2/2)z^{2q} = (1-z^2)^{-2jp-\gamma-1}$$
$$e^{-z^2(\rho^2+r^2)/[2(1-z^2)]}{}_0\mathscr{F}_1\left(2jp+\gamma+1, \frac{z^2\rho^2 r^2}{4(1-z^2)^2}\right), \quad |z|<1.$$

It follows that

$$(1-z^2)^{-\gamma-1}e^{-z^2(\rho^2+r^2)/[2(1-z^2)]}D_k^W\left(\rho,\phi,\frac{zr}{1-z^2},\theta\right) =$$
$$c_{p,k}\sum_{j,q\geq 0}\frac{q!}{\Gamma(2jp+q+\gamma+1)}N_{j,p}^{k,W}(\rho,\phi)N_{j,p}^{k,W}(r,\theta)\left(\frac{\rho r}{2}\right)^{2jp}z^{2(q+jp)}$$

where

$$N_{j,p}^{k,W}(\rho,\phi) := L_q^{2jp+\gamma}\left(\frac{\rho^2}{2}\right)P_j^{l_1,l_0}(\cos(2p\phi)).$$

This suggests that the W-invariant Dunkl-Hermite polynomials are

$$H_{\tau_1,\tau_2}(\rho,\phi) = \sqrt{\frac{q!}{\Gamma(2jp+q+\gamma+1)}}\left(\frac{\rho^2}{2}\right)^{jp}N_{j,p}^{k,W}(\rho,\phi)$$

for $\tau_1 = 2q\,(q\geq 0), \tau_2 = 2jp\,(j\geq 0)$ and zero otherwize. A proof of this claim was given to us by Professor C. F. Dunkl and is as follows: the j-th W-invariant harmonic is given by (see 3. 15 in [9])

$$h_j^W(\rho,\phi) = \rho^{2jp}P_j^{l_1,l_0}(\cos(2p\phi))$$

so that by Proposition 3.9 in [10],

$$e^{-\Delta_k/2}[\rho^{2q}h_j^W(\rho,\phi)] = e^{-\Delta_k^W/2}[\rho^{2q}h_j^W(\rho,\phi)] = (-2)^j j! L_q^{2jp+\gamma}\left(\frac{\rho^2}{2}\right)P_j^{l_1,l_0}(\cos(2p\phi)).$$

2.5. **The spherical motion and Jacobi processes: a skew-product decomposition.** The Jacobi process J of parameters d, d' is a $[0,1]$-valued process and is the unique strong solution of ([18])

$$dJ_t = 2\sqrt{J_t(1-J_t)}dB_t + (d-(d+d')J_t)dt,$$

where B is a real Brownian motion. The process J has the skew-product decomposition below ([18]):

$$\left(\frac{Z_1^2(t)}{Z_1^2(t)+Z_2^2(t)}\right)_{t\geq 0} = (J_{F_t})_{t\geq 0}, \quad F_t = \int_0^t \frac{ds}{Z_1^2(s)+Z_2^2(s)}.$$

where Z_1, Z_2 are two independent Bessel processes of dimension d, d' respectively ([15]). J is related to θ as follows: define $(H_t := -\cos 2M_t)_{t\geq 0}$ where $(M_t = p\theta_{t/p^2})_{t\geq 0}$, then an application of Itô's formula shows that $(H_t)_{t\geq 0} \overset{\mathscr{L}}{=} (Y_{2t})_{t\geq 0}$ where

$$dY_t = \sqrt{2}\sqrt{1-Y_t^2}dB_t - [(k_1-k_0)+(k_0+k_1+1)Y_t]dt$$

and that $(1-Y_{2t})/2 = (1-H_t)/2 = \cos^2(M_t)$ is a Jacobi process of parameters $d = 2k_1+1, d' = 2k_0+1$. As a result

$$(\theta_{A_t})_{t\geq 0} = \left(\frac{1}{p}\arccos(\sqrt{J_{p^2A_t}})\right)_{t\geq 0}.$$

On the one hand, It is a well known fact ([15]) that the sum of two independent squared Bessel processes of dimensions d, d' is again a squared Bessel process of dimension $d+d'$, thus $Z_1^2+Z_2^2$ is a Bessel process of index k_0+k_1. On the other hand, for any conjuguate numbers r, q and any Bessel process R_ν of index $\nu > -1/q$, there exists a Bessel process $R_{\nu q}$ of index νq such that the following holds ([15])

$$q^2 R_\nu^{2/q} \overset{d}{=} R_{\nu q}^2\left(\int_0^{\cdot} R_\nu^{-2/r}(s)ds\right). \tag{5}$$

Specializing (5) with $\nu = k_0+k_1, q = p, R_{\nu q} = |X|$, there exists a Bessel process Z of index k_0+k_1 such that

$$Z_t^{2/p} = \frac{|X^W|^2}{p^2}\left(\int_0^t \frac{ds}{Z_s^{2(p-1)/p}}\right) := \frac{1}{p^2}|X_{\tau_t}|^2, \quad r = \frac{p}{p-1}. \tag{6}$$

Let $L_t := \inf\{s, \tau_s > t\}$ be the inverse of τ, then

$$p^2 A \overset{d}{=} \int_0^{\cdot} \frac{ds}{Z_{L_s}^{2/p}} = \int_0^{L.} \frac{ds}{Z_s^2} \overset{d}{=} F_{L.}. \tag{7}$$

As a result

$$(\theta_{A_t})_{t\geq 0} = \left(\frac{1}{p}\arccos(\sqrt{J_{F_{L_t}}})\right)_{t\geq 0} = \frac{1}{p}\left(\arccos\sqrt{\frac{Z_1^2}{Z_1^2+Z_2^2}}(L_t)\right)_{t\geq 0}.$$

Finally

Proposition 2.2. *The time-changed process X_τ associated with the even Dihedral group $\mathcal{D}_2(2p)$ can be realized as the two-dimensional process*

$$\left[(Z_1^2+Z_2^2)^{1/2p}, \arccos\sqrt{\frac{Z_1^2}{Z_1^2+Z_2^2}}\right]$$

where Z_1, Z_2 are two independent Bessel processes of dimensions $d = 2k_1+1, d' = 2k_0+1$ respectively. In particular, this is true for radial Dunkl processes associated with odd dihedral groups for which Z_1 is a reflected Brownian motion.

3. On the first hitting time of a wedge

3.1. **Even dihedral groups: first formula.** Let $X_0 = x \in C$ and let

$$T_0 := \inf\{t,\, X_t \in \partial C\}$$

be the first hitting time of ∂C. Recall that for dihedral groups $\mathcal{D}_2(2p)$, C is a wedge of angle $\pi/(2p)$. Write $x = \rho\cos\phi, \rho > 0, 0 < \phi < \pi/(2p)$ and let $1/2 < k_0, k_1 \leq 1$. The index function is then defined by $l = (l_0 := k_0 - 1/2, l := k_1 - 1/2)$. From results in [4], the radial Dunkl process of index function $l' = -l$ hits ∂C a.s. so that $T_0 < \infty$ a.s. Moreover, using Proposition 2. 15 part (c) in [4], the tail distribution of T_0 is given by:

$$\mathbb{P}_x^{-l}(T_0 > t) = \mathbb{E}_x^l\left[\left(\prod_{\alpha\in R_+}\frac{<\alpha, X_t>}{<\alpha, x>}\right)^{-2l(\alpha)}\right].$$

From (1), one gets

$$\mathbb{P}_x^{-l}(T_0 > t) = \frac{e^{-\rho^2/2t}}{c_k}\left(\frac{\rho}{\sqrt{t}}\right)^{2pl}\sin^{2l}(2p\phi)g\left(\frac{\rho}{\sqrt{t}},\phi\right),$$

where

$$g(\rho,\phi) = \int_0^\infty\int_0^{\pi/n} e^{-r^2/2}D_k^W(\rho,\phi,r,\theta)r^{2p+1}\sin(2p\theta)drd\theta.$$

With regard to (2.1), it amounts to evaluate

$$\begin{aligned} I_1(j) &= \int_0^\infty e^{-r^2/2}I_{b_j}(\rho r)r^{2p+1-\gamma}dr, \\ I_2(j) &= \int_0^{\pi/2p} P_j^{-1/2,k-1/2}(\cos(2p\theta))\sin(2p\theta)d\theta \end{aligned}$$

for every $j \geq 0$, where $b_j := (2j+k_0+k_1)p$. In order to evaluate I_1, we use the expansion ([14])

$$I_{b_j}(\rho r) = \sum_{q\geq 0}\frac{1}{\Gamma(b_j+q+1)}\left(\frac{\rho r}{2}\right)^{2q+b_j}$$

and exchange the order of integration to get

$$I_1(j) = 2^{(p-\gamma)/2}\frac{\Gamma(a_j+1)}{\Gamma(b_j+1)}\left(\frac{\rho}{\sqrt{2}}\right)^{b_j}{}_1\mathscr{F}_1\left(a_j+1, b_j+1, \frac{\rho^2}{2}\right)$$

where

$$a_j = \frac{(2j+k_0+k_1)p + 2p - \gamma}{2} = (j+1)p.$$

Using the identity $\cos(2\theta) = 2\cos^2(\theta) - 1$ and the variable change $z = \cos(\theta)$, I_2 transforms to

$$I_2(j) = \frac{1}{p}\int_0^1 P_j^{-1/2,k-1/2}(2z^2-1)dz = \frac{(-1)^j}{p}\int_0^1 P_j^{k-1/2,-1/2}(1-2z^2)dz$$

which is easily computed using the expansion p. 21 in [8]. As a result, the tail distribution writes

Proposition 3.1.

$$\mathbb{P}_x^{-l}(T_0 > t) = \frac{e^{-\rho^2/2t}}{c_k}\left(\frac{\rho}{\sqrt{t}}\right)^{2pl}\sin^{2l}(2p\phi)$$

$$\sum_{j\geq 0} I_2(j)\frac{\Gamma(a_j+1)}{\Gamma(b_j+1)}\left(\frac{\rho}{\sqrt{2}}\right)^{2jp} {}_1\mathscr{F}_1\left(a_j+1, b_j+1, \frac{\rho^2}{2t}\right) P_j^{l_1,l_0}(\cos(2p\phi))$$

$$= \frac{1}{c_k}\left(\frac{\rho}{\sqrt{t}}\right)^{2pl}\sin^{2l}(2p\phi)$$

$$\sum_{j\geq 0} I_2(j)\frac{\Gamma(a_j+1)}{\Gamma(b_j+1)}\left(\frac{\rho}{\sqrt{2}}\right)^{2jk} {}_1\mathscr{F}_1\left(b_j - a_j, b_j+1, -\frac{\rho^2}{2t}\right) P_j^{l_1,l_0}(\cos(2p\phi))$$

by Kummer's transformation ([14]*).*

For odd dihedral systems, the expression easily follows by substituting in the above formula $k_1 = 0, k_0 = k$ and $p = n$.

3.2. **Even dihedral groups: second formula.** Let us assume that $0 \leq k_0 < 1/2$ and $k_1 \geq 1$. In this case, we follow another way which may be used in the previous case. In fact, we rely on results on Jacobi processes since we already proved that

$$(\theta_{A_t})_{t\geq 0} = \left(\frac{1}{p}\arccos(\sqrt{J_{p^2A_t}})\right)_{t\geq 0}$$

where J is a Jacobi process of parameters $d = 2k_1+1, d' = 2k_0+1$ and starting at $\cos\phi$. Besides, J is independent of $|X|$ and thereby independent from A. As a result,

$$\mathbb{P}_x(T_0 > t) = \mathbb{P}_x(0 < \theta_{A_t} < \pi/(2p)) = \mathbb{P}(0 < J_{p^2A_t} < 1) = \mathbb{P}_x(T_J > p^2A_t)$$

where

$$T_J := \inf\{t,\ J_t = 0\} \wedge \inf\{t,\ J_t = 1\}$$

is the first exit time from the interval $[0,1]$ by a Jacobi process. Now, use the absolute-continuity relation in [7] p. 139 with

Remarks. 1/The value $k \equiv 1$ corresponds to the first exit time of a Brownian motion from a wedge and our result fits the one in [3]. Moreover $b_j = 2a_j$ and one may use the duplication formula to simplify the above ratio of Gamma functions and use some argument simplifications for the confluent hypergeometric function ([14]).
2/For even dihedral groups with indices of different signs ($l_0l_1 < 0$), one may use results on Jacobi processes in [7] p. 140. In fact, since we already showed that

$$(\theta_{A_t})_{t\geq 0} = \left(\frac{1}{p}\arccos(\sqrt{J_{p^2A_t}})\right)_{t\geq 0}$$

where J is a Jacobi process of parameters $d = 2k_1+1, d' = 2k_0+1$ and is independent from $|X|$ (thereby from A), then one has

$$\mathbb{P}_x(T_0 > t) = \mathbb{P}_x(0 < \theta_{A_t} < \pi/(2p)) = \mathbb{P}(0 < J_{p^2A_t} < 1) = \mathbb{P}_x(T_J > p^2A_t)$$

where

$$T_J := \inf\{t,\ J_t = 0\} \wedge \inf\{t,\ J_t = 1\}$$

is the first exit time from the interval $[0, 1]$ by a Jacobi process. Note that this line of thinking is also valid in the first case and allow to recover the above result by the use of Corollary 9.4.6. p. 140 in [7].

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