Convergent noncommutative power series algebras and their finite dimensional representations

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Abstract

We define "convergence" for noncommutative power series and construct two topologies on the algebra of power series, convergent with respect to a positive radius. We indicate all finite dimensional continuous representations of this algebra and prove completeness for both topologies.

Contents

Introduction

In [11], the author constructed a sheaf of associative $\mathbb{C}$-algebras on the space $\mathbb{C}^n$ with the sheaf of holomorphic functions as abelianization and "convergent" noncommutative power series algebras as local rings. This construction led naturally to a definition of "noncommutative analytic spaces" as a generalization of (real or complex) analytic spaces. Examples for noncommutative analytic spaces are analytic super-manifolds and noncommutative links.

This paper starts with a more refined definition of "convergence" for noncommutative power series. A series, convergent in the sense of [11], is called "absolutely convergent" here and in the future. To explain the refined definition, consider the space $\mathbb{M}$ of complex $k \times k$-matrices. The star shaped, open subsets

$$P^{\mathbb{M}}(r)$$

of $\mathbb{M}^n$ of all n-tuples of matrices with "joint spectral radius" smaller than a positive r, play a crucial role. Roughly speaking, a noncommutative power series f is convergent with respect to r, if all the series $f(M)$ obtained by plugging in elements M of $P^{\mathbb{M}}(r)$ in f, converge, for arbitrary matrix length k. (In the subsequent paper [12], we show that this notion of convergence can be used to define a structure sheaf $\mathcal{O}$ on $\mathbb{C}^n$ which is slightly richer than the sheaf defined in [11].) We prove (Theorem 1.9) that the algebra

$$\mathbb{C}\{x\}_r = \mathbb{C}\{x_1|...|x_n\}_r$$

of powers series, convergent with respect to a given $r > 0$, is Frêchet in the "topology of compact convergence", where "compact" refers to subsets of $P^{\mathbb{M}}(r)$. We show (Theorem 2.3) that $\mathbb{C}\{x\}_r$ is complete with respect to the filtration by its "ideals of matrix identities". An element $f \in \mathbb{C}\{x\}_r$ is a "matrix identity" for the matrix algebra $\mathbb{M}$, if $f(M) = 0$, for each $M \in P^{\mathbb{M}}(r)$. On the geometric side, we prove (Theorem 2.1) that the continuous k-dimensional representation of $\mathbb{C}\{x\}_r$ are just those of the form $f \mapsto f(M)$, where M is in $P^{\mathbb{M}}(r)$.

Conventions: The letter $\mathbb{C}$ denotes weather the fiel of real or the field of complex numbers. We will indicate explicitly if a statement only holds for the complex numbers. An *algebra* is an associative, unitary $\mathbb{C}$-algebra. *Modules* and *ideals* are two-sided. Left-modules only appear under the name "representations". An element a of an algebra A is a *unit*, if there exists an element $b \in A$ such that $ab = 1 = ba$. A *maximal ideal* is maximal in the set of (two-sided) ideals. An algebra A with a single maximal ideal $\mathfrak{m}$ is $\mathbb{C}$-*local*, if the natural map $\mathbb{C} \to A/\mathfrak{m}$ is an isomorphism.

1 Convergent NC power series

1.1 Absolutely convergent NC power series

Refer to the Appendix for the notions of indices, maps between indices, noncommutative power series algebras, their morphisms and abelianization. For each point $p \in \mathbb{C}^n$, consider $x_i - p_i$ as a formal variable and set

$$\mathbb{C}[[x-p]] := \mathbb{C}[[x_1 - p_1| \dots |x_n - p_n]].$$

A noncommutative power series $f = \sum_I a_I (x-p)^I$ in $\mathbb{C}[[x-p]]$ is called **absolutely convergent**, if the abelianization f_{abs} of $\sum_I |a_I|(x-p)^I$ is convergent. Write

$$\mathbb{C}\{x-p\}^{\text{abs}}$$

for the local subalgebra of $\mathbb{C}[[x-p]]$ of absolutely convergent noncommutative power series, and for $r > 0$, write

$$\mathbb{C}\{x-p\}_r^{\text{abs}}$$

for the subalgebra of $\mathbb{C}\{x-p\}^{\text{abs}}$ of those f such that f_{abs} converges on the open polydisk $P(p,r)$ in $\mathbb{C}^n$ with poly-radius $(r,\ldots,r)$. A commutative power series $g = \sum_\nu b_\nu x^\nu$ converges on the open polydisk $P(r) = P(0,r)$, if and only if the series $g_{\text{abs}} := \sum_\nu |a_\nu| x^\nu$ converges on $P(r)$. If a noncommutative power series $f = \sum_I a_I x^I$ is absolutely convergent on $P(r)$ and if $|b_I| \leq |a_I|$, for each multi-index I, then the series $\sum_I b_I (x-p)^I$ converges absolutely on $P(r)$.

Observation 1.1. *If g and each $f_1, \ldots, f_n$ are absolutely convergent power series, $g(f^1, \ldots, f^n)$ is absolutely convergent.*

Proof. This follows by $g(f^1,\ldots,f^n)_{\text{ab}} = g_{\text{ab}}(f^1_{\text{ab}},\ldots,f^n_{\text{ab}})$ and equation (3.5). □

1.2 The $\mathbb{M}$-spectrum

Consider $\mathbb{C}^k$ as normed space with respect to the maximum norm and equip the matrix algebras $\mathbb{M} = M_k(\mathbb{C})$ over $\mathbb{C}$ with the corresponding operator norm $||\cdot||$.

Let $\mathcal{C}$ be a category of topological algebras. For objects A and $\mathbb{M}$, we call

$$\text{Spec}^{\mathbb{M}}(A) := \text{Hom}_{\mathcal{C}}(A, \mathbb{M})$$

the $\mathbb{M}$-**spectrum** of A. For $\mathbb{M} = M_k(\mathbb{C})$, the $\mathbb{M}$-spectrum is just the set $\text{rep}_k(A)$ of continuous k-dimensional representations of A. To each element $a \in A$, we assign the canonical map

$$\tilde{a} : \text{Spec}^{\mathbb{M}}(A) \to \mathbb{M}.$$

The ideal

$$I_A(\mathbb{M}) := \{a \in A : \ \tilde{a} \equiv 0\}$$

is called **ideal of identities for** $\mathbb{M}$. For $\mathbb{M} = M_k(\mathbb{C})$, we equip $\text{Spec}^{\mathbb{M}}(A)$ with the weakest topolology making the maps

$$\begin{aligned} \tilde{a} : \text{Spec}^{\mathbb{M}}(A) &\to \mathbb{M} \\ \chi &\mapsto \chi(a) \end{aligned}$$

continuous, for each $a \in A$.

1.3 The joint spectral radius

For an n-tuple $M = (M_1, \dots, M_n)$ in $\mathbb{M}^n$ the value

$$|M| := \limsup_{k\to\infty}\{ \sqrt[|I|]{||M^I||} : \ |I| \le k\}$$

is called **joint spectral radius** of $M_1, \dots, M_n$ (see [13]). For elements $Q \in \mathbb{C}^n$, we have $|Q| = \max_{i=1,\dots,n} |Q_i|$. Observe that, for $k > 1$, the function $|\cdot|$ is not a semi-norm on $M_k(\mathbb{C})^n$. Let, for example, E_{ij} be the elementary matrices in $M_2(\mathbb{C})$, then $|E_{12} + E_{12}| = 1 > 0 = |E_{12}| + |E_{21}|$.

Theorem 1.2. *The joint spectral radius* $|\cdot| : \mathbb{M}^n \to \mathbb{R}_{\ge 0}$ *has the following properties:*

(1) It is continuous.

(2) $|M| = \lim_{k\to\infty} \max_{|I|=k} \sqrt[k]{||M^I||}$.

Proof. Both statements are proved in [13]. The continuity was first shown by [2]. □

Proposition 1.3. *For elements* Q *in* $\mathbb{C}^n$ *and* M *in* $\mathbb{M}^n$*, we have the inequality*

$$|M + Q| \le m + |Q|,$$

where $m := \max_{i=1,\dots,n} ||M_i||$.

Proof. Set $q := |Q|$. Since for central elements $y_1, \dots, y_n$,

$$(x + y)^I = \sum_{\alpha: J \to I} x^J y^{I-\alpha J}, \tag{1.1}$$

we have

$$(x + \mathbf{1})^I = \sum_{J \le I} \binom{I}{J} x^J, \tag{1.2}$$

where $\mathbf{1} = (1, \dots, 1)$. By equation (1.1),

$$\begin{aligned} ||(M + Q)^I|| &= || \sum_{\alpha: J\to I} M^J Q^{I-\alpha J} || \\ &\le \sum_{\alpha: J \to I} ||M^J|| \cdot q^{|I|-|J|} \\ &= \sum_{J \le I} \binom{I}{J} ||M^J|| \cdot q^{|I|-|J|} \\ &\le \sum_{J \le I} \binom{I}{J} ||M||^J \cdot q^{|I|-|J|}, \end{aligned}$$

where $||M|| := (||M_1||, \dots, ||M_n||)$. Thus,

$$\sqrt[|I|]{||(M+Q)^I||} \leq q \cdot \sqrt[|I|]{\sum_{J \leq I} \binom{I}{J} ||\frac{M}{q}||^J}$$

By equation (1.2), the last expression equals

$$q \cdot \sqrt[|I|]{(||\frac{M}{q}|| + \mathbf{1})^I} \leq q(\frac{m}{q} + 1) = m + q.$$

□

For $P \in \mathbb{M}^n$ and $r > 0$, the subset

$$P^{\mathbb{M}}(P, r) := \{M \in \mathbb{M}^n : \ |M - P| < r\}$$

of $\mathbb{M}^n$ will be called **stable region in** $\mathbb{M}^n$ with radius r and center P. By Theorem 1.2, stable regions are open in $\mathbb{M}^n$. Like open polydisks in $\mathbb{C}^n$, they play the role of smooth affine spaces. There are algorithms [3] for arbitrary good approximations of the joint spectral radius of n-tuples M of matrices with rational entries. It is not known, so far, if the question weather or not an element $M \in \mathbb{M}^n$ satisfies $|M| < 1$, is decidable. However, there is the following negative result [14]:

Theorem 1.4 (Tsitsikilis-Blondel)**.** *It is undecidable, if* $|M| \leq 1$*, for a given n-tuple* M *of square matrices with rational entries.*

1.4 Convergence

For $\mathbb{M} = M_k(\mathbb{C})$, we fix the basis $E_1, \dots, E_{k^2}$, where E_1 is the identity matrix and $E_2, \dots, E_{k^2}$ are the lexicographically ordered elementary matrices E_{ij} with $(i,j) \neq (1,1)$.

The identification $\tau : \mathbb{M}^n \to \mathbb{C}^{k^2 \cdot n}$ is a map of topological vector spaces. Consider the algebra $\mathbb{C}\{x_{11}, \dots, x_{nk^2}\}_{\mathbb{M},r}$ of all commutative power series in $x_{11}, \dots, x_{nk^2}$, absolutely convergent at each point of $\tau(P^{\mathbb{M}}(r))$. The algebra $\mathbb{C}\{x_{11}, \dots, x_{nk^2}\}$ is Frêchet with respect to the topology of compact convergence. The elements of the algebra

$$\mathbb{M}[[x_{11}, \dots, x_{nk^2}]] := \mathbb{C}[[x_{11}, \dots, x_{nk^2}]] \otimes_{\mathbb{C}} \mathbb{M}$$

correspond to k^2-tuples of elements in $\mathbb{C}[[x_{11}, \dots, x_{nk^2}]]$ via the coordinate projections $\mathbb{M} \to \mathbb{C}$. The algebra

$$\mathbb{M}\{x_{11}, \dots, x_{nk^2}\}_r := \mathbb{C}\{x_{11}, \dots, x_{nk^2}\}_{\mathbb{M},r} \otimes_{\mathbb{C}} \mathbb{M}$$

is again Frêchet. It plays the role of a universal object, in the sense that any k-dimensional representation of the "convergent power series algebra" (see below) factors through it.

Consider the map

$$\phi^{\mathbb{M}} : \mathbb{C}[[x_1|\dots|x_n]] \to \mathbb{M}[[x_{11},\dots,x_{nk^2}]],$$

sending x_i to the "generic matrix" $\sum_{j=1}^{k^2} E_j \cdot x_{ij}$. Set

$$\mathbb{C}\{x\}_r^{\mathbb{M}} := (\phi^{\mathbb{M}})^{-1}(\mathbb{M}\{x_{11},\dots,x_{nk^2}\}_r).$$

and

$$\mathbb{C}\{x\}_r := \cap_{\mathbb{M}}\mathbb{C}\{x\}_r^{\mathbb{M}}.$$

The elements of

$$\mathbb{C}\{x\} := \cup_{r>0}\mathbb{C}\{x\}_r.$$

will be called **convergent** noncommutative power series.

Observe that, for $M \in P^{\mathbb{M}}(r) := P^{\mathbb{M}}(r,0)$ and $f = \sum_I a_I x^I$ in $\mathbb{C}\{x\}_r^{\text{abs}}$, the sum $\sum_I a_I M^I$ converges absolutely in $\mathbb{M}$. Thus, each absolutely convergent noncommutative power series is convergent. By Example 1.7 below, the converse is not true:

1.5 Ideals of matrix identities

The kernel of the restriction of $\phi^{\mathbb{M}}$ to the noncommutative polynomial ring $\mathbb{C}[x]$ is the ideal of matrix identities $I_{\mathbb{C}[x]}(\mathbb{M})$, that was studied by Procesi [10] and others. Since $\phi^{\mathbb{M}}$ respects total degrees, an NC polynomial f is in $I_{\mathbb{C}[x]}(\mathbb{M})$, if and only if each of its homogeneous components f_d of total degree d is in $I_{\mathbb{C}[x]}(\mathbb{M})$. By the Amitsur-Levitzki Theorem [1] (see Example 1.7 below), there is an identity of degree $2k$, for $\mathbb{M} = M_k(\mathbb{C})$. It is known [10] that $I_{\mathbb{C}[x]}(\mathbb{M})$ contains no homogeneous polynomial of degree $\leq 2k$.

For $A \in \{\mathbb{C}\{x\}_r, A = \mathbb{C}\{x\}_r^{\text{abs}}, \mathbb{C}[x]\}$, we just write $I(\mathbb{M})$ for the ideal $I_A(\mathbb{M})$ of matrix identities, or just I_k if $\mathbb{M} = M_k(\mathbb{C})$. Again,

$$I(\mathbb{M}) = \text{Kern}(\phi^{\mathbb{M}})|_A.$$

Proposition 1.5. $I_k \subseteq (x)^{2k}$

Proof. For $A = \mathbb{C}[x]$, the statement is know to be true. By homogenity of the maps $\phi^{\mathbb{M}}$, a power series f with homogeneous components f_d of total degree d belongs to I_k, if and only if each f_d belongs to d. Thus if $f \in I_k$, each f_d with $d < 2k$ is trivial. □

Corollary 1.6. $\cap_k I_k = 0$.

Example 1.7. For $l \geq 1$, let $s_l \in \mathbb{Z}[x_1|\dots|x_l]$ be the standard identity

$$s_l := \sum_{\sigma \in S_l} \mathrm{sgn}(\sigma) x_{\sigma(1)} \cdot \dots \cdot x_{\sigma(l)}.$$

One can verify as an exercise that each $l-1$-dimensional $\mathbb{C}$-algebra satisfies

$$\forall x_1, \dots, x_l : \; s_l(x_1, \dots, x_l) = 0. \tag{1.3}$$

The Amitsur-Levitzki Theorem [1] states that $M_k(\mathbb{C})$ satisfies (1.3), for $l = 2k$. The algebra map

$$\begin{aligned} \mathbb{C}[z_1|\dots|z_l] &\to \mathbb{C}[x|y] \\ z_i &\to xy^i \end{aligned}$$

is injective (see Lam). In consequence, $f_l := s_l(xy, \dots, xy^l)$ is a non-trivial identity of degree $\frac{l(l+1)}{2}$ for all $\mathbb{C}$-algebras of dimension $\leq l-1$ (and for $M_{l/2}(\mathbb{C})$, if l is even). We have that

$$f_l = \sum_{|I|} a_I (x, y)^I,$$

where $a_I \in \{0, \pm 1\}$ is non-trivial for exactly $l!$ many indices. The series

$$f := \sum_{l \geq 1} f_l$$

is not absolutely convergent, since $f_{\text{abs}} = \sum_l l! x^l y^{l(l-1)/2}$. It is convergent, since $\phi^{\mathbb{M}}(f)$ is a polynomial, for each $\mathbb{M} = M_k(\mathbb{C})$.

1.6 The topology of compact convergence

For compact subsets $K \subseteq P^{\mathbb{M}}(r)$, we define a seminorm $|| \cdot ||_K$ on $\mathbb{C}\{x\}_r$ via

$$||f|| := ||\phi^{\mathbb{M}}(f)||_K.$$

The induced topology is already defined by a countable subfamily of these seminorms and is called **topology of compact convergence**. By the following theorem, it is a Frêchet-topology.

Lemma 1.8. *If a sequence $(f_m)_m$ in $\mathbb{C}\{x\}_r$ with $f_m = \sum_I a_{m,I} x^I$ is Cauchy with respect to the topology of compact convergence, then, for each I, the coefficient sequence $(a_{m,I})_m$ converges in $\mathbb{C}$.*

Proof. The corresponding statements for commutative convergent power series is known to be true. Since $I(\mathbb{M}) \subseteq (x)^{2k}$, for $\mathbb{M} = M_k(\mathbb{C})$, the restriction of $\phi^{\mathbb{M}}$ defines a monomorphism

$$\mathbb{C}[x_1|...|x_n]_{\leq 2k-1} \to \mathbb{M}[x_{11}, \dots, x_{nk^2}]_{\leq 2k-1} \tag{1.4}$$

of finite dimensional vectorspaces. By assumption, $(\phi^{\mathbb{M}}(f_m))_m$ is a Cauchy sequence in $\mathbb{M}\{x_{11},\ldots,x_{nk^2}\}_r$, thus it converges coefficientwise, since $\phi^{\mathbb{M}}(f_m)$ is just a matrix of commutative convergent powers series. Injectivity of the morphism (1.4) implies that the series $(a_{m,I})_m$ converges, for each $|I| \leq 2k-1$. □

Theorem 1.9. *The algebra $\mathbb{C}\{r\}_r$ is complete with respect to the topology of compact convergence.*

Proof. Consider a Cauchy sequence f_k in $\mathbb{C}\{x\}_r$ with $f_k = \sum_I a_{k,I} x^I$. By Lemma 1.8, for each multi-index I, the sequence $(a_{k,I})$ has a limit $a_I \in \mathbb{C}$. Set $f := \sum_I a_I x^I$. Since f_k converges coefficientwise to f, the Cauchy sequence $\phi^{\mathbb{M}}(f_k)$ converges coefficientwise to $\phi^{\mathbb{M}}(f)$. By completeness, the series $\phi^{\mathbb{M}}(f)$ belongs to $\mathbb{M}\{x_{11},\ldots,x_{nk^2}\}_r$. □

2 Representations

2.1 $\mathbb{M}$-spectra of the convergent power series algebra

Consider the induced topology on the subalgebra $\mathbb{C}\{x\}_r^{\mathrm{abs}} \subseteq \mathbb{C}\{x\}_r$ of absolutely convergent power series. Recall that the $\mathbb{M}$-spectrum $\mathrm{Spec}^{\mathbb{M}}(\mathbb{C}\{x\}_r^{\mathrm{abs}})$ is the set of all continuous algebra homomorphisms $f : \mathbb{C}\{x\}_r^{\mathrm{abs}} \to \mathbb{M}$. Any such f is uniquely defined by the values $f(x_1),\ldots,f(x_n)$ on the variables.

Theorem 2.1. *The natural map $P^{\mathbb{M}}(r) \to \mathrm{Spec}^{\mathbb{M}}(\mathbb{C}\{x\}_r^{\mathrm{abs}})$ ist a homeomorphism.*

Proof. To prove bijectivity, it suffices to show that if $\mathcal{M} \in \mathbb{M}^n$ is not in $P^{\mathbb{M}}(r)$, there is an element $f \in \mathbb{C}\{x\}_r^{\mathrm{abs}}$ such that the sequence

$$(||\sum_{|I|=k} a_I M^I||)_{k\geq 0}$$

does not converge to zero. If $|M| \geq r$, we may assume that there exists a sequence $(I(m))_m$ in G with $|I(m+1)| > |I(m)|$, for each $m \geq 0$, such that the sequence $(\sqrt[|I(m)|]{||M^{I(m)}||})_m$ is monotouneously increasing and converges to r. The one variable power series

$$\sum_m \frac{1}{||M^{I(m)}||} z^{|I(m)|}$$

converges for $0 < z < r$: Assume that $z < r - \delta$, for a $\delta \in (0, r)$. For $m >> 0$, we have that $||M^{I(m)}||^{-1} < (\frac{1}{r-\delta})^{|I(m)|}$, thus

$$\frac{z^{|I(m)|}}{||M^{I(m)}||} < (\frac{z}{r-\delta})^{|I(m)|}.$$

In consequence, if we set $a_I := \frac{1}{||M^{I(m)}||}$, for $I = I(m)$ and $a_I = 0$, if I is not in $\{I(m) : m \geq 0\}$, the series $\sum_I a_I x^I$ belongs to $\mathbb{C}\{x\}_r^{\text{abs}}$. On the other hand-side, the sequence

$$(||\sum_{|I|=k} a_I M^I||)_{k\geq 0}$$

does not converge to zero. It is left to the reader to verify that the map is bicontinuous. □

Observe that the inclusion $\mathbb{C}\{x\}_r^{\text{abs}} \to \mathbb{C}\{x\}_r$ induces an isomorphism on each $\mathbb{M}$-spectrum.

Corollary 2.2. *For $r > 0$ and $\mathbb{M} = M_k(\mathbb{C})$, the topological space $\text{Spec}^{\mathbb{M}}(\mathbb{C}\{x\}_r)$ of continuous k-dimensional representations of $\mathbb{C}\{x\}_r$ is just the stable region $P^{\mathbb{M}}(r)$ in $\mathbb{M}^n$.*

2.2 The $I_\bullet$-topology

For $A \in \{\mathbb{C}\{x\}_r, A = \mathbb{C}\{x\}_r^{\text{abs}}, \mathbb{C}[x]\}$, the filtration

$$\dots I_3 \subseteq I_2 \subseteq I_1 = [A, A] \subseteq A$$

defines the $\boldsymbol{I_\bullet}$**-topology** on A.

Theorem 2.3. *The algebra $\mathbb{C}\{x\}_r$ is complete with respect to the $I_\bullet$-topology.*

Proof. Consider an $I_\bullet$-Cauchy sequence (f_k) in $\mathbb{C}\{x\}_r$. Denote $f_k^{(m)}$ the homogeneous part of f_k of total degree m. Replacing the sequence by an appropriate subsequence, we may assume that $f_k - f_j \in I_{k+1} \subseteq (x)^{2(k+1)}$ for $j \geq k$. This implies the following identities:

(1) $\phi^{\mathbb{M}}(f_k) = \phi^{\mathbb{M}}(f_j)$, for $\mathbb{M} = M_k(\mathbb{C})$ and $j \geq k$.

(2) $f_j^{(j)} = f_k^{(j)}$, for $j \leq k$.

Set $f^{(k)} := f_k^{(k)}$, for $k \geq 0$ and $f := \sum_{k\geq 0} f^{(k)}$. First we show that $f - f_k \in I_k$, for each $k \geq 0$, and in consequence, that the series (f_k) converges to f in the $\boldsymbol{I_\bullet}$-topology of $\mathbb{C}[[x]]$. By equation (1), for $\mathbb{M} = M_k(\mathbb{C})$ and $j > k$, we have that

$$\phi^{\mathbb{M}}(f - f_k)^{(j)} = \phi^{\mathbb{M}}(f_j^{(j)} - f_k^{(j)}) = \phi^{\mathbb{M}}(f_j - f_k)^{(j)} = 0.$$

By equation (2), for $\mathbb{M} = M_k(\mathbb{C})$ and $j \leq k$. we have that

$$\phi^{\mathbb{M}}(f - f_k)^{(j)} = \phi^{\mathbb{M}}(f_j^{(j)} - f_k^{(j)}) = 0.$$

In consequence, $\phi^{\mathbb{M}}(f-f_k)=0$. To conclude the proof, we must show that f belongs to $\mathbb{C}\{x\}_r^{\mathbb{M}}$, for each $\mathbb{M}=M_l(\mathbb{C})$: For arbitrary $P\in P^{\mathbb{M}}(r)$ and $\epsilon>0$, we can find a $k_0\geq l$ such that, for each $k_1>k_o$, we have that

$$\sum_{k=k_0}^{k_1}\phi^{\mathbb{M}}(f_l)^{(k)}_{\mathrm{abs}}(P)\leq\epsilon.$$

Since

$$\begin{aligned}\phi^{\mathbb{M}}(f_l)^{(k)}&=\phi(\mathbb{M})(f_k)^{(k)}\qquad\text{(by equation (1))}\\&=\phi^{\mathbb{M}}(f_k^{(k)})\\&=\phi^{\mathbb{M}}(f^{(k)})=\phi^{\mathbb{M}}(f)^{(k)},\end{aligned}$$

we have that

$$\sum_{k=k_0}^{k_1}\phi^{\mathbb{M}}(f)^{(k)}_{\mathrm{abs}}(P)\leq\epsilon.$$

This finishes the proof. □

Let $\mathcal{C}$ be the category all Frêchet algebras of the form $\mathbb{C}\{x\}_r/J$, where J is a Frêchet-closed ideal.

Corollary 2.4.

$$\mathbb{C}\{x\}_r=\lim_k{}^{\mathcal{C}}\mathbb{C}\{x\}_r/I_k.$$

3 Appendix: Formal NC power series

3.1 Multi-indices

Fix a dimension $n\geq 0$. Let G be the semi-group of **(noncommutive) multi-indices,** i.e. of tuples $I=(i_1,\ldots,i_{|I|})$ with $i_k\in\{1,\ldots,n\}$ of length $|I|\geq 0$. The addition of multi-indices I,J is defined by $I+J:=(i_1,\ldots,i_{\#I},j_1,\ldots,j_{\#J})$. The zero element is the empty multi-index denoted by 0.
Consider the partial relation on G defined by $I<J$ iff and only if $|I|<|J|$ and I is obtained from J by deletion of indices. If $I\leq J$ (i.e. $I<J$ or $I=J$), there exists an order-preserving, injective map α from the set $\{1,\ldots,\#I\}$ into $\{1,\ldots,\#J\}$ such that $i_k=j_{\alpha(k)}$, for $k=1,\ldots,\#I$. We say that α is a map from I to J. Set $\binom{J}{I}$ to be the number of such maps $\alpha:I\longrightarrow J$. By definition, $\binom{J}{0}=1$, for any J. We have $\binom{J}{I}\leq\binom{|J|}{|I|}$. Denote $J-_\alpha I$ be the multi-index obtained from J by deletion of $j_{\alpha(1)},\ldots,j_{\alpha(\#I)}$. Let G_{ab} denote the commutative semi-group of n-tuples $\nu=(\nu_1,\ldots,\nu_n)$ in $\{0,1,2,\ldots\}$. We have an epimorphism $\mathrm{ab}:G\to G_{\mathrm{ab}}$ of semigroups sending $I\in G$ to the multi-index $\nu\in G_{\mathrm{ab}}$ with $\nu_k=\{l:i_l=k\}$.

3.2 The category of formal NC power series

For $n > 0$, write $\mathbb{C}[[x_1|\dots|x_n]]$ or just $\mathbb{C}[[x]]$ for the noncommutative power series algebra in formal variables x_i. For each multi-index $I \in G$, set

$$x^I := x_{i_1} \cdot \dots \cdot x_{i_k} \quad \text{and}$$
$$x^0 := 1.$$

If $g = \sum_J b_J x^J$ is a power series in $\mathbb{C}[[x]]$ and if the power series $f^1, \dots, f^n$ with $f^k = \sum_I a_I^k y^I$ belong to the maximal ideal (y) of $\mathbb{C}[[y]] = \mathbb{C}[[y_1|\dots|y_m]]$, we can form the power series

$$g(f^1, ..., f^n) := \sum_J (\sum b_K \cdot a_{I_{k_1}}^{k_1} \cdot \dots \cdot a_{I_{k_{|K|}}}^{k_{|K|}}) y^J, \tag{3.5}$$

where the sum in the bracket is over all multi-indices $K, I_1, \dots, I_{|K|}$ such that $J = I_{k_1} + \dots + I_{k_{|K|}}$. A **morphism** $\mathbb{C}[[x]] \longrightarrow \mathbb{C}[[y]]$ of noncommutative power series algebras is a local algebra homomorphism of the form $g \mapsto g(f_1, \dots, f_n)$, for a given n-tuple $(f^1, \dots, f^n)$ of elements of (y).

Observation 3.1. *Let* $f : \mathbb{C}[[x]] \longrightarrow \mathbb{C}[[y]]$ *and* $g : \mathbb{C}[[y]] \longrightarrow \mathbb{C}[[z]]$ *be given by* $f(x_s) = \sum_I a_I^s y^I$ *and* $g(y_t) = \sum_J b_J^t z^J$*. Then,* $g \circ f(x_s) = \sum_K c_K z^K$*, with*

$$c_K = \sum_I a_I^s \sum b_{J_1}^{i_1} \cdot \dots \cdot b_{J_{|I|}}^{i_{|I|}},$$

where the second sum is taken over all multi-indices $J_1, \dots, J_{|I|}$ *such that* $J_1 + \dots + J_{|I|} = K$*.*

For an endomorphism f of $\mathbb{C}[[x]]$, with $f(x_k) = \sum_I a_I^k x^I$, set Jf to be the $n \times n$-matrix $(a_{(i)}^k)_{k,i}$.

Theorem 3.2. *An endomorphism* f *of* $\mathbb{C}[[x]]$ *is an automorphism, if and only if* Jf *is invertible.*

Proof. Suppose that Jf is invertible. Inductively, for $k \geq 1$, $l = 1, \dots, k$ and $J \in \{1, \dots, n\}^l$, we will define coefficients $b_J^l \in \mathbb{C}$ such that the endomorphism g^k of $\mathbb{C}[[x]]$ with $g^k(x_l) = \sum_{|J| \leq k} b_J^l x^J$ is inverse to f modulo $(x)^{k+1}$. For $k = 1$, by Observation 3.1, the necessary (and sufficient) conditions are

$$\sum_{i=1}^n a_{(i)}^s b_{(s)}^i = 1 \quad \text{for all } s,$$
$$\sum_{i=1}^n a_{(i)}^s b_{(s)}^i = 0 \quad \text{for all } s \neq l.$$

We can find such coefficients, if and only if Jf is invertible. Now suppose that Jf is invertible and that the coefficients b^l_J are constructed adequately, for $|J| \leq k-1$. For fixed J with $|J| = k$, we have to find b^l_J such that

$$\sum_{i=1}^{n} a^s_{(i)} b^i_J + \sum_{|I|\geq 2} a^l_I \sum b^{i_1}_{J_1} \cdot \ldots \cdot b^{i_{|I|}}_{J_{|I|}} = 0.$$

The second sum is known, and since Jf is invertible, we can find adequate $b^1_J, \ldots, b^n_J$. □

We have a canonical epimorphism ab from $\mathbb{C}[[x]] = \mathbb{C}[[x_1|\ldots|x_n]]$ to the commutative power series algebra $\mathbb{C}[[x]]_{\mathrm{ab}} = \mathbb{C}[[x_1, \ldots, x_n]]$. We write f_{ab} instead of $\mathrm{ab}(f)$.

Corollary 3.3. *Each lift of an automorphism of a commutative formal power series algebra to an endomorphism of the non-commutative formal power series algebra is again an automorphism.*

3.3 Products

For power series algebras $\mathbb{C}[[x]] = \mathbb{C}[[x_1|...|x_n]]$ and $\mathbb{C}[[y]] = \mathbb{C}[[y_1|...|y_m]]$, we define the **free product**

$$\mathbb{C}[[x]] * \mathbb{C}[[y]] := \mathbb{C}[[x_1|...|x_n|y_1|...|y_n]]$$

and the **complete tensor product** $\mathbb{C}[[x]]\hat{\otimes}_{\mathbb{C}}\mathbb{C}[[y]]$ as the power series algebra in $x_1, ..., x_n$ and $y_1, ..., y_m$, where the y_j are assumed to commute with the x_i.

3.4 Finitely generated ideals

For noncommutative power series algebras, the concept of finitely generated two-sided ideals has to be slightly adapted. As a reason, we give the following example:

Example 3.4. Let (x) be the two-sided ideal of $\mathbb{C}[[x|y]]$, consisting of all noncommutative power series where at least one factor x arises in each monomial. Observe that (x) is not the two-sided ideal generated by x in the algebraic sense.

Proof. Assume that (x) is the two-sided ideal generated by x in the algebraic sense. Then we can find power series $f_i = \sum_j a_{ij} y^j$ and $g_i = \sum_j b_{ij} y^j$ in $\mathbb{C}[[y]]$, $i = 1, \ldots, N$ such that

$$xyx + y^2xy^2 + ... = \sum_{i=1}^{N} f_i(y) \cdot x \cdot g_i(y).$$

The right hand-side takes the form $\sum_{j,k}(\sum_i a_{ij} b_{ik}) y^j x y^k$. In particular, for $j, k \leq N+1$, we would get $\sum_i^N a_{ij} b_{ik} = \delta_{j,k}$, which is impossible, since the left hand-side is a product of two matrices of rank at most N. □

By definition, the opposite algebra A^{op} of an algebra A is the set $\{a^{\mathrm{op}}: \ a \in \mathbb{C}[[x]]\}$ with operations $a_1^{\mathrm{op}} + a_2^{\mathrm{op}} = (a_1 + a_2)^{\mathrm{op}}$ and $a_1^{\mathrm{op}} \cdot a_2^{\mathrm{op}} = (a_2 \cdot a_1)^{\mathrm{op}}$. Observe that $\mathbb{C}[[x]]^{\mathrm{op}}$ is naturally isomorphic to the power series algebra $\mathbb{C}[[x_1^{\mathrm{op}}|...|x_n^{\mathrm{op}}]]$ and the assignment $x \mapsto x^{\mathrm{op}}$ defines an isomorphism $\mathrm{OP} : \mathbb{C}[[x]] \longrightarrow \mathbb{C}[[x]]^{\mathrm{op}}$. Attention, in general, $\mathrm{OP}(f) \neq f^{\mathrm{op}}$, for example, $\mathrm{OP}(x_1x_2) = x_1^{\mathrm{op}}x_2^{\mathrm{op}} = (x_2x_1)^{\mathrm{op}}$. We define the **(complete) envelopping algebra** $\mathbb{C}[[x]]^{\hat{e}}$ of $\mathbb{C}[[x]]$ as $\mathbb{C}[[x]]\hat{\otimes}_{\mathbb{C}}\mathbb{C}[[x]]^{\mathrm{op}}$. Consider the natural epimorphism

$$\hat{\alpha} : \mathbb{C}[[x]]^{\hat{e}} \longrightarrow \mathbb{C}[[x]].$$

For a two-sided ideal $J \subseteq \mathbb{C}[[x]]$, the inverse image $\hat{\alpha}^{-1}(J)$ is not, in general a left ideal of $\mathbb{C}[[x]]^{\hat{e}}$, since it is not, in general, closed under left multiplication by elements of $\mathbb{C}[[x]]^{\hat{e}}$. We define the **completion** $\hat{J}$ of J as the image under $\hat{\alpha}$ of the $\mathbb{C}[[x]]^{\hat{e}}$-left ideal generated by $\hat{\alpha}^{-1}(J)$. For simplicity, for elements $f_1, ..., f_m$ in $\mathbb{C}[[x]]$, we shall write $(f_1, ..., f_m)$ for the completion of the two-sided ideal generated by the f_i. A two-sided ideal of the form $(f_1, ..., f_m)$ will be called **finitely generated**.

Proposition 3.5. *The Kernel $\mathcal{K}$ of the abelization* $\mathrm{ab} : \mathbb{C}[[x_1|...|x_n]] \longrightarrow \mathbb{C}[[x_1, \dots, x_n]]$ *is finitely generated by the commutators $[x_i, x_j] = x_ix_j - x_jx_i$, for $1 \leq i < j \leq n$.*

Proof. We show that for each noncommutative power series f, the difference $f - f_{\mathrm{ab,unab}}$ is in the image under $\hat{\alpha}$ of the $\mathbb{C}[x]^{\hat{e}}$-left ideal generated by the commutators $[x_i, x_j]$, for $i < j$. Without restriction, let f be the sum of its homogeneous components f_k of degree $k \geq 2$. Each difference $f_k - f_{\mathrm{ab,unab},k}$ is of the form $\hat{\alpha}(\sum_{i<j} c_k^{ij} \cdot [x_i, x_j])$, for certain homogeneous $c_k^{i,j}$ in $\mathbb{C}[x]^{\hat{e}}$ of degree $k - 2$. Thus $f - f_{\mathrm{ab,unab}}$ is the image under $\hat{\alpha}$ of $\sum_{i<j}(\sum_k c_k^{ij}) \cdot [x_i, x_j]$. □

3.5 Locality

A family $(h_\alpha)_{\alpha \in A}$ of power series in $\mathbb{C}[[x]]$ is called **summable**, if, for each multi-index I, there are only finitely many $\alpha \in A$ such that $h_{\alpha,I} \neq 0$. In this case, we can form the power series

$$\sum_{\alpha \in A} h_\alpha := \sum_I (\sum_\alpha h_{\alpha,I})(x - p)^I.$$

Proposition 3.6. *The algebra $\mathbb{C}[[x]]$ is local with maximal ideal (x) generated by $x_1, \dots, x_n$.*

Proof. It suffices to show that each element f of $\mathbb{C}[[x]] \setminus (x)$ is a unit. Without restriction, say $f_0 = 1$. Then the family $(1 - f)^j; j \geq 0$ is summable. We have

$$f \cdot \sum_{j=0}^{\infty} (1 - f)^j = (1 - (1 - f)) \sum_{j=0}^{\infty} (1 - f)^j =$$

$$= \sum_{j=0}^{\infty} (1 - f)^j - \sum_{j=1}^{\infty} (1 - f)^j = 1.$$

Thus f is a left unit. In the same way, we show that f is a right unit. If f is convergent, the sum $\sum(1-f)^j$ is also convergent. This follows exactly as in the commutative case. □

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