Monopole Floer homology
for rational homology 3–spheres

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Abstract

We give a new construction of monopole Floer homology for spin^c rational homology 3–spheres. As applications we define two invariants of certain 4–manifolds with $b_1 = 1$ and $b^+ = 0$.

Contents

1   Introduction

In their recent book [10] Kronheimer and Mrowka constructed monopole Floer homology groups for arbitrary closed oriented 3–manifolds by applying a new kind of Morse homology to certain blown-up configuration spaces. In this paper we will develop the theory from a different angle in the case of an oriented rational homology 3–sphere Y equipped with a spin^c structure. Our starting-point will be the "irreducible" Floer cohomology $\mathrm{HF}^*(Y, m)$ which is the result of adapting Floer's original definition [4] to monopoles. Here the parameter m, which runs through a set $\mathfrak{m}(Y)$ of the form $m_0 + \mathbb{Z}$ with $m_0 \in \mathbb{Q}$, indicates which chamber for the metric and perturbation is being used. (Because of our choice of grading conventions we find it more natural to work with Floer *co*homology.)

By taking suitable limits of $\mathrm{HF}^*(Y, m)$ as $m \to \pm\infty$ we obtain invariants $\overline{\mathrm{HF}}^*(Y)$, $\underline{\mathrm{HF}}^*(Y)$ of the spin^c manifold Y. These "equivariant" Floer cohomology groups are modules over a polynomial ring and are related by a long exact sequence (which we call the *fundamental sequence*) involving a third module which is essentially the module of Laurent polynomials. Exact sequences of this kind are well known from Ozsváth and Szabó's Heegaard Floer homology [15] and also appear in [10]. In fact, it seems likely that our equivariant groups are isomorphic to the ($*$-versions of the) monopole Floer groups of Kronheimer and Mrowka (and their Heegaard cousins) when differences in grading conventions are taken into account.

A perhaps unusual feature in our model is that the fundamental sequence does not arise from a short exact sequence of cochain complexes. Exactness is established by geometric means, without recourse to homological algebra.

In the case of coefficients in a field $\mathbb{F}$ of characteristic p we give a precise description of how $\mathrm{HF}^*(Y, m; \mathbb{F})$ depends on m, in terms of an invariant $h_p(Y) \in \mathfrak{m}(Y)$. For a fixed chamber m the integer $m - h_p(Y)$ measures interaction between irreducible critical points and the reducible critical point of the Chern–Simons–Dirac functional, much in the same way as the instanton h–invariant. The invariant $h_p(Y)$ can also be read off from the fundamental sequence.

Our approach to understanding the relationship between the groups $\mathrm{HF}^*(Y, m)$ for different m involves a trick to avoid obstructed gluing prob-

lems. The idea is to study moduli spaces not only over the cylinder $\mathbb{R} \times Y$ but also over a manifold obtained by removing one point from $\mathbb{R} \times Y$, thereby creating an additional end $\mathbb{R}_- \times S^3$. By placing a suitable metric and perturbation on this new end we are able to correct the index of the Dirac operator. In this way we construct certain chain homomorphisms which are also essential to the proof of the exactness of the fundamental sequence.

In a slightly different direction we give some applications of Floer cohomology to spin^c smooth closed 4–manifolds X with $b_1(X) = 1$ which contain a non-separating smoothly embedded rational homology 3–sphere Y. In the case $b^+(X) > 1$ we express the Seiberg–Witten invariant of X as the Lefschetz number of a certain endomorphism of the *reduced* Floer cohomology of Y. If $b^+(X) = 0$ then, for certain spin^c structures on X, this Lefschetz number yields an invariant of X, i.e. is independent of Y. If in addition Y is an integral homology sphere or $b_2(X) = 0$ then $h_p(Y)$ is an invariant of (X, e), where e denotes the generator of $H_3(X; \mathbb{Z})$ represented by Y.

2 Statement of main results

2.1 Irreducible Floer groups

By a spin^c manifold we shall mean a smooth, oriented manifold with a spin^c structure. The spin^c structure will usually be suppressed from notation, except in explicit computations. Let Y be a spin^c rational homology 3–sphere. There is a classical invariant $\mathfrak{m}(Y) \in \mathbb{Q}/\mathbb{Z}$ which may be characterized as follows: If X is any compact spin^c 4–manifold with $\partial X = Y$ (as spin^c manifolds) then

$$\mathfrak{m}(Y) \equiv \frac{1}{8}(\sigma(X) - c_1(\mathcal{L}_X)^2) \mod \mathbb{Z}.$$

Here $\mathcal{L}_X$ is the determinant line bundle of the spin^c structure, and $\sigma(X)$ is the signature of X. We will think of $\mathfrak{m}(Y)$ as a set of rational numbers of the form $m_0 + \mathbb{Z}$, $m_0 \in \mathbb{Q}$. Note that $\mathfrak{m}(Y) = \mathbb{Z}$ if Y is an integral homology sphere. For any $m \in \mathfrak{m}(Y)$ we will define an *irreducible* Floer cohomology group $\text{HF}^*(Y, m)$ which is a finitely generated, graded Abelian group, where the grading runs through $2m + \mathbb{Z}$. This is the monopole analogue of the instanton homology groups introduced by Floer [4]. We will use the set-up of [5], see in particular Section 1.2 and Chapter 3 of that book. Choose a

Riemannian metric g and 1–form ν on Y, and define the Chern–Simons–Dirac functional ϑ as in [5, Section 3.2] with $\eta = d\nu$. Let $\mathcal{R} = \mathcal{R}_{(g,\nu)}$ be the set of gauge equivalence classes of critical points of ϑ. There is a unique reducible point in $\mathcal{R}$, which we denote by θ. Set $\mathcal{R}^* := \mathcal{R} \setminus \{\theta\}$. If $\alpha \in \mathcal{R}$ is non-degenerate then we define the index (or degree) of α as follows: Let X be any connected spin^c Riemannian 4–manifold with one tubular end $\mathbb{R}_+ \times Y$, and let $M(X, \alpha)$ be the moduli space of Seiberg–Witten monopoles over X which are asymptotic to α over the end. We define $\operatorname{ind}(\alpha) \in 2\mathfrak{m}(Y) + \mathbb{Z}$ and $\operatorname{ind}_2(\alpha) \in \mathbb{Z}/2\mathbb{Z}$ by

$$
\begin{aligned}
\dim M(X, \alpha) &= \operatorname{ind}(\alpha) + \frac{1}{4}\left(c_1(\mathcal{L}_X)^2 - \sigma(X)\right) - n_\alpha + b_1(X) - b^+(X), \\
\dim M(X, \alpha) &\equiv \operatorname{ind}_2(\alpha) - n_\alpha + b_1(X) - b^+(X) \mod 2,
\end{aligned}
\tag{1}
$$

where $n_\alpha = 0$ if α is irreducible and $n_\alpha = 1$ otherwise. Thus, $\operatorname{ind}_2(\alpha) \equiv \operatorname{ind}(\alpha) \mod 2$ if Y is an integral homology sphere. Moreover, $\operatorname{ind}_2(\theta) = 0$, whereas if α is irreducible then

$$
\operatorname{ind}_2(\alpha) \equiv \dim M(\theta, \alpha) \mod 2.
$$

For generic ν, all critical points of ϑ are non-degenerate, in which case $\mathcal{R}$ is a finite set. We can then form a cochain complex CF^* where CF^q is the free Abelian group generated by the elements of $\mathcal{R}^*$ of index q, and the differential is defined in terms of Seiberg–Witten moduli spaces over $\mathbb{R} \times Y$. The cohomology group of CF^* turns out to depend only on the spin^c manifold Y and $m := \frac{1}{2}\operatorname{ind}(\theta)$. This cohomology group is what we denote by $\mathrm{HF}^*(Y, m)$. It is defined for all $m \in \mathfrak{m}(Y)$ because every such m can be realized for some pair (g, ν), as we will show in Section 3.

The reason for our preference for Floer cohomology rather than homology is that we want $\operatorname{ind}(\alpha)$ to appear with a positive sign in (1), in which case the Seiberg–Witten invariants of a 4–manifold with boundary lie in the Floer *co*homology of the boundary.

Let G be any Abelian group and $\mathrm{HF}^*(Y, m; G)$ the irreducible Floer cohomology group with coefficients in G. If $m_1 \leq m_2$ then there is a canonical homomorphism

$$
J_{m_1}^{m_2} : \mathrm{HF}^*(Y, m_1; G) \to \mathrm{HF}^*(Y, m_2; G)
$$

which is an isomorphism in degree q unless $2m_1 \leq q < 2m_2$. These maps obey the composition law $J_{m_2}^{m_3} \circ J_{m_1}^{m_2} = J_{m_1}^{m_3}$.

Theorem 1 *Let $\mathbb{F}$ be a field of characteristic p. Then there is an element $h_p(Y) \in \mathfrak{m}(Y)$ such that for Floer cohomology with coefficients in $\mathbb{F}$ the following hold:*

(i) *Let $m_1 \leq m_2 \leq h_p(Y)$. Then $J_{m_1}^{m_2}$ is surjective, and its kernel has dimension 1 in degree $2m_1 + 2k - 1$ for $k = 1, \ldots, m_2 - m_1$ and is zero in all other degrees.*

(ii) *Let $h_p(Y) \leq m_1 \leq m_2$. Then $J_{m_1}^{m_2}$ is injective, and its cokernel has dimension 1 in degree $2m_2 - 2k$ for $k = 1, \ldots, m_2 - m_1$ and is zero in all other degrees.*

A precise description of the kernel and cokernel of $J_{m_1}^{m_2}$ is provided by Proposition 13 below. Some related material can be found in [12, Section 7].

It follows from Proposition 21 below that if $\mathrm{HF}^*(Y, h_0(Y); \mathbb{Z})$ is torsion-free then $\mathrm{HF}^*(Y, m; \mathbb{Z})$ is torsion-free for all m. Now, if Y admits a metric of positive scalar curvature, then $\mathrm{HF}^*(Y, h_0(Y); \mathbb{Z}) = 0$. Applying Theorem 1 with rational coefficients then yields a complete description of $\mathrm{HF}^*(Y, m; \mathbb{Z})$ for all m.

2.2 Equivariant Floer groups

We define the equivariant Floer cohomology groups $\overline{\mathrm{HF}}^q(Y; G)$ and $\underline{\mathrm{HF}}^q(Y; G)$ to be the direct and inverse limits, respectively, of the sequence of homomorphism

$$\cdots \to \mathrm{HF}^q(Y, m-1; G) \to \mathrm{HF}^q(Y, m; G) \to \mathrm{HF}^q(Y, m+1; G) \to \cdots.$$

Clearly, there are natural homomorphisms

$$\underline{\mathrm{HF}}^q(Y; G) \to \mathrm{HF}^q(Y, m; G) \to \overline{\mathrm{HF}}^q(Y; G)$$

for any q, m. The first of these maps is an isomorphism when $2m \leq q$, the second one when $q \leq 2m - 1$. Hence $\underline{\mathrm{HF}}^q(Y; G)$ vanishes for $q \gg 0$, and $\overline{\mathrm{HF}}^q(Y; G)$ vanishes for $q \ll 0$. We define the homology groups $\overline{\mathrm{HF}}_q(Y; G)$ and $\underline{\mathrm{HF}}_q(Y; G)$ in a similar fashion as limits of the groups $\mathrm{HF}_q(Y, m; G)$ as $m \to \mp\infty$.

The equivariant groups $\underline{\mathrm{HF}}^*(Y; G)$ and $\overline{\mathrm{HF}}^*(Y; G)$ both come equipped with a natural degree 2 endomorphism u, which may be thought of as "cup product" with a certain 2–dimensional cohomology class. There is also a natural homomorphism of $\mathbb{Z}[u]$–modules

$$J = J_Y : \underline{\mathrm{HF}}^*(Y; G) \to \overline{\mathrm{HF}}^*(Y; G)$$

whose image we denote by $\widehat{\mathrm{HF}}^*(Y;G)$ and call the *reduced* Floer cohomology of Y.

The two kinds of equivariant Floer groups are related by a long exact sequence (the *fundamental sequence*) which we now describe. This sequence involves the graded $\mathbb{Z}[u]$–module $P^*(Y)$ of "generalized polynomials"

$$\sum_{m \in \mathfrak{m}(Y)} a_m x^m$$

where the coefficients a_m are integers and only finitely many of them are non-zero. The grading and module structure are given by

$$\deg(x^m) = 2m, \qquad u \cdot x^m = x^{m+1}.$$

If Y is an integral homology sphere then $\mathfrak{m}(Y) = \mathbb{Z}$, so in this case $P^*(Y)$ is the module of Laurent polynomials. There is a canonical long exact sequence of $\mathbb{Z}[u]$–homomorphisms

$$\cdots \xrightarrow{D'} \underline{\mathrm{HF}}^*(Y;G) \xrightarrow{J} \overline{\mathrm{HF}}^*(Y;G) \xrightarrow{D} P^*(Y) \otimes G \xrightarrow{D'} \underline{\mathrm{HF}}^{*+1}(Y;G) \xrightarrow{J} \cdots$$
$$(2)$$

where J, D have degree 0 and D' has degree 1. The maps D, D', which are defined at the end of Section 9, in some sense measure interaction between reducible and irreducible critical points of the Chern–Simons–Dirac functional. In our set-up this sequence does not arise from a short exact sequence of cochain complexes. Instead, exactness is proved by means of an "almost inverse" of the map J_m^{m+1}.

Theorem 2 *If $\mathbb{F}$ is any field of characteristic p then $2h_p(Y)$ is the lowest degree in which $D : \overline{HF}^*(Y;\mathbb{F}) \to P^*(Y) \otimes \mathbb{F}$ is non-zero.*

Thus, with field coefficients the irreducible Floer groups contain no information that cannot be read off from the sequence (2). (With integral coefficients the situation might be different.)

Recall that reversing the orientation of Y reverses the sign of the Chern–Simons–Dirac functional. This gives rise to canonical "Poincaré duality" isomorphisms

$$\mathrm{HF}^*(-Y, -m; G) = \mathrm{HF}_{-1-*}(Y, m; G),$$
$$\overline{\mathrm{HF}}^*(-Y; G) = \underline{\mathrm{HF}}_{-1-*}(Y; G), \quad \underline{\mathrm{HF}}^*(-Y; G) = \overline{\mathrm{HF}}_{-1-*}(Y; G),$$

the latter two being isomorphisms of $\mathbb{Z}[u]$–modules.

2.3 The h–invariant

In this section p will be fixed and we mostly write h instead of h_p.

Theorem 3 *For all spin^c rational homology 3–spheres Y_1, Y_2 one has*

$$h(Y_1 \# Y_2) = h(Y_1) + h(Y_2).$$

Theorem 4 *Let W be a smooth, compact spin^c 4–manifold whose boundary is a disjoint union of rational homology spheres $Y_1, \ldots, Y_r$. If W has negative definite intersection form then*

$$-\sum_{j=1}^{r} h(Y_j) \geq \frac{1}{8}(b_2(W) + c_1(\mathcal{L}_W)^2).$$

Note that if d denotes the correction term in Heegaard Floer homology then the above two theorems hold with $-d/2$ in place of h, see [14].

If each Y_j is an integral homology sphere then by a theorem of Elkies [3] there is a spin^c structure on W for which the right hand side of the inequality in Theorem 4 is non-negative, and positive if the intersection form is not diagonalizable over the integers.

In the case of binary polyhedral spaces, [6, Proposition 8] holds with $-8h$ in place of γ. In particular, for the Poincaré homology sphere oriented as the boundary of the negative definite E_8–manifold one has

$$h(\Sigma(2, 3, 5)) = -1.$$

For simple lens spaces $L(q, 1)$, $q \geq 2$, Theorem 4 and the property $h(-Y) = -h(Y)$ suffice for the computation of h. To see this, let Σ be an internal connected sum of q disjoint (-1)–spheres in the q–fold connected sum $W := -q\mathbb{CP}^2$. Then the link of Σ, a copy of $L(q, 1)$, separates W into two pieces W_1, W_2. Now observe that every spin^c structure on $L(q, 1)$ is the restriction of a spin^c structure $\mathfrak{s}$ on W with $c_1(\mathfrak{s})^2 = -q$. Applying Theorem 4 to each of W_1, W_2 then gives

$$h(L(q, 1), \mathfrak{s}_j) = \frac{1}{8}\left[\frac{(q - 2j)^2}{q} - 1\right]$$

for a suitable labelling $\mathfrak{s}_0, \ldots, \mathfrak{s}_{q-1}$ of the spin^c structures on $L(q, 1)$.

This example shows that if W is a negative definite spin^c cobordism from one rational homology 3–sphere Y_1 to another one Y_2 (ie $\partial W = (-Y_1) \cup Y_2$)

then one need not have $h(Y_1) \geq h(Y_2)$ unless Y_1, Y_2 are integral homology spheres.

There is a relationship between the h–invariant and Casson's invariant which we will now describe. It follows from Theorem 1 that $\tilde\lambda(Y) := \chi(\mathrm{HF}^*(Y, m)) - m$ is independent of m and therefore an invariant of the spin^c–manifold Y. (Here χ denotes the Euler characteristic with respect to the mod 2 grading.) This result was proved with different methods by Lim [11]. He also showed that, when Y is an integral homology sphere, $-\tilde\lambda$ agrees with Casson's invariant λ (normalized so that $\lambda(\Sigma(2,3,5)) = -1$), thereby confirming a conjecture by Kronheimer. Since $\mathrm{HF}^*(Y, h_p(Y); \mathbb{F})$ maps isomorphically onto $\widehat{\mathrm{HF}}^*(Y; \mathbb{F})$, we obtain:

Theorem 5 *Let $\mathbb{F}$ be a field of characteristic p. Then for every oriented integral homology 3–sphere Y one has*

$$h_p(Y) - \chi(\widehat{HF}^*(Y; \mathbb{F})) = \lambda(Y),$$

where λ denotes Casson's invariant.

Similar results hold for the instanton and Heegaard Floer homologies, see [7, Section 8] and [14, Theorem 1.3].

2.4 Invariants of 4–manifolds

The results in this subsection will only be stated in their simplest form, ignoring the 1–dimensional μ–classes, and using integral coefficients for the Floer groups unless otherwise specified. Throughout the subsection Y, Y' will denote rational homology 3–spheres. All 4–manifolds will be smooth.

Let Z be a compact, homology oriented spin^c 4–manifold with $\partial Z = Y$. Then Z has a "relative" Seiberg–Witten invariant

$$\overline\psi(Z) \in \overline{\mathrm{HF}}^{-d}(Y),$$

where

$$d = \frac{1}{4}\left(c_1(\mathcal{L}_Z)^2 - \sigma(Z)\right) + b_1(Z) - b^+(Z).$$

If moreover $b^+(Z) > 1$ then there is also an invariant

$$\underline\psi(Z) \in \underline{\mathrm{HF}}^{-d}(Y)$$

with $J\underline\psi(Z) = \overline\psi(Z)$.

Theorem 6 *Let Z be a closed, connected spin^c 4–manifold which is separated by an embedded rational homology 3–sphere Y:*

$$Z = Z_0 \cup_Y Z_1.$$

Let Z_0 and Z_1 be homology oriented and let Z be equipped with the corresponding glued homology orientation as defined in [5, Section 12.6]. If $b^+(Z_1) > 1$ and $\dim M(Z) = 2n \geq 0$, then the Seiberg–Witten invariant of Z is

$$SW(Z) = u^n \overline{\psi}(Z_0) \cdot \underline{\psi}(Z_1).$$

We now turn to cobordisms. Let W be a compact, homology oriented spin^c 4–manifold with $\partial W = (-Y) \cup Y'$. Then W induces graded homomorphisms of $\mathbb{Z}[u]$–modules

$$\underline{\psi}(W) : \underline{\mathrm{HF}}^*(Y) \to \underline{\mathrm{HF}}^{*-d}(Y'), \qquad \overline{\psi}(W) : \overline{\mathrm{HF}}^*(Y) \to \overline{\mathrm{HF}}^{*-d}(Y')$$

which intertwine with the J–maps and therefore induce a homomorphism

$$\hat{\psi}(W) : \widehat{\mathrm{HF}}^*(Y) \to \widehat{\mathrm{HF}}^{*-d}(Y').$$

Here d is defined as above with W in place of Z. In general, the $\overline{\psi}$–invariant of a composite cobordism or of a composite manifold $Z \cup_Y W$ is the composition (or product) of the $\overline{\psi}$–invariants of the two pieces, provided the homology orientations are related as in Theorem 6. The same holds for the $\underline{\psi}$–invariants as long as these are defined. If $b^+(W) > 0$ then $\underline{\psi}(W)$ factors through J_Y.

If $b_1(W) = 0$ or $b^+(W) > 0$ we define the $\mathbb{Z}[u]$–homomorphism

$$P(W) : P^*(Y_1) \to P^{*-d}(Y_2)$$

as follows: If $b^+(W) > 0$ set $P(W) := 0$. If $b^+(W) = 0 = b_1(W)$ set $P(W) \cdot x^m := x^{m-d/2}$. Then we have a commutative diagramme

$$
\begin{array}{ccccccc}
\underline{\mathrm{HF}}^*(Y_1) & \longrightarrow & \overline{\mathrm{HF}}^*(Y_1) & \longrightarrow & P^*(Y_1) & \longrightarrow & \underline{\mathrm{HF}}^{*+1}(Y_1) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\underline{\mathrm{HF}}^{*-d}(Y_2) & \longrightarrow & \overline{\mathrm{HF}}^{*-d}(Y_2) & \longrightarrow & P^{*-d}(Y_2) & \longrightarrow & \underline{\mathrm{HF}}^{*+1-d}(Y_2)
\end{array}
\tag{3}
$$

where the vertical maps are the ones induced by W and the horizontal ones are as in (2). We expect that there is a similar commutative diagramme when $b^+(W) = 0$, $b_1(W) > 0$ but have not yet computed the map $P(W)$ in that case. (The left-most square of the diagramme commutes for any W, as already mentioned.)

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The homomorphisms induced by cobordisms can be defined more generally for Floer groups with coefficients in any Abelian group, and have similar properties.

We will now describe an analogue of Theorem 6 for a non-separating hypersurface Y.

If $V = V_0 \oplus V_1$ is any finitely generated mod 2 graded Abelian group then the *Lefschetz number* of a degree preserving endomorphism $f = f_0 \oplus f_1$ of V is defined as

$$L(f) := \operatorname{tr}(f_0) - \operatorname{tr}(f_1).$$

If Y is an oriented compact hypersurface in an oriented manifold X then we denote by $X_{\|Y}$ the oriented manifold with boundary $(-Y) \cup Y$ obtained by cutting X open along Y.

Theorem 7 *Let X be a closed, connected, homology oriented spin^c 4–manifold with $b^+(X) > 1$. Let Y be a non-separating, smoothly embedded oriented rational homology 3–sphere in X and set $W = X_{\|Y}$. If $\dim M(X) = 2n \geq 0$ then*

$$SW(X) = L(u^n \hat{\psi}(W)), \tag{4}$$

where $\hat{\psi}(W) : \widehat{HF}^*(Y) \to \widehat{HF}^{*-2n}(Y)$.

Here, as well as in Theorem 8 below, W should have the homology orientation determined by the homology orientation of X and the orientation of Y as specified in [5, Section 12.5].

The fact that the right hand side of (4) is defined whenever $\dim M(X)$ is non-negative and even can be exploited to extend the Seiberg–Witten invariant to a class of 4–manifolds for which the usual definition is not available.

Theorem 8 *Let X be a closed, connected, homology oriented spin^c 4–manifold with $b_1(X) = 1$, $b^+(X) = 0$, and $c_1(\mathcal{L}_X)^2 = \sigma(X)$, so that $\dim M(X) = 0$. For $j = 0, 1$ let Y_j be a non-separating, smoothly embedded oriented rational homology 3–sphere in X. Suppose Y_0 and Y_1 represent the same generator of $H_3(X; \mathbb{Z})$, and set $W_j = X_{\|Y_j}$. Then*

$$L(\hat{\psi}(W_0)) = L(\hat{\psi}(W_1)).$$

If Y_0 and Y_1 are in fact integral homology spheres, or if $b_2(X) = 0$, then

$$h_p(Y_0) = h_p(Y_1)$$

for all p.

Assuming the appropriate hypersurface $Y \subset X$ exists (which is not always the case) we can therefore define invariants

$$L(X) := L(\hat{\psi}(W)), \qquad h_p(X, e) := h_p(Y),$$

where $W = X_{\|Y}$, and e is the generator of $H_3(X; \mathbb{Z})$ represented by Y.

There is an alternative description of $L(X)$ as the number of irreducible monopoles over X, counted with sign, for a suitable choice of metric and perturbation (using a long neck $[-T, T] \times Y$). However, it was shown by Okonek–Teleman [13] that this number in general depends on the chamber of the metric and perturbation.

The reader may wish to compare the definition of $L(X)$ with Ruberman–Saveliev's description [17] of the Furuta–Ohta invariant [8] in the case of mapping tori.

2.5 Outline

Here is an outline of the remainder of this paper.

Section 3 provides a proof of a technical fact which underpins our whole approach to Floer homology, namely the non-emptiness of every chamber for the metric and perturbation on a spin^c rational homology 3–sphere. The proof relies on Bär's gluing theorem for spectra of operators of Dirac type. The proof may safely be skipped on first reading, since the techniques involved are rather different from those in the rest of the paper. Section 4 shows that certain properties of reducible monopoles over 4–manifolds are stable under small perturbations. In Section 5 we specify our orientation conventions and describe the behaviour of orientations under gluing maps. In Section 6 the irreducible Floer cohomology $\mathrm{HF}^*(Y, m)$ is defined and a first comparison of these groups for different m is made. Section 7 introduces the u–map and establishes its compatibility with cobordism-induced homomorphisms. The subject of Section 8 is the extra structure on the irreducible Floer groups that arises from the presence of the reducible critical point, and the way this structure relates to maps induced by cobordisms. In Section 9 the trick mentioned in the introduction is used to determine the kernel and cokernel of $J_{m_1}^{m_2}$. At the end of the section the maps D, D' are defined and the exactness of the fundamental sequence ist established. In Section 10 it is shown how the reduced Floer cohomology may be computed in an arbitrary chamber for the metric and perturbation. This is then used to prove the Lefschetz trace formula of Theorem 7. Section 11 contains all material on the h–invariant. Most of this section is logically independent

from Sections 8–10. The last section is then devoted to negative definite 4–manifolds with $b_1 = 1$.

3 Chambers for the metric and perturbation

Let Y be a spin^c rational homology 3–sphere with Riemannian metric g and perturbation 1–form ν. Let ϑ be the corresponding Chern–Simons–Dirac functional. Any representative of the reducible critical point θ has the form $(B - i\nu, 0)$, where B is a spin connection over Y with $\check{B}$ flat. We will say (g, ν) is an *M-pair* if θ is non-degenerate (ie if the kernel of the Dirac operator $\partial_{B-i\nu}$ is zero) and an *N-pair* if all critical points of ϑ are non-degenerate. According to [5, Proposition 8.1.1], if g is any metric on Y then (g, ν) will be an N-pair for generic ν.

To any M-pair (g, v) we associate an index $I(g, v) \in \mathfrak{m}(Y)$ as follows. Let X be any spin^c Riemannian 4–manifold with one tubular end $\mathbb{R}_+ \times Y$. Let A be a spin connection over X whose restriction to the end agrees with the spin connection induced by $B - i\nu$. Now set

$$I(g, v) = \mathrm{ind}_{\mathbb{C}}(D_A) - \frac{1}{8}(c_1(\mathcal{L}_X)^2 - \sigma(X)), \tag{5}$$

where $D_A : L_1^2 \to L^2$, cf. [5, Chapter 9]. Then $m := I(g, \nu)$ depends only on g, ν and the orientation and spin^c structure on Y. In situations where we are dealing with several pairs (g, ν) simultaneously we will often write θ_m instead of θ, so that

$$\mathrm{ind}(\theta_m) = 2m.$$

We may also write $\mathcal{R}(Y, m)$ instead of $\mathcal{R}$.

The following theorem is crucial to this whole paper:

Theorem 9 *For any $m \in \mathfrak{m}(Y)$ there exists a metric g and 1–form ν on Y such that $m = I(g, \nu)$.*

The idea of the proof, which occupies the remainder of this section, is to graft onto Y a family of Berger metrics on S^3 and then apply Hitchin's computation [9] of the spectrum of the Dirac operator for these metrics together with Bär's theorem [1] on spectra of twisted Dirac operators over connected sums.

Let $\mathfrak{m}'(Y)$ be the set of all $m \in \mathfrak{m}(Y)$ that can be realized as $I(g, v)$ for some pair (g, v). Theorem 9 is an immediate consequence of the following lemma.

Lemma 1 *(i) The set $\mathfrak{m}'(Y)$ is unbounded both below and above.*

(ii) If $m_1, m_3 \in \mathfrak{m}'(Y)$ and $m_2 \in \mathfrak{m}(Y)$ satisfy $m_1 < m_2 < m_3$ then $m_2 \in \mathfrak{m}'(Y)$.

The proof will involve the notion of spectral flow, which for the present purposes we will interpret as an index (cf. [2, 16]). To explain this, suppose Z is a closed manifold and $E \to Z$ a Hermitian vector bundle. Let $P_t : \Gamma(E) \to \Gamma(E)$, $a \leq t \leq b$ be a smooth family of first-order, self-adjoint, elliptic operators such that the kernels of P_a and P_b are zero and $\frac{\partial}{\partial t} + P_t$ is an elliptic operator over $Z \times [a, b]$. Choose a smooth function $f : \mathbb{R} \to [a, b]$ such that $f(t) = a$ for $t \leq -1$ and $f(t) = b$ for $t \geq 1$. We then define the spectral flow SF of the family $\{P_t\}_{a \leq t \leq b}$ to be the index of

$$\frac{\partial}{\partial t} + P_{f(t)} : L_1^2 \to L^2,$$

which is independent of f. We will make use of the following simple fact:

Observation 1 *If $h : [a, b] \to [0, \infty)$ is a continuous function such that $\ker(P_t - h(t)) = 0$ for all t then $SF = n_b - n_a$, where n_t is the number of eigenvalues of P_t in the interval $(0, h(t))$, counted with multiplicity.* $\square$

Proof of Lemma 1: (i) Hitchin [9] computed the spectrum of the Dirac operator over S^3 for a family of metrics $\{g_T\}_{T>0}$ known as Berger metrics, g_1 being the standard metric. Let D_T denote the Dirac operator over S^3 for the metric g_T. The eigenvalues of D_T are given by a countable family of smooth functions of T, each function describing an eigenvalue of a fixed multiplicity. Each function is either $\geq \sqrt{2}$ for all T, or has positive derivative and one zero. Moreover, there are infinitely many functions of the second kind. The set of those T for which D_T has non-trivial kernel is a closed and discrete subset of $\mathbb{R}$ whose smallest element is 4.

Fix $\tau > 1$ with $\ker(D_\tau) = 0$. The idea now is to graft the family $\{g_T\}_{1 \leq T \leq \tau}$ onto Y and apply Bär's gluing theorem for spectra of twisted Dirac operators over connected sums [1]. To this end we fix $x_0 \in S^3$ and construct a new family of metrics $g_{T,r}$ on S^3 by flattening g_T in the geodesic ball of radius r around x_0. To make this precise, choose $r_0 > 0$ such that S^3 contains a g_T–geodesic ball U_T of radius $2r_0$ around x_0 for $1 \leq T \leq \tau$. For the remainder of this paragraph we fix $T \in [1, \tau]$ and use g_T to define exponential maps etc. Let $\bar{g}_T$ denote the flat metric on U_T which under the exponential map corresponds to the constant metric $g_T(x_0)$ on $T_{x_0} S^3$.

Choose a smooth function $\beta : \mathbb{R} \to \mathbb{R}$ such that $\beta(t) = 0$ for $t \leq 1$ and $\beta(t) = 1$ for $t \geq 2$. For $0 < r < r_0$ define $\beta_r : U_T \to \mathbb{R}$ by

$$\beta_r(\exp(v)) = \beta(|v|/r)$$

for $v \in T_{x_0} S^3$ of norm $< 2r_0$. Let $g_{T,r}$ be the metric on S^3 which agrees with g_T outside U_T and such that

$$g_{T,r} = \beta_r g_T + (1 - \beta_r)\bar{g}_T \quad \text{on } U_T.$$

Then $g_{T,r} = \bar{g}_T$ on the ball of radius r around x_0.

We will need to compare Dirac operators for different metrics on S^3. In general, if $g, \tilde{g}$ are Riemannian metrics on a spin manifold Z, one can identify the spin bundles such that $\tilde{g}$–Clifford multiplication with a tangent vector e corresponds to g–Clifford multiplication with $v(e)$, where v is the positive g–symmetric bundle automorphism of the tangent bundle TZ with $v^*g = \tilde{g}$. Let $\nabla, \tilde{\nabla}$ be the Riemannian connections and $D, \tilde{D}$ the Dirac operators on Z for the metrics $g, \tilde{g}$, resp. Then for any spinor s one has a pointwise estimate

$$|(\tilde{D} - D)s| \leq C\left(\alpha \cdot |\nabla s| + (1 + \alpha) \cdot |\tilde{\nabla} - \nabla| \cdot |s|\right),$$

where $\alpha = |\tilde{g} - g|$, norms on tensors are taken with respect to g, and the constant C depends only on the dimension of Z.

Returning to our main discussion, let $D_{T,r}$ be the Dirac operator on S^3 for the metric $g_{T,r}$. It is a simple exercise to show that

$$\sup_{1 \leq T \leq \tau} \|g_{T,r} - g_T\|_{C^1} \to 0 \quad \text{as } r \to 0^+.$$

Recalling the explicit formula for the Riemannian connection in terms of the metric, we conclude that

$$\sup_{1 \leq T \leq \tau} \|D_{T,r} - D_T\| \to 0 \quad \text{as } r \to 0^+, \tag{6}$$

where we use the operator norm for bounded operators $L_1^2 \to L^2$.

We now introduce a suitable family of metrics on the cylinder $\mathbb{R} \times S^2$. As in [1, p 230] choose a smooth function $\rho : \mathbb{R} \to \mathbb{R}$ such that

- $\rho(t) = |t|$ for $|t| \geq 1$,

- $0 < \rho(t) \leq 1$ for $|t| \leq 1$,

- $|\rho'(t)| \leq 1$ for all t.

For $u > 0$ set $\rho_u(t) = u\rho(t/u)$. Let g_u^c denote the warped product metric on $\mathbb{R} \times S^2$ given by

$$ds^2 = dt^2 + \rho_u(t)^2 d\sigma^2,$$

where $d\sigma^2$ is the standard metric on S^2 and t is the $\mathbb{R}$–coordinate.

Choose a metric g on Y for which Y contains a Euclidean ball of radius r_0 around some point y_0. Fix $1 \leq T \leq \tau$ and $0 < u < r < r_0$. In this paragraph, S^3 and $Z_r := (-r, r) \times S^2$ will have the metrics $g_{T,r}$ and g_u^c, respectively, and $B_u(p)$ will denote the open geodesic ball of radius u around a point p. Let $A^\pm$ denote the region in Z_r defined by the inequality $u < \pm t < r$. Choose an orientation preserving isometry ι^+ between A^+ and the corresponding annulus in Y around y_0. Choose a smooth family $\{\phi_T\}_{T>0}$ of orientation reversing orthogonal transformations $\mathbb{R}^3 \to T_{x_0}S^3$, where $\mathbb{R}^3$ has the Euclidean metric. Identifying A^- with an annulus in $\mathbb{R}^3$ in the obvious way we obtain from ϕ_T an orientation preserving isometry ι^- between A^- and the corresponding annulus in S^3 around x_0. We can now introduce the glued smooth manifold

$$Y_T := S^3 \backslash \overline{B_u(x_0)} \ \cup_{\iota^-} \ (-r, r) \times S^3 \ \cup_{\iota^+} \ Y \backslash \overline{B_u(y_0)},$$

which inherits a metric $\tilde{g}_{T,r,u}$ from $g_{T,r}, g_u^c, g$. Set $Y^- := Y \backslash B_{r_0}(y_0)$.

Note that Y^- can also be regarded as a subset of Y_T. Since g_1 is the standard metric on S^3, there is clearly a diffeomorphism $q : Y \to Y_1$ which is the identity on Y^-.

The disjoint union

$$\tilde{Y} := \bigcup_{1 \leq T \leq \tau} Y_T \times \{T\}$$

has a natural smooth structure such that the projection $\pi : \tilde{Y} \to [1, \tau]$ is a submersion. Let Ξ be a vector field on $\tilde{Y}$ whose restriction to $Y^- \times [1, \tau]$ is $\frac{\partial}{\partial T}$ and such that $\pi_* \Xi = \frac{\partial}{\partial T}$. The flow of Ξ together with the diffeomorphism $q : Y \to Y_1$ yields a diffeomorphism $Y \times [1, \tau] \to \tilde{Y}$ which commutes with the projections onto $[1, \tau]$ and is the identity on $Y^- \times [1, \tau]$. Let $g_{T,r,u}$ be the pull-back of the metric $\tilde{g}_{T,r,u}$ by the induced diffeomorphism $Y \to Y_T$.

Choose a connection in the determinant line bundle $\mathcal{L}_0 \to Y$ which is flat outside Y^- and such that the kernel of the corresponding Dirac operator D for the metric g is zero (see [5, Lemma 8.1.2]). In this proof, all Dirac operators over Y will be defined using this connection. Let $D_{T,r,u}$ denote the Dirac operator on Y for the metric $g_{T,r,u}$. Choose $\Lambda > 0$ such that neither of the operators D_1 and D has any eigenvalue in $[-\Lambda, \Lambda]$. Because of (6) we

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can find an $r_1 \in (0, r_0)$ such that for $0 < r < r_1$ the operator $D_{1,r}$ has no eigenvalue in $[-\Lambda, \Lambda]$. By the claim on [1, p 230] and the explicit expression for δ given after that claim, there is a positive, increasing function f with the following significance: Set

$$K = \left\{ (r, u) \in \mathbb{R}^2 \,\middle|\, 0 < r < r_1, \ 0 < u < f(r) \right\}.$$

Then for $(r, u) \in K$ the kernel of $D_{1,r,u}$ is zero. Since any two points in K can be joined by a smooth path in K, it follows that the spectral flow of the Dirac operator on Y along a path of metrics from g to $g_{1,r,u}$ is independent of $(r, u) \in K$.

Let $T_1 < \cdots < T_N$ be the values of $T \in (1, \tau)$ for which $\ker(D_T) \neq 0$. Set $T_0 = 1$, $T_{N+1} = \tau$, $d_k = \dim \ker(D_{T_k})$. Choose c, λ such that

$$0 < c < \min_{0 \leq k \leq N} (T_{k+1} - T_k), \qquad 0 < \lambda \leq \Lambda,$$

and

(a) $\ker(D_T - \lambda) = 0$ if $|T - T_k| \leq c$ for some k,

(b) $D_{T_k - c}$ has no eigenvalue in $(0, \lambda)$,

(c) $D_{T_k + c}$ has exactly d_k eigenvalues in $(0, \lambda)$, counted with multiplicity.

Of course we also have

(d) $\ker(D_T) = 0$ if $1 \leq T \leq \tau$ and $|T - T_k| \geq c$ for all k.

By (6) there is an $r_2 \in (0, r_1)$ such that if $0 < r < r_2$ then (a)–(d) also hold with $D_{\bullet, r}$ in place of $D_\bullet$. Fix such an r. Another application of Bär's gluing theorem then shows that if $u > 0$ is sufficiently small then (a)–(d) also hold with $D_{\bullet, r, u}$ in place of $D_\bullet$. For such r, u the spectral flow of the path of operators $\{D_{T,r,u}\}_{1 \leq T \leq \tau}$ is therefore $\sum d_k$. (This follows from Observation 1 and the addition formula for the index [5, Corollary C.0.1].) Since this sum can be made arbitrarily large by suitable choice of τ, we conclude that the set $\mathfrak{m}'(Y)$ is unbounded above. Now recall that reversing the orientation of an odd-dimensional base-manifold reverses the sign of the Dirac operator, see [5, Section 3.1]. By replacing g_T with $\iota^*(g_T)$ for some fixed orientation reversing diffeomorphism ι of S^3 one can therefore show in a similar way that $\mathfrak{m}'(Y)$ is unbounded below.

(ii) If (g_0, v_0), (g_1, g_1) are any M-pairs on Y then these can be connected by a smooth path $\{(g_t, \nu_t)\}_{0 \leq t \leq 1}$ of metrics and 1–forms. By essentially the

same argument as in [5, Lemma 8.1.2] one can show that by replacing ν_t by $\nu_t + t(1-t)\omega$ for a generic 1–form ω on Y one can arrange that

$$\dim_{\mathbb{C}} \ker(D_{(g_t, \nu_t)}) \leq 1 \qquad \text{for } 0 \leq t \leq 1, \tag{7}$$

where $D_{(g_t, \nu_t)}$ denotes the Dirac operator on Y for the metric g_t and spin connection $B - i\nu_t$. Using Observation 1 it is easy to see that this implies statement (ii) of the lemma. $\quad\square$

4 Perturbations and reducibles

As in [5, Section 1.4] let X be a spin^c Riemannian 4–manifold with tubular ends $\mathbb{R}_+ \times Y_j$, $j = 1, \ldots, r$, where each Y_j is a rational homology sphere. Let (g_j, ν_j) be an N-pair for Y_j. By an *O-pair* for X (with respect to the above structure) we mean a pair $(\bar{g}, \mu)$ consisting of a metric $\bar{g}$ and 2–form μ on X which restrict to $1 \times g_j$ resp. $\pi_j^* d\nu_j$ over the j'th end, where $\pi_j : \mathbb{R}_+ \times Y_j \to Y_j$. By the *index* of $(\bar{g}, \mu)$ over the end $\mathbb{R}_+ \times Y_j$ we mean the index of (g_j, ν_j).

Given such $\bar{g}, \mu$ we consider moduli spaces $M(X; \vec{\alpha})$ as in [5, Section 3.4], where $\vec{\alpha} = (\alpha_1, \ldots, \alpha_r)$ and $\alpha_j \in \mathcal{R}_{Y_j}$. This moduli space also depends on $\vec{\mathfrak{p}} = (\mathfrak{p}_1, \ldots, \mathfrak{p}_r)$, where $\mathfrak{p}_j$ is a perturbation parameter for the Seiberg–Witten equations over $\mathbb{R} \times Y$. We may write $M(X; \vec{\alpha}; \vec{\mathfrak{p}})$ when we want to emphasize this dependence. In the case of reducible limits $\vec{\theta} = (\theta, \ldots, \theta)$ we will usually assume that $\bar{g}, \mu, \vec{\mathfrak{p}}$ have a certain generic form, which we now describe in the cases of relevance in this paper. Let $\bar{g}$ be given. We first take $\vec{\mathfrak{p}} = 0$. If $b^+(X) > 0$ choose μ such that $M(X; \vec{\theta})$ is free of reducibles. If $b_1(X) = 0 = b^+(X)$ then $M(X; \vec{\theta})$ contains a unique reducible point for any μ, and if in addition $\dim M(X; \vec{\theta}) \geq -1$ then by [5, Lemma 14.2.1] we may choose μ such that the reducible point is regular. Having chosen $\bar{g}, \mu$ we will now show that these properties of the moduli space are retained if we allow small perturbation parameters $\mathfrak{p}_j$.

Let $\mathfrak{P}_j$ be a Banach space of perturbation parameters $\mathfrak{p}_j$ as in [5, Section 8.2], and set $\mathfrak{P} = \mathfrak{P}_1 \times \ldots \mathfrak{P}_r$.

Proposition 1 *In the above situation the following hold when $\|\vec{\mathfrak{p}}\|$ is sufficiently small:*

(i) *If $b^+(X) > 0$ then $M(X; \vec{\theta})$ is free of reducibles.*

(ii) *If $b_1(X) = 0 = b^+(X)$ then $M(X; \vec{\theta})$ contains a unique reducible point $\omega(\vec{\mathfrak{p}})$. Moreover, if $\dim M(X; \vec{\theta}) \geq -1$ then this point is a regular point of the moduli space.*

Proof. We prove (ii) and leave the easier part (i) to the reader. We can identify the reducible part of $\mathcal{B}(X; \vec{\theta})$ with a configuration space $\mathcal{B}(\mathcal{L}_X)$ of connections in $\mathcal{L}_X$. Let A_0 be a connection in $\mathcal{L}_X$ representing $\omega(0)$. By [5, Proposition 2.3.1] any point in $\mathcal{B}(\mathcal{L}_X)$ has a representative A satisfying $d^* a = 0$, where $a = A - A_0$. Moreover, $[A]$ lies in the reducible part $M_{\vec{\mathfrak{p}}} \subset \mathcal{B}(\mathcal{L}_X)$ of the moduli space $M(X; \vec{\theta}; \vec{\mathfrak{p}})$ if and only if

$$ 0 = \frac{1}{2} F_A^+ + i\mu^+ + \epsilon(A, \vec{\mathfrak{p}}) = \frac{1}{2} d^+ a + \epsilon(A, \vec{\mathfrak{p}}), $$

where ϵ is a perturbation term. For $\delta > 0$ set

$$ U_\delta = \{ a \in L_1^{p,w}(X; i\Lambda^1) \,|\, d^* a = 0, \ \|a\|_{L_1^{p,w}} < \delta \}, $$

and consider the smooth map

$$ f : U_1 \times \mathfrak{P} \to L^{p,w}, \quad (a, \vec{\mathfrak{p}}) \mapsto \frac{1}{2} d^+ a + \epsilon(A_0 + a, \vec{\mathfrak{p}}). $$

Since $b_1(X) = 0 = b^+(X)$, the derivative in the first variable $D_1 f(0,0) = d^+$ is an isomorphism. By the implicit function theorem there exist $\epsilon, \delta > 0$ such that if $\|\vec{\mathfrak{p}}\| < \epsilon$ then f has a unique zero $(a(\vec{\mathfrak{p}}), \vec{\mathfrak{p}})$ with $a(\vec{\mathfrak{p}}) \in U_\delta$. We will show that $[A_0 + a(\vec{\mathfrak{p}})]$ is the only point in $M_{\vec{\mathfrak{p}}}$ when $\vec{\mathfrak{p}}$ is sufficiently small. So suppose $\{\vec{\mathfrak{p}}(n)\}$ is a sequence in $\mathfrak{P}$ converging to 0, and $\omega_n \in M_{\vec{\mathfrak{p}}(n)}$ for each n. After passing to a subsequence we may assume that ω_n chain-converges. (This does not quite follow from [5, Theorem 1.3.1] since we vary the perturbation parameters, but the proof of that theorem works just as well in the present situation.) For a rational homology 3–sphere there is only one reducible point in $\tilde{\mathcal{R}} = \mathcal{R}$, so the chain-limit lies in M_0 and is therefore equal to $[A_0]$. By [5, Proposition 6.4.1] the sequence $\{\omega_n\}$ actually converges in $\mathcal{B}(\mathcal{L}_X)$. Now, $A_0 + U_1 \to \mathcal{B}(\mathcal{L}_X)$ is an open map, so when n is sufficiently large there is a representative of ω_n of the form $A_0 + a_n$ such that $d^* a_n = 0$ and $\|a_n\|_{L_1^{p,w}} \to 0$. But then $a_n = a(\vec{\mathfrak{p}}(n))$ for $n \gg 0$. $\quad\square$

5 Orientations

Let Y be a spin^c rational homology 3–sphere equipped with an N-pair. By an *orientation* of a critical point $\alpha \in \mathcal{R}_Y^*$ we mean an orientation of $\mathcal{B}(\theta, \alpha) = \mathcal{B}(\mathbb{R} \times Y; \theta, \alpha)$, or more precisely, a section of the orientation cover $\lambda \to \mathcal{B}(\theta, \alpha)$ constructed in [5, Section 4.2]. We denote by $\mathcal{R}_Y^+$ the set of pairs (α, o) where $\alpha \in \mathcal{R}_Y^*$ and o is an orientation of α. Usually, o will be

suppressed from notation. An *m-orientation* of Y consists of an orientation of every point in $\mathcal{R}_Y^*$.

We will now state the conventions that we will use for orienting moduli spaces. Let X be a homology oriented Riemannian spin^c 4–manifold with tubular ends modelled on rational homology spheres. The *canonical* orientation o_{can} of $\mathcal{B}(X;\vec{\theta})$ is the one determined by the homology orientation of X as in [5, Proposition 12.2.1]. Now suppose the ends of X are $\mathbb{R}_- \times Y_j^- = \mathbb{R}_+ \times (-Y_j^-)$, $j = 1,\ldots,r^-$ and $\mathbb{R}_+ \times Y_j^+$, $j = 1,\ldots,r^+$. We will refer to these as "negative" and "positive" ends, respectively. Let $\alpha_j^\pm$ be a monopole over $Y_j^\pm$ which is irreducible and equipped with an orientation $o_j^\pm$ for $j = 1,\ldots,\rho^\pm$ and reducible for $j = \rho^\pm + 1,\ldots,r^\pm$. Let $\vec{\alpha}^\pm := (\alpha_1^\pm,\ldots,\alpha_{r^\pm}^\pm)$. We give $\mathcal{B}(X;\vec{\alpha}^-,\vec{\alpha}^+)$ the orientation o satisfying

$$o_1^- \# \cdots \# o_{\rho^-}^- \# o = o_{\mathrm{can}} \# o_1^+ \# \cdots \# o_{\rho^+}^+$$

as orientations of $\mathcal{B}(X;\vec{\theta},\vec{\alpha}^+)$, where $\#$ is as defined in [5, Section 13.6]. Note that this definition is sensitive to the orderings of the (positive and negative) ends of X. We call o the *proper* orientation of $\mathcal{B}(X;\vec{\alpha}^-,\vec{\alpha}^+)$. Recall that an orientation of $\mathcal{B}(X;\vec{\alpha}^-,\vec{\alpha}^+)$ determines an orientation of the regular and irreducible part of the moduli space $M(X;\vec{\alpha}^-,\vec{\alpha}^+)$.

If $X = \mathbb{R} \times Y$, where the negative and positive ends are the obvious ones, then we obtain orientations of $\mathcal{B}(\mathbb{R} \times Y; \alpha,\theta)$ and $\mathcal{B}(\mathbb{R} \times Y; \alpha,\beta)$ for all $\alpha,\beta \in \mathcal{R}_Y^+$. In particular, an m-orientation of Y determines an m-orientation of $-Y$ (which in turn induces the original m-orientation of Y, essentially because either $M(\alpha,\theta)$ or $M(\theta,\alpha)$ are even-dimensional).

Returning to the general case, note that changing the status of an "irreducible" end of X from negative to positive alters the orientation of $\mathcal{B}(X;\vec{\alpha}^-,\vec{\alpha}^+)$ by a sign which only depends on the parity of $b_0(X) + b_1(X) + b^+(X)$, the mod 2 indices $\mathrm{ind}_2(\alpha_j^\pm)$, and the new and old orderings of the ends of X.

Proposition 2 *Let X_1 and X_2 be two 4–manifolds as above, both connected. Suppose X_1 has only one positive end $\mathbb{R}_+ \times Y$, with limit α, and that X_2 has only one negative end $\mathbb{R}_- \times Y$, with the same limit α. Let $\vec{\alpha}^\pm$ be the limits assigned to the negative ends of X_1 and the positive ends of X_2, respectively. For $T > 0$ let $X^{(T)}$ denote the glued manifold obtained from the disjoint union $X_1 \cup X_2$ by replacing the two ends $\mathbb{R}_\pm \times Y$ with a neck $[-T,T] \times Y$. We regard $X^{(T)}$ as having the same negative (resp. positive) ends as X_1 (resp. X_2). Let $X^{(T)}$ have the glued homology orientation defined in [5, Section 13.6]. Let each limit $\alpha, \alpha_j^\pm$ be oriented if irreducible, and let*

o_1, o_2, o' *denote the proper orientations of* $\mathcal{B}(X_1; \vec{\alpha}^-, \alpha)$, $\mathcal{B}(X_2; \alpha, \vec{\alpha}^+)$, *and* $\mathcal{B}(X^{(T)}; \vec{\alpha}^-, \vec{\alpha}^+)$, *respectively. Then*

$$o_1 \# o_2 = o'. \tag{8}$$

If α *is irreducible, then any ungluing map as in [5, Theorem 11.1.1] is an orientation preserving diffeomorphism from an open subset of* $M(X^{(T)}; \vec{\alpha}^-, \vec{\alpha}^+)$ *onto an open subset of* $M(X_1; \vec{\alpha}^-, \alpha) \times M(X_2; \alpha, \vec{\alpha}^+)$.

Proof. Equation 8 is easily deduced from the properties of the #–operation mentioned in [5, Section 13.6]. The statement about the ungluing maps then follows from [5, Theorem 13.4.1] by unraveling the definition of #. $\qquad\square$

We call a diffeomorphism ϵ–*preserving* (where $\epsilon = \pm 1$) if it preserves or reverses orientations according to the sign of ϵ.

Proposition 3 *Let* X *be a 4–manifold as above which is connected and has only one negative end* $\mathbb{R}_- \times Y$ *and only one positive end* $\mathbb{R}_+ \times Y$. *Let the same oriented irreducible limit* α *be assigned to both ends. For* $T > 0$ *let* $X^{(T)}$ *denote the glued manifold obtained from* X *by replacing the two ends* $\mathbb{R}_\pm \times Y$ *with a neck* $[-T, T] \times Y$. *Let* $X^{(T)}$ *have the glued homology orientation defined in [5, Section 13.5]. Let* $\mathcal{B}(X; \alpha, \alpha)$ *have the proper orientation, and let* $\tilde{o}$ *denote the corresponding glued orientation of* $\mathcal{B}(X^{(T)})$ *defined in [5, Section 13.4]. Then*

$$\tilde{o} = (-1)^{ab+a+b} o_{can},$$

where $a = \mathrm{ind}_2(\alpha)$ *and* $b = \dim M(X^{(T)})$. *Any ungluing map as in [5, Theorem 11.1.1] is a* $(-1)^{ab+a+b}$–*preserving diffeomorphism from an open subset of* $M(X^{(T)})$ *onto an open subset of* $M(X; \alpha, \alpha)$.

Proof. The sign requires a simple computation, which we omit. The statement about the ungluing map is just a special case of [5, Theorem 13.4.1]. $\qquad\square$

6 Irreducible Floer groups

Let Y be a spin^c rational homology 3–sphere and $m \in \mathfrak{m}(Y)$. We will now define the irreducible Floer cohomology group $\mathrm{HF}^*(Y, m)$. By Theorem 9 and [5, Proposition 8.1.1] we can find an N-pair (g, ν) on Y with index m. For each pair of elements $\alpha, \beta \in \mathcal{R} := \mathcal{R}_{(g,\nu)}$, not both reducible, let $\mathbb{R}$ act on $M := M(\alpha, \beta)$ by

$$r \cdot \omega := \mathcal{T}_{-r}^* \omega, \tag{9}$$

for $(r, \omega) \in \mathbb{R} \times M$, where the translation operator $\mathcal{T}_s$ on $\mathbb{R} \times Y$ is defined by $\mathcal{T}_s(t, y) = (t + s, y)$. If moreover $\dim M(\alpha, \beta) = 1$ then the orbits of this action are precisely the connected components of M, and to each such component Γ we associate a sign ± 1 according as to whether the diffeomorphism $\mathbb{R} \to \Gamma$, $r \mapsto r \cdot \omega$ preserves or reverses orientation for $\omega \in \Gamma$.

Let CF^q be the free Abelian group generated by the elements of $\mathcal{R}^*$ of index q. We define the differential $d = d^q : \mathrm{CF}^q \to \mathrm{CF}^{q+1}$ as usual by

$$d\alpha = \sum_\beta (\# \check{M}(\alpha, \beta))\beta,$$

where $\check{M} = M/\mathbb{R}$ and the sum is taken over all $\beta \in \mathcal{R}^*$ of index $q + 1$. Here and elsewhere in this paper, $\#$ denotes a signed count. Note that the differential depends on the choice of a small perturbation of the Seiber-Witten equations over $\mathbb{R} \times Y$ as in [5]. However, this perturbation will not be reflected in our notation.

We also need to study flow-lines running between irreducibles and the reducible. Noting that

$$\dim M(\alpha, \theta_m) = 2m - 1 - \mathrm{ind}(\alpha),$$
$$\dim M(\theta_m, \alpha) = \mathrm{ind}(\alpha) - 2m,$$

we define, following [2], homomorphisms

$$\delta : \mathrm{CF}^{2m-2} \to \mathbb{Z}, \quad \alpha \mapsto \# \check{M}(\alpha, \theta_m),$$
$$\delta' : \mathbb{Z} \to \mathrm{CF}^{2m+1}, \quad 1 \mapsto \sum_\alpha (\# \check{M}(\theta_m, \alpha))\alpha,$$

where the sum is taken over all $\alpha \in \mathcal{R}^*$ of index $2m + 1$.

Proposition 4 $d \circ d = 0$, $\delta \circ d = 0$, $d \circ \delta' = 0$.

Proof. This follows from [5, Section 12.2] and Proposition 2. $\quad\square$

For a fixed Abelian group G let HF^* denote the cohomology of $\mathrm{CF}^* \otimes G$, and let

$$\mathrm{HF}^{2m-2} \xrightarrow{\delta_0} G \xrightarrow{\delta_0'} \mathrm{HF}^{2m+1}$$

be the homomorphisms induced by δ, δ'. It is convenient to extend δ_0 to all of HF^* by setting $\delta_0 := 0$ in degrees different from $2m - 2$, and similarly for δ.

We wish to understand the dependence of the group HF^* on (g, ν). (As we will see later, this problem has very much to do with δ_0, δ_0'.) So let

(g_j, ν_j), $j = 1, 2$ be two N-pairs for Y and let $m_j, \mathcal{R}_j^*, \mathrm{CF}_j, \mathrm{HF}_j$ be the corresponding data as introduced above. We will investigate for which q the usual cobordism construction yields a homomorphism $\mathrm{HF}_2^q \to \mathrm{HF}_1^q$. Let $\bar{g}$ be a Riemannian metric and μ a two-form on $\mathbb{R} \times Y$ such that $(\bar{g}, \mu)$ agrees with $(1 \times g_2, \pi_1^*(d\nu_2))$ in the region $(-\infty, -1] \times Y$ and agrees with $(1 \times g_1, \pi_2^*(d\nu_1))$ in $[1, \infty) \times Y$. Let X_{21} denote the manifold $\mathbb{R} \times Y$ equipped with the pair $\bar{g}, \mu$.

Lemma 2 *Let $\alpha \in \mathcal{R}_2^*$ and $\beta \in \mathcal{R}_1^*$ be critical points of index q. If either $q \leq 2m_1$ or $q \geq 2m_2 - 1$ then $M(X_{21}; \alpha, \beta)$ is compact.*

Proof. Since $M(X_{21}; \alpha, \beta)$ has dimension zero, it can only fail to be compact if it contains a sequence which has a chain-limit involving some monopole of negative index. The only candidate for such monopole would be the reducible point in $M(X_{21}; \theta_{m_2}, \theta_{m_1})$. For such a chain-limit to exist, however, the moduli spaces $M(\alpha, \theta_{m_2})$ and $M(\theta_{m_1}, \beta)$ would both need to have positive dimension. The lemma now follows from the dimension formulae for these spaces given above. $\quad\square$

For any q satisfying the assumptions of the previous lemma we define the homomorphism

$$k^q : \mathrm{CF}_2^q \to \mathrm{CF}_1^q, \quad \alpha \mapsto \sum_\beta (\#M(X_{21}; \alpha, \beta))\beta.$$

Lemma 3 *If $q \leq 2m_1 - 1$ or $q \geq 2m_2$ then $k^{p+1}d^p = d^p k^p$ for $p = q - 1, q$, hence k^q induces a homomorphism*

$$K^q : HF_2^q \to HF_1^q.$$

Proof. Again the point is that the assumptions on q prevent factorizations through the reducible monopole on X_{21}. $\quad\square$

Now suppose $m_1 \leq m_2$. Reversing the roles of Y_1 and Y_2 we obtain a cobordism X_{12} which by the previous lemma induces a homomorphism

$$J^q : \mathrm{HF}_1^q \to \mathrm{HF}_2^q$$

for all q. (That J^q is well-defined can also be seen by a dimension count.)

Proposition 5 *If $q \leq 2m_1 - 1$ or $q \geq 2m_2$ then J^q and K^q are inverse maps.*

Proof. Gluing the positive end of X_{12} with the negative end of X_{21} yields a cobordism $X^{(T)}$ with a band of length $2T$ where the metric and perturbation 2–form agrees with $(1 \times g_2, \pi_1^*(d\nu_2))$. Let α, β be two elements of $\mathcal{R}_1^*$ of index q. The assumptions on q mean precisely that one of the moduli spaces $M(X_{12}; \alpha, \theta_{m_2})$ and $M(\theta_{m_1}, \beta)$ have negative dimension. Therefore no sequence $\omega_n \in M(X^{(T(n))}; \alpha, \beta)$ with $T_n \to \infty$ can have a chain-limit involving the reducible monopole over X_{21}. Consequently, no factorizations through reducible critical points may appear in such chain-limits. Thus any chain-limit must lie in $M(X_{12}; \alpha, \gamma) \times M(X_{21}; \gamma, \beta)$ for some $\gamma \in \mathcal{R}_2^*$ of index q. The gluing theorem in [5] then says that for large T the endomorphism ℓ^q of CF_1^q induced by the cobordism $X^{(T)}$ is equal to the composite of the homomorphisms $\mathrm{CF}_1^q \to \mathrm{CF}_2^q \to \mathrm{CF}_1^q$ induced by X_{12} and X_{21}. Considering 1–parameter families of moduli spaces over $\mathbb{R} \times Y$ obtained by deforming the metric and perturbation form on $X^{(T)}$ to $(1 \times g_1, \pi_1^*(d\nu_1))$ we obtain a chain-homotopy between ℓ^q and the identity. Hence $K^q J^q = \mathrm{id}$, and a similar argument proves $J^q K^q = \mathrm{id}$. $\square$

A first corollary of this proposition is that if $m_1 = m_2$ then HF_1^* and HF_2^* are canonically isomorphic.

Definition 1 *For any $m \in \mathfrak{m}(Y)$ let $HF^*(Y, m; G)$ denote the cohomology group of the cochain complex $CF^* \otimes G$ defined using an N-pair (g, ν) of index m.*

Similarly, we define the Floer homology $\mathrm{HF}_*(Y, m; G)$ as the homology of the chain complex $\mathrm{Hom}(\mathrm{CF}^*, G)$.

We set $\mathrm{HF}^*(Y, m) := \mathrm{HF}^*(Y, m; \mathbb{Z})$. We will often write $\mathrm{CF}^*(Y, m)$ for CF^*, although CF^* really depends on g, ν and not just on m.

Proposition 5 now gives:

Proposition 6 *Let $m_1, m_2 \in \mathfrak{m}(Y)$ and $m_1 \leq m_2$. If either $q \leq 2m_1 - 1$ or $q \geq 2m_2$ then the natural homomorphism $HF^q(Y, m_1; G) \to HF^q(Y, m_2; G)$ is an isomorphism.* $\square$

Let $\tilde{g}$ be the round metric on S^3. Then $I(\tilde{g}, 0) = 0$ (see [5, Ch. 9]), hence

$$\mathrm{HF}^*(S^3, 0; G) = 0.$$

Combining this with Proposition 6 yields:

Proposition 7 *If $m \geq 0$ and either $q \leq -1$ or $q \geq 2m$ then $HF^q(S^3, m; G) = 0$. The same conclusion holds if $m \leq 0$ and either $q \leq 2m - 1$ or $q \geq 0$.* $\square$

7 The u–map and cobordisms

We begin by defining, in analogy with [7, Section 3], a natural degree 2 homomorphism $u : \mathrm{HF}^q(Y, m; G) \to \mathrm{HF}^{q+2}(Y, m; G)$ for $q \neq 2m - 2, 2m - 1$. The constraint on the degree is related to interaction with the reducible critical point. Set $\mathrm{CF}^* := \mathrm{CF}^*(Y, m)$. The u–map will be induced by a homomorphism

$$v : \mathrm{CF}^* \to \mathrm{CF}^{*+2}$$

defined in all degrees. Let $\alpha, \beta \in \mathcal{R}(Y, m)$ be distinct critical points, and let $\mathbb{E} \to M(\alpha, \beta)$ and $\mathbb{F} \to \mathcal{B}^*([-1, 1] \times Y)$ be the natural complex line bundles associated to the base-point $(0, y_0)$. (Here we use the same convention as in [5, Section 15.1].) Let $R : M(\alpha, \beta) \to \mathcal{B}^*([-1, 1] \times Y)$ be the natural restriction map. This is well defined because of the unique continuation property of Dirac operators. Note that we can identify $R^* \mathbb{F} = \mathbb{E}$. Choose a generic section s of $\mathbb{F}$ and let σ be the induced section of $\mathbb{E}$. If α, β are irreducible and $\mathrm{ind}(\beta) - \mathrm{ind}(\alpha) = 2$ then $\sigma^{-1}(0)$ is a finite set of oriented points, and we define the matrix coefficient $\langle v\alpha, \beta \rangle$ by

$$\langle v\alpha, \beta \rangle = \#\sigma^{-1}(0).$$

Proposition 8 $dv - vd + \delta'\delta = 0$.

Proof. This is similar to the proof of [7, Theorem 4(ii)]. Note however, that the modifications to the sections σ_j made in part (I) of that proof are unnecessary, as their role in ruling out cases (b), (c) can be replaced by a transversality argument. (See Part (V), Case 1 of the proof of Lemma 4 below.) This simplifies the proof somewhat. In particular, a general unique continuation property for monopoles is not needed for the proof of the proposition. (We do not know whether this property holds for the perturbed Seiberg–Witten equations defined in [5, Section 8.2], because these equations are not of gradient flow type.)

Up to sign, the coefficient of $\delta'\delta$ in the proposition is the Euler number of the rank 1 Hermitian vector bundle over $S^2 = D^2 \cup_{S^1} D^2$ whose "clutching map" $S^1 \to \mathrm{U}(1)$ has degree 1. The sign can be determined using Proposition 2. $\square$

Definition 2 *For $q \neq 2m - 2, 2m - 1$ we define*

$$u : HF^q(Y, m; G) \to HF^{q+2}(Y, m; G)$$

to be the homomorphism induced by v.

It will follow from Proposition 9 below that u is independent of the choice of section σ and of the metric and perturbations used in the definition of the Floer cohomology.

If W is any compact spin^c 4–manifold with boundary Y then we denote by

$$\hat{W} := W \cup_Y ([0, \infty) \times Y)$$

the result of attaching a half-infinite cylinder to each boundary component of W. Now suppose $Y = (-Y_1) \cup Y_2$, where each Y_j is a rational homology 3–sphere. If $\alpha_j \in \mathcal{R}(Y, m_j)$, $j = 1, 2$, then

$$\dim M(\hat{W}; \alpha_1, \alpha_2) = \mathrm{ind}(\alpha_2) - \mathrm{ind}(\alpha_1) - n_{\alpha_2} + d,$$

where

$$d = \frac{1}{4}(c_1(\mathcal{L}_W)^2 - \sigma(W)) + b_1(W) - b^+(W). \tag{10}$$

Let $m_j \in \mathfrak{m}(Y_j)$ and set $k = m_2 - m_1 + \frac{d}{2} \in \frac{1}{2}\mathbb{Z}$, so that

$$\dim M(\hat{W}; \theta_{m_1}, \theta_{m_2}) = 2k - 1.$$

If either $b^+(W) > 1$ or $k > -1$ then W gives rise to a cochain map

$$W^{\#} : \mathrm{CF}^*(Y_1, m_1) \to \mathrm{CF}^{*-d}(Y_2, m_2),$$

$$\alpha \mapsto \sum_{\beta} (\# M(\hat{W}; \alpha, \beta))\beta,$$

which in turn induces a homomorphism

$$W^* : \mathrm{HF}^*(Y_1, m_1; G) \to \mathrm{HF}^{*-d}(Y_2, m_2; G).$$

Note that W^* only depends on the spin^c manifold W.

These maps W^* are functorial with respect to composition of cobordisms as long as the composite cobordism has the glued homology orientation and also satisfies $b^+ > 1$ or $k > -1$.

If $b^+(W) \le 1$ and $k \le -1$ then W^* can still be defined under certain restrictions on the degree, cf. Lemma 3.

We will now show that W^* commutes with the u–maps, in as far as these are defined. Let $v_j : \mathrm{CF}^*(Y_j, m_j) \to \mathrm{CF}^{*+2}(Y_j, m_j)$ be a homomorphism as above.

Proposition 9 *If $b^+(W) > 1$ or $k \ge 0$ then there exists a homomorphism $\phi : CF^*(Y_1, m_1) \to CF^{*-d+1}(Y_2, m_2)$ such that*

$$W^{\#}v_1 - v_2 W^{\#} + \delta'\delta_W + \delta'_W\delta = d\phi \pm \phi d$$

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as maps $CF^*(Y_1, m_1) \to CF^{*-d+2}(Y_2, m_2)$, where

$$\delta_W : CF^{2m_2-1+d}(Y_1, m_1) \to \mathbb{Z}, \quad \alpha \mapsto \#M(\hat{W}; \alpha, \theta_{m_2}),$$

$$\delta'_W : \mathbb{Z} \to CF^{2m_1-d}(Y_2, m_2), \quad 1 \mapsto \sum_\beta (\#M(\hat{W}; \theta_{m_1}, \beta))\beta,$$

and $\delta_W := 0$ in degrees different from $2m_2 - 1 + d$.

Proof. See [7, Theorem 6]. $\square$

Corollary 1

$$W^* u = u W^* : HF^q(Y_1, m_1) \to HF^{q-d+2}(Y_2, m_2)$$

for all q for which both u–maps are defined.

The assumption that $b^+(W) > 1$ or $k \geq 0$ is only used to rule out that a 2–dimensional moduli space $M(\hat{W}; \alpha, \beta)$ with α, β irreducible may contain a chain-convergent sequence whose limit involves a reducible monopole on $\hat{W}$. Therefore, if $b^+(X) \leq 1$ and $k < 0$ then the conclusion of the proposition still holds under certain restrictions on the degree.

We will now define the u–maps on the equivariant Floer groups $\underline{HF}^*(Y; G)$ and $\overline{HF}^*(Y; G)$ (which for brevity we will here denote by $\underline{HF}^*$ and $\overline{HF}^*$). To define these maps in degree q choose $m_1, m_2 \in \mathfrak{m}(Y)$ with $2m_1 \leq q \leq 2m_2 - 3$. We define $u : \underline{HF}^q \to \underline{HF}^{q+2}$ and $u : \overline{HF}^q \to \overline{HF}^{q+2}$ by requiring that the left- and right-most squares in the following diagramme commute:

$$
\begin{array}{ccccccc}
\underline{HF}^q & \overset{\approx}{\longrightarrow} & HF^q_{m_1} & \longrightarrow & HF^q_{m_2} & \overset{\approx}{\longrightarrow} & \overline{HF}^q \\
u \downarrow & & u \downarrow & & u \downarrow & & u \downarrow \\
\underline{HF}^{q+2} & \overset{\approx}{\longrightarrow} & HF^{q+2}_{m_1} & \longrightarrow & HF^{q+2}_{m_2} & \overset{\approx}{\longrightarrow} & \overline{HF}^{q+2}
\end{array}
\qquad (11)
$$

Here $HF^p_m = HF^p(Y, m; G)$, and the horizontal maps are the natural ones. (The maps marked as isomorphisms are so by Proposition 6). This definition is independent of the choice of m_1, m_2. Since the middle square commutes, the homomorphism $J_Y : \underline{HF}^* \to \overline{HF}^*$ commutes with the u–maps.

Let W be as above, without any constraint on $b^+(W)$. We define $\underline{\psi}(W)$ in degree q by requiring that the diagramme

$$
\begin{array}{ccc}
HF^q(Y_1, m_1; G) & \overset{W^*}{\longrightarrow} & HF^{q-d}(Y_2, m_2; G) \\
& & \\
\approx\uparrow & & \approx\uparrow \\
\underline{HF}^q(Y_1; G) & \overset{\underline{\psi}(W)}{\Longrightarrow} & \underline{HF}^{q-d}(Y_2; G)
\end{array}
$$

commute, where m_1, m_2 are chosen so that $2m_1 \leq q$ and $2m_2 \leq q - d$. This definition is independent of the choice of m_1, m_2. In a similar fashion we define $\overline{\psi}(W)$ and the invariants $\underline{\psi}(Z)$, $\overline{\psi}(Z)$ of Subsection 2.4 when Z has one boundary component.

We can now easily prove Theorem 6 by observing that when stretching Z along Y and using an N-pair of index $m \gg 0$ on Y then no complications with reducibles can arise.

8   Cobordisms and reducibles

The homomorphisms

$$\delta_0 u^n : \mathrm{HF}^{2(m-n-1)}(Y, m; G) \to G, \quad u^n \delta_0' : G \to \mathrm{HF}^{2m+2n+1}(Y, m; G),$$

where $n \geq 0$, measure interaction between reducible and irreducible critical points and will play a central role in this paper. We will now give an alternative description of these maps.

First some notation. Let (g, ν) be an N-pair for Y of index m and consider the corresponding translationary invariant monopole equations on $\mathbb{R} \times Y$. Let S be a monopole over the band $(a, b) \times Y$ such that $\partial_t \vartheta(S_t) < 0$, where $S_t = S|_{\{t\} \times Y}$. (This will be the case unless S is gauge equivalent to the monopole determined by some critical point of ϑ, see [5, Section 7.1].) Let ϵ be a real number. If there exists a $\tau = \tau(S) \in (a, b)$ such that

$$\vartheta(S_\tau) = \vartheta(\theta_m) + \epsilon, \tag{12}$$

then we call τ the ϵ–slice of S.

Now set $K := [-1, 1] \times Y$ and let $\mathbb{L} \to \mathcal{B}^*(K)$ be the natural complex line bundle. Choose a generic smooth section s of $\oplus^n \mathbb{L}$. Choose $\epsilon > 0$ such that there is no monopole β over Y with $0 < |\vartheta(\beta) - \vartheta(\theta_m)| \leq \epsilon$. Then for any $\alpha \in \mathcal{R}^*(Y, m)$ and $\omega \in M(\alpha, \theta_m)$ the ϵ–slice $\tau(\omega)$ is well-defined. [5, Proposition 3.4.2 and Lemma 3.3.1] and an application of the implicit function theorem show that τ is a smooth function on $M(\alpha, \theta_m)$. Set

$$M'(\alpha, \theta_m) := \{\omega \in M(\alpha, \theta_m) \,|\, \tau(\omega) = 0, \ s(\omega|_K) = 0\}.$$

If α has index $2(m - n - 1)$ then $M'(\alpha, \theta_m)$ consists of a finite number of oriented points; let $\bar{\delta}_n(\alpha)$ denote the number of these points counted with sign. By considering the ends of the 1–dimensional spaces $M'(\beta, \theta_m)$ where $\mathrm{ind}(\beta) = 2m - 2n - 3$ one finds that $\bar{\delta}_n d = 0$. Similarly, for any $\alpha \in \mathcal{R}^*(Y, m)$ and $\omega \in M(\theta_m, \alpha)$ the $(-\epsilon)$–slice $\tau(\omega)$ is well-defined. Setting

$$M'(\theta_m, \alpha) = \{\omega \in M(\theta_m, \alpha) \,|\, \tau(\omega) = 0, \ s(\omega|_K) = 0\}$$

we obtain a homomorphism

$$\bar{\delta}'_n : \mathbb{Z} \to \mathrm{CF}^{2m+2n+1}(Y,m), \quad 1 \mapsto \sum_\alpha (\#M'(\theta_m, \alpha))\alpha$$

satisfying $d\bar{\delta}'_n = 0$. Let

$$\mathrm{HF}^{2(m-n-1)}(Y,m;G) \xrightarrow{\delta_n} G \xrightarrow{\delta'_n} \mathrm{HF}^{2m+2n+1}(Y,m;G)$$

be the homomorphisms induced by $\bar{\delta}_n, \bar{\delta}'_n$. This agrees with the previous definition in the case $n = 0$.

Proposition 10 $\delta_n = \delta_0 u^n$ and $\delta'_n = u^n \delta'_0$ for all $n \geq 0$.

Proof. Moving one base-point towards $-\infty$ (see the proof of Theorem 6 in [7]) one finds that $\delta_{n+1} = \delta_n u$. Similarly, $\delta'_{n+1} = u\delta'_n$. $\square$

We will now again consider a cobordism W as in the previous section, using the same notation.

Proposition 11 *If W is a cobordism as in Section 7 then the following hold:*

(i) *If $b^+(W) \geq 1$ then for all $j \geq 0$,*

$$\delta_j W^* = 0, \quad W^* \delta'_j = 0.$$

(ii) *If $b_1(W) = b^+(W) = 0$ and $k \geq 0$ then*

$$\begin{aligned}
\delta_j W^* = 0, & \quad W^* \delta'_j = 0 & \text{for } 0 \leq j \leq k-1, \\
\delta_j W^* = \delta_{j-k}, & \quad W^* \delta'_j = \delta'_{j-k} & \text{for } j \geq k.
\end{aligned}$$

Proof. We first introduce some notation that will be used in both parts of the proof. Choose a base-point $y_j \in Y_j$. We take $x_j := (0, y_j)$ as base-point for the band $K_j := [-1,1] \times Y_j$. The natural complex line bundle over $\mathcal{B}^*(K_j)$ will be denoted $\mathbb{L}_j$. For $j = 1,2$ choose $\epsilon_j > 0$ such that there is no monopole β over Y_j with $0 < \vartheta(\beta) - \vartheta(\theta_{m_j}) \leq \epsilon_j$. For elements ω of a moduli space $M(\hat{W}; \alpha, \theta_{m_2})$ the ϵ_1–slice on $\mathbb{R}_- \times Y_1$ and the ϵ_2–slice on $\mathbb{R}_+ \times Y_2$ (when defined) will be denoted $\tau_1(\omega)$ and $\tau_2(\omega)$, respectively. The restriction of ω to the band $[\tau_j(\omega) - 1, \tau_j(\omega) + 1] \times Y_j$ will be denoted $R_j(\omega)$.

(i) By assumption there is no reducible monopole in any moduli space over $\hat{W}$. If ϵ_2 is sufficiently small then τ_2 is defined on any moduli space $M(\hat{W}; \alpha, \theta_{m_2})$ with $\alpha \in \mathcal{R}(Y_1, m_1)$. Let s be a generic section of $\oplus^j \mathbb{L}_2$. Set

$$M'_\alpha := \{\omega \in M(\hat{W}; \alpha, \theta_{m_2}) \,|\, s(R_2(\omega)) = 0\}.$$

If $\text{ind}(\alpha) = 2m_2 - 2j - 1 + d$ then M'_α consists of a finite number of oriented points; for irreducible α let $\bar{\delta}_{W,j}(\alpha)$ denote the number of these points counted with sign. If $\text{ind}(\alpha)$ is one less, then since the number of ends of the 1–manifold M'_α, counted with sign, must be zero, we have

$$(\bar{\delta}_j W^\# - \bar{\delta}_{W,j} d)\alpha = 0,$$

hence $\delta_j W^* = 0$. Similarly one finds that $W^* \delta'_j = 0$.

(ii) We will focus on the statement about δ_j. The case of δ'_j is similar. We will write $\mathcal{R}_j = \mathcal{R}(Y_j, m_j)$ and $\text{CF}^*_j = \text{CF}^*(Y_j, m_j)$.

Let $\alpha \in \mathcal{R}^*_1$ have index $2m_2 - 2 + d = 2(m_1 + k) - 2$. Counting with sign the ends of the 1–manifold $M(\hat{W}; \alpha, \theta_{m_2})$ we find that

$$(\delta W^\# - \delta_W d)\alpha = \begin{cases} \delta\alpha & \text{if } k = 0, \\ 0 & \text{if } k \geq 1. \end{cases}$$

If $k = 0$ then $\delta_0 W^* = \delta_0$ in cohomology, and we are done. From now on assume $k \geq 1$, in which case $\delta_0 W^* = 0$. Propositions 9 and 10 yield

$$\delta_j W^* = \delta_0 u^j W^* = \delta_0 W^* u^j = 0, \quad 0 \leq j \leq k - 1.$$

For $j \geq k$, Proposition 9 yields

$$\delta_j W^* = \delta_k u^{j-k} W^* = \delta_k W^* u^{j-k}, \quad j \geq k.$$

To complete the proof of the proposition we need only prove the following lemma:

Lemma 4 $\delta_k W^* = \delta_0$.

Proof. Let $\alpha \in \mathcal{R}^*_1$ have index $\geq 2m_1 - 2$. Then the moduli space $M_\alpha := M(\hat{W}; \alpha, \theta_{m_2})$ has dimension $\leq 2k + 1$. We are going to cut down M_α, roughly speaking, by the k'th power of the Chern class of a complex line bundle $\mathbb{L}_\alpha$ to obtain a manifold Σ_α of dimension ≤ 1. This line bundle may be thought of as the natural line bundle associated to a base-point at ∞ over the end $\mathbb{R}_+ \times Y_2$. (Compare the definition of δ_n above). The proof will involve studying the ends of Σ_α. In the case when α has index $2m_1 - 2$ special care must be taken with the part of M_α consisting of monopoles whose spinors are small over W. By the gluing theorem in [5], such monopoles are obtained by gluing elements of $M(\alpha, \theta_{m_1}) := M(\mathbb{R} \times Y_1; \alpha, \theta_{m_1})$ with elements of $M_\theta := M(\hat{W}; \theta_{m_1}, \theta_{m_2})$ that are close to the reducible point ω_{red}. We will

use $c_1(\mathbb{L}_\alpha)^{k-1}$ to cut down the link of ω_{red}, a copy of $\mathbb{CP}^{k-1}$, to a point, while the last factor $c_1(\mathbb{L}_\alpha)$ will be played off against the gluing parameter.

Note that $M(\alpha, \theta_{m_1})$ is empty if $\mathrm{ind}(\alpha) > 2m_1 - 2$ or $\vartheta(\alpha) \leq \vartheta(\theta_{m_1})$. In that case

$$\inf_{[A,\phi] \in M_\alpha} \int_W |\phi| > 0$$

and the discussion below simplifies significantly.

First some notation: Let $x_0 \in W$ be a base-point for W and $\hat{W}$. The natural complex line bundle over M_α be denoted $\mathbb{L}_\alpha$. As in [5] we denote by $R_K(S)$ or $S|_K$ the restriction of a configuration S (or a gauge equivalence class of such) to a codimension 0 submanifold $K \subset \hat{W}$.

(I) This part of the proof is concerned with the framed moduli space $M_{\theta,x_0} := M_{x_0}(\hat{W}; \theta_{m_1}, \theta_{m_2})$ with base-point x_0. Let $(A_{\mathrm{red}}, 0)$ be a representative for ω_{red} and set $B_{\mathrm{red}} := A_{\mathrm{red}}|_W$. By [5, Section 2.5] there is a closed $U(1)$–invariant subspace

$$\Xi \subset L_1^p(W; i\Lambda_W^1 \oplus S_W^+)$$

such that

$$\Xi' := (B_{\mathrm{red}}, 0) + \Xi \to \mathcal{B}_{x_0}(W)$$

is a local diffeomorphism at $(B_{\mathrm{red}}, 0)$. Since the tangent space of M_{θ,x_0} at ω_{red} can be identified with $\ker D_{A_{\mathrm{red}}}$, it follows by unique continuation for harmonic spinors that the restriction map

$$R_W : M_{\theta,x_0} \to \mathcal{B}_{x_0}(W)$$

is immersive at ω_{red}. Let $G_0^+ \subset M_{\theta,x_0}$ be an open subset such that (i) G_0^+ is the image of a $U(1)$–equivariant smooth embedding $\iota : \mathbb{C}^k \to M_{\theta,x_0}$, (ii) R_W maps G_0^+ diffeomorphically onto a submanifold $H \subset \mathcal{B}_{x_0}(W)$, and (iii) there is a $U(1)$–invariant submanifold $\tilde{H} \subset \Xi'$ which maps diffeomorphically onto H. Let $\Xi_0 \subset \Xi$ be a closed linear complement of the tangent space $T_{(B_{\mathrm{red}},0)}\tilde{H}$, so that $\Xi = T_{(B_{\mathrm{red}},0)}\tilde{H} \oplus \Xi_0$. Then

$$\mu : \tilde{H} \times \Xi_0 \to \mathcal{B}_{x_0}(W), \quad (S, s) \mapsto [S + s]$$

is a $U(1)$–equivariant local diffeomorphism at $((B_{\mathrm{red}}, 0), 0)$, by the inverse function theorem. If $\pi_1 : \tilde{H} \times \Xi_0 \to \tilde{H}$ then $\pi_1 \circ \mu^{-1}$ is a $U(1)$–equivariant smooth map $V_0 \to \tilde{H}$ for some $U(1)$–invariant open neighbourhood V_0 of $\omega_{\mathrm{red}}|_W$. Let $\check{V}_0$ denote the image of V_0 in $\mathcal{B}(W)$ and $\check{V}_0^*$ the irreducible part of $\check{V}_0$. Let $f : \tilde{H} \to G_0^+$ be the obvious map, such that $[S] = f(S)|_W$ for all $S \in \tilde{H}$. Set

$$q_0 := f \circ \pi_1 \circ \mu^{-1} : V_0 \to G_0^+ \subset M_{\theta,x_0}$$

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and
$$z := \iota^{-1} : G_0^+ \to \mathbb{C}^k,$$

and let z_j be the j'th coordinate of z. If U is any open subset of an unframed moduli space over $\hat{W}$ which is mapped into $\check{V}_0^*$ by R_W then the U(1)–equivariant map
$$z_j \circ q_0 : V_0 \to \mathbb{C}$$

defines a section σ_j of the appropriate complex line bundle over U.

Let $G_0 \subset M_{\theta,x_0}$ be the image under ι of the ρ–ball about the origin in $\mathbb{C}^k$, where $\rho > 0$ is small. In particular, the closure $\overline{G}_0$ of G_0 should be contained in the image of q_0, and $R_W(\overline{G}_0) \subset V_0$.

(II) This part is concerned with the moduli space $M(\alpha, \theta_{m_1})$ and is only relevant in the case when $\mathrm{ind}(\alpha) = 2m_1 - 2$ and $\vartheta(\alpha) > \vartheta(\theta_{m_1})$. Then $M(\alpha, \theta_{m_1})$ has dimension 1. For any configuration S over K_1 set

$$J(S) := \vartheta(S_0),$$

where $S_0 = S|_{\{0\} \times Y_1}$. Because of the lack of unique conintuation for our perturbed equations we introduce a larger band $K_1' := [-\nu, \nu] \times Y_1$, where $\nu \geq 1$ is chosen so large that elements of $M(\alpha, \theta_{m_1})$ with $J = \epsilon_1$ are distinguished by their restrictions to K_1'. For small $\epsilon_1' > 0$ the set

$$G_1 := \{\omega \in M_{x_1}(\alpha, \theta_{m_1}) \,|\, |J(\omega) - \vartheta(\theta_{m_1}) - \epsilon_1| < \epsilon_1'\}$$

is precompact in $M_{x_1}(\alpha, \theta_{m_1})$ and one can find a U(1)–invariant open neighbourhood V_1' of $R_{K_1'}(\overline{G}_1) \subset \mathcal{B}_{x_1}^*(K_1')$ and a U(1)–invariant open subset $V_1 \subset \mathcal{B}_{x_1}^*(K_1)$ such that

(i) $R_{K_1}(V_1') \subset V_1$,

(ii) if $\check{V}_1 \subset \mathcal{B}(K_1)$ denotes the image of V_1, then $V_1 \to \check{V}_1$ is a trivial U(1)–bundle,

(iii) for every $\omega \in \check{V}_1$ there exists a (necessarily unique) $\tilde{\omega} \in M(\alpha, \theta_{m_1})$ with $J(\omega) = J(\tilde{\omega}|_{K_1})$,

(iv) $\check{V}_1'$ is a disjoint union of open sets, one for each component of $\check{G}_1$ (the image of G_1 in $M(\alpha, \theta_{m_1})$), such that $R_{K_1'}$ maps each of these components into the corresponding open subset of $\check{V}_1'$.

The $\tilde{\omega}$ in (iii) will be denotet $\check{q}_1(\omega)$. Choose a section e of the bundle $V_1 \to \check{V}_1$, and a section e' of the bundle $M_{x_1}(\alpha, \theta_{m_1}) \to M(\alpha, \theta_{m_1})$ which

agrees with the pull-back section $(R_{K_1})^* e$ on $\check{G}_1$. Let $q_1 : V_1 \to M_{x_1}(\alpha, \theta_{m_1})$ be the U(1)–equivariant map satisfying $q_1 \circ e = e' \circ \check{q}_1$.

(III) We will now construct the sections of $\mathbb{L}_\alpha$ that will be used to define Σ_α. For any configuration (A, Φ) over $\hat{W}$ set

$$\ell(A, \Phi) = \int_W |\Phi|.$$

If $\epsilon_0 > 0$ is sufficiently small then the region $U \subset M_\alpha$ defined by the inequality $\ell < \epsilon_0$ (which is empty if $M(\alpha, \theta_{m_1})$ is empty) can be described by the gluing theorem [5, Theorem 11.1.1] with $G = G_1 \times G_0$, $K = K_1 \times W$, $V = V_1 \times V_0$, and $q = q_1 \times q_0$. (Thus, roughly speaking, all elements of U can be obtained by gluing elements of G_1 with elements of G_0.) We will not spell this out precisely, but contend ourselves with a list of the conditions on ϵ_0 most relevant for the exposition (including two conditions not directly related to the gluing theorem):

(v) for any $\omega \in U$ the ϵ_1–slice $\tau_1(\omega)$ on $\mathbb{R}_- \times Y_1$ is defined and the restriction of ω to the band $[\tau_1(\omega) - \nu, \tau_1(\omega) + \nu] \times Y_1$ lies in $\check{V}_1'$,

(vi) $\omega|_W \in \check{V}_0$ for all $\omega \in U$,

(vii) all $\omega \in M_\theta$ with $\ell(\omega) < 2\epsilon_0$ lie in the image $\check{G}_0$ of G_0 in M_θ,

(viii) $\ell > \epsilon_0$ on any moduli space $M(\hat{W}; \beta_1, \beta_2)$ in which $\inf(\ell) > 0$.

Note that the chain-limit (see [5, Section 7.1]) of a sequence in U must either lie in M_α or have the form (ω_1, ω_0), where $\omega_1 \in M(\alpha, \theta_{m_1})$ and $\omega_0 \in \check{G}_0$. The map $U \to \check{V}_1'$ indicatet in (v) together with the decomposition of $\check{V}_1'$ in (iv) gives rise to a map $\psi : U \to \check{M}(\alpha, \theta_{m_1})$.

Let $\gamma : \mathbb{R} \to \hat{W}$ be a smooth path such that $\gamma(0) = x_0$ and

$$\gamma(t) = \begin{cases} (t, y_1) & t \leq -1, \\ (t, y_2) & t \geq 1. \end{cases}$$

Let $\mathcal{B}_x := \mathcal{B}(\hat{W}; \alpha, \theta_{m_2})$ be the framed orbit space of configurations with base-point x (see [5, Section 3.4]). For $a < b$ we define the holonomy map

$$h_a^b : \mathcal{B}_{\gamma(a)} \to \mathcal{B}_{\gamma(b)}$$

(depending on the choice of a reference spin connection A_o over $\hat{W}$) as follows: If $S = (A, \Phi)$ is a representative of $\omega \in \mathcal{B}_{\gamma(a)}$ and $u : \hat{W} \to U(1)$ an

element of the group of gauge transformations acting on the configuration space $\mathcal{C}(\hat{W}; \alpha, \theta_{m_2})$ such that

$$u(\gamma(b)) = \exp\left(\int_a^b \gamma^*(A - A_o)\right),$$

then $u(S)$ is a representative of $h_a^b(\omega)$. (Cf. [5, Eqn. 11.1].) One can easily verify that this holonomy map behaves functorially with respect to composition of paths.

That the map $R_1 : U \to \mathcal{B}^*(K_1)$ (defined in the very beginning of the proof of Proposition 11) is smooth follows from [5, Proposition 3.4.2 and Lemma 3.3.1] By means of holonomy along γ the pull-back $R_1^*(e)$ defines a smooth section σ^- (of constant norm 1) of $\mathbb{L}_\alpha|_U$. On the part of U where $\sigma_1 \neq 0$ we define a map η into $U(1)$ by the condition

$$\sigma^- = \eta \cdot \sigma_1/|\sigma_1|.$$

Choose a smooth function $c : \mathbb{R} \to (1, \infty)$ such that $c(t) = t$ for $t \geq 4$ and $c(t) = 2$ for $t \leq 3$. For $\omega \in M_\alpha$ set

$$\bar{\tau}_2(\omega) = \begin{cases} c(\tau_2(\omega)) & \text{if } \tau_2(\omega) \text{ is defined,} \\ 2 & \text{otherwise.} \end{cases} \tag{13}$$

For any $\omega \in M_\alpha$ let $\bar{R}_2(\omega) \in \mathcal{B}^*(K_2)$ denote the restriction of ω to the band $[\bar{\tau}_2(\omega) - 1, \bar{\tau}_2(\omega) + 1] \times Y_2$. Choose generic smooth sections $s_1', \ldots, s_k'$ of $\mathbb{L}_2$. By means of holonomy along γ the pull-back $(\bar{R}_2)^* s_j'$ defines a section σ_j' of $\mathbb{L}_\alpha$. In a similar way one obtains a section (also denoted σ_j') of the natural complex line bundle over M_θ^*.

Choose a smooth map $\xi : \mathbb{R} \times U(1) \to \mathbb{C}$ such that

$$\xi(t, z) = \begin{cases} 1, & t \geq 2, \\ z, & t \leq 1, \end{cases}$$

and such that 0 is a regular value of ξ. Then $\#\xi^{-1}(0)$ is independent of ξ, and a simple computation shows that $\#\xi^{-1}(0) = -1$.

Define a smooth section $\tilde{\sigma}_1$ of $\mathbb{L}_\alpha|_U$ as follows:

$$\tilde{\sigma}_1 := \begin{cases} \sigma^- & \text{where } \sigma_1 = 0, \\ \xi(-\tau_1|\sigma_1|, \eta) \cdot \sigma_1/|\sigma_1| & \text{where } \sigma_1 \neq 0. \end{cases}$$

Choose a smooth function $\tilde{\kappa} : \mathbb{R} \to \mathbb{R}$ such that $\tilde{\kappa}(t) = 1$ for $t \geq \epsilon_0/2$, $\tilde{\kappa}(t) > 0$ for $t > \epsilon_0/4$, and $\tilde{\kappa}(t) = 0$ for $t \leq \epsilon_0/4$. Set

$$\kappa := \tilde{\kappa} \circ \ell, \quad \underline{\kappa} := \tilde{\kappa} \circ (\ell/2),$$

and define smooth sections $\hat{\sigma}_1, \ldots, \hat{\sigma}_k$ of $\mathbb{L}_\alpha$ by

$$\hat{\sigma}_1 = \kappa \sigma_1' + (1 - \kappa)\tilde{\sigma}_1,$$
$$\hat{\sigma}_j = \underline{\kappa} \sigma_j' + (1 - \underline{\kappa})\sigma_j, \quad 2 \leq j \leq k.$$

(IV) We will now derive a gluing result which will be used in (V). For any $\tau > 0$ let $\hat{U}_\tau$ be the part of U where $-\tau_1 > \tau$, $0 < |\sigma_1| < \rho$, and $\sigma_j = 0$ for $j = 2, \ldots, k$. Applying the gluing theorem [5, Theorem 14.3.1] with $G = G_1 \times G_0$, $K = K_1 \times W$, $V = V_1 \times V_0$, and $q = q_1 \times q_0$ we find that if $\tau \gg 0$ then $\hat{U}_\tau$ is a smooth submanifold of M_α and the map

$$(\eta, -\tau_1, |\sigma_1|, \psi) : \hat{U}_\tau \to \mathrm{U}(1) \times (\tau, \infty) \times (0, \rho) \times \check{M}(\alpha, \theta_{m_1})$$

is an orientation reversing diffeomorphism. (Argue by contradiction to prove the map is injective for $\tau \gg 0$. Another application of the gluing theorem shows that the map is also a surjective submersion for large τ. To obtain submersiveness it is useful to vary the parameter T in the gluing theorem as well.)

(V) Let $\Sigma_\alpha \subset M_\alpha$ be the locus where $\hat{\sigma}_j = 0$ for $j = 1, \ldots, k$. If $\epsilon_0 > 0$ is sufficiently small then Σ_α is transversely cut out from M_α. If α has index $2m_1 - 1$ then Σ_α consists of a finite number of points, and we define

$$\delta_{W,k} : \mathrm{CF}_1^{2m_1 - 1} \to \mathbb{Z}, \quad \beta \mapsto \#\Sigma_\beta.$$

Now let α have index $2m_1 - 2$, in which case Σ_α has dimension 1. We will determine the ends of Σ_α. Let $\{\omega_n\}$ be a sequence in Σ_α. After passing to a subsequence we may assume that $\{\omega_n\}$ chain-converges to some triple consisting of a broken gradient line over $\mathbb{R} \times Y_1$, a monopole ω over $\hat{W}$, and a broken gradient line over $\mathbb{R} \times Y_2$. Suppose $\omega \notin M_\alpha$ (or equivalently, $\omega \notin \Sigma_\alpha$).

Case 1: $\omega \in M_\theta$. We will show shortly that ω must be the reducible point ω_{red}. Granted this we deduce from (IV) that the corresponding number of ends of Σ_α is $(\#\xi^{-1}(0))\delta\alpha = -\delta\alpha$.

Suppose now to the contrary that $\omega \neq \omega_{\mathrm{red}}$. If $\ell(\omega) > \epsilon_0/2$ then ω would satisfy the equations

$$\sigma_1' = 0,$$
$$\underline{\kappa}\sigma_j' + (1 - \underline{\kappa})\sigma_j = 0, \quad j = 2, \ldots, k.$$

However, the zero-set of these equations in M_θ^* is empty for a generic choice of sections s_j', because $\dim M_\theta^* = 2k - 1$. Hence $\ell(\omega) \leq \epsilon_0/2$, and consequently $\omega \in \check{G}_0$ by condition (vii) above. Certainly, ω must satisfy the equations

$$\sigma_j = 0, \quad j = 2, \ldots, k. \tag{14}$$

Therefore $\sigma_1(\omega) \neq 0$. Since $\tau_1(\omega_n) \to -\infty$ as $n \to \infty$, we conclude that

$$\tilde{\sigma}_1(\omega_n) = \frac{\sigma_1(\omega_n)}{|\sigma_1(\omega_n)|} \quad \text{for } n \gg 0,$$

so ω satisfies the equation

$$\kappa \sigma_1' + \frac{1 - \kappa}{|\sigma_1|} \sigma_1 = 0. \tag{15}$$

However, for dimensional reasons the common zero-set of the equations (14) and (15) in $\check{G}_0 \setminus \{\omega_{\mathrm{red}}\}$ is empty for a generic choice of section s_1'. This is a contradiction, so we conclude that $\omega = \omega_{\mathrm{red}}$ as claimed.

Case 2: $\omega \notin M_\theta$. Then $\ell(\omega) > \epsilon_0$ by condition (viii) above, so for large n one has $\sigma_j'(\omega_n) = 0$, $j = 1, \ldots, k$. There are now two possibilities: If factorization over $\mathbb{R}_+ \times Y_2$ does not occur then $\bar{\tau}_2(\omega_n) \to \bar{\tau}_2(\omega)$ and ω will lie in some $2k$–dimensional space $M(\hat{W}; \beta_1, \theta_{m_2})$ where $\beta_1 \in \mathcal{R}_1^*$. The corresponding number of ends of Σ_α is $-\bar{\delta}_{W,k} d\alpha$. If such factorization does occur then $\tau_2(\omega_n) \to \infty$ and ω must lie in some 0–dimensional space $M(\hat{W}; \alpha, \beta_2)$ where $\beta_2 \in \mathcal{R}_2^*$. The corresponding number of ends of Σ_α is $\bar{\delta}_k W^\# \alpha$.

Since the total number of ends of Σ_α must be zero, we conclude, after passing to Floer cohomology, that $\delta_0 = \delta_k W^*$. This completes the proof of the lemma, and also of Proposition 11. $\quad\square$

9 Irreducible Floer groups for different chambers

Let Y be a spin^c rational homology 3–sphere, $m \in \mathfrak{m}(Y)$, and G an Abelian group. In this section we will determine the kernel and image of the canonical homomorphism

$$\mathrm{HF}^*(Y, G; m) \to \mathrm{HF}^*(Y, G; m + \ell) \tag{16}$$

for any natural number ℓ. We first consider the case $\ell = 1$ and then do the general case by induction on ℓ. For $\ell = 1$ Proposition 11 (i) says that the map (16) induces a homomorphism

$$\bar{J} : \mathrm{coker}(\delta_0') \to \ker(\delta_0).$$

Proposition 12 $\bar{J}$ *is an isomorphism.*

The proof is divided into three lemmas. We begin by constructing two graded homomorphisms

$$Q_1, Q_2 : \mathrm{CF}^*(Y, m+1) \to \mathrm{CF}^*(Y, m).$$

such that $Q = Q_1 + Q_2$ will induce an inverse of $\bar{J}$.

Fix a point $y_0 \in Y$. Let $D \subset (-1, 1) \times Y$ be a compact 4–ball disjoint from $(-1, 1) \times \{y_0\}$, and set $W = [-1, 1] \times Y \setminus \mathrm{int}(D)$. We regard $\hat{W}$ as a manifold with two negative ends $\mathbb{R}_- \times Y$ and $\mathbb{R}_- \times S^3$ (in that order) and one positive end $\mathbb{R}_+ \times Y$. The notation $M(\hat{W}; \alpha_-, \alpha_+, \beta)$ will refer to a moduli space over $\hat{W}$ with limits $\alpha_\pm$ over $\mathbb{R}_\pm \times Y$ and β over $\mathbb{R}_- \times S^3$. Note that

$$\dim M(\hat{W}; \alpha_-, \alpha_+, \beta) = -\mathrm{ind}(\alpha_-) + (\mathrm{ind}(\alpha_+) - n_{\alpha_+}) - \mathrm{ind}(\beta).$$

If instead one regards $\hat{W}$ as a manifold with one negative end $\mathbb{R}_- \times Y$ and two positive ends $\mathbb{R}_+ \times S^3$ and $\mathbb{R}_+ \times Y$, then one obtains in general a different orientation of $M = M(\hat{W}; \alpha_-, \alpha_+, \beta)$. If α_+ and β are both irreducible then the two orientations differ by the sign $(-1)^{\mathrm{ind}_2(\beta)(d + \mathrm{ind}_2(\alpha_+))}$, where $d \equiv \dim M \mod 2$.

For the construction of Q we use an O-pair for $\hat{W}$ of index $m+1$, m, -1 on the ends $\mathbb{R}_- \times Y$, $\mathbb{R}_+ \times Y$, $\mathbb{R}_- \times S^3$, resp. An important point in the proof of the proposition will be that

$$\dim M(\hat{W}; \theta_{m+1}, \theta_m, \theta_{-1}) = -1.$$

Generically, this moduli space consists of the reducible monopole ω_{red} only, and this is a regular point.

Fix a small $\epsilon > 0$ and for any element ω of a moduli space $M(\hat{W}; \alpha, \beta, \theta_{-1})$ where α, β are not both reducible let $\tau(\omega)$ be the $(-\epsilon)$–slice over the end $\mathbb{R}_- \times S^3$, when this is defined. Let c be as in (13) and define

$$\bar{\tau}(\omega) = \begin{cases} -c(-\tau(\omega)) & \text{if } \tau(\omega) \text{ is defined,} \\ -2 & \text{otherwise.} \end{cases}$$

Set $K = [-1, 1] \times S^3$ and let $R(\omega) \in \mathcal{B}^*(K)$ denote the restriction of ω to the band $[\bar{\tau}(\omega) - 1, \bar{\tau}(\omega) + 1] \times S^3$. Choose a generic smooth section s of the natural complex line bundle over $\mathcal{B}^*(K)$ and set

$$M'(\alpha, \beta) := \{\omega \in M(\hat{W}; \alpha, \beta, \theta_{-1}) \,|\, s(R(\omega)) = 0\}.$$

If α, β are both irreducible and of the same index then $M'(\alpha, \beta)$ is a finite set of points, so we can define Q_1 on generators α by

$$Q_1 \alpha = \sum_\beta (\# M'(\alpha, \beta)) \beta.$$

According to Proposition 7 we have $\mathrm{HF}^1(S^3, -1) = 0$, so there is a cochain $z \in \mathrm{CF}^0(S^3, -1)$ such that $\bar{\delta}'_1 + dz = 0$. Define Q_2 by

$$Q_2 \alpha = \sum_\beta (\# M(\hat{W}; \alpha, \beta, z)) \beta,$$

where we regard $M(\hat{W}; \alpha, \beta, z)$ formally as a linear function in z. (To simplify language, we will often speak of such formal linear combinations of moduli spaces as a single moduli space.) Set $Q = Q_1 + Q_2$.

Lemma 5 $dQ - Qd + \delta'\delta = 0$.

Here $\delta : \mathrm{CF}^{2m}(Y, m+1) \to \mathbb{Z}$ and $\delta' : \mathbb{Z} \to \mathrm{CF}^{2m+1}(Y, m)$ are the maps defined in Section 6.

Proof. Let $\alpha \in \mathcal{R}^*(Y, m+1)$ and $\beta \in \mathcal{R}^*(Y, m)$ satisfy $\mathrm{ind}(\beta) = \mathrm{ind}(\alpha) + 1$. Choose $\epsilon_1 > 0$ such that there is no monopole $\gamma \in \mathcal{R}^*(Y, m+1)$ with $0 < \vartheta(\gamma) - \vartheta(\theta_{m+1}) \le \epsilon_1$. Choose $\epsilon_2 > 0$ such that there is no monopole $\gamma \in \mathcal{R}^*(Y, m)$ with $0 < \vartheta(\theta_m) - \vartheta(\gamma) \le \epsilon_2$. For any configuration (A, Φ) over $\hat{W}$ set

$$\ell(A, \Phi) := \int_W |\Phi|. \tag{17}$$

Note that if $\{\omega_n\}$ is any chain-convergent sequence in $M(\hat{W}; \alpha, \beta)$ with $\ell(\omega_n) \to 0$ then the chain-limit has the form $(\omega', \omega_{\mathrm{red}}, \omega'')$, where $\omega' \in \check{M}(\alpha, \theta_{m+1})$, $\omega'' \in \check{M}(\theta_m, \beta)$. Fix a small $\epsilon_0 > 0$. If $\omega \in M(\hat{W}; \alpha, \beta)$ and $\ell(\omega) < \epsilon_0$ then the ϵ_1–slice of ω on $\mathbb{R}_- \times Y$ and the ϵ_2–slice of ω on $\mathbb{R}_+ \times Y$ will be well-defined; we denote these by $\tau_1(\omega)$ and $\tau_2(\omega)$, resp. Let $R_j(\omega)$ denote the restriction of ω to the band $[\tau_j(\omega) - 1, \tau_j(\omega) + 1]$.

Let $\mathbb{L} \to M(\hat{W}; \alpha, \beta)$ be the natural complex line bundle associated to the base-point $(0, y_0)$. Let $q : (-\infty, 1] \to \hat{W}$ be a path with $q(1) = (0, y_0) \in W$ and $q(t) = (t, y_{-1})$ for $t \le 0$, where $y_{-1} \in S^3$. We use holonomy along the paths $\mathbb{R} \times \{y_0\}$ and q to identify $\mathbb{L}$ with the natural complex line bundles associated to other base-points along these paths. Much as in the proof of Lemma 4 we construct a section $\hat{\sigma}$ of $\mathbb{L}$ such that

- $\hat{\sigma}$ agrees with the pull-back section $R^* s$ in the region where $\ell \ge \epsilon_0/2$,

- For $j = 1, 2$, if $\ell(\omega) \leq \epsilon_0/4$ and $(-1)^j(\tau_1(\omega) + \tau_2(\omega)) \geq 1$ then $\hat{\sigma}(\omega)$ is non-zero and depends only on the restriction of ω to $[\tau_j(\omega) - 1, \tau_j(\omega) + 1] \times Y$.

In the intermediate regions we interpolate as in the proof of Lemma 4. Let $\Sigma \subset M(\hat{W}; \alpha, \beta, \theta_{-1})$ be the zero-set of $\hat{\sigma}$.

Now let $\{\omega_n\}$ be a chain-convergent sequence in Σ whose limit does not lie in Σ. If $\bar{\tau}(\omega_n) \to \infty$ then factorization takes place over $\mathbb{R}_- \times S^3$ through some $\gamma \in \mathcal{R}^*(S^3, -1)$ of index 1; the corresponding number of ends of Σ (counted with sign) is $\langle \chi, \bar{\delta}'_1 \rangle$, where

$$\chi = \sum_\gamma (\#M(\hat{W}; \alpha, \beta, \gamma))\gamma \in \mathrm{CF}_1(S^3, -1).$$

If $\bar{\tau}(\omega_n)$ stays bounded, then there are two cases depending on whether the chain-limit involves the reducible monopole over $\hat{W}$ or not. In the reducible case one finds as in the proof of Proposition 8 that the corresponding number of ends of Σ is $\langle \delta' \delta \alpha, \beta \rangle$. In the irreducible case factorization takes place over one of the tubular ends $\mathbb{R}_\pm \times Y$; here the corresponding number of ends of Σ is $\langle (dQ_1 - Q_1 d)\alpha, \beta \rangle$. Since the total number of ends of Σ must be zero, we conclude that

$$\langle (dQ_1 - Q_1 d + \delta' \delta)\alpha, \beta \rangle + \langle \chi, \bar{\delta}'_1 \rangle = 0.$$

Similarly, by considering the ends of $M(\hat{W}; \alpha, \beta, z)$ we find that

$$\langle (dQ_2 - Q_2 d)\alpha, \beta \rangle + \langle \chi, dz \rangle = 0.$$

Summing the last two equations and recalling that $\bar{\delta}'_1 + dz = 0$ we obtain

$$\langle (dQ - Qd + \delta' \delta)\alpha, \beta \rangle. \qquad \square$$

Consider an arbitrary $m' \in \mathfrak{m}(Y)$ and let $B : \mathrm{CF}^*(Y, m') \to \mathrm{CF}^*(Y, m')$ be the homomorphism defined in the same way as Q, but now using an O-pair for $\hat{W}$ of index m', -1 on the ends $\mathbb{R}_+ \times Y$, $\mathbb{R}_- \times S^3$, resp., and with the same element z. Arguing as in the proof of Lemma 5 we find that B is a cochain map (there are no complications from reducibles here).

Lemma 6 *B induces the identity map on $HF^*(Y, m'; G)$.*

As there are two signs in the proof which the author has not computed (and which affect the sign of I in formula (18)), this lemma as well as the

next one really hold only up to sign. However, we will ignore this, as it is irrelevant to the proof of Proposition 12.

Proof. Set $Z := [-1, 0] \times S^3$, and let S^3_{-1} and S^3_0 be two copies of the three-sphere. We will think of $\hat{Z}$ as having tubular ends $\mathbb{R}_- \times S^3_{-1}$ and $\mathbb{R}_+ \times S^3_0$, and the end $\mathbb{R}_- \times S^3$ of $\hat{W}$ will now be denoted $\mathbb{R}_- \times S^3_0$. Choose an O-pair for $\hat{W}$ which has index m' on each of the ends $\mathbb{R}_\pm \times Y$ and which agrees with $(1 \times \tilde{g}, 0)$ on $\mathbb{R}_- \times S^3_0$, where $\tilde{g}$ is the round metric on the three-sphere. Choose an O-pair for $\hat{Z}$ which has index -1 on the end $\mathbb{R}_- \times S^3_{-1}$ and which agrees with $(1 \times \tilde{g}, 0)$ on $\mathbb{R}_+ \times S^3_0$. Let $X^{(T)}$ denote the manifold with tubular ends obtained from $\hat{W} \cup \hat{Z}$ by replacing the ends $\mathbb{R}_\pm \times S^3_0$ with a neck $[-T, T] \times S^3_0$. Let $X^{(T)}$ be equipped with the O-pair inherited from the O-pairs of $\hat{W}, \hat{Z}$ in the natural way.

Now fix $\alpha, \beta \in \mathcal{R}^*(Y, m')$ of the same index and for any $\gamma \in \mathcal{R}(S^3, -1)$ define the parametrized moduli space

$$\mathcal{M}_\gamma := \bigcup_{T \geq 1} M(X^{(T)}; \alpha, \beta, \gamma) \times \{T\}.$$

Set $\mathcal{M} := \mathcal{M}_{\theta_{-1}}$. Fix a small $\epsilon_0 > 0$ and let U be the set of all $(\omega, T) \in \mathcal{M}$ with $\ell(\omega) < \epsilon_0$, where

$$\ell(A, \Phi) := \int_Z |\Phi|.$$

We now copy the construction in the proof of Lemma 4 (with $k = 1$) of a section $\hat{\sigma}_1$ of the natural complex line bundle over $\mathcal{M}$ corresponding to some base-point in Z. Here the present Z, $M(\hat{W}; \alpha, \beta, \theta_0)$, $-T$ correspond to the W, $\check{M}(\alpha, \theta_{m_1})$, τ_1 resp. in the proof of Lemma 4. Let $\mathcal{M}' \subset \mathcal{M}$ be the zero-set of $\hat{\sigma}_1$ and set

$$\mathcal{M}'' := \mathcal{M}' + \mathcal{M}_z,$$

regarded as a formal linear combination of oriented, smooth 1–manifolds with boundary. With the orientation conventions in [5] one has

$$\#\partial\mathcal{M}'' = -\langle B\alpha, \beta \rangle.$$

We will now describe the ends of $\mathcal{M}''$. Because $\dim M(\hat{Z}; z, \theta_0) = -1$ and there are no non-empty moduli spaces over $\hat{Z}$ of negative expected dimension, it follows that $M(X^{(T)}; \alpha, \beta, z)$ is empty for large T. Therefore the ends of $\mathcal{M}_z$ correspond to factorization through irreducible critical points over the tubular ends $\mathbb{R}_\pm \times Y$ or $\mathbb{R}_- \times S^3_{-1}$, which may occur for finitely may values of T.

Determining the ends of $\mathcal{M}'$ requires a little more consideration. Let $\{(\omega_n, T_n)\}$ be a sequence in $\mathcal{M}'$. By passing to a subsequence we may arrange that the sequence chain-converges. This means in particular that $\{T_n\}$ converges to some point $T_\infty \in [1, \infty]$. Suppose the chain-limit does not lie in $\mathcal{M}$ (or equivalently, not in $\mathcal{M}'$). We consider first the case $T_\infty < \infty$. Observe that

$$\dim M(X^{(T_\infty)}; \theta_{m'}, \theta_{m'}, \theta_{-1}) = 1.$$

Therefore, for dimensional reasons the limit over $X^{(T_\infty)}$ cannot be reducible. Hence factorization through an irreducible critical point has taken place over $\mathbb{R}_\pm \times Y$ or $\mathbb{R}_- \times S^3_{-1}$. We now turn to the case $T_\infty = \infty$. Since

$$\dim M(\hat{Z}; \theta_{-1}, \theta_0) = 1,$$

the chain-limit must have the form (ω, ω'), where $\omega \in M(\hat{W}; \alpha, \beta, \theta_0)$ and $\omega' \in M(\hat{Z}; \theta_{-1}, \theta_0)$. As in the proof of Lemma 4 (Part (V), Case 1) one finds that ω' must be the reducible point and that the corresponding number N of ends of $\mathcal{M}'$ is equal to $\#M(\hat{W}; \alpha, \beta, \theta_0)$. Let $W^{(T)}$ be the result of capping off the end $\mathbb{R}_- \times S^3_0 \subset \hat{W}$ with a disk, allowing for a long neck $[-T, T] \times S^3_0$. An easy version of the argument in [5, Part 3] yields

$$\#M(\hat{W}; \alpha, \beta, \theta_0) = \#M(W^{(T)}; \alpha, \beta).$$

Deforming the metric on $W^{(T)}$ back to a cylindrical one we conclude that

$$N = \langle (I + f_2 d \pm d f_2)\alpha, \beta \rangle \tag{18}$$

for some homomorphism $f_2 : \mathrm{CF}^*(Y, m') \to \mathrm{CF}^{*-1}(Y, m')$.

Summing up, recall that the number of boundary points plus the number of ends of $\mathcal{M}''$ must be zero, and note that the number of ends resulting from factorizations over $\mathbb{R}_- \times S^3_{-1}$ cancel out as in the proof of Lemma 5. It follows that

$$\langle (-B + I + f d \pm d f)\alpha, \beta \rangle = 0,$$

where f is the sum of f_2 and a homomorphism of degree -1 related to factorizations over $\mathbb{R}_\pm \times Y$. $\quad\square$

Lemma 5 shows that Q induces a homomorphism

$$\bar{Q} : \ker(\delta_0) \to \mathrm{coker}(\delta'_0).$$

Lemma 7 $\bar{J}$ *and* $\bar{Q}$ *are inverse maps.*

Proof. We show that $\bar{J}\bar{Q} = $ id, the proof of $\bar{Q}\bar{J} = $ id being similar. Set $Z := [0,1] \times Y$. Choose an O-pair for $\hat{W}$ which has index $m+1, m, -1$ over the ends $\mathbb{R}_- \times Y$, $\mathbb{R}_+ \times Y$, $\mathbb{R}_- \times S^3$, resp., and an O-pair for $\hat{Z}$ which has index $m, m+1$ over the ends $\mathbb{R}_- \times Y$, $\mathbb{R}_+ \times Y$, resp., and such that over the ends $\mathbb{R}_+ \times Y \subset \hat{W}$ and $\mathbb{R}_- \times Y \subset \hat{Z}$ these are induced from the same N-pair for Y. Let $X^{(T)}$ denote the manifold with tubular ends obtained from $\hat{W} \cup \hat{Z}$ by replacing the ends $\mathbb{R}_+ \times Y \subset \hat{W}$ and $\mathbb{R}_- \times Y \subset \hat{Z}$ with a neck $[-T,T] \times Y$. Let $X^{(T)}$ be equipped with the O-pair inherited from those of $\hat{W}, \hat{Z}$.

Now fix $\alpha, \beta \in \mathcal{R}^*(Y, m+1)$ of the same index and define the parametrized moduli spaces $\mathcal{M}_\gamma$ and $\mathcal{M}$ as in the proof of Lemma 6. Let ℓ be as in (17). Note that if $\{(\omega_n, T_n)\}$ is any chain-convergent sequence in $\mathcal{M}$ with $\ell(\omega_n) \to 0$ then $T_n \to \infty$ and the chain-limit is made up of the reducible monopole over $\hat{W}$ and one element from each of the sets $\check{M}(\alpha, \theta_{m+1})$ and $M(\hat{Z}; \theta_m, \beta)$ (which must then both be finite). We now copy the construction of $\hat{\sigma}$ in the proof of Lemma 5, with T and $M(\hat{Z}; \theta_m, \beta)$ playing the roles of τ_2 and $\check{M}(\mathbb{R} \times Y; \theta_m, \beta)$, resp. Let $\mathcal{M}' \subset \mathcal{M}$ be the zero-set of $\hat{\sigma}$ and set

$$\mathcal{M}'' := \mathcal{M}' + \mathcal{M}_z.$$

By considering the boundary points and ends of $\mathcal{M}''$ one finds that

$$JQ = B + \hat{\delta}'\delta + df \pm fd, \tag{19}$$

where $J : \mathrm{CF}^*(Y, m) \to \mathrm{CF}^*(Y, m+1)$ is a cochain map which induces the canonical map in cohomology, B is of the kind discussed in Lemma 6, and

$$\hat{\delta}'(1) = \sum_\gamma (\#M(\hat{Z}; \theta_m, \gamma))\gamma \in \mathrm{CF}^{2m}(Y, m+1).$$

Combining equation (19) with Lemma 6 we conclude that $\bar{J}\bar{Q} = $ id. This completes the proof of Lemma 7, and also of Proposition 12. $\qquad\square$

Proposition 13 *Let ℓ be any natural number. Then the kernel and image of the natural homomorphism*

$$HF^*(Y, G; m) \to HF^*(Y, G; m+\ell)$$

are

$$\sum_{j=0}^{\ell-1} \mathrm{im}(\delta_j') \qquad and \qquad \bigcap_{j=0}^{\ell-1} \ker(\delta_j),$$

respectively.

Proof. For both the kernel and the image, one inclusion follows from Proposition 11 (ii), whereas the other inclusion is easily proved by induction on ℓ using Propositions 11, 12 and the functoriality of the homomorphisms in question with respect to composition of cobordisms. $\quad\square$

We will now define the maps D, D' in the sequence (2) and prove that the sequence is exact. Fix $m \in \mathfrak{m}(Y)$ and let

$$u : \underline{\mathrm{HF}}^{2m+1}(Y; G) \xrightarrow{\approx} \mathrm{HF}^{2m+1}(Y, m; G),$$
$$v : \mathrm{HF}^{2m}(Y, m+1; G) \xrightarrow{\approx} \overline{\mathrm{HF}}^{2m}(Y; G)$$

be the canonical maps. For $z \in \overline{\mathrm{HF}}^{2m}(Y; G)$ and $g \in G$ we define

$$D(z) := x^m \otimes \delta_0 v^{-1}(z), \qquad D'(x^m \otimes g) := u^{-1} \delta_0'(g).$$

We set $D = 0$ on $\overline{\mathrm{HF}}^{2m+1}(Y; G)$. (There is little choice, since $P^{2m+1}(Y) = 0$.)

From Proposition 11 we deduce that D, D' are $\mathbb{Z}[u]$–homomorphisms and that the two right-most squares of Diagramme (3) commute.

Proposition 14 *The fundamental sequence (2) is exact.*

Proof. Exactness at the terms $\underline{\mathrm{HF}}$ and $\overline{\mathrm{HF}}$ follows from Proposition 13, and Lemma 5 yields $D'D = 0$. It remains to show that $\ker(D') \subset \mathrm{im}(D)$. Suppose $D'(x^m \otimes g) = 0$. This means there is a chain $\sigma = \sum_j \alpha_j \otimes g_j$ with $\alpha_j \in \mathcal{R}(Y, m)$ and $g_j \in G$ such that

$$\rho \otimes g + d\sigma = 0,$$

where $\rho = \delta'(1) \in \mathrm{CF}^{2m+1}(Y, m)$. The oriented 1–manifold $M(\mathbb{R} \times Y; \theta_m, \theta_{m+1})$ has (generically) exactly one boundary point (namely the reducible one). Counting with sign we have that the number of boundary points plus the number of ends of that moduli space must be zero, and similarly the number of ends of $M(\mathbb{R} \times Y; \alpha_j, \theta_m)$ is zero. Thus

$$\pm 1 = \langle \rho, \phi \rangle + \delta\psi, \qquad 0 = \langle d\alpha_j, \phi \rangle + \delta\psi_j$$

for a suitable chain $\phi \in \mathrm{CF}_{2m+1}(Y, m)$ and cochains $\psi, \psi_j \in \mathrm{CF}^{2m}(Y, m+1)$. Tensoring the second equation with g_j, summing over j, and adding the first equation tensored with g we obtain $\pm g = \delta\tau$, where

$$\tau = \psi \otimes g + \sum_j \psi_j \otimes g_j.$$

A similar argument shows that $d\tau = 0$, hence $x^m \otimes g \in \mathrm{im}(D)$. $\quad\square$

10 Reduced Floer groups

In this section we will show how the reduced Floer cohomology $\widehat{\mathrm{HF}}^*(Y; G)$ can be computed in an arbitrary chamber m by factoring out interaction in $\mathrm{HF}^*(Y, m; G)$ with the reducible critical point. We will then prove Theorem 7.

Definition 3 *We define the* reduced *Floer cohomology group* $\widehat{HF}^*(Y, m; G)$ *by*

$$\widehat{HF}^q(Y, m; G) = Z^q/B^q,$$

where

$$Z^* = \bigcap_{j \geq 0} \ker(\delta_j), \qquad B^* = \sum_{j \geq 0} \mathrm{im}(\delta'_j).$$

We define

$$u : \widehat{\mathrm{HF}}^*(Y, m; G) \to \widehat{\mathrm{HF}}^{*+2}(Y, m; G)$$

as follows: Choose a map v as in Proposition 8. Let Z'^* (resp. B'^*) be the set of cocycles in $\mathrm{CF}^*(Y, m) \otimes G$ representing an element of Z^* (resp. B^*). By Proposition 8 the endomorphism $v \otimes 1$ maps each of Z^* and B^* into itself and therefore induces the desired endomorphism of $Z'^*/B'^* = \widehat{\mathrm{HF}}^*(Y, m; G)$.

Let W be a cobordism as in Section 7. If the assumptions of either part (i) or part (ii) of Proposition 11 are satisfied then by Propositions 9, 11 the map W^* gives rise to a homomorphism

$$\widehat{\mathrm{HF}}^*(Y_1, m_1; G) \to \widehat{\mathrm{HF}}^{*-d}(Y_2, m_2; G)$$

which commutes with the u–maps.

Proposition 15 *(i) If $m_1 \leq m_2$ then the spin^c cobordism $[0, 1] \times Y$ induces an isomorphism*

$$\widehat{HF}^*(Y, m_1; G) \xrightarrow{\approx} \widehat{HF}^*(Y, m_2; G).$$

(ii) $\widehat{HF}^*(Y, m; G)$ is canonically isomorphic to $\widehat{HF}^*(Y; G)$ as a $\mathbb{Z}[u]$ module.

Proof. Statement (i) is an immediate consequence of Propositions 11 (ii) and 13. To prove (ii), recall that if $2m_1 \leq q \leq 2m_2 - 1$ then by Proposition 6 we have a sequence of homomorphisms

$$\underline{\mathrm{HF}}^q \xrightarrow{\approx} \mathrm{HF}^q_{m_1} \to \mathrm{HF}^q_{m_2} \xrightarrow{\approx} \overline{\mathrm{HF}}^q,$$

where we have used the same abbreviations as in (11). By Proposition 13 the image of the middle homomorphism is precisely $\widehat{\mathrm{HF}}^q(Y, m_2; G)$. On the other hand, that image can clearly be identified with $\widehat{\mathrm{HF}}^*(Y; G)$. $\quad\square$

Proof of Theorem 7: Choose $m \in \mathfrak{m}(Y)$ and a sequence $\{s_j\}$ of generic sections of the natural complex line bundle over $\mathcal{B}^*(W)$. For any $\alpha, \beta \in \mathcal{R}^*(Y, m)$ and non-negative integer k let

$$M_{\alpha\beta k} := \{\omega \in M(\hat{W}; \alpha, \beta) \mid s_j(\omega|_W) = 0, \quad j = 1, \ldots, k\}.$$

Define a cochain map

$$P_k : \mathrm{CF}^*(Y, m) \to \mathrm{CF}^{*-2n+2k}(Y, m)$$

on generators by

$$P_k(\alpha) = \sum_\beta (\#M_{\alpha\beta k})\beta.$$

Let ψ_k denote the induced endomorphism of $\mathrm{HF}^*(Y, m)$ (with integral coefficients). In analogy with Proposition 11 (i) one has

$$\delta_j \psi_k = 0, \quad \psi_k \delta'_j = 0$$

for all $j \geq 0$, hence ψ_k gives rise to an endomorphism $\hat{\psi}_k$ of $\widehat{\mathrm{HF}}^*(Y, m)$. Choose a map v as in Proposition 8. Then

$$P_{k+1} - vP_k + \delta'\delta_{(k)} = d\phi_k \pm \phi_k d,$$

where $\delta_{(k)}$ and ϕ_k are maps similar to δ_W and ϕ in Proposition 9. By induction on k we conclude that $\hat{\psi}_k = u^k \hat{\psi}_0$ for all k. By definition, $\hat{\psi}_0 = \hat{\psi}(W)$. Now

$$\mathrm{SW}(X) = L(P_n) = L(\psi_n) = L(\psi_n|_{Z^*}) = L(\hat{\psi}_n) = L(u^n \hat{\psi}(W)).$$

The first equality is basic gluing theory, the signs being given by Proposition 3. The second equality is the general fact that the Lefschetz number of a degree 0 chain endomorphism of a finitely generated chain complex agrees with the Lefschetz number of the induced endomorphism on homology. The third and fourth equalities hold because ψ_n takes values in Z^* and vanishes on B^*. $\quad\square$

11 The h–invariant

In this section we will work with Floer cohomology with coefficients in a fixed field $\mathbb{F}$, which will usually be suppressed from notation. Y will denote a spin^c rational homology 3–sphere. All 4–manifolds will be smooth.

Proposition 16 *Let $m \in \mathfrak{m}(Y)$. Then for the group $HF^*(Y, m)$, either $\delta_0 = 0$ or $\delta_0' = 0$.*

    Proof. This follows immediately from Proposition 8.   □

Definition 4 *For any $m \in \mathfrak{m}(Y)$ set*

$$\zeta = \zeta(Y, m) := \chi(HF^*(Y, m)) - \chi(\widehat{HF}^*(Y)),$$

where χ means the Euler characteristic with respect to the mod 2 grading. In other words:

- *If δ_0, δ_0' are both zero then $\zeta = 0$.*

- *If $\delta_0 \neq 0$ then $\zeta = \max\{n \geq 1 \,|\, \delta_{n-1} \neq 0\}$.*

- *If $\delta_0' \neq 0$ then $\zeta = -\max\{n \geq 1 \,|\, \delta_{n-1}' \neq 0\}$.*

Lemma 8 $\zeta(-Y, -m) = -\zeta(Y, m)$.

    Proof. The vector space $\mathrm{HF}^*(-Y, -m)$ is the dual of $\mathrm{HF}_*(-Y, -m) = \mathrm{HF}^{-1-*}(Y, m)$. Moreover, the u–map on the former vector space (in the degrees in which it is defined) is the dual of the u–map on the latter. Similarly, $\delta_0 : \mathrm{HF}^{-2m-2}(-Y, -m) \to \mathbb{F}$ is the dual of $\delta_0' : \mathbb{F} \to \mathrm{HF}^{2m+1}(Y, m)$. Together with Proposition 10, this proves the lemma.   □

Proposition 17 *Let W be a compact, connected spin^c 4–manifold whose boundary is a disjoint union of rational homology spheres $Y_1, \ldots, Y_r$, where $r \geq 1$, and let $m_j \in \mathfrak{m}(Y_j)$. Suppose $b^+(W) = 0$ and $\zeta(Y_j, m_j) \leq 0$ for $j > 1$. Then*

$$\zeta(Y_1, m_1) \geq \sum_{j=1}^{r} m_j + \frac{1}{8}(c_1(\mathcal{L}_W)^2 - \sigma(W))$$

provided the right hand side of the inequality is positive.

Proof. Let $\gamma_1, \ldots, \gamma_r$ be a collection of disjoint loops in W representing a basis for $H_1(W; \mathbb{Z})/$torsion, and let W' denote the manifold obtained by performing surgery on each γ_j. Then $b_1(W') = 0$. Let V_i be a small open tubular neighbourhood of γ_i, and set $W^- = W \setminus \cup_i V_i$. Then with integral coefficients there are isomorphisms

$$H^2(W) \overset{\approx}{\to} H^2(W^-) \overset{\approx}{\leftarrow} H^2(W'),$$

and similarly for H_2. Hence, W and W' have the same intersection forms, the induced spin^c structure on W^- has a unique extension to W', and the canonical classes for W and W' have the same square. Now let $X := \hat{W}$ be the result of adding half-infinite tubes to W.

Set

$$\vec{\theta} := (\theta_{m_1}, \ldots, \theta_{m_r}).$$

Then

$$\dim M(X; \vec{\theta}) = 2k + 1,$$

where

$$k + 1 = \operatorname{ind}_{\mathbb{C}}(D_{A_o}) = \sum_j m_j + \frac{1}{8}(c_1(\mathcal{L}_W)^2 - \sigma(W)).$$

Here A_o can be taken to be the reference spin^c–connection over X used in defining the moduli space $M(X; \vec{\theta})$, see [5, Section 3.4].

Suppose $k \geq 0$. By Proposition 1 we may assume that $M(X; \vec{\theta})$ contains a unique reducible point, and that this point is regular. Let $M^-(X; \vec{\theta})$ be the result of removing from $M(X; \vec{\theta})$ a small neighbourhood of the reducible point, so that the boundary of $M^-(X; \vec{\theta})$ is diffeomorphic to $\mathbb{CP}^k$.

Let $\mathbb{L} \to \mathcal{B}^*(W)$ be the natural complex line bundle. Choose a generic smooth section s of $\oplus^k \mathbb{L}$. For any moduli space M over X and subset Σ of the irreducible part of M set

$$c_1(\mathbb{L})^k \cap \Sigma := \{\omega \in \Sigma \,|\, s(\omega|_W) = 0\}.$$

For $j > 1$ the assumption $\zeta(Y_j, m_j) \leq 0$ means, by the universal coefficient theorem, that the cycle $\delta \in \mathrm{CF}_{2m_j-2}(Y_j, m_j)$ is in fact a boundary, ie there is a $\rho_j \in \mathrm{CF}_{2m_j-1}(Y_j, m_j)$ with

$$\delta + \partial \rho_j = 0,$$

where ∂ is the boundary operator of the chain complex CF_*. For any $\alpha \in \mathcal{R}_{Y_1}$ set

$$M_\alpha := M(X; \alpha, \sigma_2, \ldots, \sigma_r),$$

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where $\sigma_j = \rho_j + \theta_{m_j}$. Here we treat $M(X; \cdot)$ as a multilinear function, so that M_α is a formal linear combination of moduli spaces over X. In the case $\alpha = \theta := \theta_{m_1}$ one of the terms appearing in that formal linear combination is $M(X; \vec{\theta})$. Let M_θ^- be obtained from M_θ by replacing the term $M(X; \vec{\theta})$ with $M^-(X; \vec{\theta})$.

For any irreducible α set

$$\hat{M}_\alpha := c_1(\mathbb{L})^k \cap M_\alpha, \qquad \hat{M}_\theta^- := c_1(\mathbb{L})^k \cap M_\theta^-,$$

where $c_1(\mathbb{L})^k \cap \cdot$ is treated as a linear function. For the formal linear combination $\hat{M}_\theta^-$ of oriented 1–manifolds the number of boundary points plus the number of ends (both counted with sign) must be zero. The number of boundary points is

$$\# \left(c_1(\mathbb{L})^k \cap \partial M^-(X; \vec{\theta}) \right) = \pm 1.$$

The ends of $\hat{M}_\theta^-$ correspond to factorizations over the ends of X. Because of the choice of σ_j, only factorizations over the end $\mathbb{R}_+ \times Y_1$ contribute. Therefore, the number of ends of $\hat{M}_\theta^-$ is δa, where the cocycle a is defined by

$$a = \sum_\alpha (\# \hat{M}_\alpha) \cdot \alpha,$$

the sum being taken over all $\alpha \in \mathcal{R}_{Y_1}^*$ of index $2m_1 - 2$. Writing $[\ ]$ for cohomology classes we have $[a] = u^k[b]$, where

$$b = \sum_\beta (\# M_\beta) \cdot \beta.$$

Thus,

$$\delta_0 u^k [b] = \delta \alpha = \mp 1,$$

proving that $\zeta(Y_1, m_1) \geq k + 1$ as claimed. $\qquad \square$

Corollary 2 *Let W be a compact, connected spin^c 4–manifold whose boundary is a disjoint union of rational homology spheres $Y_1, \ldots, Y_r$, and let $m_j \in \mathfrak{m}(Y_j)$. Suppose $b_2(W) = 0$ and $\zeta(Y_j, m_j) = 0$ for all j. Then $\sum_{j=1}^r m_j = 0$.*

Proof. If $\sum_j m_j \neq 0$ then after perhaps reversing the orientation of W we may assume $\sum_j m_j > 0$, contradicting the proposition. $\qquad \square$

Lemma 9 $\zeta(S^3, m) = m$ *for all $m \in \mathbb{Z}$.*

Proof. Since $\zeta(S^3, -m) = -\zeta(S^3, m)$, we may assume $m \geq 0$. The lemma clearly holds for $m = 0$, because $\mathrm{HF}^*(S^3, 0) = 0$. Now let $m \geq 1$. Applying Proposition 17 with $W = [0, 1] \times S^3$, $m_1 = m$, $m_2 = 0$ we obtain $\zeta(S^3, m) \geq m$. On the other hand, Proposition 7 gives $\zeta(S^3, m) \leq m$. Together this proves the lemma. $\square$

Proposition 18 *Let W be a spin^c, compact, connected 4–manifold with boundary components $-Y_0, -Y_1, Y_2$, such that each Y_j is a rational homology sphere and $b_2(W) = 0$. Suppose $m_j \in \mathfrak{m}(Y_j)$, $j = 0, 1, 2$ satisfy $-m_0 - m_1 + m_2 = 0$. Then the following hold:*

(i) *If $\zeta(Y_j, m_j) > 0$ for $j = 0, 1$ then*

$$\zeta(Y_2, m_2) \geq \zeta(Y_0, m_0) + \zeta(Y_1, m_1).$$

(ii) *If $\zeta(Y_0, m_0) = 0$ then*

$$\zeta(Y_1, m_1) = \zeta(Y_2, m_2).$$

Proof. The proofs are similar to those of [7, Lemma 6] and [7, Theorem 14], respectively. $\square$

Corollary 3 *If $\zeta(Y_1, m_1) = \zeta(Y_2, m_2) \neq 0$ then $\zeta(Y_1 \# (-Y_2), m_1 - m_2) = 0$.*

Proof. Apply part (i) of the proposition to W or $-W$, where W is the standard cobordism from $Y_1 \cup (-Y_2)$ to $Y_1 \# (-Y_2)$. $\square$

Proposition 19 *For any Y there exists an $m \in \mathfrak{m}(Y)$ such that $\zeta(Y, m) = 0$.*

Proof. Pick any $m' \in \mathfrak{m}(Y)$ and set $k := \zeta(Y, m')$. If $k = 0$ then we are done. Otherwise, set $m = m' - k$. Then Lemma 9 and Corollary 3 give

$$\zeta(Y, m) = \zeta(Y \# (-S^3), m' - k) = 0. \quad \square$$

Proposition 20 *If $\zeta(Y, m) = 0$ then $\zeta(Y, m + k) = k$ for all integers k.*

Proof. Let $D \subset (0, 1) \times Y$ be an embedded closed 4–ball. Applying Proposition 18 (ii) to $W = ([0, 1] \times Y) \setminus \mathrm{int}(D)$ we get

$$\zeta(Y, m + k) = \zeta(S^3, k) = k. \quad \square$$

Definition 5 *For any spin^c rational homology 3–sphere Y we define $h(Y) = h(Y;\mathbb{F})$ to be the unique $m \in \mathfrak{m}(Y)$ such that $\zeta(Y,m) = 0$.*

It follows from Propositions 19 and 20 that $h(Y)$ is well defined, and that

$$h(Y) = m - \zeta(Y,m) \tag{20}$$

for all $m \in \mathfrak{m}(Y)$.

Theorem 1 is now an immediate consequence of Propositions 13, 19, and 20, and Theorem 2 follows from (20).

Proof of Theorem 3: Set $m_j := h(Y_j)$ for $j = 1, 2$ and $m_3 := h(Y_1 \# Y_2)$. Applying Corollary 2 to the standard cobordism W from $Y_1 \cup Y_2$ to $Y_1 \# Y_2$ we obtain $-m_1 - m_2 + m_3 = 0$, which is the assertion of the theorem. $\square$

Proof of Theorem 4: Set $m_j := h(Y_j)$ for $j > 1$ and choose $m_1 \in \mathfrak{m}(Y_1)$ with

$$\sum_{j=1}^{r} m_j + \frac{1}{8}(c_1(\mathcal{L}_W)^2 - \sigma(W)) > 0.$$

The theorem then follows from Proposition 17. $\square$

We now address the dependence of h on the coefficient field $\mathbb{F}$. Let $\mathbb{F}$ have characteristic p and let $\mathbb{K}$ be the prime field of $\mathbb{F}$, ie $\mathbb{K} = \mathbb{Q}$ if $p = 0$ and $\mathbb{K} = \mathbb{Z}/p\mathbb{Z}$ if $p > 0$. Then

$$\mathrm{HF}^*(Y;\mathbb{F}) = \mathrm{HF}^*(Y;\mathbb{K}) \otimes_{\mathbb{K}} \mathbb{F},$$

from which one easily deduces that

$$h(Y;\mathbb{F}) = h(Y;\mathbb{K}).$$

Thus we may write $h_p(Y)$ instead of $h(Y;\mathbb{F})$.

Note that if Y admits a metric g of positive scalar curvature then $(g, 0)$ is an N-pair for Y for which the corresponding Floer cochain complex vanishes. Hence $h_p(Y) = I(g, 0)$ for all p. (The index I was defined in (5)).

If $h_p(Y) \neq h_0(Y)$ for some p then it is not hard to see that $\mathrm{HF}^*(Y, h_0(Y); \mathbb{Z})$ contains an element of order p (in degree $2h_0(Y) \pm 1$). The following proposition shows that this group is in a sense the universal home for torsion:

Proposition 21 *For any m, q let T_m^q be the torsion subgroup of $HF^q(Y, m; \mathbb{Z})$. For $m_1 \leq m_2$ the canonical homomorphism*

$$HF^q(Y, m_1; \mathbb{Z}) \to HF^q(Y, m_2; \mathbb{Z})$$

restricts to a homomorphism

$$T^q_{m_1} \to T^q_{m_2}$$

which is injective if $m_2 \leq h_0(Y)$ and surjective if $h_0(Y) \leq m_1$.

Proof. The element $\delta'_j \in \mathrm{HF}^*(Y, m_1; \mathbb{Z})$ has infinite order for $j = 0, \ldots, h_0(Y) - m_1 - 1$ if $m_2 \leq h_0(Y)$ and finite order for all j if $h_0(Y) \leq m_1$. The proposition now follows from Proposition 13. $\square$

12   Proof of Theorem 8

Lemma 10 *Let A, B be $r \times r$ complex matrices and m a natural number such that $tr(A^n) = tr(B^n)$ for $m \leq n < 2r + m$. Then A and B have the same characteristic polynomial. In particular, $tr(A) = tr(B)$.*

Proof. Let E denote the set of all the roots of the characteristic polynomials of A and B. For any square matrix C and complex number λ let $m_C(\lambda)$ denote the multiplicity of λ as a root in the characteristic polynomial of C. It suffices to show that $m_A(\lambda) = m_B(\lambda)$ for all $\lambda \neq 0$. So suppose $m_A(\lambda) > 0$ and consider the polynomial

$$P(z) = z^m \prod_{a \in E \setminus \{\lambda\}} (z - a).$$

Putting A and B in triangular form we obtain

$$m_A(\lambda) \cdot P(\lambda) = \mathrm{tr}(P(A)) = \mathrm{tr}(P(B)) = m_B(\lambda) \cdot P(\lambda).$$

But $P(\lambda) \neq 0$, so $m_A(\lambda) = m_B(\lambda)$. $\square$

Corollary 4 *Let $U = U_0 \oplus U_1$ and $V = V_0 \oplus V_1$ be finite-dimensional mod 2 graded complex vector spaces. Let $f = f_0 \oplus f_1 \in End(U)$ and $g = g_0 \oplus g_1 \in End(V)$ preserve degrees. If m is a natural number such that the Lefschetz numbers $L(f^n) = L(g^n)$ for all $n \geq m$, then $L(f) = L(g)$.*

Proof. Apply the lemma to $f_0 \oplus g_1$ and $g_0 \oplus f_1$. $\square$

Proof of Theorem 8: Because Y_j is non-separating, there is a loop in X whose intersection number with Y_j is ± 1. Hence each of Y_0 and Y_1 represents a generator of $H_3(X; \mathbb{Z})$, and by assumption these two generators are equal. Let $f_j : X \to S^1$ be a smooth map such that 1 is a regular value of f_j and

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$(f_j)^{-1}(1) = Y_j$ as oriented manifolds. Then f_0 and f_1 represent the same generator of $[X, S^1] \approx H^1(X; \mathbb{Z})$. Set

$$X_{j,\infty} = \{(x, t) \in X \times \mathbb{R} \mid f_j(x) = e^{2\pi i t}\}.$$

Then the projection

$$\pi_{j,\infty} : X_{j,\infty} \to X$$

is a covering whose group of deck transformations is canonically isomorphic to $\mathbb{Z}$. Let

$$q_j : X_{j,\infty} \to \mathbb{R}$$

be the projection onto the second factor. We may take

$$W_j := (q_j)^{-1}[0, 1]$$

as the definition of W_j. Then

$$W_{j,n} := (q_j)^{-1} \left[- \lfloor n/2 \rfloor, n - \lfloor n/2 \rfloor \right]$$

is the result of gluing together n copies of W_j in a chain.

Since f_0 and f_1 are homotopic, the coverings $\pi_{0,\infty}$ and $\pi_{1,\infty}$ are isomorphic. In particular, $\pi_{0,\infty}$ restricts to a trivial covering of Y_1. Let B be a component of $(\pi_{0,\infty})^{-1}(Y_1)$. Since B is compact, there is a natural number m such that B is contained in the interior of $W_{0,m}$. For every natural number n set $X_n := X_{0,\infty}/n\mathbb{Z}$ and let

$$\pi_n : X_n \to X$$

be the natural n–fold covering. For $n \geq m$ there is a component C_j of $\pi_n^{-1}(Y_j)$, $j = 0, 1$, with $C_0 \cap C_1 = \emptyset$. Let C_j be oriented such that the diffeomorphism $C_j \to Y_j$ obtained by restriction of π_n is orientation preserving. Then X_n is the union of two compact, connected, codimension 0 submanifolds Z_0, Z_1 with

$$Z_0 \cap Z_1 = C_0 \cup C_1, \qquad \partial Z_j = (-C_j) \cup C_{1-j}.$$

Moreover, $b_1(Z_j) = 0 = b^+(Z_j)$. If each Y_j is an integral homology sphere then it follows from Theorem 4 and Elkies' theorem that $h_p(Y_0) = h_p(Y_1)$. The same conclusion holds if $b_2(X) = 0$, because then $b_2(Z_j) = 0$, too.

Returning to the general case, we can identify

$$W_{j,n} = Z_{1-j} \cup_{C_{1-j}} Z_j.$$

Setting
$$A_j := \hat{\psi}(W_j) : \widehat{\mathrm{HF}}^*(Y_j) \to \widehat{\mathrm{HF}}^*(Y_j)$$
we have
$$(A_j)^n = \hat{\psi}(W_{j,n}) = \hat{\psi}(Z_{1-j}) \circ \hat{\psi}(Z_j).$$
Since $\hat{\psi}(Z_j)$ preserves the mod 2 grading, we conclude that
$$L((A_0)^n) = L((A_1)^n) \quad \text{for all } n \geq m.$$

Corollary 4 now says that $L(A_0) = L(A_1)$, and the theorem is proved. $\qquad\square$

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