Transformations of measures by Gauss maps

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The goal of this paper is to introduce a new class of transformations of measures on $\mathbb{R}^d$ having the form (heuristically) $T = \varphi \cdot \nabla\varphi/|\nabla\varphi|$, where φ is some function; our construction exhibits an interesting link between two actively developing areas: optimal transportation and geometrical glows (see [1], [2], and [3] concerning these directions).

Let $A \subset \mathbb{R}^d$, $d \geq 2$, be a compact convex set and let $\mu = \varrho_0\,dx$ be a probability measure on A equivalent to the restriction of Lebesgue measure. Let $\nu = \varrho_1\,dx$ be a probability measure on the ball $B_r := \{x\colon\ |x| \leq r\}$ equivalent to the restriction of Lebesgue measure. We prove that there exists a mapping T such that $\nu = \mu \circ T^{-1}$ and $T = \varphi \cdot \mathrm{n}$, where $\varphi\colon\ A \to [0, r]$ is a continuous potential with convex sub-level sets and n is the Gauss map of the corresponding level sets of φ. Moreover, the mapping T is invertible and essentially unique. Our proof employs the optimal transportation techniques. In the case of smooth φ the level sets of φ are governed by the Gauss curvature flow $\dot{x}(s) = -s^{d-1}\,\frac{\varrho_1(s\mathrm{n})}{\varrho_0(x)} K(x) \cdot \mathrm{n}(x)$, where K is the Gauss curvature.

Throughout we assume that $d \geq 2$ and denote by $\mathcal{H}^n$ the n-dimensional Hausdorff measure. Let $\mathrm{Int}\,A$ denote the interior of a set A.

Given a compact convex set $V \subset \mathbb{R}^d$ with boundary $M = \partial V$, for an arbitrary point $x \in M$, let us set

$$N_{M,x} := \{\eta \in S^{d-1};\ \forall z \in V,\ \langle \eta, z - x\rangle \leq 0\}.$$

If $N_{M,x}$ contains a single element $\mathrm{n}(x)$, then $\mathrm{n}(x)$ is the outer unit normal in the usual sense. It is known from the properties of convex functions that one has $\mathcal{H}^{d-1}(S) = 0$, where

$$S = \{x\colon\ N_{M,x} \text{ contains more than one element}\}.$$

Hence the Gauss map $\mathrm{n}(x)$ is well-defined $\mathcal{H}^{d-1}$-almost everywhere on M.

If the surface is smooth, let $D\mathrm{n}\colon\ TM_x \to TS^{d-1}_{\mathrm{n}(x)}$ denote the differential of n. The determinant of $D\mathrm{n}(x)$ is called the Gauss curvature and is denoted throughout by $K(x)$. It is known that in the case of a convex set with not necessarily smooth boundary M the function $K(x)$ is defined $\mathcal{H}^{d-1}$-almost everywhere on M.

Theorem 1. *Let $A \subset \mathbb{R}^d$ be a compact convex set and let $\mu = \varrho_0\,dx$ be a probability measure on A equivalent to the restriction of Lebesgue measure. Let $\nu = \varrho_1\,dx$ be a probability measure on $B_r = \{x\colon\ |x| \leq r\}$ equivalent to the restriction of Lebesgue measure. Then, there exist a Borel mapping $T\colon\ A \to B_r$ and a continuous function $\varphi\colon\ A \to [0, r]$ with convex sets $A_s = \{\varphi \leq s\}$ such that $\nu = \mu \circ T^{-1}$ and*

$$T = \varphi \cdot \mathrm{n} \quad \mathcal{H}^d\text{-almost everywhere},$$

where $\mathrm{n} = \mathrm{n}(x)$ is a unit outer normal vector to the level set $\{y\colon\ \varphi(y) = \varphi(x)\}$ at the point x.

If φ is smooth, the level sets of φ are moving according to the following Gauss curvature flow equation:

$$\dot{x}(s) = -s^{d-1} \frac{\varrho_1(s\mathrm{n})}{\varrho_0(x)} K(x) \cdot \mathrm{n}(x) \tag{1}$$

where $x(s) \in \partial A_{r-s}$, $0 \le s \le r$, $x(0) \in \partial A$ is any initial point.

To prove this theorem we develop an approach based on the optimal transportation techniques. For every $t \ge 0$, we consider a mapping T_t that takes μ to ν and maximizes the functional

$$F \mapsto \int \langle x, F(x)\rangle |F(x)|^t \, \mu(dx) \tag{2}$$

in the class of mappings F with $\mu \circ F^{-1} = \nu$. Equivalently, it minimizes the functional

$$F \mapsto \int \left|x - F(x)|F(x)|^t\right|^2 \mu(dx)$$

in the class of mappings F with $\mu \circ F^{-1} = \nu$. For $t = 0$ (2) becomes the classical Monge–Kantorovich problem. For $t \neq 0$ standard arguments from the Monge–Kantorovich theory show that the set

$$\Big\{ \big(x, T_t(x)|T_t(x)|^t\big), \ x \in A \Big\}$$

is cyclically monotone, hence supported on the graph of the gradient of some convex function W_t (see [2, Chapter 2]). If the reader does not want to be concerned with the cyclical monotonicity or calculus of variations, we note that ∇W_t is just the optimal transportation of μ to $\nu \circ S_t^{-1}$, where $S_t(x) = x|x|^t$. This can be taken for a definition of W_t.

One has the following relations:

$$T_t = \frac{\nabla W_t}{|\nabla W_t|^{\frac{t}{1+t}}}, \ \nabla W_t(x) = T_t(x)|T_t(x)|^t.$$

Clearly, $|T_t(x)| \le r$ since T_t transforms μ into ν.

Throughout the paper we choose W_t in such a way that $\min_{x\in A} W_t(x) = 0$. Define a new potential function φ_t by

$$W_t = \frac{1}{t+2}\varphi_t^{t+2}.$$

One has

$$T_t = \varphi_t \frac{\nabla \varphi_t}{|\nabla \varphi_t|^{\frac{t}{t+1}}}.$$

We show below that the limits

$$\lim_{t\to\infty} \varphi_t = \varphi, \ \lim_{t\to\infty} T_t = T$$

exist almost everywhere (for a suitable sequence $t_n \to \infty$) and then we prove that T is the desired mapping.

Lemma 1. *One has*

$$\varphi_t \le (2+t)^{\frac{1}{2+t}} \big(\mathrm{diam}(A)\big)^{\frac{1}{2+t}} r^{\frac{1+t}{2+t}},$$

$$\int_A |\nabla \varphi_t(x)| \, dx \le \int_{\partial A} \varphi_t \, d\mathcal{H}^{d-1} \le (2+t)^{\frac{1}{2+t}} \big(\mathrm{diam}(A)\big)^{\frac{1}{2+t}} r^{\frac{1+t}{2+t}} \ \mathcal{H}^{d-1}(\partial A).$$

Corollary 1. *There exists a sequence $\{t_n\} \to \infty$ such that $\{\varphi_{t_n}\}$ converges almost everywhere to a finite function φ.*

Lemma 2. *There exists a sequence $\{t_n\} \to \infty$ such that*

$$\lim_{t_n \to \infty} |\nabla \varphi_n|^{\frac{1}{1+t_n}} = 1$$

almost everywhere, where $\varphi_n := \varphi_{t_n}$.

In what follows we set $\varphi_n := \varphi_{t_n}$ and assume that $\varphi_n \to \varphi$ and $|\nabla \varphi_n|^{\frac{1}{1+t_n}} \to 1$ almost everywhere.

Lemma 3. *Let $C_n \subset B_r$ be convex sets such that $I_{C_n} \to I_C$ almost everywhere. If C is of positive measure, then $\mathrm{dist}(\partial C_n, \partial C) \to 0$.*

Note that due to convexity one has $\mathrm{dist}(\partial C_n, \partial C) = \mathrm{dist}(C_n, C)$.

Lemma 4. *The sequence of potentials φ_{t_n} converges to φ uniformly on A. In particular, φ is continuous and has convex sub-level sets $A_s = \{y\colon\ \varphi(y) \le s\}$.*

Lemma 5. *Let $N_x := N_{\partial A_{\varphi(x)}, x}$, where $A_{\varphi(x)} = \{y\colon\ \varphi(y) \le \varphi(x)\}$ and*

$$S := \{x \in A\colon\ N_x \text{ contains more than one element}\},$$

i.e. S is the set of all the points x such that the boundary of the sub-level set containing x is not differentiable at x. Then $\mathcal{H}^d(S) = 0$.

Proof of Theorem 1: According to Lemma 4 and Lemma 2, we have $\varphi_n \to \varphi$ uniformly and $|\nabla \varphi_n|^{\frac{1}{1+t_n}} \to 1$ almost everywhere. It remains to prove that $\nabla \varphi_n / |\nabla \varphi_n| \to \mathrm{n}$ almost everywhere. Let us fix $x \in A$. Since $\varphi_n(x) \to \varphi(x)$, one has $I_{A_{\varphi_n(x),n}} \to I_{A_{\varphi(x)}}$ almost everywhere, where

$$A_{\varphi_n(x),n} = \{y\colon\ \varphi_n(y) \le \varphi_n(x)\},\ A_{\varphi(x)} = \{y\colon\ \varphi(y) \le \varphi(x)\}.$$

According to Lemma 5, $\mathrm{n}(x)$ is well-defined for almost all x. The same holds for every $\nabla \varphi_n(x)/|\nabla \varphi_n(x)|$. So, without loss of generality we can fix x in the interior of A such that $\mathrm{n}(x)$ and $\nabla \varphi_n(x)/|\nabla \varphi_n(x)|$ are well-defined. If the vectors $\nabla \varphi_n(x)/|\nabla \varphi_n(x)|$ do not converge to $\mathrm{n}(x)$, then, extracting a convergent subsequence from a sequence of unit vectors $\{\nabla \varphi_n(x)/|\nabla \varphi_n(x)|\}$, we obtain a unit vector $\eta \neq \mathrm{n}(x)$. By using convergence $I_{A_{\varphi_n(x),n}} \to I_{A_{\varphi(x)}}$, one can show that $\langle \eta, z - x \rangle \le 0$ for all $z \in A$, i.e., $\eta \in N_x$, which contradicts the choice of x.

It remains to verify the evolution equation for a smooth potential φ. Indeed, let us choose an orthonormal basis $\{e_i\}$ at x such that $e_1 = \mathrm{n}$ and every vector e_i, $2 \le i \le d$, belongs to the tangent space of ∂A_t at x. Let us write the change of variables formula for $T = \varphi \cdot \mathrm{n}$. Differentiating along n we find

$$\partial_{\mathrm{n}} T = \partial_{\mathrm{n}} \varphi \cdot \mathrm{n} + \varphi \cdot \partial_{\mathrm{n}} \mathrm{n}.$$

Differentiating the identity $\langle \mathrm{n}, \mathrm{n}\rangle = 1$, we see that $\partial_{\mathrm{n}}\mathrm{n}$ belongs to the tangent space of ∂A_t at x. In addition, $\partial_{\mathrm{n}}\varphi = |\nabla\varphi|$. Next we note that

$$\partial_{e_i} T = \varphi \cdot \partial_{e_i}\mathrm{n}, \quad \langle \partial_{e_i}\mathrm{n}, \mathrm{n}\rangle = 0, \quad 1 \le i \le d.$$

Hence

$$\det DT = |\nabla\varphi|\varphi^{d-1} \det\bigl(\langle \partial_{e_i} n, e_j\rangle\bigr).$$

Since $K = \det\bigl(\langle \partial_{e_i}\mathrm{n}, e_j\rangle$, we have $\det DT = |\nabla\varphi|\varphi^{d-1}K$. Thus one obtains the following change of variables formula (the Monge–Ampère equation):

$$\varrho_0 = \varrho_1(\varphi\cdot\mathrm{n})|\nabla\varphi|\varphi^{d-1}K.$$

It remains to note that the level sets ∂A_s are shrinking with the velocity $1/|\nabla\varphi|$ in the direction of $-\mathrm{n}$. Hence (1) follows from the change of variables formula. The proof is complete.

Example 1. Let A be a convex compact set. Set

$$\varrho_1(x) := \frac{C_{d,r}}{|x|^{d-1}}, \ \ \varrho_0(x) := \frac{1}{\mathcal{H}^d(A)},$$

where $C_{d,r} = \Bigl(\int_{B_r} \frac{dx}{|x|^{d-1}}\Bigr)^{-1}$. Varying r we can show the existence of a weak solution (in the "transportation sense") to the classical Gauss curvature flow which starts from ∂A and satisfies the equation

$$\dot{x}(s) = -c\ K(x)\cdot \mathrm{n}(x),$$

where c can be chosen arbitrarily.

Certainly, a rigorous justification of this formula requires some additional work, since we have not proved that φ is differentiable.

In addition to the singular set $S \subset A$ of all points x such that $N_{\partial A_t, x}$, where $t = \varphi(x)$, contains more than one element, we introduce another set of degeneracy of n defined by

$$U = \{x \in A\backslash S\colon \text{ there is } x' \in \partial A_t,\ t = \varphi(x), \quad \text{such that } x' \ne x \text{ and } \mathrm{n}(x) \in N_{\partial A_t, x'}\}.$$

Proposition 1. (i) *Consider the set $C = \partial A_t$ for some fixed t. Then the set $\mathrm{n}(U\cap C)$ in S^{d-1} has $\mathcal{H}^{d-1}$-measure zero.*

(ii) *The sets $T(U)$ and*

$$\widetilde{T}(S) := \bigcup_{x\in S} \varphi(x)\cdot N_{\partial A_{\varphi(x)}, x}$$

have ν-measure zero.

Corollary 2. *The mapping T is injective on a set of full μ-measure. Hence there exists a measurable mapping $T^{-1}\colon B_r \to A$ such that $T(T^{-1}(y)) = y$ for ν-almost all y and $T^{-1}(T(x)) = x$ for μ-almost all x.*

Theorem 2. *The mapping T constructed above is unique in the following sense: if a measurable mapping $T_0\colon A \to B_r$ is such that $\nu = \mu \circ T_0^{-1}$ and $T_0 = \varphi_0 \cdot \mathrm{n}_0$, where $\varphi_0\colon A \to [0, r]$ is a continuous function with convex sub-level sets $A_{t,0} := \{\varphi_0 \le t\}$ and n_0 is the corresponding Gauss map, then $T = T_0$ μ-a.e.*

Now we consider certain duality properties of the potential φ. The duality principle of Kantorovich is a powerful tool for investigating the Monge–Kantorovich problem. In our case we also have a kind of the duality formula which relates the potential φ to some function ψ that can be considered as the support function of the family of level sets A_t. Recall (see [4]) that the Legendre transform of a convex function W on a convex set A is defined by

$$W^*(y) = \sup_{x \in A} \bigl(\langle x, y\rangle - W(x)\bigr).$$

Let $\partial W(x)$ denote the subdifferential of W at x.

For every $y \in B_r$ we set

$$\psi(y) = \sup_{x\colon\ \varphi(x) \le |y|} \langle x, y\rangle.$$

Note that the restriction of ψ to $\partial B_{|y|}$ coincides with the support function $S_{A_{|y|}}$ of $A_{|y|} = \{x\colon\ \varphi(x) \le |y|\}$, where the support function is defined by

$$S_{A_{|y|}}(v) := \sup_{x \in A_{|y|}} \langle v, x\rangle.$$

Lemma 6. *For ν-almost all y one has*

$$\psi(y) = \langle T^{-1}(y), y\rangle. \tag{3}$$

The function ψ can be described as a limit of certain functions depending on the potentials φ_t. One can show that W_t and W_t^* satisfy the identities

$$\nabla W_t^* \circ \nabla W_t(x) = x, \quad \nabla W_t \circ \nabla W_t^*(y) = y$$

almost everywhere on the sets A and $\nabla W_t(A)$. Since

$$\nabla W_t = |T_t|^t T_t,$$

one has

$$T_t^{-1}(y) = \nabla W_t^*(|y|^t y).$$

We have found a sequence $t_n \to +\infty$ for which the mappings T_{t_n} converge to T almost everywhere on A, hence converges in measure μ. For this sequence, the following holds (the proof employs [5, Corollary 9.9.11]).

Lemma 7. *The mappings $T_{t_n}^{-1}$ converge to T^{-1} in measure ν. Hence there exists a subsequence $t_n' \to \infty$ such that $T_{t_n'}^{-1} \to T^{-1}$ ν-almost everywhere.*

Theorem 3. *Let a function ψ_t be defined by the relation*

$$W_t^*(z) = |z|^{\frac{t}{1+t}} \psi_t\bigl(z|z|^{-\frac{t}{1+t}}\bigr).$$

Equivalently,

$$\psi_t(y) = \frac{W_t^*(y|y|^t)}{|y|^t}.$$

Then one has $\psi = \lim_{t_n\to\infty} \psi_{t_n}$ *almost everywhere for some sequence* $\{t_n\}$.

Remark 1. (i) Let us set $\partial_v\psi(y) := \lim_{t_n\to\infty} \partial_v\psi_n(y)$. In view of convergence $T_n \to T$ this definition makes sense. Moreover, we have $\partial_v\psi(y) = \langle T^{-1}(y), v\rangle$ for any $v\perp y$. Taking into account (3) we obtain the following remarkable relation:

$$T^{-1}(y) = \frac{\psi(y)}{|y|}e_1(y) + \sum_{i=2}^{d} \partial_{e_i(y)}\psi(y)\ e_i(y),$$

where $\{e_i(y)\}$ is an orthonormal system of unit vectors chosen in such a way that $e_1(y) = y/|y|$ and $e_i(y)\perp y$, $2 \le i \le d$.

(ii) Let us see what happens in the limit with the duality formula

$$W_t(x) + W_t^*(z) \ge \langle x, z\rangle.$$

It can be rewritten as

$$\frac{1}{t+2}\varphi_t^{t+2}(x) + |y|^t\psi_t(y) \ge \langle x, y\rangle|y|^t$$

by letting $z := y|y|^t$. An equality holds only at the points of the graph of T_t. Since for $y = T_t(x) = \varphi_t\frac{\nabla\varphi_t}{|\nabla\varphi_t|}|\nabla\varphi_t|^{\frac{1}{1+t}}$ we have an equality, we obtain the following duality relation:

$$\frac{1}{t+2}\varphi_t^2(x)|\nabla\varphi_t(x)|^{-\frac{t}{t+1}} + \psi_t(T_t(x)) = \langle x, T_t(x)\rangle.$$

In the limit $t \to \infty$ we find

$$\psi(T(x)) = \langle x, T(x)\rangle.$$

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References

[1] Ambrosio L. Lecture notes on optimal transport problems. In: Mathematical aspects of evolving interfaces (Funchal, 2000), Lecture Notes in Math. V. 1812. P. 1–52. Springer, Berlin, 2003.

[2] Villani C. Topics in optimal transportation. Amer. Math. Soc., Providence, Rhode Island, 2003.

[3] Giga Y. Surface evolution equations. A level sets approach. Birkhäuser Verlag, Basel, 2006.

[4] Rockafellar R.T. Convex analysis. Princeton, Princeton Univ. Press, 1970.

[5] Bogachev V.I. Measure theory. V. 1,2. Springer, Berlin – New York, 2007.