MASS TRANSPORT AND VARIANTS OF THE LOGARITHMIC SOBOLEV INEQUALITY

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ABSTRACT. We develop the optimal transportation approach to modified log-Sobolev inequalities and to isoperimetric inequalities. Various sufficient conditions for such inequalities are given. Some of them are new even in the classical log-Sobolev case. The idea behind many of these conditions is that measures with a non-convex potential may enjoy such functional inequalities provided they have a strong integrability property that balances the lack of convexity. In addition, several known criteria are recovered in a simple unified way by transportation methods and generalized to the Riemannian setting.

Keywords: logarithmic Sobolev inequality, isoperimetric inequalities, optimal transportation, Riemannian manifolds, Ricci curvature.

MSC classes: 60E15; 26D10.

CONTENTS

1. INTRODUCTION

This work deals with Sobolev inequalities and isoperimetric properties of absolutely continuous probability measures on Euclidean space or Riemannian manifolds. This subject is connected, among other fields, to analysis, probability theory, differential geometry, partial differential equations. In particular such properties are crucial in the study of the concentration of measure phenomenon and of the regularizing effect and trend to equilibrium of evolution equations. Several surveys were devoted to this quickly developing topic, see e.g. [5], [7], [24], [42], [43], [49], [58].

Besides the Poincaré or spectral gap inequality, the logarithmic Sobolev inequality is the best studied Sobolev property for probability measures. The basic example of a measure satisfying the logarithmic Sobolev inequality

$$(1) \qquad \mathrm{Ent}_\mu f^2 := \int f^2 \log\left(\frac{f^2}{\int f^2\, d\mu}\right) d\mu \leq 2C \int |\nabla f|^2\, d\mu$$

is the standard Gaussian measure on $\mathbb{R}^d$, $d\gamma(x) = (2\pi)^{-\frac{d}{2}} e^{-\frac{|x|^2}{2}}\, dx$ (with $C = 1$). It is well known, that for every measure μ satisfying (1) there exists $\varepsilon > 0$ such that $e^{\varepsilon|x|^2} \in L^1(\mu)$. This condition fails for many useful probability distributions, as for example

$$\mu_\alpha = \frac{1}{Z_\alpha} e^{-|x|^\alpha}\, dx, \quad x \in \mathbb{R}$$

where $0 < \alpha < 2$. There were many attempts to reveal which inequalities of the log-Sobolev type are the right ones for these measures when $\alpha \in (1,2)$ (the condition $\alpha \geq 1$ guarantees a spectral gap property $\mathrm{Var}_\mu(f) \leq C \int |\nabla f|^2 d\mu$). As for now, we do not know of a Sobolev type inequality with all the good features of the log-Sobolev inequality (in terms of consequences for concentration, tensorisation or semigroup properties). Instead of this, several functional inequalities were proposed, each of them having one of these good features:

  – The Beckner-Latała-Oleszkiewicz inequalities have the tensorisation property (when valid for μ they hold with the same constant for $\mu^{\otimes d}$ for all d) and yield concentration estimates with decay e^{-Kt^α}, for the ℓ_2-distance on the products. They where first mentioned by Beckner [17] for the Gaussian measure, i.e. $\alpha = 2$. Their modified version for the measures μ_α, $\alpha \in (1,2)$ is due to Latała-Oleszkiewicz [47].

  – F-Sobolev inequalities of the form

$$\int f^2 F\Big(\frac{f^2}{\int f^2\,d\mu}\Big)\,d\mu \leq C \int |\nabla f|^2\,d\mu$$

where F is some increasing function, were established for the measures μ_α and their d-dimensional analogs in [60], [1], with $F(t) = \log(t)^{2-2/\alpha}$ for large t. The recent developments can be found in papers [14], [56], [68], [45]. Inequalities of this type imply hyperboundedness of the related semigroups for certain Orlicz norms and under mild conditions isoperimetric inequalities, see [14]. In fact, they are closely related with the Sobolev inequalities for Orlicz norm $\|f - \int f\,d\mu\|_\Phi^2 \leq C \int |\nabla f|^2\,d\mu$, where Φ is some Orlicz function. Details about the connections and additional semigroups properties appear in [56], [68]. Let us also mention another important work of Wang [65] devoted to the so-called super-Poincaré inequalities. It establishes a correspondence between F-Sobolev and super-Poincaré inequalities, and gives consequences in terms of isoperimetric and Nash inequalities as well as spectral properties of semigroups.

  – Modified log-Sobolev inequality with cost function c

$$(2) \qquad\qquad\qquad \mathrm{Ent}_\mu f^2 \leq \int_{\mathbb{R}^d} f^2 c^*\Big(\frac{\nabla f}{f}\Big)\,d\mu,$$

where $c^*(x) = \sup_y \langle x, y \rangle - c(y)$ is the convex conjugated of some convex cost function $c : \mathbb{R}^d \to \mathbb{R}^+$. The first modified log-Sobolev inequality was introduced by Bobkov and Ledoux [25] for the exponential measure, for c^* quadratic on a small interval around zero and infinite otherwise. The main interest of modified log-Sobolev inequalities is to imply improved concentration inequalities for product measures as well as corresponding inequalities between entropy and transportation cost, see [62], [49], [22], [16]. Modified log-Sobolev inequalities, with appropriate cost functions, have been known for some time for the measures μ_α, $\alpha \geq 2$, see [26], [39]. They were established only recently by Gentil, Guillin and Miclo [40] to the case $1 \leq \alpha < 2$ for a functions $c_\alpha^*(x)$ comparable to $\max(x^2, |x|^{\alpha/(\alpha-1)})$. See [41], [45], [16] for other examples.

An isoperimetric inequality is a lower bound of the μ-boundary measure of sets $\mu^+(\partial A)$ in terms of their measure $\mu(A)$. Recall that for a Borel measure on a metric space (X, ρ), the boundary measure of a Borel set $A \subset X$ can be defined as the Minkowski content

$$\mu^+(\partial A) = \liminf_{h \to 0^+} \frac{\mu\Big(\big\{x \in X \setminus A;\ \rho(x, A) \leq h\big\}\Big)}{h}.$$

The isoperimetric function of a probability measure is defined for $t \in (0,1)$ by

$$\mathcal{I}_\mu(t) = \inf\big\{\mu^+(\partial A);\ \mu(A) = t\big\}.$$

One easily checks that the measure μ_α, $\alpha \geq 1$ satisfy an isoperimetric inequality of the form $\mathcal{I}_{\mu_\alpha}(t) \geq \kappa(\alpha)L_\alpha(t)$, where

$$L_\alpha(t) = \min(t, 1-t)\log^{1-\frac{1}{\alpha}}\Big(\frac{1}{\min(t, 1-t)}\Big), \quad t \in (0,1).$$

Indeed the sets of minimal boundary measure for given measure are half-lines for log-concave probability measures on the real line, see e.g. [24]. It is well known that isoperimetric inequalities often imply Sobolev type inequalities. Indeed a natural way to try and unify the above functional inequalities satisfied by μ_α is to derive them from the above isoperimetric inequality. Several papers deal with such results (see [48], [11] for the log-Sobolev inequality, [65] for F-Sobolev inequalities). The most general result in this direction is

given in [45] where inequalities of the following form (encompassing F-Sobolev and modified log-Sobolev) are deduced from isoperimetric inequalities:

$$(3) \qquad \int_{\mathbb{R}^d} f^2 F_\tau\Big(\frac{f^2}{\int_{\mathbb{R}^d} f^2\, d\mu}\Big)\, d\mu \le \int_{\mathbb{R}^d} f^2 c^*\Big(\Big|\frac{\nabla f}{f}\Big|\Big)\, d\mu + B \int_{\mathbb{R}^d} f^2\, d\mu.$$

However deriving isoperimetric inequalities is hard. In practice one often proves Sobolev inequalities first and then deduce the isoperimetric inequalities from a method of Ledoux (see [48], [11], [65], [14]) which applies when the curvature is bounded below to certain Sobolev inequalities with energy term $\int |\nabla f|^2 d\mu$.

Let us mention a few successful methods to establish Sobolev type inequalities. On the real line, thanks to Hardy type inequalities, it is possible to express simple necessary and sufficient conditions for certain Sobolev inequalities to hold. This technique was first applied to the logarithmic Sobolev inequality by Bobkov and Götze [21]. See e.g. [15], [27], [16] for further applications.

The semigroup method gives Sobolev inequalities by evolution along the semigroup e^{tL} with generator $L = \Delta - \nabla V.\nabla$, for with $d\mu(x) = e^{-V(x)}dx$ is an invariant measure. It was developed in the abstract framework of diffusion generators by Bakry and Emery [9]. These authors proved the following celebrated result: if a probability measure μ on a Riemannian manifold has a density e^{-V} with respect to the Riemannian volume and if for some $K > 0$ it holds pointwise $\mathrm{Hess}V + \mathrm{Ric} \ge K\,\mathrm{Id}$ then for all smooth f ,

$$\mathrm{Ent}_\mu(f^2) \le \frac{2}{K}\int |\nabla f|^2 d\mu.$$

Here Ric is the Ricci tensor of M. This result was complemented by the following theorem of Wang [64, 66]: denoting by ρ the geodesic distance, if $\mathrm{Hess}V + \mathrm{Ric} \ge K\,\mathrm{Id}$ with $K \le 0$ and if there exists $\varepsilon > 0$ such that

$$\int e^{\frac{|K|+\varepsilon}{2}\rho(x,x_0)^2}d\mu(x) < +\infty$$

for some $x_0 \in M$, then μ satisfies a log-Sobolev inequality.

In their seminal paper [54] Otto and Villani showed that optimal mass transportation allows to derive log-Sobolev inequalities. Their approach was streamlined by Cordero-Erausquin [33] and extended in several subsequent papers, see [34, 36, 2]. Let us define the basic objects of optimal transport theory and refer to the books [63, 55] for details. Given μ, ν two Borel probability measures on a Polish space X and a cost function $c : X \times X \to \mathbb{R}^+$ vanishing on the diagonal, the c-transportation cost from μ to ν is

$$W_c(\mu,\nu) = \inf\left\{\int_{X\times X} c(x,y)\, d\pi(x,y);\ \pi \in \Pi(\mu,\nu)\right\},$$

where $\Pi(\mu,\nu)$ is the set of probability measures on $X \times X$ with first marginal μ and second marginal ν. When an optimal π exists it is called "optimal transportation plan". When an optimal plan is supported by the graph of a function $T : X \to X$, then T pushes forward μ to ν and is called optimal transport map. The existence and the structure of such optimal plans and maps is by now quite developed, see [63] and the reference therein.

The purpose of this paper is twofold. Firstly we present several new sufficient conditions for variants of log-Sobolev inequalities to hold, in Euclidean space and on Riemannian manifolds. The idea behind many of them is that measures on $\mathbb{R}^d$ with density e^{-V} verify a variant of the log-Sobolev inequality provided the lack of convexity of V is balanced by an appropriate integrability condition. This principle appears clearly in Wang's theorem as well as the following result about log-concave measures on $\mathbb{R}^d$ (for absolutely continuous measures, this means that the density is of the form e^{-V} where V is convex with values in $(-\infty, +\infty]$): every log-concave probability measure μ on $\mathbb{R}^d$ such that $\int \exp(\varepsilon|x|^\alpha)\, d\mu < +\infty$ for some $\varepsilon > 0$ and $\alpha \ge 1$ satisfies up to constant the same isoperimetric inequality as μ_α, namely $\mathcal{I}_\mu \ge \kappa L_\alpha$. This was proved by Bobkov [20] for $\alpha \in \{1, 2\}$ and was extended in [13] to $\alpha \in [1, 2]$ with an argument that actually applies to $\alpha \ge 1$.

A second purpose of this work is to develop the mass transport approach in order to get new and old results in a unified manner. To do this we had to introduce several new ways of handling the terms involved in optimal transport. Let us mention that transportation is intimately linked with the entropy functional, and therefore naturally yields modified log-Sobolev inequalities. Among other things this paper shows how to recover F-Sobolev inequalities, and therefore isoperimetric inequalities.

Next we describe the structure of the paper and highlight some of the main results and techniques. Section 2 is devoted to tightening techniques. A functional inequality is tight when it becomes an equality for constant functions. It is called defective otherwise. Tightness it crucial in applications of Sobolev inequalities to concentration or to hypercontractivity properties. A classical method of Rothaus allows to transform a defective log-Sobolev inequality into a tight one, by means of a Poincaré inequality. It does not apply to the modified inequalities. Theorem 2.4 develops a new simple method for tightening general "modified F-Sobolev inequalities" (3). This result encompasses and simplifies several existing tightness lemma for F-Sobolev inequalities. We also collect known facts about how to derive global Poincaré inequalities from local ones.

Section 3 gives a short account of the consequences of isoperimetric inequalities in terms of Sobolev type inequalities, with emphasis on the measures satisfying the same isoperimetric inequalities as the model measures μ_α. To do this we combine the main result of [45] with our new tightening results.

In Section 4 we introduce a new variant of the log-Sobolev inequality, which plays a crucial role in the paper. For $\tau \in (0,1]$ we say that μ satisfies $I(\tau)$ if there exists numbers B, C such that every smooth function f verifies

$$I(\tau) \qquad \operatorname{Ent}_\mu f^2 \leq B \int f^2 \, d\mu + C \int_{\mathbb{R}^d} |\nabla f|^2 \log^{1-\tau}\left(e + \frac{f^2}{\int f^2 \, d\mu}\right) d\mu.$$

Further results of the paper show that for $\alpha \in (1,2)$, μ_α satisfies $I(\tau)$ for $\tau = 2 - 2/\alpha$. The main result of the section is that $I(\tau)$ implies appropriate F-Sobolev inequalities and modified log-Sobolev inequalities.

In Section 5 we develop the transportation techniques and establish variants of log-Sobolev inequalities and isoperimetric inequalities for measures on $\mathbb{R}^d$. Let $d\mu = e^{-V(x)} \, dx$ be a probability measure, denote by $f \cdot \mu$ the measure with density f with respect to μ. As in previous contributions, the starting point is the "above-tangent lemma": if a map $T(x) = x + \theta(x)$ is the optimal transport, for a strictly convex cost, pushing forward a probability measure $f \cdot \mu$ to μ then

$$(4) \qquad \operatorname{Ent}_\mu f \leq -\int_{\mathbb{R}^d} \langle \nabla f, \theta \rangle \, d\mu + \int_{\mathbb{R}^d} \mathcal{D}_V(x, T(x)) f \, d\mu(x),$$

where the convexity defect is

$$\mathcal{D}_V(x,y) = -\Big(V(y) - V(x) - \langle \nabla V(x), y - x \rangle\Big).$$

Under additional integrability assumptions we prove corresponding modified log-Sobolev inequalities and Inequalities $I(\tau)$. The main part of the work consists in estimating the last term involving the convexity defect. This can be done when

1) $\mathcal{D}_V(x,y)$ has an upper bound of the type $c_0(x - y)$ and μ satisfied certain integrability assumption,
2) V is controlled by a function of the type $G(\nabla V)$ and ΔV grows slower than $|\nabla V|^2$,
3) V satisfies $V(x) \leq C_1 \langle \nabla V(x), x \rangle + C_2$ and certain integrability assumption on ∇V,
4) V is obtained by a perturbation of some convex potential.

We recover and extend the Euclidean version of Wang's result. A simple new result asserts that when $\mathcal{D}_V(x,y)$ is upper bounded by $\lambda c(x - y)$ where c is a strictly convex cost and $\lambda \geq 0$ then μ satisfies a defective modified log-Sobolev inequality with cost c provided there exists $\varepsilon > 0$ such that

$$\int e^{(\lambda + \varepsilon)c(x-y)} d\mu(x) d\mu(y) < +\infty.$$

We also extend a theorem of Bobkov on the isoperimetric inequalities for log-concave measures. Our results imply in particular that when $D^2 V \geq -K \operatorname{Id}$ and $\int \exp(\varepsilon |x|^\alpha) d\mu(x) < +\infty$ for some $K \geq 0$, $\varepsilon > 0$, $\alpha > 2$ then μ satisfies an isoperimetric inequality on the model of μ_α. We generalize this result for the Riemannian case in Section 7.

Section 6 provides improved bounds for specific measures on $\mathbb{R}$. For instance we recover by transportation techniques the modified log-Sobolev inequalities satisfied by the exponential measure. We propose an interpretation in terms of transport of the condition $|f'/f| < c$ that appears in the result of Bobkov and Ledoux. This is related to a simple fact in the spirit of the Caffarelli's contraction theorem [29].

In Section 7 we generalize some results obtained in this paper to Riemannian manifolds. We apply the manifold version of (4) obtained by Cordero-Erausquin, McCann and Schmuckenschläger in [36] for quadratic

transportation cost. We consider a smooth complete connected Riemannian manifold without boundary M with a probability measure $\mu = e^{-V} d$vol. In particular, we establish Sobolev type inequalities and isoperimetric inequalities for μ such that $D^2 V + \mathrm{Ric} \geq 0$ and $e^{\varepsilon \rho^\alpha(x_0,x)} \in L^1(\mu)$, where ρ is the Riemannian distance, $\varepsilon > 0$ and x_0 is an arbitrary point on M and $\alpha \in (1,2]$.

Most results of the paper apply to measures with the tail behavior of the order $e^{-|x|^\alpha}$ with $1 < \alpha \leq 2$ (apart from Section 3, Subsection 5.4 (Corollary 5.14), Theorem 5.16 and Theorem 7.2). Nevertheless, some of our results for $\alpha \leq 2$ can be adapted to $\alpha \geq 2$.

Dealing with $\alpha > 2$ differs from the opposite situation in several respects. First of all, unlike the case $\alpha < 2$, we don't have I_τ-inequality which allows to prove both F-Sobolev and modified log-Sobolev inequality in a suitable form. Nevertheless, estimating the linear term in the same way as in Lemma 5.2, Lemma 5.3, we can prove in many cases the defective modified log-Sobolev inequality with $c = |x|^\alpha$. The tightening procedure can be done with the help of Propositions 2.1 and 2.10 due to the fact that the modified log-Sobolev inequality for the cost function $|x|^\alpha$ is equivalent to the corresponding q-log Sobolev inequality with $q = \alpha^*$. However, in this case one has to prove (or assume) local q-Poincaré inequalities. In the case of $\mathbb{R}^d$ and locally bounded potential V this can be shown by Lemma 2.9, since the Cheeger inequality implies q-Poincaré inequalities for $q > 1$. Finally, we note that the reader can easily check that Theorem 5.27 a) and Theorem 5.25 hold also for $\alpha > 2$ in the case of modified log-Sobolev inequality.

List of the main objects considered in this paper

- Ambient space: We work in the Euclidean space $(\mathbb{R}^d, \langle \cdot, \cdot \rangle, |\cdot|)$ or on a Riemannian manifold (M, g) for which the geodesic distance is denoted by ρ.

- Duality: If $\alpha > 1$ we denote by α^* the number such that $\frac{1}{\alpha} + \frac{1}{\alpha^*} = 1$. This is consistent with the definition of the convex conjugate (or Fenchel-Legendre transform) of a function $c^*(x) = \sup_y \langle x, y \rangle - c(y)$, since the conjugate of $x \mapsto |x|^\alpha/\alpha$ is $x \mapsto |x|^{\alpha^*}/\alpha^*$.

- Special cost functions: for $\alpha > 1$, $t \in \mathbb{R}$, $c_\alpha(t) = \begin{cases} \frac{t^2}{2} & \text{if } |t| \leq 1, \\ \frac{|t|^\alpha}{\alpha} + \frac{\alpha-2}{2\alpha} & \text{if } |t| \geq 1. \end{cases}$
  Note that $c_\alpha^* = c_{\alpha^*}$. For $\alpha \in (1,2]$, up to multiplicative constants $c_\alpha(t) \approx \min(t^2, |t|^\alpha)$ whereas for $\alpha > 2$, $c_\alpha(t) \approx \max(t^2, |t|^\alpha)$.

- Special generalized entropies: $F_\tau(t) = \log^\tau(1+t) - \log^\tau(2)$, $t \geq 0$.

- Modified log-Sobolev inequality (**MLSI**) for a cost function c:

$$\mathrm{Ent}_\mu f^2 \leq \int f^2 c^* \Big(\frac{|\nabla f|}{f} \Big) \, d\mu$$

  More general functions of $\frac{\nabla f}{f}$ are sometimes considered.
- F-Sobolev inequality (**FSI**):

$$\int f^2 F \Big(\frac{f^2}{\int f^2 \, d\mu} \Big) \, d\mu \leq C \int |\nabla f|^2 \, d\mu$$

  The q-F-Sobolev inequality (**qFSI**) is defined with the same formula, replacing f^2 by $|f|^q$ and $|\nabla f|^2$ by $|\nabla f|^q$. When $F = \log$ this is the classical log-Sobolev inequality (**LSI**).
- Inequality $I(\tau)$:

$$\mathrm{Ent}_\mu(f^2) \leq B \int f^2 d\mu + C \int \Big| \nabla f \Big|^2 \log^{1-\tau} \Big(e + \frac{f^2}{\int f^2 \, d\mu} \Big) \, d\mu.$$

- Poincaré inequality (**P**)

$$\int \Big| f - \int f \, d\mu \Big|^2 d\mu \leq E \int |\nabla f|^2 d\mu.$$

  For the q-Poincaré inequality (**qP**), write q instead of 2.

2. How to tighten the inequalities

The results of this section apply in rather general settings. For simplicity we assume that μ is a probability measure on a Riemannian manifold, and is absolutely continuous with respect to the volume measure.

2.1. **Translation invariant energies.** The following result of Bobkov and Zegarlinski [27] is an extension to $q \neq 2$ of an argument going back to Rothaus [59].

Proposition 2.1. *Let* $q \in (1, 2]$. *Assume that a probability measure* μ *satisfies a defective* q-*log-Sobolev inequality as well as a* q-*Poincaré inequality:*

$$\mathrm{Ent}_\mu(|f|^q) \leq C \int |\nabla f|^q d\mu + D \int |f|^q d\mu \quad \text{and} \quad \int \left| f - \int f \, d\mu \right|^q d\mu \leq E \int |\nabla f|^q d\mu.$$

Then it automatically satisfies a tight q-*logarithmic Sobolev inequality*

$$\mathrm{Ent}_\mu(|f|^q) \leq 16 \big(C + (D+1)E \big) \int |\nabla f|^q d\mu.$$

This is a simple consequence from the following inequality (see [59, 27] for its proof)

$$\mathrm{Ent}_\mu(|f|^q) \leq 16 \Big(\mathrm{Ent}_\mu(|f - \int f \, d\mu|^q) + \int |f - \int f \, d\mu|^q d\mu \Big).$$

Remark 2.2. For $q > 2$ it is not possible to have $\mathrm{Ent}_\mu(|f|^q) \leq K \int |\nabla f|^q d\mu$, as for $f = 1 + \varepsilon g$ where $\varepsilon \to 0$ the left-hand side behaves like ε^2 whereas the energy term is of order ε^q.

Remark 2.3. The change of functions $f^q = g^2$ turns the q-log-Sobolev inequality into a modified-log Sobolev inequality with function $c^*(t)$ proportional to $|t|^q$.

2.2. **Modified energies.** The method of Rothaus relies on the invariance the energy term $\int |\nabla f|^q d\mu$ under translations $f \mapsto f + t$, $t \in \mathbb{R}$. In general, this property fails for the modified energy $\int f^2 H(|\nabla f|/f) d\mu$. This quantity may be very different for f and $\tilde{f} = f - \mu(f)$. This is why another approach is needed. The next theorem allows to tighten quite general inequalities. It encompasses several tightening results for F-Sobolev inequalities given in [14].

Theorem 2.4. *Let* H *be an even function on* $\mathbb{R}$, *which is increasing on* $\mathbb{R}^+$ *and satisfies* $H(0) = 0$ *and* $H(x) \geq cx^2$. *Assume that there exists* $q \geq 2$ *such that* $x \mapsto H(x)/x^q$ *is non-increasing on* $(0, +\infty)$.
Let $F : (0, +\infty) \to \mathbb{R}$ *be an non-decreasing function with* $F(1) = 0$, *such that* $x \mapsto xF(x)$ *is bounded from below and one of the following properties is verified for some* $A > 1$
  (i) the function $\Phi(x) = xF(x)$ *extends to a* $\mathcal{C}^2$-*function on* $[0, A^2]$,
  (ii) there exists a constant $d \geq 0$ *such that for all* $x \in (0, A^2]$, $F(x) \leq d(x - 1)$,
Assume that a probability measure μ *satisfies a defective modified* F-*Sobolev inequality: for all* f,

$$\int f^2 F \left(\frac{f^2}{\mu(f^2)} \right) d\mu \leq \int f^2 H \Big(\frac{|\nabla f|}{f} \Big) d\mu + D \int f^2 d\mu$$

If μ *also satisfies a Poincaré inequality, then there exists a constant* C *such that for every* f,

$$\int f^2 F \left(\frac{f^2}{\mu(f^2)} \right) d\mu \leq C \int f^2 H \Big(\frac{|\nabla f|}{f} \Big) d\mu.$$

The proof requires some preparation.

Lemma 2.5. *Let* f *be a function such that* $\int f^2 d\mu = 1$. *Let* $A > 1$. *It holds*

$$\int_{f^2 \geq A^2} f^2 d\mu \leq \left(\frac{A}{A-1} \right)^2 \mathbf{Var}_\mu(f). \tag{5}$$

If F is as in Theorem 2.4 above, then there exists a constant γ depending on A and F only such that

$$(6) \qquad \int f^2 F(f^2)\, d\mu \leq \gamma \mathbf{Var}_\mu(f) + \int_{f^2 \geq A^2} f^2 F(f^2)\, d\mu.$$

Proof. Since $\mathbf{Var}_\mu(|f|) \leq \mathbf{Var}_\mu(f)$ we may assume that $f \geq 0$. In this case, when $f \geq A$

$$f - \mu(f) \geq f - \mu(f^2)^{\frac{1}{2}} = f - 1 \geq f - \frac{f}{A} = \frac{A-1}{A} f.$$

Inequality (5) follows by integration.

Next we establish Inequality (6) when F satisfies Hypothesis (i) of Theorem 2.4. Let $f \geq 0$ with $\mu(f^2) = 1$. Noting that $\Phi(1) = 0$ and $\Phi'(1) \geq 0$, we have by Taylor's formula

$$
\begin{aligned}
\int f^2 F(f^2)\, d\mu &= \int \left(\Phi(f^2) - \Phi(1) - \Phi'(1)(f^2 - 1) \right) d\mu \\
&\leq \int_{f^2 < A^2} \left(\max_{[0,A^2]} \Phi'' \right) \frac{(f^2 - 1)^2}{2}\, d\mu + \int_{f^2 \geq A^2} \left(\Phi(f^2) - \Phi(1) - \Phi'(1)(f^2 - 1) \right) d\mu \\
&\leq \left(\max_{[0,A^2]} \Phi'' \right) \frac{(A+1)^2}{2} \int (f - 1)^2 d\mu + \int_{f^2 \geq A^2} \Phi(f^2)\, d\mu \\
&\leq \left(\max_{[0,A^2]} \Phi'' \right) (A+1)^2 \mathbf{Var}_\mu(f) + \int_{f^2 \geq A^2} f^2 F(f^2)\, d\mu,
\end{aligned}
$$

where the latter inequality follows from the bound $\int \left(f - \mu(f^2)^{\frac{1}{2}} \right)^2 d\mu \leq 2\mathbf{Var}_\mu(f)$, which is readily checked by expanding the square.

Finally, if the function F satisfies Hypothesis (ii) of Theorem 2.4 we simply observe that $\int f^2 F(f^2)\, d\mu = \int f^2 F(f^2) - d(f^2 - 1)\, d\mu$ and note that on $\{f^2 < A^2\}$,

$$f^2 F(f^2) - d(f^2 - 1) \leq f^2 d(f^2 - 1) - d(f^2 - 1) = d(f^2 - 1)^2 \leq d(A+1)^2 (f - 1)^2.$$

The claim follows by the same method. $\qquad\square$

Lemma 2.6. *Consider functions F and H as in Theorem 2.4 and set $m = -\inf_{t \in (0,1]} tF(t) \geq 0$. Let μ be a probability measure μ such that for all f,*

$$\int f^2 F\left(\frac{f^2}{\mu(f^2)} \right) d\mu \leq \int f^2 H\left(\frac{|\nabla f|}{f} \right) d\mu + D \int f^2 d\mu.$$

Then for $\eta > 0$ and all functions f with $\mu(f^2) = 1$ it holds

$$\int_{f^2 \geq (1+2\eta)^q} f^2 F(f^2)\, d\mu \leq \int f^2 H\left(\frac{2|\nabla f|}{f} \right) d\mu + (D + m) \int_{f^2 \geq (1+\eta)^q} f^2 d\mu.$$

Proof. It is enough to work with non-negative functions. The change of function $f^2 = g^q$ yields

$$\int g^q F\left(\frac{g^q}{\mu(g^q)} \right) d\mu \leq \int g^q H\left(\frac{q|\nabla g|}{2g} \right) d\mu + D \int g^q d\mu.$$

Since for $t > 0$, $tF(t) \geq tF_+(t) - m$, we get

$$(7) \qquad \int g^q F_+\left(\frac{g^q}{\mu(g^q)} \right) d\mu \leq \int g^q H\left(\frac{q|\nabla g|}{2g} \right) d\mu + (D + m) \int g^q d\mu.$$

Given a non-negative function φ with $\mu(\varphi^q) = 1$, we apply the latter inequality to $g = \theta(\varphi)$ where for $x \geq 0$

$$\theta(x) = \frac{1 + 2\eta}{\eta}(x - 1 - \eta) \mathbf{1}_{x \in [1+\eta, 1+2\eta)} + x \mathbf{1}_{x \geq 1+2\eta}.$$

Obviously for $x \geq 0$, $\theta(x) \leq x\mathbf{1}_{x \geq 1+\eta} \leq x$. Hence

$$\mu(g^q) \leq \mu(\varphi^q \mathbf{1}_{\varphi \geq 1+\eta}) \leq \mu(\varphi^q) = 1.$$

This estimate, together with the fact that $\varphi = g$ when $\gamma \geq 1 + 2\eta$, yields

$$\int g^q F_+\left(\frac{g^q}{\mu(g^q)} \right) d\mu \geq \int_{\varphi \geq 1+2\eta} \varphi^q F_+\left(\frac{\varphi^q}{\mu(g^q)} \right) d\mu \geq \int_{\varphi \geq 1+2\eta} \varphi^q F_+\left(\frac{\varphi^q}{\mu(\varphi^q)} \right) d\mu = \int_{\varphi \geq 1+2\eta} \varphi^q F(\varphi^q)\, d\mu.$$

Finally, since $\nabla g = 0$ when $\varphi < 1 + \eta$, and $|\nabla g| \le 2|\nabla \varphi|$ when $\varphi \ge 1 + \eta$

$$\int g^q H\Big(\frac{q|\nabla g|}{2g}\Big)\, d\mu \le \int_{\varphi \ge 1+\eta} g^q H\Big(\frac{q|\nabla \varphi|}{g}\Big)\, d\mu \le \int_{\varphi \ge 1+\eta} \varphi^q H\Big(\frac{q|\nabla \varphi|}{\varphi}\Big)\, d\mu,$$

where the last inequality follows from $g \le \varphi$ and $x \mapsto x^q H(1/x)$ non-decreasing on $(0, +\infty)$. From the above three estimates, Inequality (7) gives for φ with $\mu(\varphi^q) = 1$

$$\int_{\varphi \ge 1+2\eta} \varphi^q F(\varphi^q)\, d\mu \le \int_{\varphi \ge 1+\eta} \varphi^q H\Big(\frac{q|\nabla \varphi|}{\varphi}\Big)\, d\mu + (D+m)\int_{\varphi \ge 1+\eta} \varphi^q\, d\mu.$$

The claim follows from the change of functions $f^2 = \varphi^q$. $\qquad\square$

Proof of Theorem 2.4. By homogeneity we may assume that $\int f^2\, d\mu = 1$. Let $\eta > 0$ such that $A^2 = (1+2\eta)^q$. Combining the previous two lemmas

$$
\begin{aligned}
\int f^2 F(f^2)\, d\mu &\le \gamma \operatorname{\mathbf{Var}}_\mu(f) + \int_{f^2 \ge (1+2\eta)^q} f^2 F(f^2)\, d\mu \\
&\le \gamma \operatorname{\mathbf{Var}}_\mu(f) + \int f^2 H\Big(\frac{2|\nabla f|}{f}\Big)\, d\mu + (D+m)\int_{f^2 \ge (1+\eta)^q} f^2 d\mu \\
&\le \Big(\gamma + (D+m)\Big(\frac{A}{A-1}\Big)^2\Big) \operatorname{\mathbf{Var}}_\mu(f) + 2^q \int f^2 H\Big(\frac{|\nabla f|}{f}\Big)\, d\mu,
\end{aligned}
$$

where we have used again that $H(x)/x^q$ is non-increasing. Finally we apply Poincaré's inequality and the bound $H(x) \ge cx^2$,

$$\operatorname{\mathbf{Var}}_\mu(f) \le C_P \int |\nabla f|^2\, d\mu \le \frac{C_P}{c} \int f^2 H\Big(\frac{|\nabla f|}{f}\Big)\, d\mu.$$

$\qquad\square$

2.3. Tightening for free: local inequalities.
Local inequalities are easy to derive for locally bounded potentials by standard perturbation techniques. In many cases they allow to tighten defective inequalities. They are defined below.

Definition 2.7. Let $q \ge 1$ and μ be a probability measure. One says that μ satisfies a local q-Poincaré inequality if for every $\eta \in (0, 1)$, there exists a set A with $\mu(A) \ge \eta$ such that the measure $\mu_A = \frac{\mathbf{1}_A}{\mu(A)} \cdot \mu$ satisfies a q-Poincaré inequality, meaning that there exists $C_A < +\infty$ such that for every smooth f,

$$(8) \qquad \int \big|f - \mu_A(f)\big|^q d\mu_A \le C_A \int |\nabla f|^q d\mu_A.$$

When $q = 2$ we just say that μ verifies a local Poincaré inequality.

2.3.1. Isoperimetric inequalities. The goal of this paragraph is to show how to extend isoperimetric inequalities when they are known only for sets of small or large measure. The argument is based on local Cheeger's inequalities (which are equivalent to local 1-Poincaré inequalities). One gets the following convenient result.

Proposition 2.8. *Let $I : [0, 1/2] \to \mathbb{R}^+$ be an non-decreasing function with $I(t) > 0$ for $t > 0$. Let $\varepsilon \in (0, 1/2)$. Assume that a probability measure $\mu = e^{-V(x)}dx$ on $\mathbb{R}^d$ admits a locally bounded potential V and satisfies for every set A*

$$\mu^+(\partial A) \ge I(a), \quad \text{when} \quad a = \min(\mu(A), \mu(A^c)) < \varepsilon.$$

Then there exists a constant c such that arbitrary sets satisfy $\mu^+(\partial A) \ge c\, I(\min(\mu(A), \mu(A^c)))$.

The proof is based on the following easy fact:

Lemma 2.9. *Let $\mu = e^{-V(x)}dx$ be a probability measure on $\mathbb{R}^d$. Assume that V is locally bounded. Then for every $r > 0$ there exists a constant C_r such that the measure $\mu_{B_r} = \frac{\mathbf{1}_{B_r}}{\mu(B_r)} \cdot \mu$ satisfies for every set A,*

$$\mu^+_{B_r}(\partial A) \ge C_r \min(\mu_{B_r}(A), \mu_{B_r}(A^c)).$$

Proof. First recall that a probability measure ν satisfies Cheeger's isoperimetric inequality with constant c means that for every set $c\nu^+(\partial A) \geq \min(\nu(A), \nu(A^c))$. This is equivalent to the functional inequality

$$\int |f - \mathrm{med}_\nu(f)|d\nu \leq c \int |\nabla f|\, d\nu.$$

Using the variational expression of the median

$$\int |f - \mathrm{med}_\nu(f)|d\nu = \inf_{a \in \mathbb{R}} \int |f - a|\, d\nu,$$

one easily checks that the above inequality for ν can be transfered to any perturbed probability $\eta = e^g \cdot \nu$ as

$$\int |f - \mathrm{med}_\eta(f)|d\eta \leq c e^{\sup g - \inf g} \int |\nabla f|\, d\eta.$$

Since the uniform probability measure on B_r satisfies Cheeger's isoperimetric inequality, so does the measure μ_{B_r} (indeed V is bounded from above and below on B_r). $\square$

Proof of Proposition 2.8. Consider an arbitrary set A with $\mu(A) \in [\varepsilon, 1-\varepsilon]$. It is enough to find a universal constant $C > 0$ for which $\mu^+(\partial A) \geq C$. To do this, choose R such that $\mu(B_R) = 1 - \varepsilon/2$. Plainly $\mu_{B_R}(A) \leq (1-\varepsilon)/(1-\varepsilon/2) < 1$ and

$$\mu_{B_R}(A) = \frac{\mu(A) + \mu(B_R) - \mu(A \cap B_R)}{\mu(B_R)} \geq \frac{\varepsilon + 1 - \frac{\varepsilon}{2} - 1}{\mu(B_R)} = \frac{\varepsilon}{2 - \varepsilon} > 0.$$

By the previous lemma, μ_{B_R} satisfies Cheeger's isoperimetric inequality. Hence there is a constant $K > 0$ (depending only on ε and μ) such that $\mu_{B_R}^+(\partial A) \geq K$. Finally $\mu^+(\partial A) \geq \mu(B_R)\mu_{B_R}^+(\partial A) \geq (1-\varepsilon/2)K$. $\square$

2.3.2. *Sobolev inequalities.* Next we deal with defective F-Sobolev inequalities. In the case $q = 2$ the following result is a consequence of several existing results in the literature (Röckner-Wang [57] show that a local Poincaré inequality implies a weak Poincaré inequality, Wang [65] shows that a weak Poincaré inequality and a specific super Poincaré inequality implies a Poincaré inequality, and that defective F-Sobolev inequalities imply super-Poincaré inequalities. See also Aida [3].) However these results do not provide explicit constants. The next proposition gives a concrete bound with a straightforward proof.

Proposition 2.10. *Let $q > 1$. Let $F : (0, \infty) \to \mathbb{R}$ be a non-decreasing function with $F(1) = 0$, $F(+\infty) = +\infty$ and such that for all $x \in (0, 1)$, $xF(x) \geq -M$, where $M \in [0, +\infty)$. Let μ be a probability measure. Assume that μ satisfies a local q-Poincaré inequality (8), and the following defective F-Sobolev inequality: for all smooth function f*

$$\int |f|^q F\left(\frac{|f|^q}{\mu(|f|^q)}\right) d\mu \leq C \int |\nabla f|^q d\mu + D \int |f|^q d\mu.$$

Then it satisfies the following q-Poincaré inequality: for all smooth f

$$\int |f - \mu(f)|^q\, d\mu \leq 6 \left(2^{q-1}K + \frac{C}{4(D+M)}\right) \int |\nabla f|^q d\mu,$$

with $K = \kappa\left(\max\left(\left(1 + (2 \cdot 3^{1/(q-1)})^{-1}\right)^{-1}, 1 - \left(4F_+^{-1}(4(D+M))\right)^{-1}\right)\right)$, where $\kappa(r) = \inf\{c_A; \ \mu(A) \geq r\}$ for $r \in (0, 1)$, and F_+^{-1} is the generalized left inverse of $F_+ = \max(F, 0)$.

Proof. Since $F \geq F_+ - M$ the hypothesis implies, for all f

$$(9) \qquad \int |f|^q F_+\left(\frac{|f|^q}{\mu(|f|^q)}\right) d\mu \leq C \int |\nabla f|^q d\mu + (D + M) \int |f|^q d\mu.$$

Without loss of generality we consider a function f with $\mu(f) = 0$ and $\mu(|f|^q) = 1$. Given a set A to be specified later, we write

$$(10) \qquad 1 = \int |f - \mu(f)|^q d\mu = \int |f|^q d\mu = \int |f|^q \mathbf{1}_A\, d\mu + \int |f|^q \mathbf{1}_{A^c}\, d\mu.$$

We bound the first term by means of the local q-Poincaré inequality, noting that $\int f\,d\mu = 0$ implies that $\int f\mathbf{1}_A\,d\mu = -\int f\mathbf{1}_{A^c}\,d\mu$. By the convexity relation $|x+y|^q \le 2^{q-1}(|x|^q + |y|^q)$, we get for any probability measure

$$\text{(11)} \qquad \int |g - \nu(g)|^q d\nu \ge \frac{1}{2^{q-1}} \int |g|^q d\nu - |\nu(g)|^q.$$

The local q-Poincaré inequality hence guarantees

$$
\begin{aligned}
\int |f|^q \mathbf{1}_A d\mu &\le\; 2^{q-1} \frac{\left|\int f\mathbf{1}_A\,d\mu\right|^q}{\mu(A)^{q-1}} + 2^{q-1} c_A \int |\nabla f|^q \mathbf{1}_A\,d\mu \\
&=\; 2^{q-1} \frac{\left|\int f\mathbf{1}_{A^c}\,d\mu\right|^q}{\mu(A)^{q-1}} + 2^{q-1} c_A \int |\nabla f|^q \mathbf{1}_A\,d\mu \\
&\le\; 2^{q-1} \int |f|^q d\mu \left(\frac{1-\mu(A)}{\mu(A)}\right)^{q-1} + 2^{q-1} c_A \int |\nabla f|^q d\mu.
\end{aligned}
$$

The second term in Equation (10) is estimated using duality, and the defective F_+-Sobolev inequality (9). For a non-negative non-decreasing function G on $\mathbb{R}^+$ we apply the inequality $xy \le xG(x) + yG^{-1}(y)$ (This is obvious if $y \le G(x)$. If on the contrary $y > G(x)$ then $x \le \inf\{u;\, G(u) \ge y\} = G^{-1}(y)$). For $\varepsilon > 0$,

$$
\begin{aligned}
\int |f|^q \mathbf{1}_{A^c}\,d\mu &=\; \varepsilon \int |f|^q \frac{\mathbf{1}_{A^c}}{\varepsilon}\,d\mu \le \varepsilon \int |f|^q F_+\big(|f|^q\big)\,d\mu + \varepsilon \int \frac{\mathbf{1}_{A^c}}{\varepsilon} F_+^{-1}\left(\frac{\mathbf{1}_{A^c}}{\varepsilon}\right) d\mu \\
&\le\; \varepsilon C \int |\nabla f|^q d\mu + \varepsilon(D+M) \int |f|^q d\mu + (1-\mu(A)) F_+^{-1}\Big(\frac{1}{\varepsilon}\Big).
\end{aligned}
$$

Using both estimates and recalling that $\int |f|^q d\mu = 1$ gives

$$
1 \le \left(2\frac{1-\mu(A)}{\mu(A)}\right)^{q-1} + \varepsilon(D+M) + \big(1-\mu(A)\big) F_+^{-1}\Big(\frac{1}{\varepsilon}\Big) + \big(2^{q-1}C_A + \varepsilon C\big) \int |\nabla f|^q d\mu.
$$

To conclude we choose $\varepsilon = 1/(4(D+M))$, and A large enough to ensure $(1 - \mu(A)) F_+^{-1}\left(\frac{1}{\varepsilon}\right) \le \frac{1}{4}$ and $\left(2\frac{1-\mu(A)}{\mu(A)}\right)^{q-1} \le \frac{1}{3}$. Using again $\int |f|^q d\mu = 1$ we obtain for f with $\int f\,d\mu = 0$

$$
\int |f|^q d\mu \le 6\Big(2^{q-1}C_A + \frac{C}{4(D+M)}\Big) \int |\nabla f|^2 d\mu
$$

provided $\mu(A) \ge 1 - 1/\big(4F_+^{-1}(4(D+M))\big)$ and $\mu(A) \ge 1/\big(1 + (2 \cdot 3^{1/(q-1)})^{-1}\big)$. Optimizing on such sets yields the claimed result. $\qquad\square$

Remark 2.11. When $q = 2$ the estimates can be improved since (11) can be replaced by the variance identity. Also when $F = \log$, the duality of entropy may be used to get a more precise bound

$$
\int |f|^q \mathbf{1}_{A^c} \le \varepsilon \operatorname{Ent}_\mu(|f|^q) + \varepsilon \log \left(\int e^{\frac{\mathbf{1}_{A^c}}{\varepsilon}}\,d\mu\right) = \varepsilon \operatorname{Ent}_\mu(|f|^q) + \varepsilon \log \Big(\mu(A) + \big(1-\mu(A)\big) e^{\frac{1}{\varepsilon}}\Big).
$$

Remark 2.12. The translation invariance of the energy term was implicitly but crucially used. If μ satisfies a local Poincaré inequality and a defective modified log-Sobolev inequality with function $H(x) \ge cx^2$ then the above method yields $\int f^2 d\mu \le D \int f^2 H(|\nabla f|/f)\,d\mu$ for functions f with $\mu(f) = 0$.

Here is a direct consequence of Propositions 2.10 and 2.1:

Corollary 2.13. *Let $q \in (1, 2]$. If a probability measure μ satisfies a defective q-log-Sobolev inequality as well as a local q-Poincaré inequality, then it satisfies a tight q-log-Sobolev inequality.*

The next classical result yields local Poincaré inequalities under mild conditions.

Proposition 2.14. *Let (M, g) be a connected smooth and complete Riemannian manifold. Let $d\mu(x) = e^{-V(x)} dv(x)$ be a Borel probability measure on M (here v is the Riemannian volume). If V is locally bounded, then μ enjoys a local Poincaré inequality.*

Proof. In Euclidean space, we could proceed like in Lemma 2.9. In the general, we use the following fact, known as the Calabi lemma (see e.g. [10]): let $x_0 \in M$, then $D = M \setminus \text{Cut}(x_0)$ is an x_0- star-shaped domain. Moreover there is a sequence of pre-compact x_0-star-shaped domains D_n with smooth boundary such that $\bar{D}_n \subset D_{n+1}$ and $D = \bigcup_n D_n$. Since the Neumann Laplacian of a compact manifold with boundary has a spectral gap (see e.g. [38]), the uniform probability measure on each D_n satisfies a Poincaré inequality. Next the measure $d\mu_{D_n}(x) = \frac{\mathbf{1}_{D_n}(x)}{\mu(D_n)} d\mu(x) = \frac{\mathbf{1}_{D_n}(x)}{\mu(D_n)} e^{-V(x)} dv(x)$ is a bounded (multiplicative) perturbation of the uniform probability measure on D_n. It is classical that is therefore inherits the Poincaré inequality. Finally $\lim \mu(D_n) = \mu(\bigcup_n D_n) = 1 - \mu(\text{Cut}(x_0)) = 1$ since the cut locus has volume zero. □

3. Functional inequalities via isoperimetry

Isoperimetric inequalities are known to imply Sobolev type inequalities. Next, we illustrate this principle for F-Sobolev and modified log-Sobolev inequalities. In this section $d\mu(x) = \rho(x) \, dx$ is a probability measure on $\mathbb{R}^d$ and $\mathcal{I}_\mu$ stands for its isoperimetric function:

$$\mathcal{I}_\mu(t) = \inf_{A \subset \mathbb{R}^d : \mu(A) = t} \mu^+(\partial A).$$

The next statement allows to derive general "modified F-Sobolev inequalities" from isoperimetric estimates. The first part of the theorem, dealing with defective inequalities, was established in [45]. The tight inequality of the second part easily follows from Lemma 2.5 and Lemma 2.6 We deal below with a non-negative convex cost function $c : \mathbb{R}^+ \to \mathbb{R}^+$ with $c(0) = 0$. We recall that c is called superlinear if $\lim_{x \to \infty} \frac{c(|x|)}{|x|} = \infty$.

Theorem 3.1. *Assume that μ has convex support. Let $c : \mathbb{R}^+ \to \mathbb{R}^+$ be a convex superlinear non-negative cost function, such that for some non-negative $n : \mathbb{R}^+ \to \mathbb{R}^+$ with $\lim_{k \to 0} n(k) = 0$ the following holds:*

$$\text{for any } x, k > 0 \quad c(kx) \leq n(k)c(x), \quad c^*(kx) \leq n(k)c^*(x).$$

Let F be an increasing concave function on $\mathbb{R}^+$ satisfying $F(1) = 0$, $F(+\infty) = +\infty$ and $\lim_{y \to 0} yF(y) = 0$. Let $\Phi(t) = \sup_{s>0}(st - sF(s) + s)$. Assume that there exist $\delta > 0$, $\eta \in (0,1)$ such that

$$\int_0^\eta \Phi\left(\delta \, c\left(\frac{tF(\frac{1}{t})}{\mathcal{I}_\mu(t)}\right)\right) dt < \infty.$$

Then there exist $C, B > 0$ such that for every locally Lipschitz f

$$(12) \qquad \int_{\mathbb{R}^d} f^2 F\left(\frac{f^2}{\int_{\mathbb{R}^d} f^2 \, d\mu}\right) d\mu \leq C \int_{\mathbb{R}^d} f^2 c^*\left(\left|\frac{\nabla f}{f}\right|\right) d\mu + B \int_{\mathbb{R}^d} f^2 \, d\mu.$$

If in addition, there exists $q > 0$ such that $x \to c^*(x)/x^q$ is non-increasing, then the inequality can be made tight in the following way: the last term $\int f^2 d\mu$ can be replaced by $\text{Var}_\mu(f)$.

Let us give a concrete example, which is central in our study. In what follows, $\tilde{F}_\tau$ is any concave increasing function on $\mathbb{R}^+$ vanishing at 0, behaving like $\log^\tau$ for large values and with $\lim_{y \to 0} y\tilde{F}(y) = 0$. It can be F_τ for $\tau \in (0,1]$, but for $\tau > 1$ the definition has to be modified to ensure concavity.

Corollary 3.2. *Assume that the probability measure μ verifies*

$$(13) \qquad \mathcal{I}_\mu(t) \geq k \min(t, 1-t) \log^{1-\frac{1}{\alpha}}\left(\frac{1}{\min(t, 1-t)}\right), \qquad t \in [0,1].$$

If $\alpha \in (1,2]$ then there exists C such that for all f

$$\int_{\mathbb{R}^d} f^2 F_{2/\alpha^*}\left(\frac{f^2}{\int_{\mathbb{R}^d} f^2 d\mu}\right) d\mu \leq C \int_{\mathbb{R}^d} |\nabla f|^2 d\mu \quad \text{and} \quad \text{Ent}_\mu(f^2) \leq C \int_{\mathbb{R}^d} f^2 c_{\alpha^*}\left(\frac{\nabla f}{f}\right) d\mu.$$

If $\alpha \geq 2$ then there exists C such that all f verify

$$\int_{\mathbb{R}^d} f^2 \tilde{F}_{2/\alpha^*}\left(\frac{f^2}{\int_{\mathbb{R}^d} f^2 d\mu}\right) d\mu \leq C \int_{\mathbb{R}^d} |\nabla f|^2 d\mu \quad \text{and} \quad \text{Ent}_\mu\left(|f|^{\alpha^*}\right) \leq C \int_{\mathbb{R}^d} |\nabla f|^{\alpha^*} d\mu.$$

Proof. Applying Theorem 3.1 to $c(x) = x^2$ and $F = \tilde{F}_{2/\alpha^*}$ yields defective F-Sobolev inequalities. However (13) implies Cheeger's isoperimetric inequality $\mathcal{I}_\mu(t) \geq k' \min(t, 1-t)$. Hence μ satisfies a Poincaré's inequality and this allows us to tighten the inequalities by Theorem 2.4.

Applying Theorem 3.1 for $F = \log$ and $c(x) = |x|^\alpha$ shows that for all f

$$\mathrm{Ent}_\mu(f^2) \leq C \int f^2 \left|\frac{\nabla f}{f}\right|^{\alpha^*} d\mu + B \int f^2 d\mu.$$

When $\alpha \leq 2$, there exists a constant κ such that $|x|^{\alpha^*} \leq \kappa c_{\alpha^*}(x)$. Applying this bound yields an inequality which can be tightened thanks to Theorem 2.4 and the Poincaré inequality again. When $\alpha \geq 2$, $\alpha^* \in (1,2]$, making the change of function $f^2 = g^{\alpha^*}$ in the above inequality yields a defective α^*-Sobolev inequality. Cheeger's isoperimetric inequality also implies that μ satisfies α^*-Poincaré inequality (see, for example, [23]). By Proposition 2.1 this is enough to tighten the α^*-log-Sobolev inequality. $\qquad\square$

Remark 3.3. Recall the following fact that we mentioned in the introduction: every log-concave probability measure on $\mathbb{R}^d$ such that $\int \exp(\varepsilon|x|^\alpha) \, d\mu < +\infty$ for some $\varepsilon > 0$ and $\alpha \geq 1$ satisfies (13) for some $k > 0$. See also Subsection 5.4 where the log-concavity assumption is weakened.

Remark 3.4. One can also establish functional inequalities interpolating between the above F-Sobolev inequalities and modified log-Sobolev inequalities. In particular the above theorem implies the next result, which was proved in [45], with the restriction $1 < \alpha \leq 2$: if μ satisfies (13) for some $\alpha > 1$ and if $\tau\alpha^* \geq 2$ then there exists C' such that for all f

$$\int_{\mathbb{R}^d} f^2 \tilde{F}_\tau\Big(\frac{f^2}{\int_{\mathbb{R}^d} f^2 \, d\mu}\Big) \, d\mu \leq C' \int_{\mathbb{R}^d} f^2 c_{\tau\alpha^*}\left(\left|\frac{\nabla f}{f}\right|\right) \, d\mu.$$

Remark 3.5. The techniques of [45] allow to show that every measure μ satisfying the isoperimetric inequality (13) for some $\alpha \in (1,2]$ also verifies the inequality $I(\tau)$ introduced in the next section, when $\tau = 2/\alpha^*$.

4. INEQUALITY $I(\tau)$

In this section we introduce a new variant of the logarithmic Sobolev inequality. For $\tau \in (0,1]$ we say that a measure μ satisfies Inequality $I(\tau)$ if for some constants B, C and all f

$$I(\tau) \qquad \mathrm{Ent}_\mu f^2 \leq B \int_{\mathbb{R}^d} f^2 \, d\mu + C \int_{\mathbb{R}^d} |\nabla f|^2 \log^{1-\tau}\Big(e + \frac{f^2}{\int_{\mathbb{R}^d} f^2 \, d\mu}\Big) \, d\mu.$$

We show next that any probability measure satisfying $I(\tau)$ and a local Poincaré inequality, automatically satisfies an F_τ-Sobolev inequality as well as the corresponding modified log-Sobolev inequality. Recall that for $F_\tau(t) = \log^\tau(1+t) - \log^\tau(2)$ and that for $\beta \geq 2$, $c_\beta(t)$ is comparable to $\max(t^2, t^\beta)$.

Theorem 4.1. *Let $\tau \in (0,1]$ and $\alpha \in (1,2]$ be related by $\tau = \frac{2(\alpha-1)}{\alpha}$. Let μ be a probability measure satisfying Inequality $I(\tau)$. Then there exist constants B_i, C_i such that for all f*

$$(14) \qquad \int_{\mathbb{R}^d} g^2 F_\tau\Big(\frac{g^2}{\int_{\mathbb{R}^d} g^2 \, d\mu}\Big) \, d\mu \leq B_1 \int_{\mathbb{R}^d} g^2 \, d\mu + C_1 \int_{\mathbb{R}^d} |\nabla g|^2 \, d\mu,$$

$$(15) \qquad \mathrm{Ent}_\mu f^2 \leq B_2 \int_{\mathbb{R}^d} f^2 \, d\mu + C_2 \int_{\mathbb{R}^d} f^2 c_\alpha^*\left(\left|\frac{\nabla f}{f}\right|\right) \, d\mu.$$

If μ also verifies a local Poincaré inequality, then (14) and (15) can be tightened (i.e. one can take $B_i = 0$).

Proof. Let $\tau \in (0,1)$. First we deduce (15) from $\mathcal{I}(\tau)$. Assume as we may that f is non-negative with $\int f^2 d\mu = 1$. Our task is to bound from above the quantity $\int |\nabla f|^2 \log^{1-\tau}(e + f^2) \, d\mu$. Since $\tau \in (0,1)$, we may apply Young's inequality in the form $xy \leq \tau x^{1/\tau} + (1-\tau)y^{1/(1-\tau)}$ and the easy inequality $x \log(e+x) \leq$

$x \log x + e$:

$$|\nabla f|^2 \log^{1-\tau}(e + f^2) \leq \varepsilon^2 f^2 \left(\tau \left(\frac{|\nabla f|}{\varepsilon f} \right)^{\frac{2}{\tau}} + (1 - \tau) \log(e + f^2) \right)$$

$$\leq \tau \varepsilon^{2(1-\frac{1}{\tau})} f^2 \left| \frac{\nabla f}{f} \right|^{\frac{2}{\tau}} + \varepsilon^2 (1 - \tau) f^2 \log f^2 + \varepsilon^2 (1 - \tau) e.$$

Taking integrals and using the fact that up to constants $c_{\alpha^*}(t)$ is comparable to $\max(t^2, |t|^{2/\tau})$, we obtain that for some constant B_0 depending on α

$$\int |\nabla f|^2 \log^{1-\tau}(e + f^2) \, d\mu \leq \varepsilon^2 (1 - \tau) \mathrm{Ent}_\mu(f^2) + \varepsilon^2 (1 - \tau) e + \tau \varepsilon^{2(1-\frac{1}{\tau})} B_0 \int f^2 c_\alpha^* \left(\frac{|\nabla f|}{f} \right) d\mu.$$

If we choose $\varepsilon > 0$ small enough to have $C\varepsilon^2(1 - \tau) \leq 1/2$, the above inequality can be combined with Inequality $\mathcal{I}(\tau)$ to obtain the defective modified log-Sobolev inequality (15).

In order to show that $\mathcal{I}(\tau)$ implies a defective F_τ-Sobolev inequality, we consider

$$\Phi(x) = \frac{x^2}{\log^{1-\tau}(e + x^2)}, \quad x \in \mathbb{R}.$$

Let us fix a positive Lipschitz function g. We denote by L the Luxembourg norm of g related to Φ:

$$L = \inf \left\{ \lambda; \ \int \Phi\left(\frac{g}{\lambda} \right) d\mu \leq 1 \right\}.$$

Thus by definition $\int \Phi\left(\frac{g}{L} \right) d\mu = 1$. Since $\Phi(x) \leq x^2$, one has

$$L^2 \leq \int g^2 \, d\mu.$$

Set $f = \varphi(g/L) := \sqrt{\Phi(g/L)}$. Note that $\int f^2 \, d\mu = 1$. Thus by hypothesis,

$$(16) \qquad \int f^2 \log f^2 \, d\mu \leq B + C \int |\nabla f|^2 \log^{1-\tau}(e + f^2) \, d\mu.$$

The left hand side of this inequality equals to

$$\int f^2 \log f^2 \, d\mu = \int \frac{g^2}{L^2 \log^{1-\tau}(e + g^2/L^2)} \log \left(\frac{g^2}{L^2 \log^{1-\tau}(e + g^2/L^2)} \right) d\mu.$$

It is not hard to check that there exists a constant $\kappa \geq 0$ depending on $\tau \in (0, 1)$ such that for all $x \geq 0$,

$$\frac{x}{\log^{1-\tau}(e + x)} \log \left(\frac{x}{\log^{1-\tau}(e + x)} \right) \geq -\kappa + \frac{1}{2} x \log^\tau(e + x),$$

for instance the existence of a finite κ for $x \in [0, 4]$ is obvious by continuity, whereas for $x \geq 4$ one may use $\sqrt{x} \geq \log(e + x)$ and the bound $x \log x \geq x \log(e + x) - e$. Hence there are constants $\kappa_1, \kappa_2 > 0$ depending on τ such that

$$(17) \qquad \int f^2 \log f^2 \, d\mu \geq -\kappa_1 + \frac{\kappa_2}{L^2} \int g^2 F_\tau \left(\frac{g^2}{L^2} \right) d\mu$$

Now let us estimate the gradient term in (16). Recall that $f = \varphi(g/L)$, where

$$\varphi(x) = \frac{x}{\log^{\frac{1-\tau}{2}}(e + x^2)}.$$

Elementary estimates show that there exists $M > 0$ such that for every $x \geq 0$

$$|\varphi'(x)| \leq \frac{M}{\log^{\frac{1-\tau}{2}}(e + x^2)}.$$

Applying this bound together with the estimate $f^2 \leq \frac{g^2}{L^2}$ we obtain

$$\int |\nabla f|^2 \log^{1-\tau}(e + f^2)\, d\mu \leq \frac{M^2}{L^2} \int |\nabla g|^2 \frac{\log^{1-\tau}(e + f^2)}{\log^{1-\tau}\left(e + \frac{g^2}{L^2}\right)}\, d\mu \leq \frac{M^2}{L^2} \int |\nabla g|^2\, d\mu.$$

Combining the latter inequality with (16) and (17) we get that

$$\kappa_2 \int g^2 F_\tau\left(\frac{g^2}{L^2}\right) d\mu \leq \left(B + \kappa_1\right) L^2 + C M^2 \int |\nabla g|^2\, d\mu.$$

The claim follows from the estimate $L^2 \leq \int g^2\, d\mu$ and monotonicity of F_τ.

If μ satisfies a local Poincaré inequality, then the defective F_τ-Sobolev inequality is enough to apply Proposition 2.10. Hence μ satisfies a Poincaré inequality. By Theorem 2.4, this is enough to tighten both (14) and (15). $\qquad\square$

5. Optimal transportation and functional inequalities.

The optimal transport theory is widely represented in surveys and monographs and the reader can consult [4, 63, 55] for definitions and main results.

5.1. **The above-tangent lemma.** The application of the optimal transportation techniques to functional inequalities is based on the following remarkable estimate called "above-tangent lemma". It has numerous applications to functional inequalities (especially Sobolev-type inequalities). From a more general point of view this inequality comes from convexity of some special functional (the so-called "displacement convexity"). This notion has been introduced by McCann in [52]. For more details about displacement convexity, above-tangent inequalities and applications, see [4, 34, 33, 37, 36, 2, 44, 28].

Given a function V on $\mathbb{R}^d$ we define its *convexity defect*

$$(18) \qquad \mathcal{D}_V(x, y) = -\left(V(y) - V(x) - \langle \nabla V(x), y - x \rangle\right).$$

Lemma 5.1. *Let $g \cdot \mu$ and $h \cdot \mu$ be probability measures and $T : \mathbb{R}^d \to \mathbb{R}^d$ be the optimal transportation mapping pushing forward $g \cdot \mu$ to $h \cdot \mu$ and minimizing the Kantorovich functional W_c for some strictly convex superlinear function c. Then the following inequality holds:*

$$\mathrm{Ent}_\mu g \leq \mathrm{Ent}_\mu h + \int_{\mathbb{R}^d} \langle x - T(x), \nabla g(x) \rangle\, d\mu + \int_{\mathbb{R}^d} \mathcal{D}_V(x, T(x)) g\, d\mu.$$

Sketch of proof. Without loss of generality one can assume that g and h are smooth and bounded. By the change of variables formula

$$\log g = \log h(T) + V - V(T) + \log \det DT.$$

Integrating with respect to $g \cdot \mu$ and changing variables, one gets

$$\mathrm{Ent}_\mu g = \mathrm{Ent}_\mu h + \int \left(V - V(T) + \log \det DT\right) g\, d\mu$$

$$= \mathrm{Ent}_\mu h + \int \langle x - T(x), \nabla V(x) \rangle g\, d\mu + \int \mathcal{D}_V(x, T(x)) g\, d\mu + \int \log \det DT\ g\, d\mu$$

$$\leq \mathrm{Ent}_\mu h + \int \langle x - T, \nabla g \rangle\, d\mu + \int \mathcal{D}_V(x, T(x)) g\, d\mu$$

$$+ \int \left[\mathrm{Tr}(I - DT) + \log \det DT\right]\ g\, d\mu$$

The claim follows from the fact that the last integrand is non-positive. This is due to the structure of the optimal transport T which ensures that pointwise, DT can be diagonalized, with a non-negative spectrum. $\qquad\square$

This lemma tells that the convexity type information about the potential V, i.e. an estimate of $\mathcal{D}_V$ in

$$V(y) = V(x) + \langle y - x, \nabla V(x) \rangle - \mathcal{D}_V(x, y),$$

passes to the entropy functional on the space of probability measures

$$\text{Ent}_\mu(h) \geq \text{Ent}_\mu(g) + \int \langle T(x) - x, \nabla g(x) \rangle \, d\mu - \int \mathcal{D}_V(x, T(x)) g \, d\mu.$$

The second term of the right-hand side is linear in the displacement $\theta(x) = T(x) - x$. It can be thought of as the linear part in the tangent approximation of the entropy functional. We will call it the "linear term".

Our aim is to derive modified LSI inequalities using the "above-tangent" lemma for $g = f^2$ and $h = 1$:

$$(19) \qquad \text{Ent}_\mu(f^2) \leq 2 \int \langle x - T(x), \nabla f(x) \rangle f(x) \, d\mu(x) + \int \mathcal{D}_V(x, T(x)) f(x)^2 \, d\mu(x).$$

We will show that the "linear" term in the above inequality can be estimated by assuming the integrability of $\exp(\varepsilon |x|^p)$ for some $\varepsilon > 0, p > 1$. Estimating the term involving $\mathcal{D}_V$ is more difficult and can be done by different methods, under various assumptions. When $\mathcal{D}_V(x, y) \leq c(y - x)$, the integral involving $\mathcal{D}_V$ can be upper-bounded by the transportation cost from $f^2 \cdot \mu$ to μ when the unit cost is c. This argument was already used many times. We will see in the next subsections that estimates of the form $\mathcal{D}_V(x, y) \leq \varphi(x) + \psi(y)$ are even more convenient.

5.2. **Estimation of the linear term.** The classical estimate is recalled in the next two lemmas

Lemma 5.2. *Let c be a convex cost function. Assume that T pushes forward $f^2 \cdot \mu$ to μ, and is optimal for the cost c. Then for every $\alpha > 0$,*

$$\int_{\mathbb{R}^d} 2 \langle \nabla f, x - T(x) \rangle f \, d\mu \leq \alpha \int_{\mathbb{R}^d} c^* \left(\frac{-2\nabla f}{\alpha f} \right) f^2 d\mu + \alpha W_c(f^2 \cdot \mu, \mu).$$

Proof. We simply apply Young's inequality $\langle u, v \rangle \leq \alpha c(u) + \alpha c^*(v/\alpha)$ to $u = T(x) - x$ and $v = -2\nabla f(x)/f(x)$, and integrate with respect to $f^2.\mu$. The conclusion comes from $\int c(T(x) - x) f(x)^2 \, d\mu(x) = W_c(f^2 \cdot \mu, \mu)$. $\square$

It is well known that the transportation cost W_c in the above lemma can be estimated in terms of the entropy of f^2 if μ has strong integrability properties. This is recalled now:

Lemma 5.3. *Let μ, $f \cdot \mu$ and $g \cdot \mu$ be probability measures and c a cost function. Then for all $\alpha > 0$,*

$$W_c(f \cdot \mu, g \cdot \mu) \leq \alpha \log \left(\int_{\mathbb{R}^d} e^{\frac{c(x,y)}{\alpha}} d\mu(x) d\mu(y) \right) + \alpha (\text{Ent}_\mu f + \text{Ent}_\mu g).$$

In particular for any Borel sets A, B,

$$W_c(\mu_A, \mu_B) \leq \alpha \log \left(\int_{\mathbb{R}^d} e^{\frac{c(x,y)}{\alpha}} d\mu(x) d\mu(y) \right) + \alpha \log \left(\frac{1}{\mu(A)\mu(B)} \right),$$

where μ_A is the conditional measure $\mu_A = \frac{\mu|_A}{\mu(A)}$.

Proof. We bound the transportation cost from above by using the product coupling:

$$W_c(f \cdot \mu, g \cdot \mu) \;=\; \inf \left\{ \int c(x, y) \, d\pi(x, y); \; \pi \text{ with marginals } f.\mu \text{ and } g.\mu \right\}$$

$$\leq \;\; \alpha \int \frac{c(x, y)}{\alpha} f(x) g(y) \, d\mu(x) d\mu(y)$$

The classical inequality $\int \varphi \psi \, d\nu \leq (\int \varphi \, d\nu) \log(\int e^\psi \, d\nu) + \text{Ent}_\nu(\varphi)$ yields

$$W_c(f \cdot \mu, g \cdot \mu) \leq \alpha \log \left(\int e^{\frac{c(x,y)}{\alpha}} d\mu(x) d\mu(y) \right) + \alpha \text{Ent}_{\mu \otimes \mu}(f(x)g(y)).$$

The claim follows. $\square$

The next result provides a new way to deal with the linear term in the above tangent inequality, in relation with Inequality $I(\tau)$. It is quite flexible, as it does not require the transport to be optimal.

Proposition 5.4. *Let $\alpha \in (1,2], \delta > 0$ and let μ be a probability measure such that*

(20)
$$\int_{\mathbb{R}^d} e^{\delta |x|^\alpha} \, d\mu < \infty,$$

Let T be a map which pushes forward a probability measure $f^2 \cdot \mu$ to μ. Then for all $\varepsilon > 0$ there exist $C_1, C_2 > 0$ depending on δ, α and the above integral such that

$$2\int_{\mathbb{R}^d} \langle \nabla f(x), x - T(x)\rangle f(x) \, d\mu(x) \leq C_1 + C_2 \int_{\mathbb{R}^d} |\nabla f|^2 \log^{1-\tau}(e + f^2) \, d\mu + \varepsilon \mathrm{Ent}_\mu f^2,$$

where we have set $\tau = \frac{2\alpha - 2}{\alpha} = \frac{2}{\alpha^*} \in (0, 1]$.

Proof. Note that $2\int \langle \nabla f(x), x - T(x)\rangle f(x)\, d\mu(x)$ is not bigger than

$$\varepsilon \int \frac{f^2(x)}{\log^{1-\tau}\left(e + f^2(x)\right)} |x - T(x)|^2 \, d\mu(x) + \frac{1}{\varepsilon} \int |\nabla f|^2 \log^{1-\tau}(e + f^2) \, d\mu$$

for arbitrary ε. Since $|x - T(x)|^2 \leq 2|x|^2 + 2|T(x)|^2$ and $1 - \tau \geq 0$, we get

(21)
$$\int \frac{f^2(x)}{\log^{1-\tau}\left(e + f^2(x)\right)} |x - T(x)|^2 \, d\mu(x)$$

$$\leq 2\int \frac{f^2(x)}{\log^{1-\tau}\left(e + f^2(x)\right)} |x|^2 \, d\mu(x) + 2\int f^2(x)|T(x)|^2 \, d\mu(x).$$

We apply the inequality $ab \leq a\varphi(a) + b\varphi^{-1}(b)$, $a, b \geq 0$ for the function $\varphi(t) = e^{\frac{\delta}{2}t^{\alpha/2}} - 1$, and get

$$\int \frac{f^2(x)}{\log^{1-\tau}\left(e + f^2(x)\right)} |x|^2 \, d\mu(x)$$

$$\leq \int \frac{f^2}{\log^{1-\tau}(e + f^2)} \left(\frac{2}{\delta} \log\left(1 + \frac{f^2}{\log^{1-\tau}(e + f^2)}\right)\right)^{\frac{2}{\alpha}} d\mu + \int |x|^2 \left(e^{\frac{\delta}{2}|x|^\alpha} - 1\right) d\mu(x)$$

$$\leq \int \frac{f^2}{\log^{1-\tau}(e + f^2)} \left(\frac{2}{\delta} \log\left(1 + f^2\right)\right)^{\frac{2}{\alpha}} d\mu + \int |x|^2 e^{\frac{\delta}{2}|x|^\alpha} \, d\mu(x).$$

Using $1 + f^2 \leq e + f^2$ with the relation $\frac{2}{\alpha} + \tau - 1 = 1$ and Assumption (20), we get constants κ_i depending on α, δ, μ but not on f such that the above quantity is at most

$$\kappa_1 \int f^2 \log(e + f^2) \, d\mu + \kappa_2 \leq \kappa_1 \mathrm{Ent}_\mu(f^2) + \kappa_3.$$

The last term in (21) is controlled by the change of variable formula and the integrability assumption again:

$$\int f^2(x)|T(x)|^2 \, d\mu(x) = \int |x|^2 \, d\mu(x) \leq \kappa_4 < \infty.$$

The proof is complete. $\qquad\square$

5.3. **Basic facts about convexity defect.** In the next two subsections, we will work with potentials V such that there exists a function c such that

(22)
$$V(x + u) \geq V(x) + \langle u, \nabla V(x)\rangle - c(u).$$

In other words, the defect of convexity satisfies $\mathcal{D}_V(x, y) \leq c(y - x)$. When c is a negative function, V is uniformly convex. We will focus on the case when c is positive. In this case we say that V is weakly convex. This subsection provides concrete examples of such potentials.

The first example is as follows: if V is $\mathcal{C}^2$, the condition $D^2 V \geq -\lambda$ for some $\lambda \geq 0$, is equivalent to condition (22) with $c(u) = \frac{\lambda}{2}|u|^2$. It is also equivalent to the fact that the function $V(x) + \frac{\lambda}{2}|x|^2$ is convex. Next we present other possible conditions, extending the latter.

Choose $c(x) = \|x\|^p$ where $\|\cdot\|$ is a strictly convex norm and $p > 1$. Note that $p > 2$ is not very interesting in our case since for a smooth V

$$V(x + u) - V(x) - \langle u, \nabla V(x)\rangle \sim \frac{1}{2}D^2 V(x).u.u$$

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dominates $-\lambda\|u\|^p$ only when $\lambda = 0$ and V is convex since $-|u|^p >> -|u|^2$ for small u. The case $p \in (1,2)$ contains new examples. We need some preparation.

For $p \in [1,2]$, a norm on a vector space X has a modulus of smoothness of power-type p or for short is p-smooth with constant S if for all $x,y \in X$ it satisfies

$$\|x+y\|^p + \|x-y\|^p \le 2\|x\|^p + 2S^p\|y\|^p.$$

As shown in [12], this formulation is equivalent to the more standard definition given in [50]. We need the following classical fact, see Lemma 4.1 in [53]

Lemma 5.5. *Let $(X\|\cdot\|)$ be a Banach space with a p-smooth norm with constant S. Let X be a random vector with values in X such that $E\|Z\|^p < +\infty$. Then*

$$E\|Z\|^p \le \|EZ\|^p + \frac{S^p}{2^{p-1}-1}E\|Z - EZ\|^p.$$

Applying the above lemma when the law of Z is $(1-t)\delta_x + t\delta_y$ for $t \in (0,1]$, $x,y \in X$ yields

$$(1-t)\|x\|^p + t\|y\|^p \le \|(1-t)x + ty\|^p + \frac{S^p}{2^{p-1}-1}t(1-t)\big((1-t)^p + t^p\big)\|y - x\|^p,$$

which is equivalent to

$$\|y\|^p - \|x\|^p \le \frac{\|x + t(y-x)\|^p - \|x\|^p}{t} + \frac{S^p}{2^{p-1}-1}\|y - x\|^p\big(1 + o(1)\big).$$

Letting t to zero yields for almost every x and for all u

(23) $$\|x+u\|^p \le \|x\|^p + \langle u, \nabla\|\cdot\|^p(x)\rangle + \frac{S^p}{2^{p-1}-1}\|u\|^p.$$

In this form the meaning of p-smoothness is very clear: the application $\|\cdot\|^p$ is not too much above its tangent map and the distance is measured by $\|u\|^p$. Therefore the application $-\|\cdot\|^p$ is not too much below its tangent. Hence we obtain

Corollary 5.6. *Let $p \in (1,2]$ and $\|\cdot\|$ be a p-smooth norm on $\mathbb{R}^d$, with constant S. Let $V : \mathbb{R}^d \to \mathbb{R}$ such that $V(x) + \lambda\|x\|^p$ is convex in x. Then for all almost every x and every u*

$$V(x+u) \ge V(x) + \langle u, \nabla V(x)\rangle - \lambda\frac{S^p}{2^{p-1}-1}\|u\|^p.$$

For $p \in (1,2]$ the L_p-norm on $\mathbb{R}^d$ denoted as $\|\cdot\|_p$ is p-smooth, and the optimal constants S have been calculated. However, the use of Lemma 5.5 introduces the poor constant $2^{p-1}-1$. A better result than (23) is obtained by hands:

Lemma 5.7. *Let $p \in (1,2]$, then for all x,u in $\mathbb{R}^d$,*

$$\|x+u\|_p^p \le \|x\|_p^p + \langle u, \nabla\|\cdot\|_p^p(x)\rangle + 2^{2-p}\|u\|_p^p.$$

Proof. First note that it is enough to prove the inequality in dimension one. Indeed all the term are sums of n corresponding terms involving only one coordinate. The inequality is obvious for $x = 0$ and both terms are p-homogeneous in (x,u). Hence it is enough to deal with the case $x = \pm 1$. Finally the case $(x = -1, u)$ can be deduced from $(1, -u)$ and all we have to do is to show that for all $u \in \mathbb{R}$, it holds

(24) $$|1+u|^p \le 1 + pu + 2^{2-p}|u|^p.$$

Actually when $u > -1$ it is even true that $|1+u|^p \le 1 + pu + |u|^p$. To see this we start with the case $u \ge 0$ and consider the function φ defined on $[0, +\infty)$ by $\varphi(u) = 1 + pu + u^p - (1+u)^p$. Clearly $\varphi(0) = \varphi'(0) = 0$. Moreover φ is convex since $\varphi''(u) = p(p-1)\big(u^{p-2} - (1+u)^{p-2}\big) \ge 0$, using $p - 2 \le 0$. Hence φ is nonnegative.

Next we prove the stronger inequality when $u \in [-1, 0]$. Setting $t = -u$ we have to show that the function ψ defined on $[0,1]$ by $\psi(t) = 1 - pt + t^p - (1-t)^p$ is nonnegative. This is clear since $\psi(0) = 0$ and $\psi'(t) = p\big(u^{p-1} + (1-u)^{p-1} - 1\big) \ge 0$ since $u, 1 - u$ and $p - 1$ are in $[0,1]$ (so $u^{p-1} \ge u$, $(1-u)^{p-1} \ge 1 - u$).

Finally we prove (24) when $u = -t \in (-\infty, -1]$ by studying the function ξ defined on $[1, +\infty)$ by $\xi(t) = 1 - pt + 2^{2-p}t^p - (t-1)^p$. First $\xi(1) \ge 0$ and we shall prove that ξ is nondecreasing. To see this we compute

$$\xi'(t) = p\big(2^{2-p}t^{p-1} - 1 - (t-1)^{p-1}\big), \quad \xi''(t) = p(p-1)\big(2^{2-p}t^{p-2} - (t-1)^{p-2}\big).$$

The latter quantity is nonpositive on $[1, 2]$ and nonnegative on $[2, +\infty)$. Therefore ξ' achieves its minimum at $t = 2$ where $\xi'(2) = 0$. So ξ is nondecreasing as claimed. The proof is complete. $\qquad\square$

Corollary 5.8. *Let $p \in (1, 2]$. Assume that there exists $\lambda \geq 0$ such that the function $x \mapsto V(x) + \lambda\|x\|_p^p$ is convex. Then for almost every x and every u,*

$$V(x + u) \geq V(x) + \langle u, \nabla V(x) \rangle - \lambda 2^{2-p}\|u\|_p^p.$$

In other words $\mathcal{D}_V(x, y) \leq \lambda 2^{2-p}\|y - x\|_p^p$.

5.4. **Isoperimetric inequalities for weakly convex potentials.** It follows from the work of Bobkov [20], extended in [13], that for log-concave probability measures on $\mathbb{R}^d$, the isoperimetric profile is somehow governed by the decay of the measure outside large balls. The goal of this subsection is to show that the log-concavity assumption may be weakened.

In what follows we consider a function $c : \mathbb{R}^d \to \mathbb{R}^+$ which may be identically zero or strictly convex superlinear . For every Borel A let us by denote μ_A the conditional measure $\mu_A = \frac{\mu|_A}{\mu(A)}$. The set $\mathbb{R}^d$ is equipped with a (general) norm $\|\cdot\|$. The dual norm is $\|y\|_* = \sup\{\langle x, y \rangle; \ \|x\| \leq 1\}$. The boundary measure in the sense of μ is computed for the distance induced by the norm, and denoted by $\mu_{\|\cdot\|}^+$ to avoid ambiguities. Let $B_r = \{x : \|x - x_0\| \leq r\}$. The next lemma extends an isoperimetric inequality proved by Bobkov for log-concave measures.

Lemma 5.9. *Let $c : \mathbb{R}^d \to \mathbb{R}^+$ be a strictly convex superlinear cost function. Let $\mu = e^{-V} dx$ be a probability measure on $\mathbb{R}^d$ with $\mathcal{D}_V(x, y) \leq c(y - x)$. Then for every $r > 0$ and every Borel set A,*

$$\mu(A) \log \frac{1}{\mu(A)} + \mu(A^c) \log \frac{1}{\mu(A^c)} + \log \mu(B_r)$$

$$\leq \ 2r\mu_{\|\cdot\|}^+(\partial A) + \mu(A) \cdot W_c(\mu_A, \mu_{B_r}) + \mu(A^c) \cdot W_c(\mu_{A^c}, \mu_{B_r}).$$

Proof. Let T be the optimal map pushing forward $(f/\mu(f)).\mu$ to μ_{B_r} for the cost function $c(y - x)$. If we apply Lemma 5.1, we get after multiplication by $\int f \, d\mu$

$$\mathrm{Ent}_\mu f \ \leq \ \log \frac{1}{\mu(B_r)} \int f \, d\mu - \int \langle T(x) - x, \nabla f \rangle \, d\mu + \left(\int f \, d\mu \right) W_c\big((f/\mu(f)).\mu, \mu_{B_r}\big)$$

$$\leq \ \log \frac{1}{\mu(B_r)} \int f \, d\mu + r \int \|\nabla f\|_* \, d\mu + \int \langle x - x_0, \nabla f \rangle \, d\mu + \left(\int f \, d\mu \right) W_c\big((f/\mu(f)).\mu, \mu_{B_r}\big),$$

where we have used that $\|T(x) - x_0\| \leq r$ since the range of T is in B_r. If we sum up the latter upper bound on $\mathrm{Ent}_\mu f$ with the corresponding one for $\mathrm{Ent}_\mu(1 - f)$, the terms $\int \langle x - x_0, \nabla f \rangle \, d\mu$ cancel out. The conclusion follows from letting f tend to $\mathbf{1}_A$. $\qquad\square$

Proposition 5.10. *Let $c(x) = \tilde{c}(\|x\|)$ where $\tilde{c}$ is identically zero or is a strictly convex superlinear cost function, increasing on $\mathbb{R}^+$. Let $\mu = e^{-V} dx$ be a probability measure on $\mathbb{R}^d$ with $\mathcal{D}_V(x, y) \leq c(y - x)$. Assume that for some $\varepsilon > 0$,*

$$\exp\big((3 + \varepsilon)c(x - y)\big) \in L_1(\mu \otimes \mu).$$

Then there exist $D > 0$ and $a_0 > 0$ such that the following is true:
for any Borel set A such that $a := \min\big(\mu(A), \mu(A^c)\big)$ verifies $a \leq a_0$, it holds

$$a \log\Big(\frac{1}{a}\Big) \leq Dr \, \mu_{\|\cdot\|}^+(\partial A),$$

where r is chosen so that $a = \mu(B_r^c)$.

Proof. Let us assume that $\mu(A) \leq 1/2$ (the case $\mu(A^c) < 1/2$ follows from the same method since A and its complement play symmetric roles in our estimates). Hence by hypothesis $a = \mu(A) = \mu(B_r^c)$. We apply Lemma 5.9. Our task is to bound the transportation costs involved in its conclusion. Set $\eta = 1/(3 + \varepsilon)$ and $K(\eta) = \eta \log \big(\int \exp(c(y - x)/\eta) \, d\mu(x) d\mu(y) \big)$. It is finite by hypothesis. Lemma 5.3 gives

$$(25) \qquad \mu(A)W_c(\mu_A, \mu_{B_r}) \leq K(\eta)\mu(A) + \eta\mu(A)\log(1/\mu(A)\mu(B_r^c)) = K(\eta)a + \eta a \log \frac{1}{a} + \eta a \log \frac{1}{1-a}.$$

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Applying the corresponding bound for $\mu(A^c)W_c(\mu_{A^c}, \mu_{B_r})$ would give a term of order $1 - a$ which is too big. To avoid this problem, we consider another coupling. Let S be the c-optimal map pushing forward $\mu_{A^c \cap B_r^c}$ to $\mu_{A \cap B_r}$. We define the map $T : A^c \to B_r$ by $T(x) = x$ when $x \in A^c \cap B_r$ and by $T(x) = S(x)$ for $x \in A^c \cap B_r^c$. One readily checks that T pushes μ_{A^c} forward to μ_{B_r} (this uses the relation $\mu(A^c) = \mu(B_r)$ and its consequence $\mu(A^c \cap B_r^c) = \mu(A \cap B_r)$). Hence

$$
\begin{aligned}
W_c(\mu_{A^c}, \mu_{B_r}) &\leq \frac{1}{\mu(A^c)} \int_{A^c} c\big(S(x) - x\big)\, d\mu(x) = \frac{1}{\mu(A^c)} \int_{A^c \cap B_r^c} c\big(S(x) - x\big)\, d\mu(x) \\
&= \frac{\mu(A^c \cap B_r^c)}{\mu(A^c)} W_c(\mu_{A^c \cap B_r^c}, \mu_{A \cap B_r}).
\end{aligned}
$$

Apply Lemma 5.3 to the latter transportation cost yields

$$
\begin{aligned}
\mu(A^c)W_c(\mu_{A^c}, \mu_{B_r}) &\leq \mu(A^c \cap B_r^c)\left(K(\eta) + \eta \log \frac{1}{\mu(A^c \cap B_r^c)\mu(A \cap B_r)} \right) \\
&= \mu(A \cap B_r)K(\eta) + 2\eta\mu(A \cap B_r) \log \frac{1}{\mu(A \cap B_r)}.
\end{aligned}
$$

Note that $-x \log x$ is increasing for $x \leq 1/e$. Thus $\mu(A) = a \leq 1/e$ ensures that

(26) $$\mu(A^c)W_c(\mu_{A^c}, \mu_{B_r}) \leq K(\eta)a + 2\eta a \log \frac{1}{a}.$$

Combining Lemma 5.9 with (25), (26) and the relation $\eta = 1/(3 + \varepsilon)$ yields

$$\frac{\varepsilon}{3 + \varepsilon} a \log \frac{1}{a} \leq 2r\mu^+(\partial A) + 2K(\eta)a + \eta a \log \frac{1}{1 - a}.$$

When a is small enough, $2K(\eta)a + \eta a \log \frac{1}{1-a}$ is less than half of the left hand side, hence $\frac{\varepsilon}{3+\varepsilon} a \log \frac{1}{a} \leq 4r\mu^+(\partial A)$. $\qquad\square$

Theorem 5.11. *Let $c(x) = \tilde{c}(\|x\|)$ be identically zero or a strictly convex superlinear cost function on $\mathbb{R}^d$. Let $d\mu(x) = e^{-V(x)}dx$ be a probability measure on $\mathbb{R}^d$ with for all x, y and some $\varepsilon > 0$*

$$\mathcal{D}_V(x, y) \leq c(x - y) \quad and \quad \exp\big((3 + \varepsilon)c(x - y)\big) \in L_1(\mu \otimes \mu).$$

Let $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ be strictly increasing. Assume in addition that for some x_0,

$$\int_{\mathbb{R}^d} e^{\psi(\|x - x_0\|)}\, d\mu(x) \leq K < \infty,$$

then there exist $D, a_0 >$ such that every Borel set A such that $a := \min\big(\mu(A), \mu(A^c)\big) \leq a_0$ verifies

$$D\mu^+_{\|\cdot\|}(\partial A) \geq a \frac{\log \frac{1}{a}}{\psi^{-1}\big(\log \frac{K}{a}\big)}.$$

Proof. Proposition 5.10 gives $Dr\mu^+(\partial A) \geq a \log \frac{1}{a}$ when $a \leq a_0$ is such that $a = \mu(B_r^c)$. Using Markov's inequality in exponential form gives

$$a = \mu\big(\{x \in \mathbb{R}^d;\ \|x - x_0\| > r\}\big) \leq Ke^{-\psi(r)},$$

hence $r \leq \psi^{-1}(\log \frac{K}{a})$. $\qquad\square$

Remark 5.12. The restriction on the value of a may be weakened or removed by making more precise calculations in concrete situations, or in general situation by applying Proposition 2.8.

Remark 5.13. Assume that c is not the zero function. Jensen's inequality yields

$$\int e^{(3+\varepsilon)c\left(x - \int y\, d\mu(y)\right)}\, d\mu(x) \leq \int \int e^{(3+\varepsilon)c(x-y)}\, d\mu(y) d\mu(x).$$

Hence the above theorem for $\psi = (3 + \varepsilon)\tilde{c}$ gives the following result: if μ satisfies

$$\mathcal{D}_V(x, y) \leq c(x - y) \quad and \quad \exp\big((3 + \varepsilon)c(x - y)\big) \in L_1(\mu \otimes \mu),$$

then there exist $D, a_0 >$ such that every Borel set A such that for $a \in (0, a_0)$, $\mathcal{I}_\mu(a) \geq D\, a \frac{\log \frac{1}{a}}{\tilde{c}^{-1}\big(\frac{1}{3+\varepsilon} \log \frac{K}{a}\big)}$.

Next we consider the Euclidean norm and we give an application to potentials with Hessian bounded from below and with a strong integrability property. A similar statement holds when $\int e^{\psi(|x|)}d\mu(x) < \infty$ for an increasing ψ with $\lim_{+\infty}\frac{\psi(t)}{t^2} = +\infty$.

Corollary 5.14. *Let $d\mu(x) = e^{-V(x)}dx$ be a probability measure on the Euclidean space $(\mathbb{R}^d, |\cdot|)$. Assume that there exits $K \geq 0$, $\varepsilon > 0$, $\alpha > 2$ and $x_0 \in \mathbb{R}^d$ such that*

$$D^2V(x) \geq -K\,\mathrm{Id},\ x \in \mathbb{R}^d \quad \text{and} \quad \int_{\mathbb{R}^d} e^{\varepsilon|x-x_0|^\alpha}d\mu(x) < +\infty.$$

There there exists $\kappa > 0$ such that the isoperimetric profile of μ satisfies

$$\mathcal{I}_\mu(t) \geq \kappa \min(t, 1-t) \log^{1-\frac{1}{\alpha}}\left(\frac{1}{\min(t, 1-t)}\right), \quad t \in (0,1).$$

Proof. By hypothesis $\mathcal{D}_V(x, y) \leq \frac{K}{2}|x-y|^2$. We need to check that $\int \exp\left(\beta|x-y|^2\right)d\mu(x)d\mu(y)$ is finite for some $\beta > 3K/2$. However this is true for every β. Indeed for every $\delta > 0$ there is a constant such that for all x, $|x|^2 \leq \delta|x|^\alpha + N(\alpha, \delta)$ (e.g. using Young's inequality $xy \leq x^p/p + y^{p^*}/p^*$ for $p = \alpha/2 > 1$). Hence

$$\int\int e^{\beta|x-y|^2}d\mu(x)d\mu(y) \leq \left(\int e^{2\beta|x-x_0|^2}d\mu(x)\right)^2 \leq \left(\int e^{2\beta\left(\delta|x-x_0|^\alpha + N(\alpha,\delta)\right)}d\mu(x)\right)^2$$

is finite by choosing $\delta < \varepsilon/(2\beta)$. Therefore we may apply the previous corollary with $\psi(t) = t^\alpha$. This gives the claimed isoperimetric inequalities for small values of t. Since V is locally bounded we apply Proposition 2.8 to extend the result to all $t \in (0,1)$. $\qquad\square$

Remark 5.15. Modified log-Sobolev inequalities are established in the next subsection under the weaker integrability assumption $\exp((1+\varepsilon)c(x-y)) \in L^1(\mu \otimes \mu)$ (see Theorem 5.16). Thus, one may ask whether the results of this subsection remain valid when $3 + \varepsilon$ is replaced by $1 + \varepsilon$. This is indeed the case for the statement of Remark 5.13 when $c(x) = \frac{K}{2}|x|^2$. If $D^2V \geq K$ and $\exp((\varepsilon + K/2)|x-y|^2) \in L_1(\mu \otimes \mu)$, then Wang's result yields a logarithmic Sobolev inequality. But when the Hessian is bounded from below, Ledoux [48] showed that an appropriate (Gaussian) isoperimetric inequality follows. Apart from the factor $3 + \varepsilon$, another feature of the method of this subsection is not completely satisfactory: it does not seem to extend to the Riemannian setting.

5.5. **Modified LSI via weak convexity and integrability.** In this section we derive log-Sobolev inequalities when the potential V satisfies $\mathcal{D}_V(x, y) \leq c_0(y-x)$, or equivalently

$$V(x+u) \geq V(x) + \langle u, \nabla V(x)\rangle - c_0(u).$$

If c_0 is negative then V is strictly uniformly convex and log-Sobolev inequalities have been proved (Bakry-Emery [9] for quadratic c_0, Bobkov and Ledoux [26] in general). When c_0 is positive, V is not convex anymore and an additional integrability assumption is needed to balance the convexity defect.

Theorem 5.16. *Let $d\mu(x) = e^{-V(x)}dx$ be a probability measure on $\mathbb{R}^d$. Assume that there exists $\lambda \geq 0$ and an even strictly convex function $c : \mathbb{R} \to \mathbb{R}^+$ with $c(0) = 0$ such that for all x, u,*

$$V(x+u) \geq V(x) + u \cdot \nabla V(x) - \lambda c(u).$$

If there exists $\varepsilon > 0$ such that

$$\int_{\mathbb{R}^{2d}} e^{(\lambda+\varepsilon)c(y-x)}d\mu(x)d\mu(y) < +\infty;$$

then there exists $K_1, K_2, K_3 \geq 0$ such that every nonnegative smooth function f verifies,

$$\mathrm{Ent}_\mu(f^2) \leq K_1 \int f^2 c^*\left(\frac{\nabla f}{K_2 f}\right)d\mu + K_3 \int f^2 d\mu.$$

Proof. Let $\eta_1, \eta_2 \in (0,1)$. Assume that $\int f^2 \, d\mu = 1$. Let $T(x) = x + \theta(x)$ be the optimal transport from $f^2 \cdot \mu$ to μ for the unit cost $c(x - y)$. Applying Lemma 5.1 to $g = f^2$ and $h = 1$ and Young's inequality as in Lemma 5.2 gives

$$
\begin{aligned}
\mathrm{Ent}_\mu(f^2) &\leq \eta_1 \int \left\langle \frac{-2\nabla f}{\eta_1 f}, \theta \right\rangle f^2 \, d\mu + \lambda W_c(f^2 d\mu, \mu) \\
&\leq \eta_1 \int \varphi \, c^* \left(\frac{2\nabla f}{\eta_1 f} \right) d\mu + \eta_1 \int c(\theta) f^2 \, d\mu + \lambda W_c(f^2 \, d\mu, \mu) \\
&= \eta_1 \int \varphi \, c^* \left(\frac{2\nabla f}{\eta_1 f} \right) d\mu + (\eta_1 + \lambda) W_c(f^2 d\mu, \mu) \\
&\leq \eta_1 \int \varphi \, c^* \left(\frac{2\nabla f}{\eta_1 f} \right) d\mu + (1 - \eta_2) \mathrm{Ent}_\mu(f^2) + (1 - \eta_2) \log \left(\int e^{\frac{\lambda + \eta_1}{1 - \eta_2} c(x-y)} \right) d\mu(x) d\mu(y),
\end{aligned}
$$

where the last inequality comes from Lemma 5.3 for $\alpha = (1 - \eta_2)/(\lambda + \eta_1)$. Rearranging the entropy terms and tuning η_1, η_2 to ensure that $\frac{\lambda + \eta_1}{1 - \eta_2} \leq \lambda + \varepsilon$ completes the proof. $\qquad \square$

Theorem 5.16 yields defective modified log-Sobolev inequalities. Under suitable conditions, the methods of Section 2 allow to tighten them. This is illustrated by two of the following corollaries.

Corollary 5.17. *Let $p \in (1, 2]$ and $q = p/(p-1) \in [2, +\infty)$ be its dual exponent. Let $d\mu(x) = e^{-V(x)} dx$ be a probability measure on $\mathbb{R}^d$. Assume that there exists $\lambda \geq 0$ such that the function $x \mapsto V(x) + \lambda \|x\|_p^p$ is convex and that there exists $\varepsilon > 0$ such that*

$$
\int_{\mathbb{R}^{2d}} e^{(\lambda 2^{2-p} + \varepsilon)\|x-y\|_p^p} d\mu(x) d\mu(y) < +\infty.
$$

Then there exists constants K_1, K_2 such that for every nonnegative smooth function g it holds

$$
\mathrm{Ent}_\mu(g^q) \leq K_1 \int \|\nabla g\|_q^q d\mu + K_2 \int g^q d\mu.
$$

Proof. The convexity type hypothesis on V and Corollary 5.8 ensure that $\mathcal{D}_V(x, y) \leq 2^{2-p} \|y - x\|_p^p$. The integrability condition allows to apply Theorem 5.16 with $c(u) = \|u\|_p^p$ for which $c^*(u) = \|u\|_q^q$. We conclude the change of function $f = g^{\frac{q}{2}}$. $\qquad \square$

Corollary 5.18 (Wang [64, 66]). *Let $d\mu(x) = e^{-V(x)} dx$ be a probability measure on $\mathbb{R}^d$. Assume that V is $\mathcal{C}^2$ and there exists $\lambda \geq 0$ such that pointwise $D^2 V \geq -\lambda \, \mathrm{Id}$. If there exists $\varepsilon > 0$ and x_0 such that $\int \exp(\frac{\lambda + \varepsilon}{2} |x - x_0|^2) \, d\mu(x) < +\infty$ then for some K and all smooth functions*

$$
\mathrm{Ent}_\mu(f^2) \leq K \int_{\mathbb{R}^d} |\nabla f|^2 d\mu.
$$

Proof. Combining Theorem 5.16 for $c(u) = u^2/2$ and Corollary 2.13 gives the claim inequality under the slightly stronger assumption $\int \exp\left(\frac{\lambda + \varepsilon}{2} |x - y|^2\right) d\mu(x) d\mu(y) < +\infty$. Following the proof of Theorem 5.16 in our specific context, we come across a term $(\eta_1 + \lambda) \int \frac{|x - T(x)|^2}{2} f(x)^2 d\mu(x)$ where T is the optimal map from $f^2 \cdot \mu$ to μ for the quadratic cost. In order to get the full result we estimate it a bit differently. In particular, the optimality of T is not used. Since for all $\eta_2 > 0$, $|x + y|^2 \leq (1 + \eta_2)|x|^2 + (1 + \eta_2^{-1})|y|^2$:

$$
\begin{aligned}
&\frac{\lambda + \eta_1}{2} \int |x - T(x)|^2 f(x)^2 d\mu(x) \\
&\leq \frac{\lambda + \eta_1}{2} \left(\int (1 + \eta_2)|x - x_0|^2 f(x)^2 d\mu(x) + (1 + \eta_2^{-1}) \int |T(x) - x_0|^2 f^2(x) d\mu(x) \right) \\
&\leq (1 - \eta_3) \left(\mathrm{Ent}_\mu(f^2) + \log \left(\int e^{\frac{(\lambda + \eta_1)(1 + \eta_2)}{2(1 - \eta_3)} |x - x_0|^2} d\mu(x) \right) \right) + \frac{\lambda + \eta_1}{2}(1 + \eta_2^{-1}) \int |y - x_0|^2 d\mu(y).
\end{aligned}
$$

For small enough $\eta_i > 0$ the first term is finite. The second one is finite by the stronger integrability condition. Hence a defective LSI has been proved. It can be tightened since the potential is locally bounded. $\qquad \square$

Remark 5.19. Wang's original proof yields a better control on the constant. It is based on semigroup interpolation and seems hard to apply for integrability conditions of the form $\int \exp c(x-y) \, d\mu(x) d\mu(y) < +\infty$ with non quadratic c. This is possible with the transportation approach, but a limitation remains: the function c in the integrability condition is the same as the one which controls the lack of convexity of V. Nevertheless when the potential is convex, $\lambda = 0$ and c disappears from the convexity hypothesis, hence any integrability assumption can be used. This is similar to what happened with applications of Bobkov's isoperimetric inequality.

The techniques of the above proof also have the advantage to work in more general conditions:

Theorem 5.20. *Let $d\mu(x) = e^{-V(x)}dx$ be a probability measure on $\mathbb{R}^d$. Assume that there exists $K, L \geq 0$ and $\varepsilon, p > 0$ and such that*

$$D^2V(x) \geq -\big(K + L|x|^p\big)\,\mathrm{Id} \quad and \quad \int_{\mathbb{R}^d} e^{\frac{L+\varepsilon}{p+2}|x|^{p+2}} d\mu(x) < +\infty,$$

then μ satisfies a log-Sobolev inequality as well as **(q-LSI)** *for $q = \frac{p+2}{p+1}$.*

Proof. Applying Taylor's formula with integral remainder

$$
\begin{aligned}
\mathcal{D}_V(x,y) &\leq -\int_0^1 (1-u)\big\langle D^2V\big((1-u)x+uy\big) \cdot (y-x), y-x\big\rangle du \\
&\leq \int_0^1 (1-u)\Big(K + L|(1-u)x+uy|^p\Big)|y-x|^2 du \\
&\leq \frac{K}{2}|y-x|^2 + L|y-x|^2\left(\frac{(1+\eta)}{p+2}|x|^p + \frac{N(p,\eta)}{(p+1)(p+2)}|y|^p\right),
\end{aligned}
$$

where we have applied for $\eta > 0$ the bound $|a+b|^p \leq (1+\eta)|a|^p + N(p,\eta)|b|^p$ and have computed the integrals. Next we apply the bound $|y-x|^2 \leq (1+\eta)|x|^2 + (1+\eta^{-1})|y|^2$ and develop all the products. The terms of the form $|y|^2, |x|^2, |x|^2|y|^p$ or $|x|^p|y|^2$ are controlled by applying Young's inequality in the form $a^2b^p \leq \eta a^{p+2} + M(p,\eta)b^{p+2}$ or the similar upper bound of $a^p b^2$ (but each time the small η factor should appear in front of $|x|$). Eventually

$$\mathcal{D}_V(x,y) \leq \frac{L+\varphi(\eta)}{(p+2)}|x|^{p+2} + M_1(p,\eta)|y|^{p+2} + M_2(p,\eta),$$

where $\varphi(\eta)$ tends to zero as η does and all other parameters are fixed. Hence, integrating against the probability measure $f^2 \cdot \mu$ and using the change of variables by T,

$$\int \mathcal{D}_V(x,T(x))f(x)^2 d\mu(x) \leq \int \frac{L+\varphi(\eta)}{(p+2)}|x|^{p+2}f(x)^2 d\mu(x) + M_1(p,\eta)\int |y|^{p+2}d\mu(y) + M_2(p,\eta).$$

In view of the strong integrability of μ this can be bounded by $(1-\eta_1)\mathrm{Ent}_\mu(f^2) + B(\eta_1)$ for $\eta_1 > 0$ small enough. It remains to bound from above the linear term, using for $\alpha \in \{2, p+2\}$ and any $\eta_2 > 0$ the inequality

$$2\int \big\langle \frac{\nabla f(x)}{f(x)}, x - T(x)\big\rangle f(x)^2 d\mu(x) \leq M_3(\alpha,\eta_2)\int \left|\frac{\nabla f}{f}\right|^{\alpha^*} f^2 d\mu + \eta_2 \int |x - T(x)|^\alpha f(x)^2 d\mu(x).$$

Hence the techniques already used allow to bound the latter integral by an arbitrary small fraction of the entropy plus a constant. For $\alpha = 2$ we get a defective **(LSI)**, for $\alpha = p+2$ we get a defective modified log-Sobolev inequality with cost t^{p+2}, or a defective **(qLSI)** by a change of function. They may be tightened by Corollary 2.13. Indeed μ has a locally bounded potential, hence it satisfies a local Cheeger inequality by Lemma 2.9, which implies local q-Poincaré (see e.g. [23]). □

Note that the transportation argument was used in [40] to prove a defective modified log-Sobolev inequality adapted to a given log-concave measure on $\mathbb{R}$. Our contribution here is rather in the tightening techniques of Section 2 which yield a soft proof of the main results of [40, 41] with slightly relaxed conditions:

Corollary 5.21. *Let $d\mu(x) = e^{-\Phi(x)}dx/Z$ where Φ is an even non-negative convex function on $\mathbb{R}$ with $\Phi(0) = 0$. Assume in addition that for some $\alpha, \eta, x_0 > 0$ and $x \geq x_0$, $\Phi(x) \leq \alpha x^2$ and $\frac{\Phi(x)}{x^{1+\eta}}$ increases. Then for all smooth f*

$$\mathrm{Ent}_\mu(f^2) \leq \int_{\mathbb{R}} H\left(\frac{f'}{f}\right) f^2 d\mu,$$

where $H(x) = c_1 x^2$ for $|x| \leq c_2$ and $H(x) = c_1 \Phi^*(c_3 x)$ otherwise. Here c_i are constants depending on Φ.

Proof. Assume as we may that $\eta \in (0, 1]$, and consider the function

$$h(x) = |x|^{1+\eta} \mathbf{1}_{|x|<x_0} + \frac{\Phi(x)}{\Phi(x_0)} x_0^{1+\eta} \mathbf{1}_{|x|\geq x_0}.$$

By hypothesis $x_0 \Phi'(x_0) \geq (1+\eta)\Phi(x_0)$ which ensures convexity of h. On easily verifies that for $x \geq 0$, $h(x)/x^{1+\eta}$ is non-decreasing. This implies that $h^*(x)/x^\beta$ is non-increasing on $\mathbb{R}^+$ where $\beta \geq 2$ is the dual exponent of $1 + \eta$. Also note that for small x, $h^*(x) = |x|^\beta \leq x^2$ whereas for large x, $h^*(x) = c_4 \Phi^*(c_5 x) \geq c_6 x^2$. Consequently the function $\tilde{h}(x) = \max(x^2, h^*(x))$ is bounded above by the function $H(x)$ of the Corollary for a suitable choice of the constants. Moreover $\tilde{h}(x)/x^\beta$ is non-increasing on $\mathbb{R}^+$.

We apply Theorem 5.16 with the convex potential $V = \Phi + \log Z$, the cost function $c = h(\cdot/2)$ and $\varepsilon = x_0^{-1-\eta}\Phi(x_0)$. By convexity and parity

$$\int e^{\varepsilon h\left(\frac{y-x}{2}\right)} d\mu(x)d\mu(y) \leq \left(\int_{\mathbb{R}} e^{\frac{\varepsilon}{2}h(x)-\Phi(x)} \frac{dx}{Z}\right)^2,$$

which is finite since εh coincides with Φ in the large, where Φ grows at least linearly. Therefore any smooth function verifies

$$\mathrm{Ent}_\mu(f^2) \leq \kappa_1 \int f^2 d\mu + \kappa_2 \int h^*\left(\frac{f'}{\kappa_3 f}\right) f^2 d\mu \leq \kappa_1 \int f^2 d\mu + \kappa_4 \int \tilde{h}\left(\frac{f'}{\kappa_5 f}\right) f^2 d\mu.$$

The measure μ being log-concave, it satisfies a Poincaré inequality, see [20]. Therefore the latter inequality may be tightened using Theorem 2.4. $\square$

In the case of non quadratic cost functions, the previous method provides tight inequalities only when a spectral gap inequality is known by other means. To avoid this problem we may also work with Inequalities $I(\tau)$; as explained before they imply F-Sobolev inequalities which may be easily tightened using only local Poincaré property.

Theorem 5.22. *Let $d\mu(x) = e^{-V(x)}dx$ be a probability measure on $\mathbb{R}^d$. Assume that there exists a strictly convex superlinear function c_0 on $\mathbb{R}^d$ such that V satisfies*

$$V(x+u) \geq V(x) + u \cdot \nabla V(x) - c_0(u), \quad x, u \in \mathbb{R}^d$$

and for some $\varepsilon > 0$

$$(27) \qquad \int_{\mathbb{R}^{2d}} e^{(1+\varepsilon)c_0(x-y)} d\mu(x)d\mu(y) < \infty.$$

If there exists $\eta > 0$ and $\alpha \in (1, 2]$ such that $\int_{\mathbb{R}^d} e^{\eta|x|^\alpha} d\mu(x) < +\infty$ then μ satisfies Inequality $I(\tau)$ for $\tau = \frac{2\alpha-2}{\alpha} = \frac{2}{\alpha^*}$: for all smooth f,

$$\mathrm{Ent}_\mu f^2 \leq B \int_{\mathbb{R}^d} f^2 d\mu + C \int_{\mathbb{R}^d} |\nabla f|^2 \log^{1-\tau}\left(e + \frac{f^2}{\int_{\mathbb{R}^d} f^2 d\mu}\right) d\mu.$$

If, in addition, we assume the local Poincaré inequality, then μ satisfies
  1) a modified log-Sobolev inequality with $c = c_\alpha$,
  2) an F-inequality with $F = F_\tau$.

Proof. Let T be the optimal transport minimizing W_{c_0} and sending $f^2 \cdot \mu$ to μ. We apply the "above tangent" lemma 5.1 in this situation and estimate the linear term by Proposition 5.4. It remains to estimate $\int_{\mathbb{R}^d} \mathcal{D}_V(x, T(x)) f^2 \, d\mu$. Since T minimizes the c_0-Kantorovich functional, one obtains

$$
\begin{aligned}
\int_{\mathbb{R}^d} \mathcal{D}_V(x, T(x)) f^2 \, d\mu \quad &\leq \quad \int_{\mathbb{R}^d} c_0(x - T(x)) f^2 \, d\mu = W_{c_0}(\mu, f^2 \cdot \mu) \\
&\leq \quad \frac{1}{1+\varepsilon} \log\left(\int e^{(1+\varepsilon)c_0(x-y)} d\mu(x) d\mu(y) \right) + \frac{1}{1+\varepsilon} \mathrm{Ent}_\mu(f^2),
\end{aligned}
$$

where the latter inequality follows from Lemma 5.3. This proves Inequality $I(\tau)$. By Theorem 4.1, the measure μ satisfies a defective F_τ-Sobolev inequality as well as a defective modified Sobolev inequality with function c_α. However the local Poincaré inequality and the defective F_τ-Sobolev inequality yield a Poincaré inequality, as follows from Proposition 2.10 for $q = 2$, $F = F_\tau$. This spectral gap inequality allows to tighten the two inequalities by Theorem 2.4. □

Corollary 5.23. *Let the assumptions of Corollary 5.17 hold. Assume in addition the local Poincaré inequality. Then μ satisfies Inequality $I(\tau)$ and an F-inequality with $F = F_\tau$, where $\tau = \frac{2p-2}{p}$.*

5.6. Modified LSI for perturbations of convex potentials.

Theorem 5.24. *Let $\alpha \in (1, 2]$. Let $d\mu(x) = e^{-V(x)} dx$ be a probability measure on $\mathbb{R}^d$ such that $V = V_0 + V_1$, where V_0 is convex and V_1 is continuously differentiable. Assume that there exists a function $p \in L^1(\mu)$ and a constant $\varepsilon > 0$ such that*

$$
\exp\Big((1+\varepsilon) \big[V_1(x) - \langle x, \nabla V_1(x) \rangle + (V_1 + p)^*(\nabla V_1(x)) \big] \Big) \in L^1(\mu),
$$

where $(V_1 + p)^*$ is the Legendre transform of $V_1 + p$. If for some $\eta > 0$, $\int_{\mathbb{R}^d} \exp\big(\eta |x|^\alpha\big) d\mu < +\infty$ then μ satisfies Inequality $I\big(2/\alpha^*\big)$.

Proof. Let $f^2 \cdot \mu$ be a probability measure and T be the optimal transport (e.g. for the quadratic cost) pushing this measure forward to μ. Lemma 5.1 gives

$$
\tag{28} \mathrm{Ent}_\mu(f^2) \leq 2 \int \langle x - T(x), \nabla f(x) \rangle f(x) \, d\mu(x) + \int \mathcal{D}_V(x, T(x)) f^2(x) \, d\mu(x).
$$

Since V_0 is convex, the convexity defect of V is controlled by the one of V_1

$$
\mathcal{D}_V(x, T(x)) \leq \mathcal{D}_{V_1}(x, T(x)) = V_1(x) - V_1(T(x)) - \langle \nabla V_1(x), x - T(x) \rangle.
$$

The definition of the Legendre transform gives $\langle \nabla V_1(x), T(x) \rangle \leq (V_1 + p)(T(x)) + (V_1 + p)^*(\nabla V_1(x))$. Hence

$$
\mathcal{D}_V(x, T(x)) \leq V_1(x) - \langle \nabla V_1(x), x \rangle + (V_1 + p)^*(\nabla V_1(x)) + p(T(x)).
$$

By the change of variables $\int p(T(x)) f^2(x) \, d\mu(x) = \int p \, d\mu < \infty$. By the duality of entropy and the exponential integrability assumption, there exists a constant C such that

$$
\int_{\mathbb{R}^d} \Big[V_1(x) - \langle \nabla V_1(x), x \rangle + (V_1 + p)^*(\nabla V_1(x)) \Big] f^2 \, d\mu \leq C + \frac{1}{1+\varepsilon} \mathrm{Ent}_\mu f^2.
$$

This gives an upper bound on $\int \mathcal{D}_V(x, T(x)) f^2(x) \, d\mu(x)$ by a constant plus $1/(1+\varepsilon)$ times the entropy of f^2. We apply Proposition 5.4 in order to bound the remaining term in (28) by an arbitrarily small multiple of the entropy plus a gradient term. This completes the proof of $I(2/\alpha^*)$. □

Next, we give a better result for a concrete potential V_0.

Theorem 5.25. *Let $\alpha \in (1, 2]$. Let $d\mu(x) = e^{-V(x)} dx$ be a probability measure on $\mathbb{R}^d$ with potential*

$$
V(x) = N\Big(\frac{|x|^\alpha}{\alpha} + V_1(x) \Big), \quad x \in \mathbb{R}^d
$$

where $N > 0$ is a constant. If V_1 is continuously differentiable and if there exits $C > 0$ and $\delta < \frac{\alpha}{2+\alpha}$ such that

$$
|\nabla V_1(x)| \leq \delta |x|^{\alpha-1} + C, \quad x \in \mathbb{R}^d
$$

then μ satisfies the modified log-Sobolev inequality with $c = c_\alpha$, as well as an F_{2/α^*}-Sobolev inequality.

Proof. Since V is locally bounded, μ satisfies a local Poincaré inequality. In view of Theorem 4.1, it is enough to establish Inequality $I(\tau)$ for $\tau = 2/\alpha^*$. The scheme of the proof is the same as for the previous theorem: let T pushing forward $f^2 \cdot \mu$ to μ, then the bound (28) is available. First note that there exists a constant D such that

$$|V_1(x)| \le \frac{\delta}{\alpha}|x|^\alpha + C|x| + D, \quad x \in \mathbb{R}^d.$$

Hence $\int \exp(\kappa|x|^\alpha)\,d\mu(x)$ is finite provided $\kappa < N(1-\delta)/\alpha$. In particular, by Proposition 5.4 for all $\varepsilon > 0$ there are constants C_i such that

$$2\int \langle \nabla f(x), x - T(x)\rangle f(x)\,d\mu(x) \le C_1 + C_2 \int |\nabla f|^2 \log^{1-\tau}(e + f^2)\,d\mu + \frac{\varepsilon}{2}\mathrm{Ent}_\mu f^2.$$

It remains to show that $\int \mathcal{D}_V(x, T(x))f^2 d\mu \le (1-\varepsilon)\mathrm{Ent}_\mu f^2 + C_3$ for a small enough $\varepsilon > 0$. Set $V_0(x) = |x|^\alpha/\alpha$. By linearity $\mathcal{D}_V(x,y) = N\big(\mathcal{D}_{V_0}(x,y) + \mathcal{D}_{V_1}(x,y)\big)$. Plainly

$$\begin{aligned}
\mathcal{D}_{V_0}(x,y) &= \frac{|x|^\alpha}{\alpha} - \frac{|y|^\alpha}{\alpha} + \langle |x|^{\alpha-2}x, y - x\rangle \\
&\le \frac{1-\alpha}{\alpha}|x|^\alpha - \frac{|y|^\alpha}{\alpha} + |x|^{\alpha-1}|y| \\
&\le \left(\frac{1-\alpha}{\alpha} + \varepsilon_0\right)|x|^\alpha + N_1(\alpha, \varepsilon_0)|y|^\alpha,
\end{aligned}$$

for arbitrary $\varepsilon_0 > 0$, where we have used Young's inequality in the form $uv = \eta(u\frac{v}{\eta}) \le \eta\frac{u^{\alpha/(\alpha-1)}}{\alpha/(\alpha-1)} + \eta\frac{(v/\eta)^\alpha}{\alpha}$. One obtains a similar estimate for the convexity defect of V_1 by using the previous bounds on $|V_1|$, $|\nabla V_1|$,

$$\begin{aligned}
\mathcal{D}_{V_1}(x,y) &\le |V_1(x)| + |V_1(y)| + |x|\,|\nabla V_1(x)| + |y|\,|\nabla V_1(x)| \\
&\le \left(\frac{\delta(1+\alpha)}{\alpha} + \varepsilon_0\right)|x|^\alpha + N_2(\alpha,\varepsilon_0)|y|^\alpha + C_2.
\end{aligned}$$

Here we have used Young's inequality as before to separate variables in the product term and also to absorb the linear terms $|x| \le \eta|x|^\alpha + N_3(\alpha,\eta)$. Finally

$$\mathcal{D}_V(x, T(x)) \le \kappa|x|^\alpha + N_4(\alpha,\varepsilon_0)|T(x)|^\alpha + C_3.$$

where $\kappa = N\left(\frac{(1-\alpha)+\delta(1+\alpha)}{\alpha} + 2\varepsilon_0\right)$. Since $\delta < \alpha/(2+\alpha)$ it is possible to find $\varepsilon_0, \varepsilon > 0$ small enough so that $\int \exp(\frac{\kappa}{1-\varepsilon}|x|^\alpha)d\mu(x) < +\infty$. Hence the duality of entropy, the change of variable formula and the strong integrability of μ yield a bound of the form $\int \mathcal{D}_V(x, T(x))f(x)^2 d\mu(x) \le (1-\varepsilon)\mathrm{Ent}_\mu(f^2) + C_4$, as needed. $\quad\square$

5.7. Modified LSI via integration by parts.
The technique developed here is close to the Lyapunov function method (see e.g. [8]). We estimate the convexity defect by the divergence of a special vector field (usually x or ∇V) and apply integration by parts.

The next lemma follows immediately from integration by parts

Lemma 5.26. *Let ω be a locally Lipschitz vector field. Assume that there exist $t, s, C \in \mathbb{R}$ such that*

$$sV \le (1-t)\langle \nabla V, \omega\rangle - \mathrm{div}(\omega) + C.$$

Then for every smooth g

$$t\int_{\mathbb{R}^d} \langle \nabla V, \omega\rangle g e^{-V}\,dx + s\int_{\mathbb{R}^d} V g e^{-V}\,dx \le C\int_{\mathbb{R}^d} g\,d\mu + \int_{\mathbb{R}^d} \langle \nabla g, \omega\rangle e^{-V}\,dx.$$

Theorem 5.27. *Let $d\mu(x) = e^{-V(x)}dx$ be a probability measure on $\mathbb{R}^d$, such that $\int_{\mathbb{R}^d} e^{\eta|x|^\alpha}\,d\mu(x) < +\infty$ for some $\eta > 0$, $\alpha \in (1,2]$.*

a) *Assume that V is continuously differentiable, that there exist $C_1, C_2, \varepsilon > 0 \in \mathbb{R}$ such that*

$$V(x) \le C_1\langle x, \nabla V(x)\rangle + C_2,$$

$$\exp\big(\varepsilon|\nabla V|\log^{\frac{1}{\alpha}}|\nabla V|\big) \in L^1(\mu)$$

and $-V \le g$ with $g \in L^1(\mu)$. Then μ satisfies Inequality $I(2/\alpha^*)$.

b) *Assume that $\alpha = 2$, V is twice continuously differentiable, $-V \le g$ such that $g \in L^1(\mu)$ and there exist $s_0 > 0, 1 > t > 0$ such that for every $0 < s < s_0$ there exists $C = C(s,t)$ satisfying*

$$sV \le (1-t)|\nabla V|^2 - \Delta V + C.$$

Then μ satisfies the defective log-Sobolev inequality.

In particular, the result holds if V is bounded from below and $sV \le (1-t)|\nabla V|^2 - \Delta V + C$ for some $s > 0, 1 > t > 0$

Proof. To prove a) we apply a bit more general estimate than the above-tangent lemma. Namely, let T be the optimal transport sending $f^2 \cdot \mu$ to μ. Then in the same way as above (changing variables, taking logarithm and integrating with respect to μ) we get

$$\text{Ent}_\mu f^2 = \int f^2 V \, d\mu - \int_{\mathbb{R}^d} V \, d\mu + \int \log \det DT \, f^2 \, d\mu.$$

By the concavity of logarithm

$$\text{Ent}_\mu f^2 \le \int f^2 V \, d\mu - \int V \, d\mu + d \log\Big(\int \frac{\text{Tr} DT}{d} \, f^2 \, d\mu \Big).$$

First we note that $-\int V \, d\mu \le \int g \, d\mu < \infty$. Applying the assumption of the theorem and integration by parts, we get

$$\int f^2 V \, d\mu \le C_2 + dC_1 + 2C_1 \int f(x)\langle x, \nabla f(x)\rangle \, d\mu(x).$$

Note that

$$\int \text{Tr} DT \, f^2 \, d\mu = -2 \int \langle T, \nabla f\rangle f \, d\mu + \int \langle T, \nabla V\rangle f^2 \, d\mu.$$

Then we estimate

$$\int f(x)\langle \nabla f(x), x\rangle \, d\mu(x) \quad \text{and} \quad \int \langle T, \nabla f\rangle f \, d\mu$$

exactly in the same way as in Proposition 5.4. Next

$$(29) \qquad \int \langle \nabla V, T\rangle f^2 \, d\mu \le \int e^{\delta |T|^{\frac{2}{2-\tau}}} f^2 \, d\mu + N_1 \int |\nabla V| \log^{\frac{2-\tau}{2}} |\nabla V| f^2 \, d\mu + N_2.$$

Using the assumption on ∇V one can easily estimate the right-hand side by $N_1 \text{Ent}_\mu f^2 + C$. It remains to note that logarithm grows slowly than any linear function. The proof of a) is complete.

For the proof of b) apply Lemma 5.26 with $\omega = \nabla V$. One obtains

$$s \int V f^2 \, d\mu + t \int |\nabla V|^2 f^2 \, d\mu \le C + 2 \int f\langle \nabla f, \nabla V\rangle \, d\mu.$$

for some $t > 0$ and every $0 < s < s_0$. By the Cauchy inequality

$$\frac{t}{2} \int |\nabla V|^2 f^2 \, d\mu + s \int V f^2 \, d\mu \le C + \frac{4}{t} \int |\nabla f|^2 \, d\mu.$$

Choosing arbitrary small s we obtain that for every $N > 0$

$$(30) \qquad N \int |\nabla V|^2 f^2 \, d\mu + \int V f^2 \, d\mu \le C(N)\Big(1 + \int |\nabla f|^2 \, d\mu \Big)$$

if $C(N)$ is sufficiently big. This gives the desired bound for

$$\int \Big(V(x) - \langle \nabla V(x), x - T(x)\rangle \Big) f(x)^2 \, d\mu(x).$$

Indeed, the latter is not bigger than

$$\int \Big(V(x) + N|\nabla V(x)|^2 \Big) f(x)^2 \, d\mu(x) + \frac{4}{N} \int |T(x) - x|^2 f(x)^2 \, d\mu(x),$$

where the first term is estimated by (30) and the second term one can easily estimate by $C(\varepsilon_0) + \varepsilon_0 \text{Ent}_\mu f^2$ for arbitrary ε_0 by choosing appropriate N. Finally, $-\int V(T) f^2 \, d\mu = -\int V \, d\mu \le |g|_{L^1(\mu)}$. The proof of b) is complete. $\square$

The analog of b) holds also for $1 < \alpha < 2$. However, we need more restrictive assumptions.

Theorem 5.28. *Let $d\mu(x) = e^{-V(x)}dx$ be a probability measure on $\mathbb{R}^d$, such that $\int_{\mathbb{R}^d} e^{\eta|x|^\alpha} d\mu(x) < +\infty$ for some $\eta > 0$, $\alpha \in (1, 2]$. Assume that V is twice continuously differentiable and bounded from below and that there exist $s > 0, 1 > t > 0$ such that for some $C = C(s, t)$ one has*

(31) $$ s\max(V, 1)^\tau \le (1 - t)|\nabla V|^2 - \Delta V + C, $$

where $\tau = 2/\alpha^* \in (0, 1]$. Then μ satisfies Inequality $I(\tau)$.

Proof. The proof is similar to the proof of Theorem 5.27, but more involved. First we multiply (31) by $\max(V, 1)^{1-\tau}$ and apply integration by parts. We get

$$ s\int Vf^2 \, d\mu + t\int |\nabla V|^2 \max(V, 1)^{1-\tau} f^2 \, d\mu $$

$$ \le 2\int f\langle \nabla f, \nabla V\rangle \max(V, 1)^{1-\tau} \, d\mu + (1 - \tau)\int \max(V, 1)^{-\tau}|\nabla V|^2 f^2 \, d\mu + C_1. $$

First we note that

$$ \int \max(V, 1)^{-\tau}|\nabla V|^2 f^2 \, d\mu \le \int |\nabla V|^2 f^2 \, d\mu \le -\frac{1}{t}\int (|\nabla V|^2 - \Delta V) f^2 \, d\mu + C $$

$$ = \frac{2}{t}\int f\langle \nabla f, \nabla V\rangle \, d\mu + C. $$

Hence one can write

$$ s\int Vf^2 \, d\mu + t\int |\nabla V|^2 \max(V, 1)^{1-\tau} f^2 \, d\mu $$

$$ \le C_2 \int f|\langle \nabla f, \nabla V\rangle| \max(V, 1)^{1-\tau} \, d\mu + C_3. $$

Let us estimate the first term in the right-hand side:

$$ 2\int f|\langle \nabla f, \nabla V\rangle| \max(V, 1)^{1-\tau} \, d\mu \le N(\varepsilon)\int |\nabla f|^2 \log^{1-\tau}(e + f^2) \, d\mu $$

$$ + \varepsilon \int \frac{f^2}{\log^{1-\tau}(e + f^2)}|\nabla V|^2 \big(\max(V, 1)\big)^{2(1-\tau)} \, d\mu. $$

Let $M_\delta = \{x : f^2 \le e^{\delta V}\}$. Then for every $\delta' > \delta$ there exists $C(\delta, \delta', \tau)$ such that

$$ \int_{M_\delta} \frac{f^2}{\log^{1-\tau}(e + f^2)}|\nabla V|^2 \big(\max(V, 1)\big)^{2(1-\tau)} \, d\mu \le C(\delta, \delta', \tau)\int_{M_\delta} e^{\delta' V}|\nabla V|^2 \, d\mu. $$

In the other hand

$$ \int_{M_\delta^c} \frac{f^2}{\log^{1-\tau}(e + f^2)}|\nabla V|^2 \big(\max(V, 1)\big)^{2(1-\tau)} \, d\mu $$

$$ \le C(\delta, \tau, \inf V)\int_{\mathbb{R}^d} |\nabla V|^2 \max(V, 1)^{1-\tau} f^2 \, d\mu. $$

Choosing a sufficiently small δ' and ε one gets the following:

$$ s\int_{\mathbb{R}^d} Vf^2 \, d\mu + \frac{t}{2}\int |\nabla V|^2 \max(V, 1)^{1-\tau} f^2 \, d\mu $$

$$ \le C_4 \int |\nabla f|^2 \log^{1-\tau}(e + f^2) \, d\mu + C_5 \int e^{\delta' V}|\nabla V|^2 \, d\mu + C_6. $$

Let us show that

$$ \int e^{\delta' V}|\nabla V|^2 \, d\mu $$

is finite for a sufficiently small δ'. First we note that $\int e^{\delta' V} \, d\mu$ is a finite measure for sufficiently small δ'. This can be easily proved by Hölder's inequality since $\int \exp(\eta|x|^\alpha) d\mu(x) < \infty$. Next, integrating inequality

$$ t|\nabla V|^2 \le |\nabla V|^2 - \Delta V + C' $$

over $e^{\delta' V} \cdot \mu$ and integrating by parts we easily get the claim. Thus, we obtain that there exists $\tilde{C}$ depending on s, τ, δ' such that

$$s \int V f^2 \, d\mu + \frac{t}{2} \int |\nabla V|^2 \max(V, 1)^{1-\tau} f^2 \, d\mu$$

(32)
$$\leq \tilde{C} \int |\nabla f|^2 \log^{1-\tau}(e + f^2) \, d\mu + \tilde{C}.$$

Since the function V is bounded from below, we get the desired bounds for the terms $\int V f^2 \, d\mu$ and $\int |\nabla V|^2 \max(V, 1)^{1-\tau} f^2 \, d\mu$. The estimates of $-\int \langle \nabla V(x), x \rangle f(x)^2 \, d\mu(x)$ and $-\int V(T) f^2 \, d\mu$ are the same as in Theorem 5.27. Finally,

$$\int \langle \nabla V, T \rangle f^2 \, d\mu \leq 2 \int |\nabla V|^2 f^2 \, d\mu + 2 \int |T|^2 f^2 \, d\mu$$

$$\leq 2 \int |\nabla V|^2 f^2 \max(V, 1)^{1-\tau} \, d\mu + 2 \int |x|^2 \, d\mu(x).$$

The latter can be estimated by (32). The proof is complete. $\qquad\square$

Corollary 5.29. *Under assumptions of Theorems 5.27, 5.28 the tight modified log-Sobolev inequality with $c = c_\alpha$, $\alpha = \frac{2}{2-\tau}$ as well as F-inequality with $F = F_\tau$ holds.*

Proof. By Theorem 4.1 it suffices to prove the local Poincaré inequality. This follows from Proposition 2.14 since the potential V locally bounded. $\qquad\square$

Remark 5.30. Let us compare this result with the known ones. Theorem 5.28 is not completely new for F-inequalities This type of criteria for F-inequalities have been already considered in work of Rosen [60] (note, however, that assumptions on the potential from [60] are stronger). Kusuoka and Stroock [46] proved different types of hyperboundedness of semigroups using Lyapunov function techniques. We note that assumptions on the potential in Theorem 5.28 and in Theorem 5.27 a) can be viewed as special cases of some Lyapunov function-type assumptions. Nevertheless, such kind of criteria are not known for modified log-Sobolev inequalities. Also the transportation approach for this kind of results is new. Some related results can be also found in [31], [14], [30].

A less general but more beautiful sufficient condition is the following: V is bounded from below, for some $s > 0$

$$s|V|^\tau \leq |\nabla V|^2 + C, \quad \lim_{|x| \to \infty} \frac{\Delta V(x)}{|\nabla V(x)|^2} = 0.$$

It appears in many works as a sufficient condition for log-Sobolev type inequalities (see [5, 27, 56, 16]).

Corollary 5.31. *Let V be a continuously differentiable function such that $V(tx)$ is convex as a function of $t \in [0, \infty)$ for every x. Assume that $\int_{\mathbb{R}^d} e^{\eta |x|^\alpha} \, d\mu(x) < +\infty$ for some $\eta > 0$, $\alpha \in (1, 2]$ and*

$$\exp(\varepsilon |\nabla V| \log |\nabla V|^{\frac{1}{\alpha}}) \in L^1(\mu).$$

Then the tight modified log-Sobolev inequality with $c = c_\alpha$, as well as the F-inequality with $F = F_{2/\alpha^*}$ hold.

Proof. Since $\varphi(t) = V(tx)$ is convex, it holds $\varphi(0) \geq \varphi(1) - (0 - 1)\varphi'(1)$. In other words,

$$V(0) \geq V(x) - \langle \nabla V(x), x \rangle.$$

The result follows from Theorem 5.27, Corollary 5.29. $\qquad\square$

6. Improved bounds in dimension 1

We start with a precised version of the "above tangent" lemma. We omit the proof which is similar to the one of Lemma 5.1. The only difference is that the term $\theta' - \log(1 + \theta')$ is not lower bounded by 0. The goal of this section is to develop applications of sharper estimates of this quantity.

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Lemma 6.1. *Let $d\mu(x) = e^{-V(x)}dx$ be a probability measure on $\mathbb{R}$, with V smooth. Let $f \cdot \mu$ and $g \cdot \mu$ be two probability measures with smooth and compactly supported densities f, g. Let $T(x) = x + \theta(x)$ be the monotone map pushing forward $f \cdot \mu$ to $g \cdot \mu$. Then*

$$\mathrm{Ent}_\mu(f) + \int_{\mathbb{R}} \Big(V\big(x + \theta(x)\big) - V(x) - \theta(x)V'(x)\Big)f(x)\,d\mu(x)$$

$$+ \int_{\mathbb{R}} \big(\theta' - \log(1 + \theta')\big)f\,d\mu = \mathrm{Ent}_\mu(g) - \int_{\mathbb{R}} f'\theta\,d\mu.$$

If $d\mu(x) = e^{-V(x)}\mathbf{1}_{x \geq 0}dx$ where V is smooth and convex, and f, g are smooth with finite entropy, then provided $\lim_{+\infty} f\theta e^{-V} = 0$, the equality is valid with an additional term $-f(0)\theta(0)e^{-V(0)}$ on the right-hand side.

6.1. Inequalities for the exponential law. Let $d\mu(t) = e^{-t}\mathbf{1}_{t>0}\,dt$ be the exponential measure. Our goal is to provide a simple transportation proof of the modified log-Sobolev inequality for μ due to Bobkov and Ledoux [25]. We also discuss related transportation cost inequalities. One should note that in an unpublished work Schukenscläger [61] establishes an inequality for the exponential measure which extends Maurey's τ-property [51], with the same method. Taking suitable limits he recovers transportation cost inequalities and modified log-Sobolev inequalities. His proof is quite involved, and it is not clear at all whether the argument coming next can be viewed as a limit of it.

We start with recalling useful Sobolev type inequalities for μ. The first part of the next lemma is a particular case of a result of Bobkov and Houdré [23], for which we provide a streamlined proof. The second part is Lemma 2.2 of Talagrand's paper [62].

Lemma 6.2.

1) Let $N : \mathbb{R} \to [0, +\infty)$ be an even differentiable convex function with $N(0) = 0$. Assume that there exists $c \geq 1$ such that $xN'(x) \leq cN(x)$ for all x. Let $\varphi : [0, +\infty) \to [0, +\infty)$ be a locally Lipschitz function with $\varphi(0) = 0$, then

$$\int_{\mathbb{R}^+} N(\varphi)\,d\mu \leq \int_{\mathbb{R}^+} N(c\varphi')\,d\mu.$$

2) Let $M(x) = x - \log(1 + x)$, $x > -1$ and $S(x) = x - 1 + e^{-x}$ and $\alpha \in (0, 1)$. Let $\varphi : [0, +\infty) \to [0, +\infty)$ as above with $\varphi(0) = 0$ and $\varphi' \geq -1$, then

$$\frac{1 - \alpha}{\alpha} \int_{\mathbb{R}^+} S(\alpha\varphi)\,d\mu \leq \int_{\mathbb{R}^+} M(\varphi')\,d\mu.$$

Proof. First we assume that φ is also bounded. For $a > 0$ an integration by parts yields

$$\int_0^a N\big(\varphi(x)\big)e^{-x}dx - N\big(\varphi(a)\big)e^{-a} = \int_0^a \varphi'(x)N'\big(\varphi(x)\big)e^{-x}dx.$$

Let N^* be the Legendre transform of N, defined by $N^*(v) = \sup_u\{uv - N(u)\}$. Then the following inequalities hold pointwise:

$$\varphi'N'(\varphi) \leq \frac{1}{c}\Big(N(c\varphi') + N^*\big(N'(\varphi)\big)\Big) = \frac{1}{c}\big(N(c\varphi') + \varphi N'(\varphi) - N(\varphi)\big)$$

$$\leq \frac{1}{c}\big(N(c\varphi') + (c-1)N(\varphi)\big).$$

Plugging this inequality in the above integral equality and rearranging gives

$$\int_0^a N\big(\varphi(x)\big)e^{-x}dx - cN\big(\varphi(a)\big)e^{-a} \leq \int_0^a N\big(c\varphi'(x)\big)e^{-x}dx.$$

Letting a to $+\infty$, we obtain the claimed inequality for bounded functions. If φ is unbounded we apply the inequality to $\min\big(|\varphi|, n\big)$ for n growing to infinity and conclude by monotone convergence.

The proof of the second inequality is similar. It uses the remarkable relation $M^*(S'(x)) = S(x)$. $\qquad\square$

Next, we state the transportation inequality for the exponential law with a cost function comparable to $\min(x^2, |x|)$. It is the analogue of Talagrand's inequality for the symmetric exponential law [62].

Proposition 6.3. *Let $\alpha \in (0,1)$ and $c_\alpha(x) = \frac{1-\alpha}{\alpha}\Big(\alpha x - 1 + \exp(-\alpha x)\Big)$. Let $g \cdot \mu$ be a probability measure. Then*

$$\mathrm{Ent}_\mu(g) \geq T_{c_\alpha}(g \cdot \mu, \mu).$$

Proof. Let $T(x) = x + \theta(x)$ be the non-decreasing map transporting μ to $g\,d\mu$. Lemma 6.1 with $f = 1$ gives

$$\mathrm{Ent}_\mu(g) - \theta(0) = \int \big(\theta' - \log(1+\theta')\big)\,d\mu.$$

The term $\theta(0)$ is the displacement that is applied to the origin. It corresponds to the first point of the support of $g\,d\mu$. One way to get rid of this term is to approximate $g\,d\mu$ by a measure with support starting at 0. Another way is to translate g: let a be the first point of the support of $g\,d\mu$, then let $g_a(x) = e^{-a}g(x + a)$. It is easy to check that $g_a\,d\mu$ is a probability measure, that $T - a$ pushes forward μ to $g_a d\mu$ and $\mathrm{Ent}_\mu(g_a) = \mathrm{Ent}_\mu(g) - a$. So without loss of generality, we can assume that $\theta(0) = 0$ and by Lemma 6.2

$$\mathrm{Ent}_\mu(g) = \int \big(\theta' - \log(1+\theta')\big)\,d\mu = \int M(\theta')\,d\mu \geq \frac{1-\alpha}{\alpha}\int S(\alpha\theta)\,d\mu.$$

$\square$

In order to recover the modified log-Sobolev inequality for μ, we need the following lemma:

Lemma 6.4. *Let $d\mu(x) = e^{-x}\mathbf{1}_{x>0}\,dx$ be the exponential law and let $d\nu = e^g\,d\mu$ be a probability measure. Assume that g is locally Lipschitz and satisfies $|g'| < c$ a.e. for some constant $c < 1$. Then the monotone map T which transports ν to μ verifies*

$$T'(x) \in [1-c, 1+c], \quad \text{for all } x \geq 0.$$

The reciprocal map $S = T^{-1}$ transports μ to ν and satisfies $S'(x) \leq \frac{1}{1-c}$ for $x \geq 0$.

Proof. The map T is actually given by

$$T(x) = -\log\left(\int_x^\infty e^{g(u)-u}du\right).$$

Indeed, this expression is strictly increasing and satisfies for $x \geq 0$

$$\mu\big((-\infty, T(x)]\big) = \int_0^{T(x)} e^{-u}du = 1 - e^{-T(x)} = \int_0^x e^{g(u)-u}du = \nu\big((-\infty, x]\big).$$

Hence

$$T'(x) = \frac{e^{g(x)-x}}{\int_x^\infty e^{g(u)-u}du} = \frac{\int_x^\infty \big(1-g'(u)\big)e^{g(u)-u}du}{\int_x^\infty e^{g(u)-u}du} \in [1-c, 1+c].$$

Since $1 - c > 0$, it follows that its reciprocal bijection $S = T^{-1}$ satisfies $0 < S'(x) \leq \frac{1}{1-c}$. $\square$

Remark 6.5. The above statement is an elementary companion to Caffarelli's celebrated theorem [29]: if γ is a Gaussian measure on $\mathbb{R}^d$ and and $d\mu = e^{-W}d\gamma$ with $W'' \geq 0$ then μ is the image of γ by a contraction. The following heuristic argument allows to understand better the similarities. We work in dimension 1 with two probability measures $d\mu = e^{-V(x)}dx$ and $d\sigma = e^{-W}d\mu$. The monotone transport S from μ to σ satisfies $e^{-V(x)} = e^{-W\left(S(x)\right)-V\left(S(x)\right)}S'(x)$. If S is smooth enough, taking logarithms and differentiating gives

$$(33) \qquad S'(x)V'\big(S(x)\big) + S'(x)W'\big(S(x)\big) = V'(x) + \frac{S''(x)}{S'(x)}.$$

Following Caffarelli, we assume that S' achieves its maximum at an interior point x_0. Then $S''(x_0) = 0$ and the latter equality yields $S'(x_0)\big(V'\big(S(x_0)\big) + W'\big(S(x_0)\big)\big) = V'(x_0)$. For the exponential law, $V' = 1$ on the image of S. If $W' \geq -c$, the function S' satisfies at its maximum

$$S'(x_0) = \frac{1}{1 + W'\big(S(x_0)\big)} \leq \frac{1}{1-c}.$$

In the case $V(x) = x^2/2$, it is natural to differentiate (33) in order to get constant terms $V'' = 1$

$$S''(x)V'\big(S(x)\big) + S'(x)^2 V''\big(S(x)\big) + S''(x)W'\big(S(x)\big) + S'(x)^2 W''\big(S(x)\big) = V''(x) + \frac{S'''(x)}{S'(x)} - \frac{S''(x)^2}{S'(x)^2}.$$

At any point x_0 where S' reaches its maximum, $S''(x_0) = 0$ and $S'''(x_0) \leq 0$, hence

$$S'(x_0)^2\Big(1 + W''\big(S(x_0)\big)\Big) \leq 1.$$

Finally $W'' \geq 0$ implies $S'(x_0) \leq 1$ and S is a contraction.

Proposition 6.6. *Let $c \in (0,1)$ and $f : [0, +\infty) \to (0, +\infty)$ such that $|f'/f| < c$, then*

$$\mathrm{Ent}_\mu(f) \leq \frac{4}{(1-c)^2} \int_{\mathbb{R}} \frac{f'^2}{f} \, d\mu.$$

Proof. By homogeneity we may assume that $\int f \, d\mu = 1$. Let $T(x) = x + \theta(x)$ be the monotone map pushing forward $f \cdot \mu$ to μ. By Lemma 6.4 we know that $|\theta'| \leq c$. This allows to check the growth conditions needed to apply Lemma 6.1 with f and $g = 1$ and to obtain

$$(34) \qquad -\int f'\theta \, d\mu = \mathrm{Ent}_\mu(f) + \int \big(\theta' - \log(1 + \theta')\big) f \, d\mu \geq \mathrm{Ent}_\mu(f) + \frac{1}{4} \int (\theta')^2 f \, d\mu,$$

where we have used that for $|x| \leq c < 1$, $N(x) = x - \log(1+x) \geq x^2/4$. To conclude the argument we need to get rid of the θ' term by means of a Sobolev type inequality for the measure $f \cdot \mu$. But thanks to Lemma 6.4, the hypothesis $|f'/f| < c$ guarantees the existence of a $\frac{1}{1-c}$-Lipschitz map $S = T^{-1}$ pushing forward μ to $f \cdot \mu$. This classically implies that Sobolev type inequalities enjoyed by μ transfer to $f \cdot \mu$. Indeed assume that every smooth function φ with $\varphi(0) = 0$ satisfies $\int N_1(\varphi) \, d\mu \leq \int N_2(\varphi') \, d\mu$ where N_2 is non-increasing on $\mathbb{R}^-$ and non-decreasing on $\mathbb{R}^+$. Then for any smooth ψ vanishing at 0, and since $S(0) = 0$, we may apply the Sobolev inequality to $\varphi = \psi \circ S$. Since the law of S under μ is $f \cdot \mu$, we obtain

$$\int N_1(\psi) f \, d\mu \;=\; \int N_1(\psi \circ S) \, d\mu \leq \int N_2(S'\psi' \circ S) \, d\mu$$

$$\leq \; \int N_2\Big(\frac{1}{1-c}\psi' \circ S\Big) \, d\mu = \int N_2\Big(\frac{1}{1-c}\psi'\Big) f \, d\mu.$$

By Lemma 6.2, we recover the classical Poincaré inequality for the exponential law: if $\varphi(0) = 0$ then $\int \varphi^2 \, d\mu \leq 4 \int (\varphi')^2 d\mu$ from which we deduce

$$\int \theta^2 f \, d\mu \leq \frac{4}{(1-c)^2} \int (\theta')^2 f \, d\mu.$$

Plugging this inequality in the above entropy estimate yields

$$\begin{aligned}
\mathrm{Ent}_\mu(f) &\leq \int |f'|\,|\theta| \, d\mu - \frac{(1-c)^2}{16} \int \theta^2 f \, d\mu \\
&\leq \int f \sup_u \Big\{ \frac{|f'|}{f} u - \frac{(1-c)^2}{16} u^2 \Big\} = \frac{4}{(1-c)^2} \int \frac{(f')^2}{f} \, d\mu.
\end{aligned}$$

$\square$

Remark 6.7. The interest of the above proof lies in the interpretation of the condition $|f'/f| < c$ in terms of transport. It does not provide very good constants. Bobkov and Ledoux obtain a constant of the form $\frac{2}{1-c}$ which captures the right order in c as one can check with the function $f(t) = (1-c)e^{ct}$.

6.2. **Inequalities for the Laplace distribution.** Let $d\nu_1(t) = e^{-|t|} \, dt/2$, $t \in \mathbb{R}$ be the symmetric exponential law. We use the relation $\nu_1 = \mu * \check{\mu}$, where $d\check{\mu}(t) = e^t \mathbf{1}_{t<0} \, dt$ is the image of the exponential law by a reflection. Log-Sobolev and transportation cost inequalities easily pass to product measures. Hence the previous results on μ transfer to $\mu \otimes \mu$, and to ν_1 by considering functions of (x,y) depending only of $x - y$:

Proposition 6.8. *For $\alpha \in (0, 1)$, and any probability measure of the form $f \cdot \nu_1$,*

$$\text{Ent}_{\nu_1}(f) \geq 2\frac{1-\alpha}{\alpha} \inf_{\pi \in \Pi(\nu_1, f \cdot \nu_1)} \int_{\mathbb{R}^2} \log \cosh\left(\frac{\alpha}{2}(x-y)\right) d\pi(x, y).$$

Let $c \in (0, 1)$, $f : \mathbb{R} \to \mathbb{R}^+$ be a smooth function with $|f'/f| < c$ then

$$\text{Ent}_{\nu_1} f \leq \frac{8}{(1-c)^2} \int_{\mathbb{R}} \frac{f'^2}{f} d\nu_1.$$

Remark 6.9. Talagrand actually proved a slightly stronger transportation inequality, but his proof is a lot more involved. Bobkov and Ledoux also had a better constant in the log-Sobolev inequality.

6.3. **Gaussian transportation cost inequality for measures with median at 0.** Applying Lemma 6.1 to $\mu = \gamma$ the standard Gaussian measure on $\mathbb{R}$, $f = 1$ amounts to reproducing Talagrand's proof of the Gaussian transportation cost inequality [62]:

$$\text{Ent}_\gamma(g) = \int \frac{\theta^2}{2} d\gamma + \int \left(\theta' - \log(1 + \theta')\right) d\gamma \geq \int \frac{\theta^2}{2} d\gamma = \frac{1}{2} T_2(\gamma, g \cdot \gamma).$$

This inequality is known to be stronger than the Poincaré inequality for γ, and strictly weaker than the Gaussian logarithmic Sobolev inequality. A natural question asks for improvements of these inequalities for even functions, or for centered functions (i.e. $\int x f(x) d\gamma(x) = 0$). It is known that the Poincaré constant may be improved by a factor 2 for centered functions, whereas the log-Sobolev inequality does not improve for even functions. It was recently understood that the τ-property can be improved for centered functions [51, 6]. The constant $\frac{1}{2}$ in the above transportation cost inequality cannot be improved for symmetric measures, as shown by the following example. Consider for $a \geq 0$ the probability measure m_a defined by

$$dm_a(x) = (2\pi)^{-1/2} \left(e^{-\frac{(x-a)^2}{2}} \mathbf{1}_{x<0} + e^{-\frac{(x+a)^2}{2}} \mathbf{1}_{x>0}\right) dx, \qquad x \in \mathbb{R}.$$

Clearly the map T defined by $T(x) = x - a$ for $x \leq 0$ and $T(x) = x + a$ for $x > 0$ pushes γ forward to m_a. It is monotone and therefore optimal form the quadratic cost. Since for all x, $|T(x) - x| = a$, it follows that $T_2(\gamma, m_a) = a^2$. Let $g_a = \frac{dm_a}{d\gamma}$. By a straightforward calculation

$$\text{Ent}_\gamma(g_a) = \frac{a^2}{2} + \sqrt{\frac{2}{\pi}} a \sim_{a \to +\infty} \frac{a^2}{2}.$$

However there is room for an improvement of lower order.

Proposition 6.10. *Let $N(t) = |t| - \log\left(1 + |t|\right)$. For all probability measures $g \cdot \gamma$ on $\mathbb{R}$ with median at 0,*

$$\text{Ent}_\gamma(g) \geq T_w(m, \gamma),$$

where the cost function is $w(x) = x^2/2 + N\left(x/\sqrt{2\pi}\right)$. For large x, $w(x) = \frac{x^2}{2} + \frac{x}{\sqrt{2\pi}} + o(x)$.

Proof. Assume that g is positive and continuous. Since the median of $g \cdot \gamma$ is zero, the monotone transport T from γ to $g \cdot \gamma$ satisfies $T(0) = 0$, hence the displacement vanishes at the origin: $\theta(0) = 0$. Recall

$$\text{Ent}_\gamma(g) \geq \int \frac{\theta^2}{2} d\gamma + \int \left(\theta' - \log(1 + \theta')\right) d\gamma = \frac{1}{2} T_2(\gamma, g \, d\gamma) + \int \left(\theta' - \log(1 + \theta')\right) d\gamma.$$

We need a Sobolev type inequality to lower bound the latter term. Given a smooth function φ on $\mathbb{R}$ vanishing at 0, we apply Lemma 6.2 to $\varphi(x)$ and $\varphi(-x)$ and obtain

$$\int N(\varphi) \, d\nu_1 \leq \int N(2\varphi') \, d\nu_1.$$

One can check that the monotone transport S from ν_1 to γ is $\sqrt{\frac{\pi}{2}}$-Lipschitz. Reasoning as in the proof of Proposition 6.6, the latter inequality applied to $\varphi = \theta \circ S$, which vanishes at 0, yields

$$\int \left(\theta' - \log(1 + \theta')\right) d\gamma \geq \int N(\theta') \, d\gamma \geq \int N(\theta/\sqrt{2\pi}) \, d\gamma.$$

$\square$

7. Generalizations to Riemannian manifolds

Some results of this paper can be obtained in the Riemannian setting. This is illustrated in this section. Several lemmata obviously extend since they do not use the geometric structure of the space; we shall use them in the Riemannian setting without further explanation.

As in the flat case, the starting point here is the above-tangent lemma. Let (M, g) be a smooth, complete, connected Riemannian manifold without boundary. The geodesic distance on M is denoted by ρ and the Riemannian volume by v. The following theorem is an adapted version of the result from [35], [36]. The proof is almost the same as the original one and we omit it here.

Theorem 7.1. *Let $d\mu(x) = e^{-V(x)}\, dv(x)$ be a probability measure on M, g and h two compactly supported non-negative functions such that $g \cdot \mu$ and $h \cdot \mu$ are probability measures. Let $T(x) = \exp_x(\nabla\theta(x))$ be the optimal transport minimizing the quadratic transportation cost and pushing forward $g \cdot \mu$ to $h \cdot \mu$. Then it holds*

$$\mathrm{Ent}_\mu g \leq \mathrm{Ent}_\mu h - \int_M \langle \nabla\theta, \nabla g \rangle\, d\mu + \int_M \mathcal{D}_V(x, T(x)) g\, d\mu,$$

where

$$\mathcal{D}_V(x, T(x)) = V(x) + \langle \nabla\theta(x), \nabla V(x) \rangle - V(T(x)) - \int_0^1 (1-t)\mathrm{Ric}_{\gamma(t)}\big(\dot\gamma(t), \dot\gamma(t)\big)\, dt,$$

where γ is the geodesic joining x and $T(x)$ given by $\gamma(t) = \exp(t\nabla\theta(x))$.

If V is twice continuously differentiable, one has

$$\mathcal{D}_V(x, T(x)) = -\int_0^1 (1-t)\Big(\mathrm{Hess}_{\gamma(t)}V + \mathrm{Ric}_{\gamma(t)}\Big)\big(\dot\gamma(t), \dot\gamma(t)\big)\, dt.$$

The next statement is obtained as an application. It nicely complements Wang's theorem:

Theorem 7.2. *Let $d\mu(x) = e^{-V(x)}\, dv(x)$ be a probability measure on M, with a twice continuously differentiable potential V. Let $\alpha > 1$ and suppose that there exists $x_0 \in M$ and $\varepsilon > 0$ such that*

$$\exp\big(\varepsilon\rho(x_0, x)^\alpha\big) \in L^1(\mu).$$

Assume that one of the following two conditions is satisfied
  (i) $\alpha \in (1, 2]$ and pointwize $\mathrm{Hess}V + \mathrm{Ric} \geq 0$
  (ii) $\alpha > 2$ and there exists $K \in \mathbb{R}$ such that pointwize $\mathrm{Hess}V + \mathrm{Ric} \geq K\,\mathrm{Id}$.
Then there exists $\kappa > 0$ such that μ satisfies the isoperimetric inequality

$$\mathcal{I}_\mu(t) \geq \kappa \min(t, 1-t)\log^{1-\frac{1}{\alpha}}\left(\frac{1}{\min(t, 1-t)}\right), \quad t \in (0, 1).$$

Remark 7.3. By Corollary 3.2, various functional inequalities follow. Also note that the results of Wang [66] and Ledoux [48] provide the case $\alpha = 2$: the isoperimetric inequality is valid provided $\mathrm{Hess}V + \mathrm{Ric} \geq K\,\mathrm{Id}$ and $\exp\big((\varepsilon + |K|/2)\rho(x, x_0)^2\big) \in L^1(\mu)$. Unfortunately our method does not reach $\alpha = 1$.

Proof. First we assume (i). Let $\tau = 2/\alpha^*$. We establish Inequality $I(\tau)$ for μ along the same lines as in $\mathbb{R}^d$: We apply Theorem 7.1 and note that $\mathcal{D}_V \leq 0$. Hence we only have to deal with the linear term. This can be done exactly as in Proposition 5.4.

By Proposition 2.14, μ satisfies a local Poincaré inequality. By Theorem 4.1, Inequality $I(\tau)$ implies the corresponding tight F_τ-Sobolev inequality. As an intermediate step of this argument, it has been established that μ satisfies a Poincaré inequality. An argument of Ledoux [48] shows that when $\mathrm{Hess}V + \mathrm{Ric}$ is uniformly bounded from below, the spectral gap inequality yields an isoperimetric inequality of Cheeger $\mathcal{I}_\mu(t) \geq c \min(t, 1-t)$. In particular, it is enough to prove the claimed isoperimetric bound for $\min(t, 1-t)$ small. Ledoux's argument has been adapted to other functional inequalities: the F_τ inequality implies an isoperimetric inequality of the form $\mathcal{I}_\mu(t) \geq c' \min(t, 1-t)\, F_\tau\big(1/\min(t, 1-t)\big)^{1/2}$ when $\min(t, 1-t)$ is small. This is explained in Section 4 and 8 of [14]; it also follows from different arguments of [65]. The proof is complete under Condition (i).

For (ii), assume that $K \le 0$. Reasoning as in the proof of Corollary 5.14, we get that

$$\int e^{\frac{K+\varepsilon}{2}\rho(x,x_0)^2} d\mu(x) < +\infty.$$

Hence Wang's theorem applies and gives in particular that μ satisfies a Poincaré inequality (note that the new proof that we gave in the Euclidean case is easily adapted to the Riemannian setting). By the result of Ledoux [48], μ satisfies Cheeger's isoperimetric inequality, and consequently it is enough to prove the claimed isoperimetric inequality for small values of $\min(t, 1-t)$. Our strategy is to prove a (defective) modified LSI with cost t^α. To do this, we apply the above tangent lemma with $f^2 \cdot \mu$ and μ. The linear term is estimated as in Lemma 5.2: since $|\nabla\theta(x)| = \rho(x, T(x))$, for any $\eta > 0$

$$2\int \langle \nabla f, -\nabla\theta \rangle f \, d\mu \le \frac{\eta}{\alpha^*} \int \left|\frac{2\nabla f}{\eta f}\right|^{\alpha^*} f^2 d\mu + \frac{\eta}{\alpha} \int \rho(x, T(x))^\alpha f(x)^2 d\mu(x).$$

On the other hand

$$\mathcal{D}_V(x, y) \le \frac{K}{2}\rho(x, T(x))^2 \le \eta\rho(x, T(x))^\alpha + N(K, \alpha, \eta).$$

Hence we are done if we show that for some $\eta, \delta \in (0, 1)$ and $C \in \mathbb{R}$,

$$\eta(1 + \alpha^{-1}) \int \rho(x, T(x))^\alpha f(x)^2 d\mu(x) \le (1 - \delta)\text{Ent}_\mu(f^2) + C.$$

This is done as in the proof of Corollary 5.18 using $\rho(x, T(x))^\alpha \le (1+\eta_1)\rho(x, x_0)^\alpha + M(\eta_1)\rho(T(x), x_0)^\alpha$, the duality of entropy and the integrability property. The **(MLSI)** with cost t^α, or equivalently the (α^*-**LSI**), implies the claimed isoperimetric inequality for small values. This is explained in the next lemma. $\square$

The next result extends to $q \in (1, 2)$ a statement of Ledoux [48] for $q = 2$.

Lemma 7.4. *Let $q \in (1, 2)$. Let $\mu = e^{-V(x)} \cdot dv(x)$ be a probability measure on M. Assume that there exists $K \in \mathbb{R}$ such that $\text{Hess}V + \text{Ric} \ge K\,\text{Id}$ and that μ satisfies a possibly defective q-log-Sobolev inequality. Then there exists $t_0 \in (0, \frac{1}{2}], \kappa > 0$ such that*

$$\mathcal{I}_\mu(t) \ge \kappa \min(t, 1-t) \log^{\frac{1}{q}}\left(\frac{1}{\min(t, 1-t)}\right), \quad t \in (0, t_0) \cup (1 - t_0, 1).$$

Proof. The **(qLSI)** is equivalent by a change of functions to the following defective **MSLI**

$$\text{Ent}_\mu(f^2) \le B \int f^2 \left|\frac{\nabla f}{f}\right|^q d\mu + D \int f^2 d\mu.$$

Since $2/q > 1$, Young's inequality yields for $t \ge 0, \eta > 0$ that $t^{q/2} \le \frac{q}{2}\eta t + \frac{2-q}{2}\eta^{\frac{-q}{2-q}}$. For $t = |\nabla f/f|^2$, $\eta = 2\varepsilon/q$ we get for some $m > 0$ and all $\varepsilon > 0$

$$\text{Ent}_\mu(f^2) \le \varepsilon \int |\nabla f|^2 d\mu + \left(D + m\varepsilon^{\frac{-q}{2-q}}\right) \int f^2 d\mu.$$

Set $\beta(\varepsilon) = D + m\varepsilon^{\frac{-q}{2-q}}$. By a celebrated theorem of Gross, any log-Sobolev inequality satisfied by μ implies continuity properties of the semigroup (P_t) generated by $L = \Delta - \nabla V \cdot \nabla$, see e.g. [43]. Denoting $\|f\|_p = \left(\int |f|^p d\mu\right)^{1/p}$, this theorem yields for all $\varepsilon, t > 0$, and all f,

$$(35) \qquad \|P_t f\|_2 \le \exp\left(\frac{\beta(\varepsilon)}{2} \cdot \frac{e^{4t/\varepsilon} - 1}{e^{4t/\varepsilon} + 1}\right) \|f\|_{1+e^{-4t/\varepsilon}}.$$

On the other hand a theorem of Ledoux [48] improved in [11] shows that under the condition $\text{Hess}V + \text{Ric} \ge K\,\text{Id}$, there exists a constant $C > 0$ such that for all $t \in (0, 1/|K|)$, and all Borel sets $A \subset M$,

$$\mu^+(\partial A) \ge \frac{C}{\sqrt{t}}\left(\mu(A) - \|P_t \mathbf{1}_A\|_2^2\right).$$

Combining this fact with (35) for $f = \mathbf{1}_A$ gives for $t < |K|^{-1}, \varepsilon > 0$

$$(36) \qquad \mu^+(\partial A) \ge C\frac{\mu(A)}{\sqrt{t}}\left[1 - \exp\left((\beta(\varepsilon) + \log(\mu(A)))\frac{e^{4t/\varepsilon} - 1}{e^{4t/\varepsilon} + 1}\right)\right].$$

It remains to make a good choice of ε, t. The idea is to fix ε so that $\beta(\varepsilon) \sim \frac{1}{2}\log(1/\mu(A))$ and then to choose t so that $t/\varepsilon \sim 1/\log(1/\mu(A))$, which is small if we consider sets of small measure. More precisely, we set

$$\varepsilon = \left(\frac{1}{2m}\log\frac{1}{\mu(A)}\right)^{\frac{q-2}{q}} \qquad \text{and} \qquad t = \left(\log\frac{1}{\mu(A)}\right)^{-\frac{2}{q}}.$$

When $\mu(A)$ is small enough this is compatible with the constraint $t < |K|^{-1}$. In particular this choice implies $\beta(\varepsilon) + \log\mu(A) = D - \frac{1}{2}\log\frac{1}{\mu(A)}$ and $t/\varepsilon = (2m)^{(q-2)/q}\left(\log\frac{1}{\mu(A)}\right)^{-1}$. Consequently, the quantity in brackets in (36) has a strictly positive limit when $\mu(A)$ tends to zero and there exists $C' > 0$ such that

$$\mu^+(\partial A) \geq C'\frac{\mu(A)}{\sqrt{t}} = C'\mu(A)\log^{\frac{1}{q}}\frac{1}{\mu(A)},$$

when $\mu(A)$ is small enough. The same argument gives a similar bound for large sets since $\mu(A) - \|P_t\mathbf{1}_A\|_2^2 = \mu(A^c) - \|P_t\mathbf{1}_{A^c}\|_2^2$. $\qquad\square$

We will need more detailed description of the optimal transport of measures on manifolds. See [35], [36] for details.

Let $T(x) = \exp(\nabla\theta(x))$ be the quadratic optimal transportation mapping pushing forward $g \cdot \mu$ to $f \cdot \mu$. Set: $T_t(x) = \exp(t\nabla\theta(x))$. The change of variables formula reads as

$$(37) \qquad\qquad\qquad\qquad g = f(T)J_1,$$

where

$$J_t = \det Y(t)\big(H(t) + t\mathrm{Hess}\theta\big),$$

$Y(t) = \mathrm{D}(\exp_x)_{t\nabla\theta}$, $H(t) = \frac{1}{2}\mathrm{Hess}_x\rho^2(x, T_t(x))$. Here $\mathrm{Hess}\theta$ is understood in the sence of Alexandrov due to a local semiconvexity of θ. There exist the following relations between the volume distortion coefficients $v_t(x, y)$ and $Y(t)$, $H(t)$ (see [35], [36] for the precise definition)

$$(38) \qquad\qquad\qquad\qquad v_t(x, T(x)) = \det Y(t)Y^{-1}(1),$$

$$(39) \qquad\qquad\qquad\qquad v_{1-t}(T(x), x) = \det\frac{Y(t)\big(H(t) - tH(1)\big)}{1 - t}.$$

If $\mathrm{Ric} \geq k(d-1)\mathrm{Id}$, where $k \leq 0$, the volume distortion coefficients can be estimated by the Bishop's comparison theorem

$$(40) \qquad\qquad\qquad\qquad v_t(x, y) \geq \left(\frac{S_k(t\rho(x, y))}{S_k(\rho(x, y))}\right)^{d-1},$$

where $S_k(t) = \frac{\sinh(k^{1/2}t)}{k^{1/2}t}$.

The following theorem is a generalization of Theorem 5.27 a) and Theorem 5.28 (see also Remark 5.30).

Theorem 7.5. *Let M satisfy* $\mathrm{Ric} \geq K\mathrm{Id}$, $K \leq 0$. *Assume that $\mu = e^{-V}dv$ is a probability measure with twice continuously differentiable potential such that one of the following assumptions is fulfilled*

a)
$$\exp(\delta\rho(x, x_0)^\alpha) \in L^1(\mu)$$

for some $\delta > 0$, $1 < \alpha \leq 2$,

$$V \leq N_1\big(-\Delta\phi + \langle\nabla V, \nabla\phi\rangle\big) + N_2,$$

where $N_1 > 0$, $\phi = \rho^2(x, x_0)$ for some $x_0 \in M$. In addition, assume that

$$\exp\big(\varepsilon|\nabla V|\log^{\frac{1}{\alpha}}|\nabla V|\big) \in L^1(\mu)$$

and $-V \leq g$ with $g \in L^1(\mu)$

b)

$$\exp(\delta\rho(x,x_0)^\alpha) \in L^1(\mu)$$

for some $\delta > 0$, $1 < \alpha \leq 2$, $x_0 \in M$ and V is bounded from below and for $s > 0$, $0 < t < 1$

$$s \max(1, V)^\tau \leq (1-t)|\nabla V|^2 - \Delta V + C,$$

where $\tau = \frac{2}{\beta}$.

Then μ satisfies inequality (I_τ), F_τ -inequality and modified Sobolev inequality with $c = c_\alpha$.

Proof. Exactly as in the previous theorem it is sufficient to prove the defective (I_τ)-inequality. In order not to repeat lengthy arguments, we prove only the case $\alpha = 2$ in b). The proofs of the case $\alpha \neq 2$ and the item a) can be obtained from the proofs of Theorem 5.28 and Theorem 5.27 respectively by the similar modifications.

Set: $r(x) = \rho(x_0, x)$. First we note that by a comparison theorem the volume of the ball $\{x : \rho(x, x_0) \leq r\}$ grows mostly exponential as a function of r. Hence $\exp(-tr^2)\,dv$ is a finite measure for every $t > 0$. Consider the quadratic transportation T of $f^2 \cdot \mu$ to μ, where f is smooth and compactly supported.

Applying (37) and integrating the logarithm of both sides with respect to μ, we get

$$\int_M f^2 \log f^2 d\mu = \int_M V f^2 d\mu - \int_M V d\mu + \int_M \log \det\Big(H(1) + \mathrm{Hess}\theta\Big) f^2 d\mu + \int_M \log \det Y(1) f^2 d\mu.$$

The term $\int_M V f^2 d\mu$ can be estimated as in the flat case (see Theorem 5.27 and Theorem 5.28). Further applying (38) with $t = 0$ and (40), we get that $\log \det Y(1)$ can be estimated by $C\rho(x, T(x))$ for big values of $\rho(x, T(x))$ and by $C\rho^2(x, T(x))$ for small values. Applying the triangle inequality

$$\rho(x, T(x)) \leq r + \rho(T(x), x_0),$$

the Young inequality and change of variables, it is easy to show that the term $\int_M \log \det Y(1) f^2 d\mu$ is dominated by $\varepsilon \mathrm{Ent}_\mu f^2$ for any small ε. Further, by the Jensen inequality

$$\int_M \log \det\Big(H(1) + \mathrm{Hess}\theta\Big) f^2 d\mu \leq d \log \int_M \mathrm{Tr}\Big(\frac{\frac{1}{2}\mathrm{Hess}_x \rho^2(x, T(x)) + \mathrm{Hess}\theta}{d}\Big) f^2 d\mu.$$

Next we note that

$$\frac{\mathrm{Tr}\ \mathrm{Hess}_x \rho^2(x, T(x))}{2} = |\nabla_x \rho(x, T(x))|^2 + \rho(x, T(x))\Delta_x \rho(x, T(x)).$$

By a comparison result $\Delta_x \rho(x, T(x))$ is dominated by $\Delta_H r(o, x)|_{r(o,x) = \rho(x, T(x))}$, where $r(o, x)$ is the distance in the model space with constant curvature K from some fixed pont o. This implies, in particular, that $\Delta_x \rho(x, T(x))$ is bounded for big values of $\rho(x, T(x))$. Since $\rho^2(x, x_0)$ is smooth in the neighborhood of x_0, hence

$$\frac{\mathrm{Tr}\ \mathrm{Hess}_x \rho^2(x, T(x))}{2} \leq C\big(1 + \rho(x, T(x))\big)$$

and

$$\int_M \frac{\mathrm{Tr}\ \mathrm{Hess}_x \rho^2(x, T(x))}{2} f^2 d\mu \leq C\Big(1 + \int_M \rho(x, T(x)) f^2 d\mu\Big).$$

In addition,

$$\int_M \frac{\Delta\theta}{d} f^2 d\mu \leq -\frac{2}{d}\int_M \langle\nabla\theta, \nabla f\rangle f d\mu$$

Here we estimate the Alexandrov's Laplacian by the distributional one from above. This is possible since θ is locally semi-convex and the singular part of its Laplacian is non-negative. Then we apply integration by parts. Using $|\nabla\theta| = \rho(x, T(x))$, we arrive at the following estimate

$$\Big|\int_M \langle\nabla\theta, \nabla f\rangle f d\mu\Big| \leq 2\int_M \rho^2(x, T(x)) f^2 d\mu + 2\int_M |\nabla f|^2 d\mu.$$

The rest proceeds in the standard way. This means that we apply the triangle inequality $\rho(x, T(x)) \leq r + \rho(T(x), x_0)$ and make the change of variables for $\rho(T(x), x_0)$. Then we estimate $\log x$ by $\varepsilon x + N(\varepsilon)$, apply the standard Young inequality and choose a sufficiently small small ε. This completes the proof. $\square$

Corollary 7.6. *Let M satisfy* $\mathrm{Ric} \geq K\mathrm{Id}$, $K \in \mathbb{R}$ *and*

$$V = \delta\rho^{\alpha}(x_0, x) + N$$

for some $x_0 \in M$, $1 < \alpha \leq 2$, $\delta > 0$, $N \in \mathbb{R}$. Then μ satisfies inequality (I_{τ}), F_{τ}-inequality and modified log-Sobolev inequality with $c = c_{\alpha}$, where $\tau = \frac{2}{\beta}$.

Proof. As we know from the previous proof

$$\Delta r^2 \leq C'(1 + r)$$

for some C' in points of differentiability of r^2. In addition, $|\nabla r| = 1$ almost everywhere. Function r^2 is differentiable outside of C_{x_0}, where C_{x_0} is the cut-locus of x_0. It is known that C_{x_0} has measure zero. Formally applying Theorem 7.5 b), we get the result. Nevertheless, since V is not smooth everywhere, the proof needs some justification. Analyzing the proof of Theorem 7.5, we see that the estimate

$$\int_M r^2 f^2 d\mu \leq C_1 + C_2 \int_M rf^2 d\mu + C_3 \int_M r|\nabla f|f d\mu$$

should be justified. This can be done by integration by parts formula with the help of Calabi lemma (see, for instance, [10]) : there exists an increasing sequence of precompact starshaped domains D_n with smooth boundaries which union is $M \setminus C_{x_0}$. In addition, r^2 is smooth in every D_n and $\langle \nabla r, \nu \rangle > 0$ on ∂D_n where ν is the normal outward vector field on ∂D_n. $\qquad\square$

Remark 7.7. Corollary 7.6 for the case of F-inequalities has been proved by Wang in [65] for $\alpha > 1$. It follows from his more general result obtained from a Nash-type inequality by perturbation techniques.

Finally, we show that the Euclidean logarithmic Sobolev inequality implies modified log-Sobolev inequalities for special types of measure on manifolds with the lower Ricci curvature bound.

Theorem 7.8. *Assume that $\mathrm{Ric} \geq K\mathrm{Id}$ and the Riemannian volume measure satisfies the logarithmic Sobolev inequality in the Euclidean form:*

$$\tag{41} \int_M f^2 \log\Big(\frac{f^2}{\int_M f^2 dv}\Big) dv \leq A \ln\Big(B \int_M |\nabla f|^2 dv\Big),$$

where A and B are positive constants. Let μ be a probability measure of the type

$$\mu = \frac{1}{Z_{\alpha,N}} \exp(-N\rho^{\alpha}(x, x_0)) dv,$$

where $1 < \alpha \leq 2$, $N > 0$ and x_0 is a fixed point. Then μ satisfies (I_{τ})-inequality with $\tau = 2\big(1 - \frac{1}{\alpha}\big)$.
In addition, μ satisfies the F_{τ} -inequality and modified Sobolev inequality with $c = c_{\alpha}$.

Proof. As above, applying the tightening techniques, it is sufficient to show (I_{τ})-inequality. Without loss of generality assume that $N = 1$. In addition, since inequalities of this type are stable under bounded perturbations, it is sufficient to prove the result for the measure $\nu = \frac{1}{A_{\alpha}}\exp\big(-p\big)dv$, where $p = \varphi(\rho(x, x_0))$ and $\varphi(t)$ is a twice continuously differentiable function which is equal to t^{α} for $t \geq 1$ and quadratic for small values of t. Let g be a smooth function such that $\int_M g^2 d\nu = 1$. Set:

$$f = \frac{1}{A_{\alpha}^{1/2}}\, g\, \exp\Big(-\frac{p}{2}\Big).$$

Obviously, $\int_M f^2 dv = 1$. Applying (41), one gets

$$\tag{42} \int_M g^2 \Big(\ln g^2 - p - \ln A_{\alpha}\Big) d\nu \leq A \ln\Big(B \int_M \Big[\nabla g - \frac{\nabla p}{2}g\Big]^2 d\nu\Big).$$

Hence

$$\int_M g^2 \ln g^2 d\nu \leq \int_M g^2 p\, d\nu + \ln A_{\alpha} + A \ln\Big(B \int_M \Big[\nabla g - \frac{\nabla p}{2}g\Big]^2 \Big(1 + \log^{1-\tau}(e + g^2)\Big) d\nu\Big).$$

Now we want to apply integrations by parts to the term

$$- \int_M \big\langle \nabla g, \nabla p \big\rangle g\Big(1 + \log^{1-\tau}(e + g^2)\Big) d\nu.$$

This can not be done directly, since the function p is differentiable only outside of cut locus of x_0. Nevertheless, proceeding as above with the Calabi lemma and taking into account that p is increasing function of the distance, we get that the right-hand side of (42) can be estimated by

$$A\ln\Big(B\int_M\Big(|\nabla g|^2-\frac{|\nabla p|^2}{4}g^2+\frac{\Delta p}{2}g^2\Big)\big(1+\log^{1-\tau}(e+g^2)\big)d\nu+B\int_M g^2\psi(g^2)g\langle\nabla g,\nabla p\rangle d\nu\Big)$$

where

$$\psi=\frac{d}{dx}\Big[1+\log^{1-\tau}(e+x)\Big].$$

Further we note that $x\psi(x)$ is bounded and since the Ricci curvature is bounded from below, one has

$$\Delta p\le C_1+C_2 r^{\alpha-1}$$

outside of cut locus, where $r=\rho(x,x_0)$. Since $|\nabla p|^2\sim(\alpha-1)^2 r^{2(\alpha-1)}$, $|\nabla p|^2$ dominates Δp for big values of r. Applying Cauchy inequality one easily gets that the right-hand side of (42) can be estimated by

$$A\ln\int_M\Big(N_1|\nabla g|^2-\frac{(\alpha-1)^2}{5}r^{2(\alpha-1)}g^2+N_2 g^2\Big)\big(1+\log^{1-\tau}(e+g^2)\big)d\nu$$

for sufficiently big N_1,N_2.

Applying estimate $\ln x\le Cx+D(C)$ for a sufficiently big C, one finally obtains

$$\int_M g^2\ln g^2\,d\nu\le\int_M g^2 p\,d\nu+\int_M\Big(C_1|\nabla g|^2-C_2\frac{(\alpha-1)^2}{5}r^{2(\alpha-1)}g^2+C_3 g^2\Big)\big(1+\log^{1-\tau}(e+g^2)\big)d\nu+C_4,$$

where C_2 can be chosen arbitrary big. It remains to note that $p\sim r^{\alpha-1}$ for big r, hence by the Young inequality $\int_M g^2 p\,d\nu$ can be estimated by

$$\int\Big[N(\varepsilon)g^2\big(1+\log^{1-\tau}(e+g^2)\big)+e^{\varepsilon r^{\frac{2-\alpha}{1-\tau}}}+1\Big]r^{2(\alpha-1)}d\nu.$$

Note that $\frac{2-\alpha}{1-\tau}=\alpha$, hence $e^{\varepsilon r^{\frac{2-\alpha}{1-\tau}}}r^{2(\alpha-1)}\in L^1(\nu)$. Finally, choosing a sufficiently big C_2, we make the term $\int_M g^2 p\,d\nu$ disappear. Since $\int_M g^2\big(1+\log^{1-\tau}(e+g^2)\big)d\nu$ is dominated by $\varepsilon\mathrm{Ent}_\nu g^2$ for any positive ε, we immediately get the desired estimate. $\qquad\square$

Remark 7.9. It is known that inequality (41) holds for the hyperbolic space H^d (see [18]). Thus, we get another proof of a partial case of Corollary 7.6.

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