

A NOTE ON THE DYNAMICAL ZETA FUNCTION OF GENERAL TORAL ENDOMORPHISMS

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ABSTRACT. It is well-known that the Artin-Mazur dynamical zeta function of a hyperbolic or quasi-hyperbolic toral automorphism is a rational function, which can be calculated in terms of the eigenvalues of the corresponding integer matrix. We give an elementary proof of this fact that extends to the case of general toral endomorphisms without change. The result is a closed formula that can be calculated by integer arithmetic only. We also address the functional equation and the relation between the Artin-Mazur and Lefschetz zeta functions.

1. INTRODUCTION

Any d -dimensional toral endomorphism is represented by an integer matrix, $M \in \text{Mat}(d, \mathbb{Z})$, with action mod 1 on the d -torus $\mathbb{T}^d \simeq \mathbb{R}^d / \mathbb{Z}^d$; see [2] and [15, Ch. 1.8] for background and [1, Ex. 1.16] for an illustration. Important aspects of the dynamical system (\mathbb{T}^d, M) are related to its periodic orbits and their distribution over \mathbb{T}^d ; compare [11, 24]. The Artin-Mazur [3] dynamical zeta function provides a generating function for the orbit counts that is interesting both from an arithmetic and from a topological point of view [13, 20]. The latter was also Smale's approach [22], who related the Artin-Mazur and Lefschetz zeta functions of a *hyperbolic* toral automorphism and calculated both in terms of eigenvalues.

In this note, we explain a different approach via elementary geometry and linear algebra, which bypasses more advanced topological methods as well as the need to calculate eigenvalues. A key observation is that all arguments apply to general toral endomorphisms without additional effort. We also treat the connection between the Artin-Mazur and the Lefschetz zeta function and their functional equations. As we have learned along the way, most arguments we use appear already in the literature, notably in [13], but at least their combination seems to be new. Also, we make several steps explicit to facilitate their computational use.

For $M \in \text{Mat}(d, \mathbb{Z})$ and $m \geq 1$, let a_m be the number of *isolated* fixed points in \mathbb{T}^d of the m -th iterate M^m . The starting point of our considerations is the identity

$$(1) \quad a_m = |\det(\mathbb{1} - M^m)|.$$

This formula is well-known [24, 4] when no eigenvalue of M^m is 1; it then follows from counting the number of points of \mathbb{Z}^d in a fundamental domain of the lattice $(\mathbb{1} - M^m)\mathbb{Z}^d$. Otherwise, (1) is true because both sides are zero. Indeed, since the fixed points of M^m form a closed subgroup of \mathbb{T}^d , they are either all isolated, or they form entire subtori of positive dimension; see [4, Appendix] for a detailed discussion of the subtorus case. Incidentally, when no eigenvalue of M^m is 1, a_m is also the Reidemeister number of a toral endomorphism, see [13, Thm. 22 and p. 33], while, in general, a_m is its Nielsen number.

Following [3], the Artin-Mazur zeta function of a general $M \in \text{Mat}(d, \mathbb{Z})$ is defined as

$$(2) \quad \zeta_M(z) := \exp\left(\sum_{m=1}^{\infty} \frac{a_m}{m} z^m\right) = \prod_{m=1}^{\infty} (1 - z^m)^{-c_m}.$$

Here, the exponents c_m of the Euler product representation are well-defined integers, see Proposition 2 below. An explicit representation of $\zeta_M(z)$ as a rational function is given below in Theorem 1. We stress that at least for hyperbolic or quasihyperbolic toral endomorphisms, this result is well-known by [22] or [13]; our focus is the elementary method.

A matrix M is called *hyperbolic* when it has no eigenvalue on the unit circle \mathbb{S}^1 . Such toral automorphisms are expansive [24, p. 143]. Note that M may possess eigenvalues on \mathbb{S}^1 other than roots of unity (for instance, if one eigenvalue of M is a Salem number; see [23, 5] for examples). Integer matrices without roots of unity in their spectrum constitute the *quasihyperbolic* cases, compare [23] and references therein, where formula (1) still counts all fixed points. For quasihyperbolic matrices M , the exponents c_m are the cycle numbers, which are related to the fixed point counts via

$$(3) \quad a_m = \sum_{\ell|m} \ell c_\ell \quad \text{and} \quad c_m = \frac{1}{m} \sum_{\ell|m} \mu\left(\frac{m}{\ell}\right) a_\ell.$$

This follows from a standard application of Möbius inversion; compare [19, 6].

When roots of unity are among the eigenvalues of M , the Euler product still exists, with the same relation between the counts of (isolated) fixed points and the exponents c_m , though the latter can now be negative. Let us briefly illustrate this phenomenon in one dimension. Endomorphisms of $\mathbb{T}^1 \simeq \mathbb{S}^1$ are represented by multiplication (mod 1) with an integer n . The dynamical zeta function reads $\zeta_0(z) = 1/(1 - z)$ and

$$(4) \quad \zeta_n(z) = \frac{1 - \text{sgn}(n)z}{1 - |n|z}$$

for $n \neq 0$, due to our Theorem 1 below (or a simple direct calculation). For $n = -1$, we get $\zeta_{-1}(z) = (1 - z^2)/(1 - z)^2$, thus $c_1 = 2$ and $c_2 = -1$, while $c_m = 0$ for all $m \geq 3$. The negative c_2 corresponds to the fact that the two isolated fixed points of the map fail to be *isolated* for any even iterate, which is the identity.

Finally, let us note that our arguments extend to the case of nilmanifolds $X = G/\Gamma$ considered in [13, Sec. 2.6], where G is a simply connected nilpotent Lie group and Γ a discrete subgroup such that X is compact. Namely, any endomorphism φ of Γ extends to an endomorphism $\tilde{\varphi}$ of X , the isolated fixed points of which are counted by $|\det(\mathbb{1} - \text{Lie}(\tilde{\varphi}))|$, analogously to (1).

2. A RELATED ZETA FUNCTION

Let us start with the numbers $\tilde{a}_m := \det(\mathbb{1} - M^m)$, which can be viewed as signed fixed point counts, and the corresponding zeta function

$$(5) \quad \tilde{\zeta}_M(z) = \exp\left(\sum_{m=1}^{\infty} \frac{\tilde{a}_m}{m} z^m\right).$$

In Section 4, we will see that this is actually a Lefschetz zeta function, see Eq. (16) below.

For $A \in \text{Mat}(d, \mathbb{R})$, let $\Lambda^k(A)$ be the induced linear map on the exterior power $\Lambda^k(\mathbb{R}^d)$. In terms of the standard basis of that space, $\Lambda^k(A)$ is represented by the matrix of all minors of A of order k ; see [14, Ch. 1.4] for details. This is an integer matrix of dimension $\binom{d}{k}$, with $\Lambda^0(A) = 1$, $\Lambda^1(A) = A$ and $\Lambda^d(A) = \det(A)$.

Proposition 1. *For $M \in \text{Mat}(d, \mathbb{Z})$, we have $\tilde{\zeta}_M(z) = \prod_{k=0}^d \det(\mathbb{1} - z\Lambda^k(M))^{(-1)^{k+1}}$.*

Since all $\Lambda^k(M)$ are integer matrices, $\tilde{\zeta}_M(z)$ is a rational function with numerator and denominator in $\mathbb{Z}[z]$. It can be calculated by integer arithmetic alone (many algebraic program packages have the matrices of minors of arbitrary order k as built-in functions). Also, since 0 is never a root of the denominator, the series (5) for $\tilde{\zeta}_M$ converges uniformly on sufficiently small disks around 0.

Proof of Proposition 1. This is analogous to [13, Lemma 17]: The assertion is immediate from the well-known formula in linear algebra

$$(6) \quad \det(\mathbb{1} - A) = \sum_{k=0}^d (-1)^k \text{tr}(\Lambda^k(A))$$

together with the power series identity

$$(7) \quad \exp\left(\sum_{m=1}^{\infty} \frac{\text{tr}(A^m)}{m} z^m\right) = \frac{1}{\det(\mathbb{1} - zA)},$$

which is omnipresent in connection with zeta functions of any kind (in particular, it appears in the calculation of dynamical zeta functions of shifts of finite type, see [7, 20]). We recall that (6) is proved by evaluating the characteristic polynomial of M at 1, while (7) is a simple consequence of the relation $\det(\exp(C)) = \exp(\text{tr}(C))$ for square matrices C , together with the Taylor series for $-\log(1 - z)$, which is the case $d = 1$ of (7). \square

Let us also note that, in terms of the d eigenvalues $\lambda_1, \dots, \lambda_d$ of M , one has the relation $\det(\mathbb{1} - z\Lambda^k(M)) = P_k(z)$ with the polynomials $P_0(z) = 1 - z$ and

$$P_k(z) = \prod_{1 \leq \ell_1 < \ell_2 < \dots < \ell_k \leq d} (1 - z\lambda_{\ell_1} \dots \lambda_{\ell_k})$$

for $1 \leq k \leq d$. This version is useful for the derivation of the functional equation of $\tilde{\zeta}_M$.

Lemma 1. *If $M \in \text{Mat}(d, \mathbb{Z})$ with $D := \det(M) \neq 0$, one has $\tilde{\zeta}_M(1/Dz) = B(\tilde{\zeta}_M(z))^{(-1)^d}$, where $B = D$ for $d = 1$ and $B = 1$ otherwise.*

Proof. First, a direct calculation shows that

$$P_k\left(\frac{1}{Dz}\right) = \frac{1}{\beta_{d-k}} \left(\frac{-1}{z}\right)^{\binom{d}{k}} P_{d-k}(z),$$

where $\beta_0 = 1$ and $\beta_k = D^{\binom{d-1}{k-1}}$ for $1 \leq k \leq d$. Note that each prefactor β_k involves products of eigenvalues, but is symmetric in them and thus simplifies to a power of the determinant.

Next, recall the binomial formula $\sum_{\ell=0}^n \binom{n}{\ell} (-1)^\ell = \delta_{n,0}$ for $n \geq 0$, and insert the previous polynomial identities into the product expression of Proposition 1. Our claim follows, because

the prefactor that contains z disappears by an application of the binomial formula, while the prefactor with the determinants simplifies to the factor B by an analogous calculation; compare [13, Lemma 19] and its proof for a related argument. \square

The special situation for $d = 1$ is also immediate from $\tilde{\zeta}_n(z) = (1 - nz)/(1 - z)$, as the determinant is n ; compare the example in the introduction.

3. THE ARTIN-MAZUR ZETA FUNCTION

To derive a formula for the dynamical zeta function, we observe that $a_m = \tilde{a}_m \operatorname{sgn}(\tilde{a}_m)$. Hence the signs of all nonzero \tilde{a}_m need to be determined. When M is quasihyperbolic, this is done in [23, Lemma 2.1], see also the proof of [13, Lemma 15], but the argument works for general $M \in \operatorname{Mat}(d, \mathbb{Z})$, too: We employ the formula

$$(8) \quad \tilde{a}_m = \det(\mathbb{1} - M^m) = \prod_{j=1}^d (1 - \lambda_j^m)$$

with the λ_j as above. It is clear that neither complex eigenvalues play a role (as they come in complex conjugate pairs, and $(1 - \lambda^m)(1 - \bar{\lambda}^m) = |1 - \lambda^m|^2 \geq 0$), nor do eigenvalues $\lambda \in [-1, 1]$ (because then $1 - \lambda^m \geq 0$). The remaining eigenvalues (evs) matter, and one finds

$$(9) \quad a_m = \tilde{a}_m \left((-1)^{\#\text{ real evs } < -1} \right)^m \left((-1)^{\#\text{ real evs outside } [-1, 1]} \right) =: \tilde{a}_m \delta^m \varepsilon.$$

Inserting this into $\zeta_M(z)$ and comparing with $\tilde{\zeta}_M(z)$ gives

$$(10) \quad \zeta_M(z) = (\tilde{\zeta}_M(\delta z))^\varepsilon$$

for the dynamical zeta function of M . Though (9) involves the eigenvalues of M , the signs δ and ε can once again be obtained by integer arithmetic alone. When no eigenvalue of M is ± 1 , they are simply given by

$$(11) \quad \delta = \operatorname{sgn}(\det(\mathbb{1} + M)) \quad \text{and} \quad \varepsilon = \delta \operatorname{sgn}(\det(\mathbb{1} - M)).$$

In general, the signs can be defined by the one-sided limits

$$\delta = \lim_{\alpha \searrow 0} \operatorname{sgn}(\det((1 + \alpha)\mathbb{1} + M)) \quad \text{and} \quad \varepsilon = \delta \lim_{\alpha \searrow 0} \operatorname{sgn}(\det((1 + \alpha)\mathbb{1} - M)),$$

which can be evaluated explicitly as follows. Factorise $\det(x\mathbb{1} - M) = (x - 1)^\sigma (x + 1)^\tau Q(x)$ with $Q \in \mathbb{Z}[x]$ and $Q(\pm 1) \neq 0$, where the non-negative integers σ, τ are unique. This implies $\det(x\mathbb{1} + M) = (x - 1)^\tau R(x)$ with $R \in \mathbb{Z}[x]$ and $R(1) \neq 0$. Consequently, one has

$$(12) \quad \delta = \operatorname{sgn} \left(\frac{\det(x\mathbb{1} + M)}{(x - 1)^\tau} \Big|_{x=1} \right) \quad \text{and} \quad \varepsilon = \delta \operatorname{sgn} \left(\frac{\det(x\mathbb{1} - M)}{(x - 1)^\sigma} \Big|_{x=1} \right),$$

which is used in our sample program in the appendix.

Let us summarise the result of our derivation so far.

Theorem 1. *Consider a general toral endomorphism, represented by a matrix $M \in \operatorname{Mat}(d, \mathbb{Z})$. The associated Artin-Mazur zeta function, defined in terms of isolated fixed points, satisfies*

$$\zeta_M(z) = \prod_{k=0}^d \det(\mathbb{1} - \delta z \wedge^k(M))^{\varepsilon(-1)^{k+1}},$$

where the signs δ and ε are given by Eq. (11) when ± 1 is not an eigenvalue of M , and by Eq. (12) in general. In particular, $\zeta_M(z)$ is a rational function. When no eigenvalue of M is a root of unity, all fixed points are covered this way. \square

For quasihyperbolic toral endomorphisms, this result follows from an analogous result for nilmanifolds [13, Thm. 45], which is proved by Reidemeister-Nielsen fixed point theory, while the case of hyperbolic toral automorphisms is already treated in [22, Prop. 4.5] in a slightly different formulation. Theorem 1 covers the special cases of automorphisms for $d = 2$ from [15, 11, 6]. Let us note that, in the hyperbolic case, the rationality of ζ_M can also be seen as a consequence of the general rationality result [17] proved by Markov partitions. Of related interest is the approach of [18], which connects the problem to an interesting class of \mathbb{Z}^d -actions.

Since δ and ε are signs, in particular $\delta = 1/\delta$, the functional equation for ζ_M is now immediate from Lemma 1 and Theorem 1.

Corollary 1. *When $D := \det(M) \neq 0$, one has $\zeta_M(1/Dz) = B^\varepsilon (\zeta_M(z))^{(-1)^d}$ with ε from Eq. (12), where $B = D$ for $d = 1$ and $B = 1$ otherwise. \square*

Let us also mention that

$$(13) \quad \sum_{m=1}^{\infty} a_m z^m = \frac{z \zeta'_M(z)}{\zeta_M(z)}$$

is the ordinary power series generating function of the sequence $(a_m)_{m \in \mathbb{N}}$, which is still a rational function. Its radius of convergence ϱ_M is always positive (it is the absolute value of the smallest root of the denominator of (13) in reduced form). Thus, when $\lim_{m \rightarrow \infty} \frac{a_{m+1}}{a_m}$ exists, $1/\varrho_M$ is the asymptotic growth rate of the fixed point counts, which provides a simple alternative to the approach in [23]. The limit exists precisely for hyperbolic (and hence expansive) endomorphisms, as follows from [10, Thm. 6.3]. The ratio as a growth measure is also employed in [16, Thm. 16], where the case of unimodular roots is briefly discussed, too.

Finally, we observe that the Artin-Mazur zeta function of a general toral endomorphism can be written as an Euler product.

Proposition 2. *For any matrix $M \in \text{Mat}(d, \mathbb{Z})$, the associated Artin-Mazur zeta function $\zeta_M(z)$ has an Euler product representation (2) with uniquely determined integers c_n .*

Proof. The Euler product representation (2) is equivalent to the relations (3). These define rational numbers c_n which we must show to be integers. When M is quasihyperbolic, this is true by their geometric interpretation as cycle numbers. The general case (including $d = 1$, which also follows from (4)) can be proved by a deformation argument as follows.

Fix n and recall that c_n is linear in the a_k , while $a_k = \delta^k \varepsilon \det(\mathbb{1} - M^k)$ from (9) with the signs δ and ε . Hence, c_n is a polynomial with rational coefficients in the entries of M , which can be written as $c_n = \varepsilon P(M)$. Here, P itself depends on δ , but neither on ε nor on M . For another matrix M' , let δ' , ε' and c'_n denote the associated signs and numbers. When $\delta' = \delta$, we thus have $c'_n = \varepsilon' P(M')$.

Let $\nu > 1$ be a common denominator of all coefficients of P and define the diagonal $d \times d$ -matrix $N = \text{diag}(2\delta, 3, 4, \dots)$, which shares the sign δ with M . Consider now $M' = M + \nu^r N =$

$\nu^r(N + \nu^{-r}M)$. Since N has distinct real eigenvalues, the eigenvalues of $N + \nu^{-r}M$, for sufficiently large r , are real and close to those of N . Then, M' is hyperbolic with $\delta' = \delta$ by construction; in particular, $c'_n = \varepsilon'P(M')$ is integral. Since the difference $P(M) - P(M')$ is integral as soon as $r \geq 1$, it follows that $c_n = \varepsilon P(M)$ is integral, too. \square

4. INTERPRETATION AS LEFSCHETZ ZETA FUNCTION

Suppose that X is a compact differentiable manifold, assumed orientable for simplicity, and $f : X \rightarrow X$ is some differentiable map. In this situation (and also more generally), there is a fixed point index $I_f \in \mathbb{Z}$ which satisfies the Lefschetz trace formula

$$(14) \quad I_f = \sum_k (-1)^k \operatorname{tr}(f_* | H_k(X; \mathbb{Q})),$$

see [12, Prop. VII.6.6] or [8, Thm. 12.9]. Here, $H_k(X; \mathbb{Q})$ denotes singular homology with coefficients in \mathbb{Q} . It is a finite-dimensional \mathbb{Q} -vector space in our situation, on which f acts by functoriality.

When all fixed points of f are isolated, we have $I_f = \sum_{x \in \operatorname{Fix}(f)} i_f(x)$, where $i_f(x) \in \mathbb{Z}$ is the local index of f at x . If x is a regular fixed point, meaning that 1 is not an eigenvalue of the tangential map $T_x(f)$, the local index is given by

$$i_f(x) = \operatorname{sgn}(\det(\mathbb{1} - T_x(f))) \in \{\pm 1\}.$$

The Lefschetz zeta function associated to f can be defined as

$$\zeta_f^L(z) = \exp\left(\sum_{n \geq 1} \frac{z^n}{n} I_{f^n}\right).$$

This definition seems to appear first in [22]; see also [13]. By using the identity (7), the trace formula (14) applied to all iterates of f implies that $\zeta_f^L(z)$ is a rational function,

$$(15) \quad \zeta_f^L(z) = \prod_k \det(\mathbb{1} - z f_* | H_k(X; \mathbb{Q}))^{(-1)^{k+1}}.$$

Let us now assume that $X = \mathbb{T}^d = \mathbb{R}^d / \mathbb{Z}^d$ as above, and that f is given by an arbitrary $M \in \operatorname{Mat}(d, \mathbb{Z})$. In this case, with the zeta function $\tilde{\zeta}_M$ of Proposition 1, we have

$$(16) \quad \tilde{\zeta}_M(z) = \zeta_f^L(z),$$

including a correspondence of all related formulas (for the case of hyperbolic toral automorphisms, this was noted in [13, p. 86, lines 7 and 16]). Let us sketch a possible line of argument. First, it is well-known that the Künneth formula [8, Thm. 3.2] gives an isomorphism

$$(17) \quad H_k(X; \mathbb{Q}) \cong \wedge^k(\mathbb{Q}^d)$$

such that the action of f on H_k corresponds to $\wedge^k(M)$. This identifies the right hand sides of (6) and (14), and similarly for Proposition 1 and Eq. (15). It follows that the left hand sides of the corresponding pairs of equations are equal as well, that is

$$(18) \quad I_f = \det(\mathbb{1} - M),$$

and similarly $I_{f^n} = \det(\mathbb{1} - M^n)$; this appears also in [22, Prop. 4.15] and in [9].

A direct proof of (18) without using the Lefschetz trace formula can be done as follows. Assume first that 1 is not an eigenvalue of M . All fixed points x of f are then regular, with the same local index $i_f(x) = \text{sgn}(\det(\mathbb{1} - M))$, because the tangent space $T_x(X)$ can be identified with \mathbb{R}^d , where the action of f is given by M . Thus (18) is immediate. When no eigenvalue of M is a root of unity, the same applies to all iterates of f .

For arbitrary M , we may use the following lemma.

Lemma 2. *Assume X to be a compact Lie group of dimension d and $f : X \rightarrow X$ a differentiable map. Let $g(x) = x \cdot f(x)^{-1}$. Then, g acts on the 1-dimensional \mathbb{Q} -vector space $H_d(X; \mathbb{Q})$ by the scalar I_f .*

Granting the lemma, (18) follows easily because, in the torus case, g is given by $\mathbb{1} - M$, and we have $H_d(X; \mathbb{Q}) \cong \wedge^d(\mathbb{Q}^d)$. We only sketch the proof of Lemma 2 and leave the details to the reader. The fixed point index can be defined as the homology intersection product in $X \times X$ of the graph of f and the diagonal, $I_f = [\Gamma_X] \cdot [\Delta]$. Since the automorphism of $X \times X$ given by $(x, y) \mapsto (x, x \cdot y^{-1})$ acts on the orientation by $(-1)^d$, we get $I_f = (-1)^d [\Gamma_g] \cdot [X \times \{1\}]$. The assertion follows by a straightforward computation based on decomposing $[\Gamma_g]$ according to the Künneth formula for $X \times X$.

APPENDIX: A SAMPLE PROGRAM FOR CALCULATING ζ_M

One can implement the explicit zeta function formulas of Proposition 1 and Theorem 1 in a simple MATHEMATICA[®] program as follows.

```
Clear[tilzeta,zeta,ord,sig,tau,del,eps,dim,one];
tilzeta[mat_] := (dim = Length[mat]; Factor[Product[Det[
  IdentityMatrix[Binomial[dim,k]] - z Minors[mat,k]]^((-1)^(k+1)), {k,0,dim}]]);
ord[pol_, x_] := (tmp = pol; i = 0; While[(tmp /. z -> x) == 0, (i ++; tmp = D[tmp,z]); i];
zeta[mat_] := (one = IdentityMatrix[Length[mat]]; pol = Det[z one - mat];
sig = ord[pol, 1]; tau = ord[pol, - 1];
del = Sign[Factor[Det[z one + mat]/(z-1)^tau] /. z -> 1];
eps = del Sign[Factor[pol/(z-1)^sig] /. z -> 1];
Factor[(tilzeta[mat] /. z -> del z)^eps]);
```

The input is an integer matrix, in the standard format of a double list. The calculation is exact and reasonably fast for small dimensions, and can be used up to dimension 8 or 10 say.

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