CHARACTERIZATION PROBLEMS FOR LINEAR FORMS WITH FREE SUMMANDS.

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Abstract. Let $T_1,\dots,T_n$ denote free random variables. For two linear forms $L_1=\sum_{j=1}^n a_jT_j$ and $L_2=\sum_{j=1}^n b_jT_j$ with real coefficients a_j and b_j we shall describe all distributions of $T_1,\dots,T_n$ such that L_1 and L_2 are free. For identically distributed free random variables $T_1,\dots,T_n$ with a distribution μ we establish necessary and sufficient conditions on the coefficients a_j,b_j, $j=1,\dots,n$, such that the statements (i) μ is the centered semicircular distribution; and (ii) L_1 and L_2 are identically distributed ($L_1\overset{D}{=}L_2$); are equivalent.

1. Introduction

The intensive research in the asymptotic theory of random matrices has motivated increased research on infinitely dimensional limiting models. Free convolution of probability measures (p-measures), introduced by D. Voiculescu, may be regarded as such a model [29], [30]. The key concept of this definition is the notion of freeness, which can be interpretated as a kind of independence for noncommutative random variables. As in the classical probability the concept of independence gives rise to the classical convolution, the concept of freeness leads to a binary operation on the probability measures on the real line, the free convolution. Many classical results in the theory of addition of independent random variables have their counterpart in this theory, such as the law of large numbers, the central limit theorem, the Lévy-Khintchine formula and others. We refer to Voiculescu, Dykema and Nica [31], Hiai and Petz [11], and Nica and Speicher [25] for an introduction to these topics.

In many problems of mathematical statistis, conclusions are based on the fact that certain special distributions have important properties which permit the reduction of

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the original problem to a substantially simpler one, for instance via the notion of sufficiency.

The simplest type of statistics of independent observations, admitting a fairly complete description of the mutual independence and identical distribution, are linear statistics.

Consider independent scalar random variables $X_1, \dots, X_n$ (not necessary identically distributed) and two linear statistics

$$L_1 := \sum_{j=1}^{n} \alpha_j X_j \quad \text{and} \quad L_2 := \sum_{j=1}^{n} \beta_j X_j, \tag{1.1}$$

where α_j, β_j are real constant coefficients. It turns out that the independence of the two linear statistics L_1 and L_2 essentially characterizes the normality of the variables X_j. To be precise, the following assertion, due to Darmois [10] and Skitovich [27], [28], holds.

Let L_1 and L_2 given by (1.1) be independent. Then the random variables X_j such that $\alpha_j \beta_j \neq 0$, i.e., which enter in both L_1 and L_2, have normal distributions.

Note that the converse proposition holds in the following form: if $\sum_{j=1}^{n} \alpha_j \beta_j Var(X_j) = 0$ and all X_j such that $\alpha_j \beta_j \neq 0$ are normal, then L_1 and L_2 are independent.

Polya [26] was the first who established that only the normal distribution leads to identically distributed linear statistics X_1 and $a_1 X_1 + a_2 X_2$, where X_1 and X_2 are independent and identically distributed. Marcinkiewicz [23] proved that distributions having moments of all orders and admitting the existence of a nontrivial pair of identically distributed linear statistics based on a random sample are normal. Yu. V. Linnik [19], [20] described the class of symmetric distributions admitting identically distributed linear statistics and studied in detail the problem of characterising the normal distribution via properties of such statistics.

In this paper we give a complete discription of those free random variables $T_1, \dots, T_n$ such that the linear statistics $a_1 T_1 + \dots a_n T_n$ and $b_1 T_1 + \dots b_n T_n$ are free.

In addition we prove an analogue of Yu. V. Linnik's results [19], [20], [13], and give the solution of the problem of characterization of the semicircular distribution via identical distribution of linear statistics $a_1 T_1 + \dots a_n T_n$ and $b_1 T_1 + \dots b_n T_n$, where $T_1, \dots, T_n$ are identically distributed free random variables.

2. RESULTS

Assume that A is a finite von Neuman algebra with normal faithful trace state τ. The paire (A, τ) will be called a *tracial W^*-probability space.* Assume that A is acting on a Hilbert space H. We will denote by $\tilde{A}$ the set of all operators on H which are affiliated with A and by $\tilde{A}_{sa}$ the set of selfadjoint operators. Recall that a (generally unbounded) selfadjoint operator X on H is affiliated with A if all the spectral projections of X belong to A. The elements of $\tilde{A}_{sa}$ will be regarded as (possibly) unbounded random variables.

Let $T \in \tilde{A}_{sa}$. The distribution μ_T of T is the unique p-measure on $\mathbb{R}$ satifying the equality

$$\tau(u(T)) = \int\limits_{\mathbb{R}} u(\lambda)\,\mu_T(d\lambda)$$

for every bounded Borel function u on $\mathbb{R}$.

Recall that a family $\{T_j\}_{j=1}^k$ of elements of $T \in \tilde{A}_{sa}$ is said to be free if for all bounded continuous functions $u_1, u_2, \dots, u_n$ on $\mathbb{R}$ we have $\tau(u_1(T_{j_1})u_2(T_{j_2})\dots u_n(T_{j_n})) = 0$ whenever $\tau(u_l(T_{j_l})) = 0$, $l = 1, \dots, n$, and all alternating sequences $j_1, j_2, \dots, j_n$ of 1's, 2's, and k's, i.e., $j_1 \ne j_2 \ne \dots \ne j_n$.

Bercovici and Voiculescu [4] proved that if $T_j \in \tilde{A}_{sa}$ are free random variables for $j = 1, \dots, n$, and Q is a selfadjoint polynomial in n noncommuting variables, then the distribution of the random variable $Q(T_1, T_2, \dots, T_n)$ depends only on the distributions of $T_1, T_2, \dots, T_n$.

If $Q(T_1, \dots, T_n) = T_1 + \dots + T_n$, then the distribution of $T_1 + \dots + T_n$ only depends on the μ_{T_j} and is called the *additive free convolution* of $\mu_{T_1}, \dots, \mu_{T_n}$. Denote this distribution by $\mu_{T_1} \boxplus \dots \boxplus \mu_{T_n}$.

Let $T_1, \dots, T_n$ denote free random variables with distributions $\mu_1, \dots, \mu_n$, respectively. Consider two linear statistics

$$L_1 := a_1T_1 + \dots + a_nT_n \quad \text{and} \quad L_2 := b_1T_1 + \dots + b_nT_n \tag{2.1}$$

with real coefficients a_j and b_j. In the sequel we assume without loss of generality that $|a_j| \le 1$ and $|b_j| \le 1$ for all $j = 1, \dots, n$.

Denote by μ_w the *standard* semicircular measure, i.e., the measure with the density $p_w(x) = \frac{1}{2\pi}\sqrt{(4-x^2)_+}$, where $a_+ = \max\{0, a\}$. We shall call measures with densities $p(x) = \frac{1}{2\pi a^2}\sqrt{(4a^2-x^2)_+}$ for some $a > 0$ semicircular.

Nica [24] established that the stability of freeness under rotations characterizes semicircular random variables. Lehner [17] proved that there are free random variables T_1, T_2, T_3 which are not semicircular and such that $L_1 := a_1T_1 + a_2T_2 + a_3T_3$ and $L_2 := b_1T_1 + b_2T_2 + b_3T_3$ are free. Hence the analogue of the Darmois–Skitovich theorem fails in the free case if there are at least three random variables involved. We can nevertheless describe all free random variables $T_1, \dots, T_n$ for which the linear statistics L_1 and L_2 in (2.1) are free. Our result extends the results of Nica and Lehner considerably. In order to formulate our first result we need the following notation.

Let $T \in \tilde{A}_{as}$ be a given random variable with a distribution μ such that $\beta_n(\mu) := \int_{\mathbb{R}} |x|^n\,\mu(dx) < \infty$ for some $n \in \mathbb{N}$ and let $T^{(k)}$, $k = 1, \dots, n$, be its free copies. Let ω be n-th primitive root of unity (e.g., $\omega = e^{2\pi i/n}$) and set

$$T^{\omega} = \omega T^{(1)} + \omega^2 T^{(2)} + \dots + \omega^n T^{(n)}. \tag{2.2}$$

Following Lehner [17], we define the nth free cumulant of the random variable T to be

$$\kappa_n(T) = \frac{1}{n}\tau((T^{\omega})^n) \tag{2.3}$$

in a short way. About the definition of free cumulants see [25] as well. Since the free cumulant depends on n and the distribution μ of T only, we will denote this cumulant by $\kappa_n(\mu)$ as well.

Theorem 2.1. *Consider free random variables* $T_1, \dots, T_n$, $n \geq 2$, *and let* a_j, b_j *be real numbers such that* $a_j b_j \neq 0$ *and* $\frac{b_j}{a_j} \neq \frac{b_s}{a_s}$ *for* $j, s = 1, \dots, m$, *where* $m \leq n$, *and* $a_j b_j = 0$ *for* $j = m+1, \dots, n$. *The linear statistics* L_1 *and* L_2 *are free if and only if the distributions* $\mu_1, \dots, \mu_m$ *have compact supports and the free cumulants* $\kappa_s(T_j)$, $j = 1, \dots, m$, *satisfy the relations:*

$$\sum_{j=1}^{m} a_j b_j \kappa_2(T_j) = 0, \quad \sum_{j=1}^{m} a_j^2 b_j \kappa_3(T_j) = 0, \quad \sum_{j=1}^{m} a_j b_j^2 \kappa_3(T_j) = 0, \quad \dots,$$

$$\sum_{j=1}^{m} a_j^{m-1} b_j \kappa_m(T_j) = 0, \quad \sum_{j=1}^{m} a_j^{m-2} b_j^2 \kappa_m(T_j) = 0, \quad \sum_{j=1}^{m} a_j b_j^{m-1} \kappa_m(T_j) = 0, \tag{2.4}$$

and $\kappa_s(T_j) = 0$ *for* $s \geq m+1$.

The following result describe all distributions $\mu_1, \dots, \mu_m$ in the previous theorem.

Theorem 2.2. *A sequence* $\{\kappa_n\}_{n=1}^{\infty}$ *of real numbers such that* $\kappa_n = 0$ *for* $n \geq m+1$, $m \geq 2$, *is a sequence of free cumulants of some p-measure with compact support if and only if for every* $\theta \in [0, \pi]$ *there exists* $r_0 = r_0(\theta) > 0$ *such that*

$$r_0^m - \kappa_2 r_0^{m-2} - \kappa_3 r_0^{m-3} \frac{\sin(2\theta)}{\sin\theta} - \dots - \kappa_m \frac{\sin((m-1)\theta)}{\sin\theta} = 0. \tag{2.5}$$

Here define $\sin(k\theta)/\sin\theta := k$, $k = 1, 2, \dots$, in (2.5) for $\theta = 0, \pi$.

Corollary 2.3. *A sequence* $0, 1, \kappa_3, \kappa_4, 0, \dots$ *is a free cumulant sequence of some p-measure if and only if* $(\kappa_3, \kappa_4) \in D$, *where*

$$D := \Big\{(x,y) \in \mathbb{R}^2 : |x| \leq f_1(y),\ -\frac{1}{12} \leq y \leq 0\Big\} \cup \Big\{(x,y) \in \mathbb{R}^2 : |x| \leq f_2(y),\ 0 < y \leq \frac{1}{4}\Big\}.$$

where

$$f_1(y) := \frac{1}{\sqrt{2}} \sqrt{\frac{1}{3} + \sqrt{\frac{1}{9} - 4y}} \Big(\frac{2}{3} - \sqrt{\frac{1}{9} - 4y}\Big),$$

$$f_2(y) := \sqrt{\frac{2}{3} - \sqrt{\frac{1}{9} + \frac{4}{3}y}} \Big(\frac{1}{3} + \sqrt{\frac{1}{9} + \frac{4}{3}y}\Big).$$

Note that the set D is not closed. Moreovere the closure of D is not a convex set.

We see from Corollary 2.3 that a sequence $0, 1, \kappa_3, 0, \dots$ is a free cumulant sequence of some p-measure if and only if $|\kappa_3| \leq \frac{1}{3\sqrt{3}}$. This assertion was obtained by Lehner (oral communication) by other means.

Corollary 2.4. *A sequence* $\{\kappa_n\}_{n=1}^\infty$ *of real numbers such that* $\kappa_2 > 0$, $\kappa_n = 0$ *for* $n \ge m+1$, $m \ge 2$, *and* $|\kappa_n| \le \varepsilon$, $n = 3, \dots, m$, *with sufficiently small* $\varepsilon > 0$, *is a sequence of free cumulants of some p-measure with compact support.*

This corollary is an obvious consequence of Theorem 2.2. Note that Bercovici and Voiculecu [5] proved a more general result than Corollary 2.4 and showed the failure of the well-known Cramér' and Marcinkiewicz' theorems in free probability theory.

Theorem 2.2 is a simple consequence of the following theorem.

Theorem 2.5. *Let* $\{\kappa_n\}_{n=1}^\infty$ *be a sequence of real numbers such that*

$$\lim_{n\to\infty} |\kappa_n|^{1/n} = 0 \tag{2.6}$$

and

$$\limsup_{\varepsilon\to 0} \min_{z\in\mathbb{C},\,|z|=\varepsilon} \Big| \sum_{n=2}^\infty \frac{\kappa_n}{z^{n-1}} \Big| = \infty. \tag{2.7}$$

The sequence $\{\kappa_n\}_{n=1}^\infty$ *is a sequence of free cumulants of some p-measure with compact support if and only if for every* $\theta \in [0,\pi]$ *there exists* $r_0 = r_0(\theta) > 0$ *such that*

$$1 - \sum_{n=2}^\infty \frac{\kappa_n}{r_0^n} \frac{\sin((n-1)\theta)}{\sin\theta} = 0. \tag{2.8}$$

This theorem gives a description of free cumulants under the assumptions (2.6) and (2.7).

Remark 2.6. *If* $\limsup_{n\to\infty} \frac{n\log n}{\log(1/|\kappa_n|)} < \frac{1}{2}$, *then the conditions* (2.6) *and* (2.7) *hold.*

Note that the assumption on $\{\kappa_n\}_{n=2}^\infty$ in this remark is sharp. Indeed, consider the sequence $\{\kappa_n = \frac{1}{(2n-2)!}\}_{n=2}^\infty$. It is easy to see that $\limsup_{n\to\infty} \frac{n\log n}{\log(1/|\kappa_n|)} = \frac{1}{2}$ for this sequence and

$$\sum_{n=2}^\infty \frac{\kappa_n}{z^{n-1}} = \cos\Big(\frac{1}{\sqrt{z}}\Big) - 1.$$

It follows from the last formula that the assumption (2.7) does not hold for the chosen sequence $\{\frac{1}{(2n-2)!}\}_{n=2}^\infty$.

Now we consider the problem of the description of identically distributed free random variables $T_1, \dots, T_n$ such that the statistics L_1 and L_2 are identically distributed as well ($L_1 \overset{D}{=} L_2$).

Following Linnik [19], we introduce two entire functions of a complex variable z:

$$\Lambda_1(z) = |a_1|^z + \dots + |a_n|^z - |b_1|^z - \dots - |b_n|^z$$

and

$$\Lambda_2(z) = a_1^z + \dots + a_n^z - b_1^z - \dots - b_n^z,$$

where $a_1, \dots, a_n$ and $b_1, \dots, b_n$ are restricted as in (2.1). It is easy to see that all zeros of the functions $\Lambda_1(z)$ lie in a strip $b_1 < \operatorname{Re} z < b_2$ with some $b_1, b_2 \in \mathbb{R}$.

We prove the following characterization of semicircular measures which is an analogue of Linnik's result [19] about a characterization of Gaussian p-measures.

Theorem 2.7. *Let* $\Lambda_1(z) \not\equiv 0$*. In the class of all probability measures* μ*, the statements:*

(1) μ *is a semicircular measure*

(2) $L_1 \overset{D}{=} L_2$

are equivalent, if and only if the following conditions are satisfied:

a) 2 *is a simple and unique positive zero of the function* $\Lambda_1(z)$*,*

b) $\Lambda_2(2m-1) \neq 0$ *for all* $m = 1, \dots$*.*

Corollary 2.8. *Assume that* $a_1 + \dots + a_n \neq 1$*,* $a_1^2 + \dots + a_n^2 = 1$*, and* $b_1 = 1$*,* $b_2 = \dots = b_n = 0$*. Furthemore, assume* $L_1 \overset{D}{=} L_2$*. Then* μ *is a semicircular measure.*

Corollary 2.8 is an analogue of a result by Polya [26]. For analogs of the Polya result in noncommutative probability theory see Lehner [16].

We prove the following result for probability measures μ with moments of finite order which is an analogue of a result by Linnik [19] in classical probability theory.

Theorem 2.9. *Assume that* $\Lambda_1(z) \not\equiv 0$ *and that* $\Lambda_1(z)$ *has zeros in* $-i\mathbb{C}^+$*. Let* γ *denote the maximum of the real parts of such zeros. In order that, for some probability measure* μ *such that* $\int\limits_{\mathbb{R}} u^{2s}\,\mu(du) < \infty$ *with* $s = [\gamma/2 + 1]$*, the statements:*

(1) μ *is a semicircular measure*

(2) $L_1 \overset{D}{=} L_2$

are equivalent, it is necessary and sufficient that $\Lambda_2(2) = 0$ *and* $\Lambda_2(m) \neq 0$ *for all other positive integers.*

We prove in Lemma 8.1 that if $L_1 \overset{D}{=} L_2$, then the function $\Lambda_1(z)$ has a real root γ such that $0 < \gamma \leq 2$.

In the case where all moments of μ exist we obtain the following theorem.

Theorem 2.10. *In order that, for some probability measure* μ *such that* $\int\limits_{\mathbb{R}} u^{2m}\,\mu(du) < \infty$ *for all* $m \in \mathbb{N}$*, the statements:*

(1) μ *is a semicircular measure*

(2) $L_1 \overset{D}{=} L_2$

are equivalent, it is necessary and sufficient that $\Lambda_2(2) = 0$ *and* $\Lambda_2(m) \neq 0$ *for all other positive integers.*

Theorem 2.10 is an analogue of a result by Marcinkiewicz [23] in classical probability theory.

3. AN ANALYTIC APPROACH TO A SOLUTION OF THE CONSIDERED PROBLEM. AUXILLIARY RESULTS.

Denote by $\mathcal{M}$ the family of all Borel p-measures defined on the real line $\mathbb{R}$. On the set $\mathcal{M}$ define two associative composition laws denoted $*$ and $\boxplus$. Let $\mu_1, \mu_2 \in \mathcal{M}$. The measure $\mu_1 * \mu_2$ will denote the classical convolution of μ_1 and μ_2. In probabilistic terms, $\mu_1 * \mu_2$ is the probability distribution of $X+Y$, where X and Y are (commuting) independent random variables with distributions μ_1 and μ_2 respectively. The measure $\mu_1 \boxplus \mu_2$ is the free (additive) convolution of μ_1 and μ_2 introduced by Voiculescu [29] for compactly supported measures. The free convolution was extended by Maassen [22] to measures with finite variance and by Bercovici and Voiculescu [4] to the whole class $\mathcal{M}$. Thus, $\mu_1 \boxplus \mu_2$ is the distribution of $X+Y$, where X and Y are free random variables with distributions μ_1 and μ_2, respectively.

Let $\mathbb{C}^+$ $(\mathbb{C}^-)$ denote the open upper (lower) half of the complex plane. If $\mu \in \mathcal{M}$, denote its Cauchy transform by

$$G(z) = \int_{-\infty}^{\infty} \frac{\mu(dt)}{z-t}, \qquad z \in \mathbb{C}^+. \tag{3.1}$$

Following Maassen [22] and Bercovici and Voiculescu [4], we introduce the *reciprocal Cauchy transform*

$$F(z) = \frac{1}{G(z)}. \tag{3.2}$$

The corresponding class of reciprocal Cauchy transforms of all $\mu \in \mathcal{M}$ we denote by $\mathcal{F}$. This class admits a simple description. Recall that the Nevanlinna class $\mathcal{N}$ is the class of analytic functions $F : \mathbb{C}^+ \to \mathbb{C}^+$. The class $\mathcal{F}$ is the subclass of Nevanlinna's functions F_μ such that $F_\mu(z)/z \to 1$ as $z \to \infty$ nontangentially to $\mathbb{R}$ (i.e., such that $\operatorname{Re} z / \operatorname{Im} z$ stays bounded), and this implies that F_μ has certain invertibility properties. (For details see Akhiezer and Glazman [2], Akhiezer [1]). To be precise, for two numbers $\alpha > 0, \beta > 0$ we set

$$\Gamma_\alpha = \{z = x+iy \in \mathbb{C}^+ : |x| < \alpha y\} \quad \text{and} \quad \Gamma_{\alpha,\beta} = \{z = x+iy \in \Gamma_\alpha : y > \beta\}.$$

Then for every $\alpha > 0$ there exists $\beta = \beta(\mu, \alpha)$ such that F_μ has the right inverse $F_\mu^{(-1)}$ defined on $\Gamma_{\alpha,\beta}$. The function $\phi_\mu(z) = F_\mu^{(-1)}(z) - z$ will be called the Voiculescu transform of μ. It is not hard to show that $\operatorname{Im} \phi_\mu(z) \le 0$ for $z \in \Gamma_{\alpha,\beta}$ where ϕ_μ is defined. Note that $\phi_\mu(z) = o(z)$ as $|z| \to \infty$, $z \in \Gamma_\eta$. In the sequel we will denote $\phi_\mu(z)$ by $\phi_T(z)$ for a random variable T with a distribution μ as well.

In the domain $\Gamma_{\alpha,\beta}$, where the functions $\phi_{\mu_1}(z)$, $\phi_{\mu_2}(z)$, and $\phi_{\mu_1 \boxplus \mu_2}(z)$ are defined, we have

$$\phi_{\mu_1 \boxplus \mu_2}(z) = \phi_{\mu_1}(z) + \phi_{\mu_2}(z). \tag{3.3}$$

This caracterization for the distribution $\mu_1 \boxplus \mu_2$ of $X+Y$, where X and Y are free random variables, is due to Voiculescu [29]. He considered compactly supported measures

μ. The result was extended by Maassen [22] to measures with finite variance; the general case was proved by Bercovici and Voiculescu [4].

We need the following auxilliary results.

The following proposition is proved in [6].

Proposition 3.1. *For every p-measure μ we have*

$$\phi_\mu(z) = z^2\Big(G_\mu(z) - \frac{1}{z}\Big)(1 + q_\mu(z)), \quad z \in \Gamma_{\alpha,\beta},$$

where $q_\mu(z) = o(1)$ as $z \to \infty$.

The following lemma is well-known, see [1].

Lemma 3.2. *Let μ be a p-measure such that*

$$m_k(\mu) := \int\limits_{\mathbb{R}} u^k \, \mu(du) < \infty, \qquad k = 1, \dots, 2n.$$

Then the following relation holds

$$\lim_{z\to\infty} z^{2n+1}\Big(G_\mu(z) - \frac{1}{z} - \frac{m_1(\mu)}{z^2} - \dots - \frac{m_{2n-1}(\mu)}{z^{2n}}\Big) = m_{2n}(\mu)$$

uniformly in the angle $\delta \le \arg z \le \pi - \delta$, where $0 < \delta < \pi/2$.

By Lemma 3.2 and the Cartier–Good formula for free random variables (see Lehner [17]), we easily obtain the expansion of the function $\phi_\mu(z)$. For a proof see for example [14].

Proposition 3.3. *For every p-measure μ such that $m_{2n}(\mu) < \infty$ for a nonnegative integer n we have*

$$\phi_\mu(z) = \kappa_1(\mu) + \frac{\kappa_2(\mu)}{z} + \dots + \frac{\kappa_{2n}(\mu)}{z^{2n-1}} + \frac{o(1)}{z^{2n-1}}, \qquad z \in \Gamma_{\alpha,\beta}, \quad z \to \infty,$$

where $\kappa_j(\mu)$, $j = 2, \dots, 2n-1$, are the free cumulants of the measure μ.

We also need the following well-known result (see for example [25]).

Proposition 3.4. *In order that a p-measure μ has a compact support it is necessery and sufficient that the sequence $\{\kappa_s(\mu)\}_{s=1}^\infty$ of free cumulants of this measure satisfies the inequality*

$$|\kappa_s(\mu)| \le c^s, \quad \text{for all } s \in \mathbb{N}, \tag{3.4}$$

with some constant $c > 0$.

We introduced the definition of free cumulants in Section 2. Let us recall the definition of mixed free cumulants as well. Let $T_1, \dots, T_n \in \tilde{A}_{as}$ be random variables and let $T_j^{(k)}$, $k = 1, \dots, n$, denote free copies. Set T_j^ω as in (2.2). Following Lehner [17], the nth mixed cumulant may be defined via

$$\kappa_n(T_1, \dots, T_n) = \frac{1}{n}\tau(T_1^\omega T_2^\omega \dots T_n^\omega)$$

in a short way. We will use the following known results, see [25] and [17].

Theorem 3.5. *Consider a non-commutative probability space* (A,τ) *and let* $(\kappa_n)_{n\in\mathbb{N}}$ *be the corresponding free cumulant functionals. Consider random variables* $(T_j)_{j\in I}$ *in* $\tilde{A}_{sa}$*. Then the following two statements are equivalent.*

(i) $(T_j)_{j\in I}$ *are freely independent.*

(ii) We have $\kappa_n(T_{j_1},\dots,T_{j_n})=0$ *for all* $n\ge 2$ *and* $j_1,\dots,j_n\in I$ *such that at least two of these* n *indices* j_h *are different.*

Proposition 3.6. *Let* $L_k=\sum_{j=1}^n a_{kj}T_j$, $k=1,\dots,m$, *be an affine transformation of* $T_1,\dots,T_n$, *then we have, for* $m\ge 2$,

$$\kappa_m(L_1,\dots,L_m)=\sum_{j_1,\dots,j_m} a_{1,j_1}\dots a_{m,j_m}\kappa_m(T_{j_1},\dots,T_{j_m}).$$

Proposition 3.7. *Mixed cumulants vanish. That is, if there is a nontrivial subset* $I\subset[n]$ *(i.e.,* $I\ne\emptyset$ *and* $I\ne[n]$, $[n]:=\{1,\dots,n\}$*) such that* $\{T_j\}_{j\in I}$ *and* $\{T_j\}_{j\in[n]\setminus I}$ *are free, then* $\kappa_n(T_1,\dots,T_n)=0$.

Now we prove an analogue of a known lemma of Linnik for characteristic functions (see [12], Ch. 1, §6).

Recall that a p-measure μ is symmetric if $\mu(S)=\mu(-S)$ for any real Borel set S. It is not difficult to verify (see [9]) that μ is symmetric if and only if $\phi_\mu(iy)$ takes imaginary values for $y>0$, where $\phi_\mu(iy)$ is defined.

Lemma 3.8. *Let* μ *be a symmetric p-measure and* $\{y_k\}$ *be a sequence of positive numbers such that* $\lim y_k\to\infty$. *If, for all* k, $\phi_\mu(iy_k)=\phi_\nu(iy_k)$, *where* ν *is a symmetric p-measure with compact support, then* $\mu=\nu$.

Proof. We shall show that μ has moments $m_n(\mu):=\int_{\mathbb{R}}u^n\,\mu(du)$ of all orders and that $m_n(\mu)=\int_{\mathbb{R}}u^n\,\nu(du)$ for all $n=1,\dots$. The proof proceeds by induction for even n.

From the assumptions of the lemma we see

$$G_\mu(it_k)=G_\nu(it_k),\quad k\ge k_0, \tag{3.5}$$

where $t_k:=-iF_\nu(iy_k)\to\infty$ as $y_k\to\infty$, and k_0 is sufficiently large positive integer. By (3.5), we obtain the following equation, using the symmetry of the measures μ and ν,

$$(it_k)^3\Big(G_\mu(it_k)-\frac{1}{it_k}\Big)=\int_{\mathbb{R}}\frac{t_k^2u^2}{u^2+t_k^2}\,\mu(du)=\int_{\mathbb{R}}\frac{t_k^2u^2}{u^2+t_k^2}\,\nu(du). \tag{3.6}$$

We shall prove by induction that $m_n(\mu)=m_n(\nu)$ for all $n=0,1,\dots$. Letting $t_k\to\infty$, we conclude that $m_2(\mu)<\infty$ and $m_2(\mu)=m_2(\nu)$. Now suppose that, for all $p<n$, $m_{2p}(\mu)$ exists and $m_{2p}(\mu)=m_{2p}(\nu)$. Using (3.5) and the formula

$$(it_k)^{2n+1}\Big(G_\mu(it_k)-\frac{1}{it_k}-\frac{m_2(\mu)}{(it_k)^3}-\dots-\frac{m_{2n-2}(\mu)}{(it_k)^{2n-2}}\Big)=\int_{\mathbb{R}}\frac{t_k^2u^{2n}}{u^2+t_k^2}\,\mu(du)$$

we arrive at the relation

$$\int\limits_{\mathbb{R}} \frac{t_k^2 u^{2n}}{u^2+t_k^2}\,\mu(du) = \int\limits_{\mathbb{R}} \frac{t_k^2 u^{2n}}{u^2+t_k^2}\,\nu(du).$$

Letting here $t_k \to \infty$, we obtain $m_{2n}(\mu) < \infty$ and $m_{2n}(\mu) = m_{2n}(\mu_w)$ that was to be proved.

It remains to note that since $m_n(\mu) = m_n(\nu)$ for all $n = 0, 1, \dots$ and the measure ν has compact support, we have $\mu = \nu$. Hence the lemma is proved. ☐

Voiculescu [31], Maassen [22] and in the general case Biane [7] proved that there exist unique functions $Z_1(z)$ and $Z_1(z)$ from the class $\mathcal{F}$ such that

$$z = Z_1(z) + Z_2(z) - F_{\mu_1}(Z_1(z)) \quad \text{and} \quad F_{\mu_1}(Z_1(z)) = F_{\mu_2}(Z_1(z)), \quad z \in \mathbb{C}^+. \tag{3.7}$$

In addition $F_{\mu_1 \boxplus \mu_2}(z) = F_{\mu_1}(Z_1(z))$. The relation (3.7) was proved by purely analytic methods by Chistyakov and Götze [8], see also Belinschi and Bercovici [3].

Introduce the class $\mathcal{K}[a,b]$ in the following way. A function $F(z)$ is in class $\mathcal{K}[a,b]$ if

1) $F(z)$ is in class $\mathcal{N}$, and

2) $F(z)$ is holomorphic and positive in the interval $(-\infty, a)$, and holomorphic and negative in the interval $(b, +\infty)$. The following theorem is due to Krein [15].

Theorem 3.9. *A function $F(z)$ is in class $\mathcal{K}[a,b]$ if and only if it admits a representation*

$$F(z) = \int\limits_a^b \frac{\sigma(dt)}{t-z},$$

where σ is a finite nonnegative measure.

Prove now free analogue of one Wintner's result, see [21], Ch. 3, §2.

Lemma 3.10. *Assume that $\mu = \mu_1 \boxplus \mu_2$, where μ has compact support. Then μ_1 and μ_2 have compact support as well.*

Proof. By symmetry it suffices to prove the lemma for the measure μ_1.

By (3.7), there exists $Z(z) \in \mathcal{F}$ such that $F_\mu(z) = F_{\mu_1}(Z(z))$, $z \in \mathbb{C}^+$. Hence we obtain the relation

$$\int\limits_{[-d,d]} \frac{\mu(du)}{z-u} = \int\limits_{\mathbb{R}} \frac{\mu_1(du)}{Z(z)-u}, \quad z \in \mathbb{C}^+, \tag{3.8}$$

with $0 < d < \infty$. Since $z/(Z(z)-u) \to 1$ as $z \to \infty$ nontangentially to $\mathbb{R}$ and $\operatorname{Im}(1/(u - Z(z))) \geq 0$ for $z \in \mathbb{C}^+$, then $1/(u - Z(z)) \in \mathcal{F}$ and we may write

$$\frac{1}{u - Z(z)} = \int\limits_{\mathbb{R}} \frac{\sigma(u, ds)}{s-z}, \quad z \in \mathbb{C}^+, \tag{3.9}$$

where $\sigma(u, ds)$ is a p-measure for every $u \in \mathbb{R}$ and $\sigma(u, S)$ is Borel function for every Borel S set in $\mathbb{R}$. Using this representation we deduce from (3.8)

$$\mu(S) = \int\limits_{\mathbb{R}} \sigma(u, S)\, \mu_1(du) \tag{3.10}$$

for every Borel S set in $\mathbb{R}$. Let u_0 be a point such that

$$\mu_1((u_0 - \varepsilon, u_0 + \varepsilon)) > 0 \quad \text{for any} \quad \varepsilon > 0 \tag{3.11}$$

and let a be a point of continuity of the function $f(a) := \sigma(u_0, [a, \infty))$. Fix such point a and consider the function $\sigma(u, [a, \infty))$ as a function of the variable u. This function is continuous at the point u_0. Therefore we conclude from (3.10) that for every point u_0 with the property (3.11) the measure $\sigma(u_0, ds)$ has support contained in $[-d, d]$.

It remains to show that μ_1 has a bounded support. Return to (3.9) with $u = u_0$. By Theorem 3.9, the function $Z(z) - u_0$ is holomorphic and real for $z = x < -d$ and for $z = x > d$. Since $Z(z)$ admits the representaion

$$Z(z) = \alpha + z + \int\limits_{\mathbb{R}} \Big(\frac{1}{t - z} - \frac{t}{1 + t^2}\Big)(1 + t^2)\, \nu(dt),$$

where $\alpha \in \mathbb{R}$ and ν is a finite nonnegative measure, it follows from the inversion formula that the measure ν has bounded support contained in $[-d, d]$. Thus

$$Z(z) - u_0 = \gamma - u_0 + z + \int\limits_{[-d,d]} \frac{(1 + t^2)\, \nu(dt)}{t - z}, \tag{3.12}$$

where $\gamma \in \mathbb{R}$. The parameter γ and the measure ν depend on Z only and do not depend on u_0. Let $u_0 > 0$ and be sufficiently large, i.e., $u_0 > c(Z) > 0$. Then, by (3.12), $Z(x) - u_0 < 0$ for $x = -u_0/2 < -2d$, a contradiction with (3.9) for $u = u_0$. An analogous argument holds for $u_0 < 0$. Hence there exists $c(Z) > 0$ such that the points u_0 with property (3.11) satisfy the inequality $|u_0| \le c(Z)$. The lemma is proved. □

4. AUXILLIARY RESULTS ON SPECIAL FUNCTIONAL EQUATIONS AND AUXILLIARY ANALYTIC RESULTS.

In this section we first describe some results (see Kagan, Linnik, Rao [13]) on continuous solutions of special equations which were used to characterize distributions via independence of linear statistics and identical distribution of linear statistics.

Lemma 4.1. *Consider the following equation, for* $|u| < \delta_0$, $|v| < \delta_0$,

$$\psi_1(u + b_1 v) + \dots + \psi_r(u + b_r v) = A(u) + B(v) + P_k(u, v),$$

where P_k *is a polynomial of degree* k*;* ψ_j, A *and* B *are complex valued functions of two real variables* u *and* v. *We assume that*

(i) without loss of generality, the numbers b_j *are all distinct*

(ii) the functions A, B, *and* ψ_j *are continuous.*

Then, in some neighborhood of the origin, all the functions A, B, *and* ψ_j *are polynomials of degree at most* $\leq \max(r,k)$.

Consider the equation

$$\int_0^1 v(st)\,dQ_1(s) = \int_0^1 v(st)\,dQ_2(s), \quad \text{for all} \quad 0<t<1, \tag{4.1}$$

for a bounded continuous function $v(t)$ defined on $(0,1)$. Here $Q_1(s)$ and $Q_2(s)$ are nondecreasing functions satisfying the condition

$$\int_0^1 s^{-b}\,d(Q_1(s)+Q_2(s)) < \infty \tag{4.2}$$

for some $b>0$. We assume that relation (4.1) is nondegenerate, i.e., $Q_2(s)-Q_1(s) \not\equiv$ *const*.

Applying a Mellin transform to (4.1) we easily obtain, for $Q(s) := Q_1(s)-Q_2(s)$,

$$\int_0^1 t^{z-1}\,dt \int_0^1 v(st)\,dQ(s) = \int_0^1 s^{-z}\,dQ(s) \int_0^s t^{z-1}v(t)\,dt = 0 \quad \text{for all} \quad \operatorname{Re} z>0. \tag{4.3}$$

In view of (4.2) we deduce from (4.3), for $0<\operatorname{Re} z<b$,

$$\Lambda(z)X(z;v) - K(z;v) = 0, \tag{4.4}$$

where

$$\begin{gathered}\Lambda(z) := \int_0^1 s^{-z}\,dQ(s), \qquad X(z;v) := \int_0^1 t^{z-1}v(t)\,dt, \\ K(z;v) := \int_0^1 s^{-z}\,dQ(s) \int_s^1 t^{z-1}v(t)\,dt.\end{gathered} \tag{4.5}$$

The functions $\Lambda(z)$ and $K(z;v)$ are analytic in the half-plane $\operatorname{Re} z<b$, and the function $X(z;v)$ is analytic in the half-plane $\operatorname{Re} z>0$. We use the relation (4.4) for analytic continuation of $X(z;v)$ into the half-plane $\operatorname{Re} z \leq 0$ as a meromorphic function. Keeping the same notation, we have

$$X(z;v) = K(z;v)/\Lambda(z), \qquad \operatorname{Re} z<b. \tag{4.6}$$

The singularities of $X(z;v)$ in the half-plane $\operatorname{Re} z \leq 0$ happen to be poles distributed among the zeros of $\Lambda(z)$.

Taking an arbitrary $\lambda > 0$, the inversion formula for the Mellin transform yields

$$\int\limits_0^t u^{\lambda-1}\log(t/u)v(u)\,du = \frac{1}{2\pi i}\int\limits_{x-i\infty}^{x+i\infty} t^{\lambda-z}\frac{K(z;v)}{(z-\lambda)^2\Lambda(z)}\,dz,\ \ 0 < x < \min(b,\lambda). \tag{4.7}$$

Let z_0 be some zero of $\Lambda(z)$ of multiplicity m_0 in the half-plane $\operatorname{Re} z \le 0$. We see that

$$Res\Big(t^{\lambda-z}\frac{K(z;v)}{(z-\lambda)^2\Lambda(z)}\Big) = P_{z_0}(\log t)t^{\lambda-z_0}, \quad 0 < t < 1, \tag{4.8}$$

where $P_{z_0}(t)$ is a polynomial of degree at most $m_0 - 1$. These residues may depend on λ, but it easily seen that the degree of the polynomial $P_{z_0}(t)$ does not depend on λ.

If $P_{z_0}(t) \not\equiv 0$, we call the number $-z_0$ an *active exponent* of the solution $v(t)$ and the number $degP_{z_0}(t) + 1$ will be called the *multiplicity* of the active exponent $\xi = -z_0$. The leading coefficient of $P_{z_0}(t)$ will be denoted $a_{-z_0}(v)$ and the degree by $m_{-z_0}(v)$. All the active exponents of $v(t)$ are located in the half-plane $\operatorname{Re} z \ge 0$.

If the number of the active exponents $\{z_k\}_{k=1}^d$ of $v(t)$ is finite, then, by Jordan's lemma on residues, it follows from (4.7) that

$$\int\limits_0^t u^{\lambda-1}\log(t/u)v(u)\,du = t^\lambda\sum_{k=1}^d P_{z_k}(\log t)t^{-z_k}, \quad 0 < t < 1. \tag{4.9}$$

We shall introduce the notation

$$\sigma_1(v) := \inf\{\operatorname{Re}\xi,\ \ \xi \text{ active exponents of } v(t)\}. \tag{4.10}$$

We need the following results on active exponents and differentiable solutions $v(t)$.

Lemma 4.2. *If $v(t) \ge 0$ is a continuous solution (4.1) such that $v(t) \to 0$ as to $0+$, then $\sigma_1(v)$ is an active exponent and*

$$\sigma_1(v) > 0 \qquad \textit{and} \qquad a_{\sigma_1}(v) > 0. \tag{4.11}$$

Lemma 4.3. *If under the hypothesis of Lemma 4.2 the function $v(t)$ has a continuous derivative $v^{(n)}(t)$ for some $n \ge 1$ and for all $0 < t < 1$ and the following limit exists and is finite*

$$\lim_{t\to 0+} v^{(n)}(t) = v_+^{(n)}(0),$$

then all the active exponents ξ of $v(t)$ which are not simultaneously integers and simple active exponents satisfy the condition $\operatorname{Re}\xi > n$.

We now formulate some results on entire functions (see [18]).

Let $f(z)$ be an entire function. Denote $M(r,f) := \max_{|z|=r}|f(z)|$ and $m(r,f) := \min_{|z|=r} = |f(z)|$. The number $\rho = \rho(f) := \limsup_{r\to\infty}\frac{\log\log M(r,f)}{\log r}$ is called the order of the function $f(z)$.

Theorem 4.4. *In order that a power series $f(z)=\sum_{n=0}^{\infty} c_n z^n$ represents an entire function of an order ρ it is necessary and sufficient that*

$$\rho = \limsup_{n\to\infty} \frac{n\log n}{\log(1/|c_n|)}.$$

Theorem 4.5. (*Wiman*) *If the order ρ of an entire function f is less than $\frac{1}{2}$, then*

$$\limsup_{r\to\infty} m(r,f) = \infty.$$

5. Proof of an analogue of Darmois–Skitovich's theorem.

In this section we prove Theorem 2.1.

In order to prove Theorem 2.1 we follow the proof of [13].

Proof. Necessity. Assume that the free random variables $T_1,\dots,T_n$ satisfy the assumptions of Theorem 2.1. Then for every pair of real numbers (u,v) the linear statistics uL_1 and vL_2 are free and we have the relation

$$\begin{aligned} L := uL_1 + vL_2 = (ua_1+vb_1)T_1 + \dots + (ua_m+vb_m)T_m \\ + (ua_{m+1}+vb_{m+1})T_{m+1} + \dots + (ua_n+vb_n)T_n. \end{aligned} \tag{5.1}$$

We deduce from (5.1) that

$$\begin{aligned} &\phi_{(ua_1+vb_1)T_1}(z) + \dots + \phi_{(ua_n+vb_n)T_m}(z) \\ &\quad = \phi_{uL_1}(z) + \phi_{vL_2}(z) - \phi_{(ua_n+vb_n)T_{m+1}}(z) - \dots - \phi_{(ua_n+vb_n)T_n}(z) \end{aligned} \tag{5.2}$$

for $z\in\Gamma_{\alpha,\beta}$ with some $\alpha>0$ and $\beta>0$, where all functions $\phi_{(ua_j+vb_j)T_j}(z), j=1,\dots,n$, and $\phi_{uL_1}(z)$, $\phi_{vL_2}(z)$ are defined. Hence (5.2) holds for $z=i$ and for $|u|\le\delta$ and $|v|\le\delta$ with sufficiently small $\delta>0$. Note that the functions $\phi_{(ua_n+vb_n)T_{m+1}}(z),\dots,\phi_{(ua_n+vb_n)T_n}(z)$ depend on u or v only. Consider the functions $\psi_j(w) := w\phi_{T_j}(i/w)$, $j=1,\dots,n$, for $w\in\Gamma_\alpha$ and $|w|\le\delta'$ with sufficiently small $\delta'>0$. Since $\phi_{wT_j}(i) = w\phi_{T_j}(i/w)$, and $w\phi_{T_j}(i/w)\to 0$ as $w\to 0$, we see that (5.2) has the form

$$\psi_1(ua_1+vb_1) + \dots + \psi_m(ua_m+vb_m) = A(u) + B(v), \quad |u|<\delta,\ |v|<\delta, \tag{5.3}$$

where ψ_j, $j=1,\dots,m$, and A,B are complex valued continuous functions, and b_j/a_j, $j=1,\dots,m$ are all distinct. Then, by Lemma 4.1, the functions ψ_j, $j=1,\dots,m$, are polynomials of degree $\le m$. Therefore we have the representation

$$\phi_{T_j}(z) = z\sum_{s=0}^{m}\frac{d_{sj}}{z^s}, \quad j=1,\dots,m, \tag{5.4}$$

for $z\in\Gamma_{\alpha',\beta'}$ with some $\alpha'>0$ and $\beta'>0$, where d_{sj} are complex valued coefficients. Since $\phi_{T_j}(iy)=o(y)$ as $y\to\infty$, all $d_{0j}=0$. By Proposition 3.3, the coefficients d_{sj} are real-valued and for every $j=1,\dots,m$ there exist p-measures μ_j such that all moments $m_k(\mu_j)$ exist and $d_{sj}=\kappa_{s+1}(\mu_j)$, $s=1,\dots,m$ and $\kappa_s(\mu_j)=0$ for $s>m$. By Proposition 3.4, the measures μ_j, $j=1,\dots,m$, have compact supports. We now return to the relation (5.3). By (5.4), the functions on both sides of (5.3) are differentiable. Differentiating sequentially both sides of (5.3) with respect to u and v we obtain a relation from which (2.4) follows immediately.

Sufficiency. Assume that the random variables $T_1, \dots, T_n$ satisfy the assumptions of Theorem 2.1. Consider mixed cumulants $\kappa_s(L_{j_1}, L_{j_2}, \dots, L_{j_s})$ such that q indices j_h are equal 1 and $s-q$ indices j_h are equal 2. Then, by Propositions 3.6, 3.7, and by (2.4), L_1 and L_2 have vanishing mixed cumulants

$$\kappa_s(L_{j_1}, L_{j_2}, \dots, L_{j_s}) = \sum_{j=1}^{m} a_j^q b_j^{s-q} \kappa_s(\underbrace{T_j, \dots, T_j}_{s \text{ times}}) = \sum_{j=1}^{m} a_j^q b_j^{s-q} \kappa_s(T_j) = 0$$

for $s = 2, \dots, m$ and $q = 1, \dots, s-1$. In addition we clearly have $\kappa_s(L_{j_1}, L_{j_2}, \dots, L_{j_s}) = 0$ for all $s \geq m+1$ and $q = 1, \dots, s-1$. Hence, by Theorem 3.5, the linear forms L_1 and L_2 are free independent.

The theorem is completely proved. □

6. CHARACTERIZATION OF FREE CUMULANTS

We noted in Section 2 that Theorem 2.2 is an obvious consequence of Theorem 2.5. Therefore we shall prove Theorem 2.5 first.

Proof. Sufficiency. By the assuptions of the theorem, for every $\theta \in [0, \pi]$, there exists $r_0 = r_0(\theta)$ such that (2.8) holds. Denote by $0 < r(\theta) < \infty$ the maximum of $r_0(\theta)$ satisfying (2.8). It is clear that $r = r(\theta)$, $\theta \in [0, \pi]$, is a continuous function on the interval $[0, \pi]$ and

$$\operatorname{Im}(r(\theta)e^{i\theta} + \varphi(r(\theta)e^{i\theta})) = 0, \quad \theta \in [0, \pi], \tag{6.1}$$

where $\varphi(z) := \sum_{s=2}^{\infty} \frac{\kappa_s}{z^{s-1}}$, $z \in \mathbb{C} \setminus \{0\}$.

Define the region: $\Omega := \{re^{i\theta} \in \mathbb{C}^+ : 0 < \theta < \pi,\ r > r(\theta)\}$. The region Ω is a Jordan domain with boundary curve $\gamma := \gamma_1 \cup \gamma_2 \cup \gamma_3$, where $\gamma_1 : x,\ x \leq r(\pi)$, $\gamma_2 : r = r(\pi - \theta)$, $0 < \theta < \pi$, and $\gamma_3 : x,\ x \geq r(0)$.

Note that the function $\varphi(z) : \Omega \to \mathbb{C}$ is analytic such that

$$\lim_{R \to +\infty} \max_{|z|=R} |\varphi(z)| = 0. \tag{6.2}$$

We shall show that the function $f : \Omega \to \mathbb{C}$ defined via $z \mapsto z + \varphi(z)$ takes every value in $\mathbb{C}^+$ precisely once. The inverse $f^{(-1)} : \mathbb{C}^+ \to \mathbb{C}^+$ thus defined is in the class $\mathcal{F}$.

Let R be a sufficiently large positive number. For every fixed $w \in \mathbb{C}^+$ we consider a closed rectifiable curve γ_4 consisting of a smooth curve $\gamma_{4,1}$, which is a part of the curve γ, connecting $-R$ to R, and the arc $\gamma_{4,2}$ in the circle $|z| = R$ connecting R to $-R$. The curve γ_4 depends on R.

We see from the construction of the curve $\gamma_{4,1}$ that if z runs through $\gamma_{4,1}$ the image $\zeta = f(z)$ lies on the interval $[A_{-R}, A_R]$, where $f(-R) = -A_{-R}$ and $f(R) = A_R$. Here $A_{\pm R} \to \infty$ as $R \to \infty$. We note as well that if z runs through $\gamma_{4,2}$ the image $\zeta = f(z)$ lies in the domain $|\zeta| \geq \min\{A_{-R}, A_R\}/2$, $\operatorname{Im} \zeta > 0$.

Hence $f(z)$ winds around w once, and it follows from the argument principle that inside the curve γ_4 there is a unique point z_0 such that $f(z_0) = w$. Since this relation holds for all sufficiently large $R > 0$ and all curves γ_4, we deduce that the point z_0 is unique in Ω.

Hence the inverse function $f^{(-1)} : \mathbb{C}^+ \to \Omega$ exists and is analytic in $\mathbb{C}^+$. By condition (6.2), $\lim_{y\to+\infty}(iy/f^{(-1)}(iy)) = 1$ and therefore $f^{(-1)} \in \mathcal{F}$. Hence there exists a p-measure μ such that $\varphi(z) = \phi_\mu(z)$ and the sufficiency of the assumptions of the theorem is proved.

Necessity. Let there exists $\theta \in [0, \pi]$ such that $\operatorname{Im}(z + \varphi(z)) > 0$ for all $r > 0$, where $z = re^{i\theta}$. We shall show that in this case it does not exist a p-measure μ such that $\varphi(z) = \phi_\mu(z)$. Assume to the contrary that there exists a p-measure μ such that $\varphi(z) = \phi_\mu(z)$ for $z \in \mathbb{C}^+$ and $|z| \ge c$ with a sufficiently large constant $c > 0$. By Proposition 3.4, μ has compact support. Consider the domain $\Omega_1 := \{z \in \mathbb{C}^+ : \operatorname{Im}(z + \varphi(z)) > 0\}$. It is obvious that the relation

$$F_\mu(z + \varphi(z)) = z \tag{6.3}$$

holds for $z \in \Omega_1$. Moreover Ω_1 contains a curve γ_5 containing 0 and ∞. By the condition (2.7), there exists a sequence $\varepsilon_k \downarrow 0$ such that $\min_{|z|=\varepsilon_k} |\varphi(z)| \to \infty$ as $\varepsilon_k \to \infty$. Let $z_k \in \gamma_5$ and $|z_k| = \varepsilon_k$. Since $\varphi(z_k) \to \infty$ as $z_k \to 0$, and $F_\mu(z) = (1 + o(1))z$ as $z \to \infty$, the relation (6.3) with $z = z_k$ leads to a contradiction. This proves the necessity of the assumptions of the theorem and completely proves the theorem. ☐

Now we prove Remark 2.6.

Proof. We need obviously to verify the condition (2.7). By Theorem 4.4, we see that under the assumption $\limsup_{n\to\infty} \frac{n \log n}{\log(1/\kappa_n)} < \frac{1}{2}$ the function $f(z) := \varphi(1/z)$, $z \in \mathbb{C}$, is an entire function of an order less than 1/2. Then, by Theorem 4.5, $\limsup_{R\to\infty} \min_{|z|=R} |f(z)| = \infty$ and the condition (2.7) holds. The remark is proved. ☐

Let us prove Corollary 2.3.

Proof. Let $|\kappa_3| \le f_2(\kappa_4)$ and $0 < \kappa_4 \le \frac{1}{4}$. We obtain from Theorem 2.2 that the sequence $0, 1, \kappa_3, \kappa_4, 0, \dots$ is not a sequence of free cumulants of some p-measure if and only if there exists $x \in [-1, 1]$ such that

$$-2\kappa_3 xr - (4x^2 - 1)\kappa_4 > r^2(1 - r^2), \quad \text{for all} \quad r > 0. \tag{6.4}$$

Note at once that if $\kappa_4 > 0$, then (6.4) does not hold for every fixed $|x| \in (1/2, 1]$.

Let $x \in [-1/2, 1/2]$. Fix a parameter $b := -(4x^2 - 1)\kappa_4$. It is clear that $0 \le b \le 1/4$. The line $y = ar + b$ is a tagent to the curve $y = r^2(1 - r^2)$ if

$$a = 2r_1(1 - 2r_1^2), \quad \text{where} \quad r_1 = r_1(x) > 0 \quad \text{and} \quad r_1^2(x) := \frac{1}{2}\Big(\frac{1}{3} + \sqrt{\frac{1}{9} - \frac{4}{3}(4x^2 - 1)\kappa_4}\Big).$$

It is easy to see that $r_1(1/2) \le r_1(x) \le r_1(0)$, where $r_1(1/2) = 1/\sqrt{3}$ and $r_1(0) \le 1/\sqrt{2}$. Hence in order that (6.4) holds for some $x \in [-1/2, 1/2]$ it is necessary and sufficient that, for some $x \in [-1/2, 1/2]$, $-\kappa_3 x > r_1(x)(1 - 2r_1^2(x))$.

We shall now prove that

$$-\kappa_3 x \le r_1(x)(1-2r_1^2(x)) \quad \text{for all} \quad x \in [-1/2, 1/2]. \tag{6.5}$$

Without loss of generality we assume that $\kappa_3 > 0$ and $x < 0$. Note that (6.5) holds at the point $x = 0$ and at the point $x = -1/2$ for $\kappa_3 \le \frac{2}{3\sqrt{3}}$ only. For $x \in [-1/2, 0)$, the inequality (6.5) is equivalent to the following one

$$g(u) := \rho(u)(1-2\rho(u))^2 - \kappa_3^2 u \ge 0, \quad u \in (0, 1/4], \tag{6.6}$$

where $\rho(u) := r_1^2(\sqrt{u})$. In order to find $\min_{u\in(0,1/4]} g(u)$ we find points $u_* \in (0, 1/4]$, where $g'(u_*) = 0$. Since $\rho'(u) = -\frac{4}{3}\kappa_4/(2\rho(u) - 1/3)$, we need to solve the following equation

$$\rho^2(u_*) - \Big(\frac{2}{3} - \frac{1}{8}\frac{\kappa_3^2}{\kappa_4}\Big)\rho(u_*) + \frac{1}{12}\Big(1 - \frac{1}{4}\frac{\kappa_3^2}{\kappa_4}\Big) = 0 \tag{6.7}$$

and choose a solution $\rho(u_*) \in [\frac{1}{3}, r_1^2(0)]$. We see that the solutions of (6.7) have the form

$$\rho(u_*) = q(t) := \frac{1}{2}\Big(\frac{2}{3} - \frac{t}{8} \pm \Big|\frac{1}{3} - \frac{t}{8}\Big|\Big), \quad \text{where} \quad t = \frac{\kappa_3^2}{\kappa_4}. \tag{6.8}$$

It follows from (6.8) that the desired solution is equal to

$$\rho(u_*) = \frac{1}{6} \quad \text{or} \quad \rho(u_*) = \frac{1}{2} - \frac{t}{8}. \tag{6.9}$$

If either $\rho(u_*) \le \frac{1}{3}$ or $\rho(u_*) \ge r_1^2(0)$, then $g(u)$ is a monotone function and (6.6) holds.

Let $\frac{1}{3} < \rho(u_*) < r_1^2(0) \le \frac{1}{2}$. We now calculate $g(u_*)$. We conclude from the definition of $\rho(u_*) := r_1^2(\sqrt{u_*})$ that $u_* = \frac{1}{4} - \frac{3}{4\kappa_4}\rho(u_*)(\rho(u_*) - \frac{1}{3})$. In order to prove (6.6) it is sufficient to establish that $g(u_*) \ge 0$ which has the form

$$\rho(u_*)(1-2\rho(u_*))^2 - \kappa_3^2 u_* = \rho(u_*)(1-2\rho(u_*))^2 - \kappa_3^2\Big(\frac{1}{4} - \frac{3}{4\kappa_4}\rho(u_*)\big(\rho(u_*) - \frac{1}{3}\big)\Big). \tag{6.10}$$

Hence we need to prove the inequality

$$q(t)(1-2q(t))^2 + \frac{t}{4}q(t)\big(q(t) - \frac{1}{3}\big) \ge \frac{1}{4}\kappa_3^2,$$

or, by (6.9),

$$q^2(t)(1-2q(t)) \ge \frac{1}{4}\kappa_3^2. \tag{6.11}$$

Since $\frac{1}{3} < q(t) < r_1^2(0)$ and $|\kappa_3| \le f_2(\kappa_4) = 2r_1^2(0)\sqrt{1-2r_1^2(0)}$, this lower bound follows from the chain of inequalities

$$q^2(t)(1-2q(t)) \ge r_1^4(0)(1-2r_1^2(0)) \ge \frac{1}{4}\kappa_3^2.$$

Therefore (6.4) does not hold in the case $|\kappa_3| \le f_2(\kappa_4)$ and $0 < \kappa_4 \le \frac{1}{4}$.

We shall assume that $\kappa_4 > \frac{1}{4}$ and $\kappa_3 \in \mathbb{R}$. In this case (6.4) holds for $x = 0$.

We now assume that $|\kappa_3| \le f_1(\kappa_4)$ and $-\frac{1}{12} \le \kappa_4 < 0$. In this case (6.4) does not hold for $x \in (-1/2, 1/2)$. We need to prove (6.5) for all $1/2 \le |x| \le 1$. Without loss of generality we assume that $\kappa_3 > 0$ and $x < 0$. We see that (6.5) is valid at the points $x = -\frac{1}{2}$ and $x = -1$ if and only if $|\kappa_3| \le f_1(\kappa_4) = r_1(1)(1 - 2r_1^2(1))$. For $x \in (-1, -\frac{1}{2})$, the inequality (6.5) is equivalent to (6.6) for $u \in (\frac{1}{4}, 1)$. Note that, $\frac{1}{3} \le \rho(u) \le r_1^2(1) \le \frac{1}{2}$ for $u \in [\frac{1}{4}, 1]$. In order to find $\min_{u\in(1/4,1)} g(u)$ we find points $u_* \in (1/4, 1)$, where $g'(u_*) = 0$. For this we solve the equation (6.7) and choose solutions $\rho(u_*) \in (\frac{1}{3}, r_1^2(1))$. But, by (6.8), the solutions of (6.7) does not belong to $(\frac{1}{3}, r_1^2(1))$ in the case $t < 0$. Therefore (6.6) holds for $u \in [\frac{1}{4}, 1]$ under the conditions $|\kappa_3| \le f_1(\kappa_4)$ and $-\frac{1}{12} \le \kappa_4 < 0$. Hence (6.4) does not hold in the case $|\kappa_3| \le r_1(1)(1 - 2r_1^2(1))$ and $-\frac{1}{12} \le \kappa_4 < 0$ as well.

Assume that $\kappa_4 < -\frac{1}{12}$ and $\kappa_3 \in \mathbb{R}$. In this case (6.4) holds for $x = -sign(\kappa_3)$ if $\kappa_3 \ne 0$ and for $x = 1$ if $\kappa_3 = 0$.

Assume finally that $\kappa_4 = 0$. We easily conclude from (6.5) for $x \in [-1, 1]$ that the condition $|\kappa_3| \le \frac{1}{3\sqrt{3}}$ is necessary and sufficient in order that the sequence $0, 1, \kappa_3, 0, 0, \dots$ is a sequence of free cumulants of some p-measure. □

7. NECESSITY OF CONDITIONS FOR THE CHARACTERIZATION OF SEMICIRCULAR MEASURES. AUXILLIARY RESULTS

In order to prove Theorem 2.1 we need the following results.

The first of them is a description of ⊞-stable distributions (see [6] and [4]).

Lemma 7.1. *Every ⊞-stable p-measure is equivalent to a unique p-measure whose Voiculescu transform is given by one of the following*

(1) $\phi(z) = z^{-1}$;
(2) $\phi(z) = e^{i(\alpha-2)\rho\pi} z^{-\alpha+1}$ *with* $1 < \alpha < 2,\ 0 \le \rho \le 1$;
(3) (*i*) $\phi(z) = 0$,
  (*ii*) $\phi(z) = -2\rho i + 2(2\rho - 1)/\pi \log z$ *with* $0 \le \rho \le 1$;
(4) $\phi(z) = -e^{i\alpha\rho\pi} z^{-\alpha+1}$ *with* $0 < \alpha < 1,\ 0 \le \rho \le 1$.

Here and in the sequel we choose the principal branch of the functions $z^{-\alpha+1}$ and $\log z$.

The stability index of a ⊞-stable p-measure is equal to 2 in case (1), to α in cases (2) and (4), and to 1 in case (3). The parameter ρ which appears in cases (2), (3) and (4) will be called the *assymmetry coefficient*, and one can see that the measure corresponding to the parameters (α, ρ) is the image of the measure with parameters $(\alpha, 1 - \rho)$ by the map $t \mapsto -t$ on $\mathbb{R}$.

The next two lemmas are an analogue of of a result by Linnik (see [19], [20]).

Lemma 7.2. *Let* $\alpha > 1$ *and* $\alpha \ne 2m + 1$, *where* $m \in \mathbb{N}$. *The function*

$$\phi(z) = \frac{1}{z} - \varepsilon \cos\big(\alpha\pi/2\big)\frac{ie^{i\alpha\pi/2}}{z^{\alpha}}, \quad z \in \mathbb{C}^+,$$

with sufficiently small parameter $\varepsilon > 0$ *is the Voiculescu transform of some symmetric p-measure.*

Proof. Define the following region:

$$\Omega_{\alpha,\varepsilon}=\Big\{re^{i\theta}\in\mathbb{C}^+:0<\theta<\pi,\ r^{\alpha}\Big(r-\frac{1}{r}\Big)>b\frac{\cos\big(\alpha(\theta-\pi/2)\big)}{\sin\theta}\Big\},$$

where $b:=\varepsilon\cos(\alpha\pi/2)$. The region $\Omega_{\alpha,\varepsilon}$ is a Jordan domain with boundary curve $\gamma:r=r(\theta)$, $0<\theta<\pi$, where r is defined by the equation:

$$r^{\alpha}\Big(r-\frac{1}{r}\Big)=b\frac{\cos\big(\alpha(\theta-\pi/2)\big)}{\sin\theta}. \tag{7.1}$$

We see that (7.1) has an unique solution $r(\theta)$ which is greater than 1 for θ such that $b\cos\big(\alpha(\theta-\pi/2)\big)>0$. If $b\cos\big(\alpha(\theta-\pi/2)\big)<0$, (7.1) has two solutions $r(\theta)<1$. We choose the larger of them. If $\cos\big(\alpha(\theta-\pi/2)\big)=0$, then $r=1$.

Note that the function $\phi(z):\Omega_{\alpha,\varepsilon}\to\mathbb{C}$ is analytic with

$$\lim_{R\to+\infty}\max_{|z|=R}|\phi(z)|=0. \tag{7.2}$$

We shall now show that the function $f:\Omega_{\alpha,\varepsilon}\to\mathbb{C}$ defined via $z\mapsto z+\phi(z)$ takes every value in $\mathbb{C}^+$ precisely once. The inverse $f^{(-1)}:\mathbb{C}^+\to\mathbb{C}^+$ thus defined is in the class $\mathcal{F}$.

Denote by a_R, $\operatorname{Re}a_n>0$, a point of an intersection of the curve γ with the circle $|z|=R$ with sufficiently large $R\ge R_0$.

For every fixed $w\in\mathbb{C}^+$ we consider a closed rectifiable curve γ_1 consisting of some smooth curve $\gamma_{1,1}$, which is a part of the curve γ, connecting $-\overline{a}_R$ to a_R, the arc $\gamma_{1,2}$ of the circle $|z|=R$ connecting a_R to $-\overline{a}_R$. The curve γ_1 depend on R.

We see from the construction of the curve $\gamma_{1,1}$ that if z runs through $\gamma_{1,1}$ the image $\zeta=f(z)$ lies on the interval $[-A_R,A_R]$, where $f(-\overline{a}_R)=-A_R$ and $f(a_R)=A_R$. Here $A_R\to\infty$ as $R\to\infty$. We note as well that if z runs through $\gamma_{1,2}$ the image $\zeta=f(z)$ lies in the domain $|\zeta|\ge A_R/2$, $\operatorname{Im}\zeta>0$.

Hence $f(z)$ winds around w once, and it follows from the argument principle that inside the curve γ_1 there is a unique point z_0 such that $f(z_0)=w$. Since this relation holds for all sufficiently large $R>0$ and all curves γ_1, we deduce that the point z_0 is unique in $\Omega_{\alpha,\varepsilon}$.

Hence the inverse function $f^{(-1)}:\mathbb{C}^+\to\mathbb{C}^+$ exists and is analytic in $\mathbb{C}^+$. By condition (7.2), $\lim_{y\to+\infty}(iy/f^{(-1)}(iy))=1$ and therefore $f^{(-1)}\in\mathcal{F}$. This proves our assertion.

□

Lemma 7.3. *The function*

$$\phi(z)=\frac{1+\varepsilon\,(\log z-i\pi/2)}{z},\quad z\in\mathbb{C}^+,$$

with sufficiently small parameter $\varepsilon>0$ *is the Voiculescu transform of some symmetric p-measure.*

Proof. Denote $z=re^{i\theta}$, $r>0, 0<\theta<\pi$, and consider the function

$$\psi(r,\theta):=r\sin\theta+\operatorname{Im}\phi(r^{i\theta})=\Big(r-\frac{1}{r}\Big)\sin\theta-\frac{\varepsilon\sin\theta}{r}\log r+\frac{\varepsilon\cos\theta}{r}(\theta-\pi/2).$$

We see from this formula that $\psi(1,\theta) \le 0$ for $0 \le \theta \le \pi$. In addition, for every fixed $\theta \in (0,\pi)$, $\psi(r,\theta) \to +\infty$ as $r \to +\infty$. Hence, for every fixed $\theta \in (0,\pi)$, there exist points $r_j \ge 1$ such that $\psi(r_j,\theta) = 0$. Denote by $r(\theta)$ their maximum. Note that $r(\theta) \to \infty$ as $\theta \to 0$ or $\theta \to \pi$.

Introduce the curve γ by the equation $\gamma : r = r(\theta)$, $0 < \theta < \pi$. Denote by Ω_ε the domain in $\mathbb{C}^+$ bounded by the curve γ.

Note that the function $\phi(z) : \Omega_\varepsilon \to \mathbb{C}$ is analytic with

$$\lim_{R\to+\infty} \max_{|z|=R} |\phi(z)| = 0. \tag{7.3}$$

We shall now show that the function $f : \Omega_\varepsilon \to \mathbb{C}$ defined via $z \mapsto z + \phi(z)$ takes every value in $\mathbb{C}^+$ precisely once. The inverse $f^{(-1)} : \mathbb{C}^+ \to \mathbb{C}^+$ thus defined is in the class $\mathcal{F}$.

We define a closed rectifiable curve γ_1 in the same way as in the proof of Lemma 7.2. Repeating the argument of the proof of Lemma 7.2, we conclude that if z runs through γ_1 the image $\zeta = f(z)$ winds around every fixed point $w \in \mathbb{C}^+$ once, and it follows from the argument principle that inside the curve γ_1 there is a unique point z_0 such that $f(z_0) = w$. Since this relation holds for all sufficiently large $R > 0$ and all curves γ_1, we deduce that the point z_0 is unique in Ω_ε.

Hence the inverse function $f^{(-1)} : \mathbb{C}^+ \to \mathbb{C}^+$ exists and is analytic in $\mathbb{C}^+$. By condition (7.3), $\lim_{y\to+\infty}(iy/f^{(-1)}(iy)) = 1$ and therefore $f^{(-1)} \in \mathcal{F}$. This proves the lemma. □

Remark 7.4. *Using similar arguments as those in the proof of Lemma* 7.3 *one can prove a more general result.*

Let m be a positive integer. The function

$$\phi(z) = \frac{1 + \varepsilon\,(\log z - i\pi/2)^m}{z}, \quad z \in \mathbb{C}^+,$$

with sufficiently small parameter $\varepsilon > 0$ is the Voiculescu transform of some symmetric p-measure.

8. CHARACTERIZATION OF SEMICIRCULAR MEASURES

In this section we shall prove Theorem 2.7 and Theorem 2.9. We use in the proof of these theorems some ideas of the papers [19], [20] and [33].

Proof of Theorem 2.7. First we note that in the case where μ is a semicircular distribution the statictics L_1 and L_2 are identically distributed if and only if $\Lambda_1(2) = 0$.

Therefore we assume that $\Lambda_1(2) = \Lambda_2(2) = 0$.

Sufficiency. Let μ be the distribution of the free random variables $T_1, \dots, T_n$ which satisfy the relation $L_1 \overset{D}{=} L_2$. The Voiculescu transform $\phi_\mu(z)$ of the probability measure μ is defined in a domain $\Gamma_{\alpha,\beta}$ with some $\alpha > 0$ and $\beta > 0$. Since one can extend the function $G_\mu(z)$ on $\mathbb{C}^-$ assuming $G_\mu(z) = \overline{G_\mu(\bar z)}$, we can extend the Voiculescu transform $\phi_\mu(z)$ on the domain $-\Gamma_{\alpha,\beta}$ assuming $\phi_\mu(z) = \overline{\phi_\mu(\bar z)}$. Now we note that the Voiculescu transform

$\phi_\mu(z)$ satisfies the following equation

$$\sum_{j=1}^{n} a_j\phi_\mu(z/a_j) = \sum_{k=1}^{n} b_k\phi_\mu(z/b_k) \quad \text{for all} \quad z \in \Gamma_{\alpha,\beta}. \tag{8.1}$$

Without loss of generality we assume that $\beta = 1$. As shown above, $\operatorname{Im}\phi_\mu(z) \le 0$ for $z \in \Gamma_{\alpha,1}$. Denote

$$v(t) := t\operatorname{Im}\phi_\mu(i/t), \qquad 0 < t < 1. \tag{8.2}$$

Note that the function $v(t), t \in (0,1)$, is infinitely differentiable and $v(t) \to 0$ as $t \to 0+$. Moreover $v(at) = v(-at)$ for real a and $t \in (0,1)$, therefore it follows from (8.1) that

$$\sum_{j=1}^{n} v(|a_j|t) = \sum_{k=1}^{n} v(|b_k|t), \qquad 0 < t < 1. \tag{8.3}$$

In the sequel we consider special solutions of these equations.

We shall apply the auxilliary results of Section 4 which describe solutions of the equation (8.3) in the case $Q := \delta_{|a_1|} + \dots + \delta_{|a_1|} - \delta_{|b_1|} + \dots + \delta_{|b_n|}$ and $v(t) := t\operatorname{Im}\phi_\mu(i/t)$. First we shall prove the following lemma.

Lemma 8.1. *The parameter* $\sigma_1(v)$*, defined in* (4.10)*, for a solution* $v(t)$ *of* (8.3) *is an active exponent and* $0 < \sigma_1(v) \le 2$, $a_{\sigma_1(v)} > 0$.

This lemma shows that if $v(t)$ from (8.2) is a solution of (8.3), then $\Lambda_1(z)$ has a root γ such that $0 < \gamma \le 2$.

Proof. By Lemma 4.2, we only need to prove the inequality $\sigma_1(v) \le 2$. Let us assume to the contrary that $\sigma_1(v) = 2+\eta$, $\eta > 0$. By the definition of $\sigma_1(v)$ we see that the function $X(z;v)$ is analytic for $\operatorname{Re} z > -2-\eta$. Since $v(t) \ge 0$, $0 < t < 1$, we conclude by Lévy's and Raikov's theorem (see [21], Ch. 2, Theorem 2.2.1) that

$$\int_0^1 t^{-3-\eta/2} v(t)\,dt < \infty.$$

It follows from this relation that there exists a sequence $\{t_l\}_{l=1}^{\infty}$ such that $t_l \to 0$ as $l \to \infty$ and for which

$$\lim_{l\to\infty} v(t_l)/t_l^2 = 0. \tag{8.4}$$

By Proposition 3.1,

$$\phi_\mu(z) = z^2\Big(G_\mu(z) - \frac{1}{z}\Big)(1 + q_\mu(z)), \quad z \in \Gamma_{\alpha_1,\beta_1}, \tag{8.5}$$

where $|q_\mu(z)| = o(1)$ as $z \to \infty$ nontangentially to $\mathbb{R}$. Denote by $\overline{\mu}$ the p-measure such that $\overline{\mu}(S) := \mu(-S)$ for any Borel set S. It is easy to see that $\operatorname{Im}\phi_\mu(iy) = \frac{1}{2}\operatorname{Im}\phi_{\mu\boxplus\overline{\mu}}(iy)$

for $y \geq y_0 > 0$. In addition the measure $\mu \boxplus \overline{\mu}$ is symmetric. Therefore it easily follows from (8.5) that the relation

$$\begin{aligned}\operatorname{Im}\phi_\mu(iy) &= -\frac{y^2}{2}\operatorname{Im}\Big(G_{\mu\boxplus\overline{\mu}}(iy)-\frac{1}{iy}\Big)(1+\operatorname{Re} q_{\mu\boxplus\overline{\mu}}(iy))\\ &= -\frac{1}{2y}\int\limits_{\mathbb{R}}\frac{u^2}{u^2+y^2}\,(\mu\boxplus\overline{\mu})(du)(1+\operatorname{Re} q_{\mu\boxplus\overline{\mu}}(iy)), \qquad (8.6)\end{aligned}$$

holds, where $\operatorname{Re} q_{\mu\boxplus\overline{\mu}}(iy) \to 0$ as $y \to \infty$. We conclude from (8.4) and (8.6) that

$$\int\limits_{\mathbb{R}}\frac{u^2}{u^2+y_l^2}\,(\mu\boxplus\overline{\mu})(du) = o(1/y_l^2), \quad l \to \infty,$$

for $y_l := 1/t_l$. This relation implies $\int\limits_{\mathbb{R}} u^2\,(\mu\boxplus\overline{\mu})(du) = 0$ and therefore the measure $\mu\boxplus\overline{\mu} = \delta_0$, where δ_0 is the Dirac measure concentrated at the point 0. Since $\phi_{\mu\boxplus\overline{\mu}}(z) = \phi_\mu(z)+\phi_{\overline{\mu}}(z) = 0$ for $z \in \Gamma_{\alpha_1,\beta_1}$, and $\operatorname{Im}\phi_\mu(z) \leq 0$ and $\operatorname{Im}\phi_{\overline{\mu}}(z) \leq 0$ for such z, we easily conclude that $\phi_\mu(z) = 0, z \in \Gamma_{\alpha_1,\beta_1}$, and $\mu = \delta_0$, a contradiction. The lemma is proved. ☐

From the definition of the active exponent $\sigma_1(v)$ (see (4.8), where $K(z;v)$ and $\sigma(z)$ are defined in (4.5) with $Q := \delta_{|a_1|}+\dots+\delta_{|a_1|}-\delta_{|b_1|}+\dots+\delta_{|b_n|}$ and $v(t) := t\operatorname{Im}\phi_\mu(i/t))$ we conclude that $\sigma_1(v)$, $0 < \sigma_1(v) \leq 2$, is a root of the function $\Lambda_1(z)$.

By the assumptions of the theorem and Lemma 4.2 it follows that $\sigma_1(v) = 2$. Consider the function $v_1(t) := v(t)-a_2t^2$, where we have choosen $a_2 := a_2(v)(2+\lambda)^2$. The coefficient $a_2(v)$ and the parameter λ were chosen in Section 4. It is clear that $v_1(t)$ is a solution of equation (8.3). Moreover

$$\begin{aligned}K(z;v_1) &:= K(z;v) - a_2\int\limits_0^1\frac{s^{-z}-s^2}{z+2}\,dQ(s)\\ &= K(z;v)-a_2\frac{\Lambda_1(-z)-\Lambda_1(2)}{z+2} = K(z;v)-a_2\frac{\Lambda_1(-z)}{z+2}.\end{aligned}$$

Therefore

$$\begin{aligned}&Res_{z=-2}\Big(t^{\lambda-z}\frac{K(z;v_1)}{(z-\lambda)^2\Lambda_1(-z)}\Big)\\ &\quad = Res_{z=-2}\Big(t^{\lambda-z}\frac{K(z;v)}{(z-\lambda)^2\Lambda_1(-z)}\Big) - a_2 Res_{z=-2}\Big(\frac{t^{\lambda-z}}{(z-\lambda)^2(z+2)}\Big)\\ &\quad = t^{\lambda+2}\Big(a_2(v)-\frac{a_2}{(2+\lambda)^2}\Big) = 0. \qquad (8.7)\end{aligned}$$

Thus, we may choose a_2 in such a way that 2 is not an active exponent of the solution $v_1(t)$. Hence $v_1(t)$ has no active positive exponents.

Hence we arrive at two cases. In the first case $v_1(t) \neq 0$ in some interval $(0, t_0)$ with $0 < t_0 \le 1$. In the second case there exists a sequence $\{t_k\}$, $0 < t_k \le 1$, $\lim_{k\to\infty} t_k = 0$, such that $v_1(t_k) = 0$.

In the first case, by Lemma 4.2, there exists a positive active exponent of the solution $v_1(t)$, a contradiction. Hence, we may consider the second case only. In this case it is easy to see that the function $\phi_{\mu\boxplus\overline{\mu}}(z)$ satifies the assumptions of Lemma 3.8 and we obtain $\mu \boxplus \overline{\mu} = \mu_w$.

Thus, we proved that $\mu \boxplus \overline{\mu} = \mu_w$. In order to complete the proof of the sufficiency of the assumptions of the theorem it remains to apply the following lemma.

Lemma 8.2. *Assume that the function $\Lambda_2(z)$ satisfies the condition: $\Lambda_2(2k-1) \neq 0$ for all $k = 1, 2, \dots$. Let the statistics L_1 and L_2 be identically distributed and let $\mu \boxplus \overline{\mu}$ be a semicircular measure. Then μ is a semicircular measure as well.*

Proof. By Lemma 3.10, the measure μ has a compact support. Hence, the Voiculescu transform $\phi_\mu(z)$ is an analytic function in the domain $|z| > R$ with some parameter $R > 0$ and it admits in this domain the following Laurent expansion

$$\phi_\mu(z) = \kappa_1 + \frac{\kappa_2}{z} + \sum_{l=3}^{\infty} \frac{\kappa_l}{z^{l-1}}.$$

Here $\kappa_2 \ge 0$. Since $\mu \boxplus \overline{\mu}$ is a semicircular measure, we have, using (3.3),

$$\sum_{l=1}^{\infty} \frac{\kappa_{2l}}{z^{2l-1}} = \frac{b}{z}, \quad |z| > R,$$

where $b \ge 0$. From this formula we deduce that $\kappa_{2l} = 0$ for $l = 2, 3, \dots$. Since the function $\phi_\mu(z)$ satisfies the equation (8.1), we obtain the relation

$$\frac{\kappa_2 \Lambda_2(2)}{z} + \sum_{l=1}^{\infty} \frac{\kappa_{2l-1}\Lambda_2(2l-1)}{z^{2l-2}} = 0, \quad |z| > R.$$

By the assumptions of the lemma $\Lambda_2(2) = 0$ and $\Lambda_2(2l-1) \neq 0$ for $l = 1, 2, \dots$, we conclude that $\kappa_{2l-1} = 0$ for $l = 1, 2, \dots$.

Thus, the lemma is proved. □

Necessity. We note that in order that statement (1) of the theorem implies statement (2) it is necessary that $\Lambda_1(2) = 0$.

We shall first assume that the function $\Lambda_1(z)$ has a root γ_1 such that $0 < \gamma_1 < 2$. Let $0 < \gamma_1 < 1$ or $1 < \gamma_1 < 2$. By Lemma 7.1, there exist a symmetric p-measure μ whose the Voiculescu transform has the form $\phi_\mu(z) = -e^{i\gamma_1\pi/2} z^{-\gamma_1+1}$. We conclude for this function that

$$\sum_{j=1}^{n} a_j\phi_\mu(z/a_j) - \sum_{k=1}^{n} b_k\phi_\mu(z/b_k) = \sum_{j=1}^{n} |a_j|\phi_\mu(z/|a_j|) - \sum_{k=1}^{n} |b_k|\phi_\mu(z/|b_k|)$$
$$= -e^{i\gamma_1\pi/2}z^{-\gamma_1+1}\Lambda_1(\gamma_1) = 0, \quad z \in \mathbb{C}^+.$$

Let $\gamma_1 = 1$. By Lemma 7.1, there exist symmetric p-measure μ whose the Voiculescu transform has the form $\phi_\mu(z) = -i$. We obtain for this function

$$\sum_{j=1}^{n} a_j\phi_\mu(z/a_j) - \sum_{k=1}^{n} b_k\phi_\mu(z/b_k) = -i\Lambda_1(1) = 0, \quad z \in \mathbb{C}^+.$$

We shall now assume that $\gamma_1 = 2$ and 2 is not a simple root of the function $\Lambda_1(z)$. By Lemma 7.3, there exist a symmetric p-measure μ whose the Voiculescu transform has the form

$$\phi_\mu(z) = \frac{1 + \varepsilon(\log z - i\pi/2)}{z}, \quad z \in \mathbb{C}^+,$$

with sufficiently small parameter $\varepsilon > 0$. It is easy to see that

$$\sum_{j=1}^{n} a_j\phi_\mu(z/a_j) - \sum_{k=1}^{n} b_k\phi_\mu(z/b_k) = \Lambda_1(2)\frac{1}{z} + \frac{\varepsilon}{z}\sum_{s=0}^{1}(-1)^s\Big(\log z - \frac{i\pi}{2}\Big)^s\Lambda_1^{(1-s)}(2) = 0.$$

Assume that $\gamma_1 > 2$ and γ_1 is not even. By Lemma 7.2, there exist a symmetric p-measure μ whose the Voiculescu transform $\phi_\mu(z)$ has the form

$$\phi_\mu(z) = \frac{1}{z} - \varepsilon\cos\big((\gamma_1-1)\pi/2\big)\frac{ie^{i(\gamma_1-1)\pi/2}}{z^{\gamma_1-1}}, \quad z \in \mathbb{C}^+,$$

with sufficiently small parameter $\varepsilon > 0$. We deduce as above that

$$\sum_{j=1}^{n} a_j\phi_\mu(z/a_j) - \sum_{k=1}^{n} b_k\phi_\mu(z/b_k) = \Lambda_1(2)\,\frac{1}{z} - \varepsilon\cos\big((\gamma_1-1)\pi/2\big)\frac{ie^{i(\gamma_1-1)\pi/2}}{z^{\gamma_1-1}}\Lambda_1(\gamma_1) = 0$$

for $z \in \mathbb{C}^+$.

We shall now show that if there exists a positive integer $m > 2$ such that $\Lambda_2(m) = 0$, then the statement (2) of the theorem does not imply the statement (1). Using Corollary 2.4 (see [5] as well), consider a p-measure μ with the Voiculescu transform

$$\phi_\mu(z) := \frac{1}{z} + \frac{\varepsilon}{z^{m-1}},$$

where $\varepsilon \in \mathbb{R}$ and is sufficiently small by modulus. We easily see that the function $\phi_\mu(z)$ satisfies the equation (8.1). Moreover, the p-measure has a compact support. In the case $\Lambda_2(1) = 0$ we note that the function $\phi_\mu(z) := c$ with $c \neq 0$ (which corresponds to the Dirac measure $\mu = \delta_c$) satisfies the equation (8.1).

Thus, we have established that if 2 is not unique simple positive zero of the function $\Lambda_1(z)$ or there exist odd positive numbers $2l+1$ such that $\Lambda_2(2l+1)=0$ the statement (2) does not imply the statement (1) of the theorem.

The theorem is completely proved. $\Box$

Proof of Theorem 2.9. We keep all previous notations. First we assume that the statistics L_1 and L_2 are identically distributed. By the assumptions of the theorem, $m_{2s}(\mu)<\infty$ with $s:=[\gamma/2+1]$, where γ is maximum of the real parts of zeros of the function $\Lambda_1(z)$. By Proposition 3.3 and (8.2), we have

$$v(t):=t\operatorname{Im}\phi_\mu(i/t)=-\kappa_2(\mu)t^2+\dots+(-1)^s\kappa_{2s}(\mu)t^{2s}+o(t^{2s}),\quad t\to+0.$$

Therefore $\lim_{t\to+0}v^{(2s)}(t)=(-1)^s(2s)!\kappa_{2s}(\mu)$. We now conclude from Lemmas 4.2 and 4.3 that all active exponents of $v(t)$ are positive integers and simultaneously simple exponents. Since the number of active exponents of $v(t)$ is finite we can use the formula (4.9). Using this identity we easily obtain the relation

$$\frac{1}{2}\operatorname{Im}\phi_{\mu\boxplus\bar\mu}(i/t)=\operatorname{Im}\phi_\mu(i/t)=\sum_{l=1}^{2s}b_lt^{l-1},\quad 0<t<1,\tag{8.8}$$

where b_l, $l=1,\dots,2s$, are real coefficients. We deduce from (8.8) that $b_l=\frac{1}{2}\kappa_l(\mu\boxplus\bar\mu)$, $l=1,\dots,2s$, and $\kappa_l(\mu\boxplus\bar\mu)=0$ for $l\ge 2s+1$. The function $\phi_\mu(z)$ satisfies the equation (8.1). Therefore, using (8.8), we get

$$\sum_{l=1}^{2s}\Lambda_2(l)\frac{\kappa_l(\mu\boxplus\bar\mu)}{z^{l-1}}=0,\quad z\in\mathbb{C}^+.\tag{8.9}$$

We conclude from (8.9) that $\kappa_{2l}(\mu\boxplus\bar\mu)=0$ for $l=2,3,\dots$. Thus, $\mu\boxplus\bar\mu$ is a semicircular measure.

From the assumption of the theorem and Lemma 8.2 it follows that μ is a semicurcular measure as well.

One can prove the necessity of the assumptions of Theorem 2.9 in the same way as in the proof of the necessity of the assumptions of Theorem 2.7.

Thus, the theorem is completely proved. $\Box$

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