On the divisibility of the discriminant of an integer polynomial by prime powers.

October 14, 2008

V. Bernik
Institute of Mathematics, Surganova str. 11, 220072, Minsk, Belarus
(e-mail: bernik@im.bas-net.by)

F. Götze
Universität Bielefeld, Fakultät für Mathematik, Postfach 10 01 31, 33501 Bielefeld,
Germany
(e-mail: goetze@math.uni-bielefeld.de)

O. Kukso
Institute of Mathematics, Surganova str. 11, 220072, Minsk, Belarus
(e-mail: olga_kukso@tut.by)

Abstract

Let \( P \in \mathbb{Z}[t] \) with \( \deg P \leq n \) and height \( H = H(P) \). Let \( |w|_p \) be the \( p \)-adic valuation of \( w \in \mathbb{Q}_p \). Also define \( K \) to be a cylinder in \( \mathbb{Q}_p \) and a set \( B \subset K \). The Haar measure of \( B \) is denoted by \( \mu B \).

In this article we obtain a lower bound for the number of polynomials with fixed degree and height whose discriminants are multiples of large prime powers. The basic idea of the proof concerns the simultaneous metric theory of the size of integer polynomials and their derivatives. Namely, it is shown that for a given number \( c_0 \) and constants \( c_1, c_2 \) with \( c_1 c_2 < c_0 \) the system of inequalities

\[
\begin{cases}
|P(w)|_p < c_1 Q^{-n-1+v}, \\
|P'(w)|_p < c_2 Q^{-v}, \\
H(P) \leq Q
\end{cases}
\]

has solutions \( P \in \mathbb{Z}[t] \) only for a set \( B = \{w\} \) with \( \mu B \leq \frac{1}{2} \mu K \).

Keywords: Diophantine approximation, discriminant, \( p \)-adic valuation, Haar measure.

The basic characteristic of a real number is considered to be its value. For natural numbers \( q \) the same we could consider one of its basic characteristics to be its arithmetic
structure. By this, we mean the canonical decomposition of $q$ into its unique product of primes. It is well-known that the factorization of sufficiently large natural numbers demands a lot of computer time. The discriminant value may grow quickly with the growth of degree and height.

Let $P \in \mathbb{Z}[x]$ be of the form

$$P(x) = a_n x^n + \cdots + a_1 x + a_0, \quad H = H(P) = \max_{0 \leq j \leq n} |a_j|. \tag{1}$$

Here, $H$ denotes the height of the polynomial $P$. Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be the roots of $P$. Then the discriminant $D(P)$ of $P$ is defined as

$$D(P) = a_n^{2n-2} \prod_{1 \leq i < j \leq n} (\alpha_i - \alpha_j)^2. \tag{2}$$

The discriminant of the polynomial is a symmetric function of its roots. As it concerns the product of the square of the distances between the roots multiplied by $a_n^{2n-2}$ it is in fact a polynomial of the coefficients of $P$ and is therefore an integer. The discriminant $D(P)$ can also be written as the determinant of a matrix of order $2n - 1$ (for details see [12]). Therefore discriminant calculating for general polynomials with large $n$ and $H$ is hard to perform. Taking this into account and assuming that $D(P) \neq 0$ we obtain that there exists a constant $c(n)$ such that

$$1 \leq |D(P)| < c(n)H^{2n-2}. \tag{3}$$

Several papers, see [2 – 4], are devoted to the study of the set of possible values of the discriminants. In this article we ask: how often is $D(P)$ divisible by the power of a chosen prime $p$. The main method used is the metric theory of $p$–adic Diophantine approximation, the basis of which was developed by V.G. Sprindžuk in [4] when he considered Mahler’s conjecture and its analogue in the field of $p$–adic numbers.

Throughout the paper we will need the following notation. Let $p \geq 2$ be a fixed prime number with $\mathbb{Q}_p$ the field of $p$–adic numbers and $|w|_p$ the $p$–adic valuation of $w \in \mathbb{Q}_p$. Let $\mathbb{A}_{p,n}$ be the set of algebraic numbers of degree $n$ lying in $\mathbb{Q}_p$, $\mathbb{A}_p$ be the set of all algebraic numbers and finally, $\mathbb{Q}_p^\ast$ be the extension of $\mathbb{Q}_p$ containing $\mathbb{A}_p$. There is a natural extension of the $p$–adic valuation from $\mathbb{Q}_p$ to $\mathbb{Q}_p^\ast$, see [11, 6, 8]. This valuation will also be denoted by $| \cdot |_p$. The disc in $\mathbb{Q}_p$ of radius $r$ centred at $\alpha$ is the set of solutions of the inequality $|x - \alpha|_p < r$. The Haar measure of a set $S \subset \mathbb{Q}_p$ will be denoted by $\mu(S)$. The notation $X \ll Y$ will mean $X = O(Y)$ and $X \asymp Y$ means $X \ll Y \ll X$.

In 1989 Bernik [3] proved A. Baker’s conjecture by showing that for almost all $x \in \mathbb{R}$ the inequality $|P(x)| < H(P)^{-n+1}\Psi(H(P))$ has only finitely many solutions in $P \in \mathbb{Z}[x]$ with $\deg P \leq n$ whenever $\Psi : \mathbb{N} \rightarrow \mathbb{R}_+$ is monotonic and the sum

$$\sum_{h=1}^{\infty} \Psi(h) \tag{4}$$
converges. In 1999 Beresnevich [1] showed that if the sum in (4) diverges then the inequality has infinitely many solutions. These results of Bernik and Beresnevich have been generalised to the \( p \)-adic case, see [9, 2].

In this article we develop methods that were initially proposed in [11, 3, 1]. More precisely, for sufficiently large \( Q \in \mathbb{R} \) we obtain a lower bound for the number of polynomials in the set
\[
P_n(Q) = \{ P(x) \mid \deg P \leq n, \, H(P) \leq Q \},
\]
whose discriminants are divisible by large powers of a fixed prime number.

**Theorem 1** There are at least \( c(n)Q^{n+1-2v} \) polynomials \( P \in P_n(Q) \) with discriminants \( D(P) \) satisfying
\[
|D(P)|_p < Q^{-2v},
\]
where \( v \in [0, \frac{1}{2}] \).

**Theorem 2** Let \( Q \) denote a sufficiently large number and \( c_1 \) and \( c_2 \) denote constants depending only on \( n \). Also, let \( K \) be a disc in \( \mathbb{Q}_p \). Assume that \( c_1c_2 < 2^{-n-9}p^{-6} \) and \( v \in [0, \frac{1}{2}] \). If \( M_{n,Q}(c_1, c_2) \) is the set of \( w \in K \subset \mathbb{Q}_p \) such that the system of inequalities
\[
\begin{align*}
|P(w)|_p &< c_1Q^{-n-1+v}, \\
|P'(w)|_p &< c_2Q^{-v}
\end{align*}
\]
has solutions in polynomials \( P \in P_n(Q) \), then
\[
\mu(M_{n,Q}(c_1, c_2)) < \frac{\mu(K)}{2}.
\]

**Auxiliary statements.** The following lemmas are necessary for the proof of the theorems. After stating them we will show how Theorem 1 can be obtained from Theorem 2.

**Lemma 1** Let \( \alpha_1 \) be the nearest root of a polynomial \( P \) to \( w \in \mathbb{Q}_p \). Then
\[
|w - \alpha_1|_p \leq |P(w)|_p|P'(w)|_p^{-1},
\]
and
\[
|w - \alpha_1|_p \leq \min_{2 \leq j \leq n} \left( |P(w)|_p|P'(w)|_p^{-1} \prod_{k=2}^{j} |\alpha_1 - \alpha_j|_p \right)^{\frac{1}{j}},
\]
where \( |\alpha_1 - \alpha_2|_p \leq |\alpha_1 - \alpha_j|_p \) for \( 2 \leq j \leq n \).

For the proof see [11, 3, 5].

**Lemma 2** Let \( \delta > 0 \) and \( E(\delta) \) be the set of \( w \in \mathbb{Q}_p \) such that the inequality
\[
|P(w)|_p < H(P)^{-n-\delta}
\]
has infinitely many solutions in reducible polynomials \( P \in \mathbb{Z}[x] \) with \( \deg P \leq n \). Then \( \mu(E(\delta)) = 0 \).
The proof can be found in [3, 5].

A polynomial \( P \in \mathbb{Z}[x] \) with leading coefficient \( a_n \) will be called leading if

\[
|a_n| \gg H(P) \quad \text{and} \quad |a_n|_p > p^{-n}.
\]

Let \( \mathcal{P}_n(H) \) be the set of irreducible primitive leading polynomials \( P \in \mathbb{Z}[x] \) of degree \( n \) with height \( H(P) = H \). Also define

\[
\mathcal{P}_n = \bigcup_{H=1}^{\infty} \mathcal{P}_n(H).
\]

The reduction to leading polynomials is completed with the help of Lemma 2 and the following lemma.

**Lemma 3** Let \( P(x) = a_n x^n + \cdots + a_1 x + a_0 \) be an integer polynomial with discriminant \( D(P) \) such that \((a_0, a_1, \ldots, a_n) = 1 \) and \( H(P) \leq Q \). Another polynomial \( T(x) = b_n x^n + \cdots + b_1 x + b_0 \) can be constructed which satisfies the conditions:

\[
H(T) \leq Q, \quad D(T) = D(P), \quad |b_n|_p \gg 1.
\]

**Proof.** To begin we will prove that there exists \( m_0, 1 \leq m_0 \leq n+1 \), such that

\[
\max_{1 \leq k \leq n+1} |P(k)|_p = |P(m_0)|_p \gg 1.
\]

Assume that for some \( c_1 = p^d \) the system of inequalities

\[
\max_{1 \leq k \leq n+1} |P(k)|_p < |c_1|_p = p^{-d}
\]

holds. The assumptions in the statement of the lemma imply that there exists \( j_0, 0 \leq j_0 \leq n \), such that \( |a_{j_0}|_p = 1 \). Replace system (10) by the system of equations relative to \( a_i, 0 \leq i \leq n \), and solve it relatively \( a_{j_0} \). We obtain

\[
\begin{aligned}
|a_n + a_{n-1} + \cdots + a_1 + a_0|_p &= c_1 \theta_1, \\
|a_n 2^n + a_{n-1} 2^{n-1} + \cdots + a_1 2 + a_0|_p &= c_1 \theta_2, \\
&\cdots \\
|a_n (n+1)^n + a_{n-1}(n+1)^{n-1} + \cdots + a_1 (n+1) + a_0|_p &= c_1 \theta_{n+1},
\end{aligned}
\]

where \( \theta_i = p^{d_i}, \quad d_i \geq 1 \) and

\[
a_{j_0} = \frac{\Delta_{j_0}}{\Delta}.
\]

The determinant \( \Delta \) in (12) is the Van der Monde determinant:

\[
\Delta = \begin{vmatrix}
1 & 1 & \cdots & 1 \\
1 & 2 & \cdots & n+1 \\
\cdots & \cdots & \cdots & \cdots \\
1 & 2^n & \cdots & (n+1)^n
\end{vmatrix} = \prod_{k=0}^{n} k!(n-k)!.
\]
The prime number \( p \) divides \( k! \) with degree at most

\[
\left\lfloor \frac{k}{p} \right\rfloor + \left\lfloor \frac{k}{p^2} \right\rfloor + \cdots \leq k \sum_{j=1}^{\infty} p^{-j} \leq k,
\]

and \( \Delta \) contains degree at most \( n^n \). For the determinant \( \Delta_{j_0} \) the inequalities \( \Delta_{j_0} = c_1 \Delta'_{j_0} \) and \( |\Delta_{j_0}|_p \leq 1 \) hold; therefore \( |\Delta_{j_0}|_p \leq p^{-d} \). Since \( |\alpha_i|_p = 1 \) equation (12) gives a contradiction for \( d > n^n \).

We denote the roots of \( P \) by \( \alpha_1, \alpha_2, \ldots, \alpha_n \) and consider the polynomial

\[
P_1(x) = P(x + m_0) = a_n x^n + a'_{n-1} x^{n-1} + \cdots + a_1 x + P(m_0).
\]

Its roots are \( \beta_j = \alpha_j - m_0, 1 \leq j \leq n \), and its discriminant satisfies

\[
D(P_1) = a_n^{2n-2} \prod_{1 \leq i < j \leq n} (\beta_i - \beta_j)^2 = a_n^{2n-2} \prod_{1 \leq i < j \leq n} (\alpha_i - \alpha_j)^2 = D(P).
\]

The polynomial \( T(x) = x^n P_1(x) = P(m_0) x^n + a''_{n-1} x^{n-1} + \cdots + a_1 x + a_n \) has roots \( \gamma_j = \frac{1}{\beta_j} = \frac{1}{\alpha_j - m_0}, 1 \leq j \leq n \). The absolute value of the discriminant is

\[
|D(T)| = |P(m_0)|^{2n-2} \prod_{1 \leq i < j \leq n} (\beta_i - \beta_j)^2 \beta_i^{-2} \beta_j^{-2}.
\]

However, \( \prod_{1 \leq i < j \leq n} \beta_i^{-2} \beta_j^{-2} = \left( \frac{a_n}{P(m_0)} \right)^{2n-2} \), so that

\[
D(T) = a_n^{2n} \prod_{1 \leq i < j \leq n} (\beta_i - \beta_j)^2 = D(P).
\]

The condition \( H(T) < c(n) H \) is obviously satisfied as \( H(T) = H(P_1) \) and \( H(P_1) \asymp H(P) \).

**Lemma 4** Let \( \alpha_1, \ldots, \alpha_n \) be the roots of \( P \in \mathcal{P}_n \). Then \( \max_{1 \leq i \leq n} |\alpha_i|_p < p^n \).

For the proof see [11, p85].

For the roots \( \alpha_1, \ldots, \alpha_n \) of \( P \) we define the sets

\[
S(\alpha_i) = \{ w \in \mathbb{Q}_p : |w - \alpha_i|_p = \min_{1 \leq j \leq n} |w - \alpha_j|_p \} \ (1 \leq i \leq n).
\]

Let \( P \in \mathcal{P}_n \). As \( \alpha_1 \) is fixed, we reorder the other roots of \( P \) so that \( |\alpha_1 - \alpha_2|_p \leq |\alpha_1 - \alpha_3|_p \leq \cdots \leq |\alpha_1 - \alpha_n|_p \). We can assume that there exists a root \( \alpha_m \) of \( P \) for which \( |\alpha_1 - \alpha_m|_p \leq 1 \) (see [5, p99]). Then we have

\[
|\alpha_1 - \alpha_2|_p \leq |\alpha_1 - \alpha_3|_p \leq \cdots \leq |\alpha_1 - \alpha_m|_p \leq 1 \leq \cdots \leq |\alpha_1 - \alpha_n|_p.
\]  

(14)

Let \( \varepsilon > 0 \) be sufficiently small, \( d > 0 \) be a large fixed number, \( \varepsilon_1 = \frac{\varepsilon}{2} \), and \( T = \lfloor \varepsilon_1^{-1} \rfloor + 1 \). We define real numbers \( \mu_j \) and integers \( \ell_j \) by the relations

\[
|\alpha_1 - \alpha_j|_p = H^{-\mu_j}, \quad \frac{\ell_j}{T} \leq \mu_j < \frac{\ell_j + 1}{T} \quad (2 \leq j \leq m).
\]

(15)

5
It follows from (14) and (15) that \( \mu_2 \geq \mu_3 \geq \cdots \geq \mu_m \geq 0 \) and \( l_2 \geq l_3 \geq \cdots \geq l_m \geq 0 \). We assume that \( \mu = 0 \) and \( l_j = 0 \) if \( m < j \leq n \).

Now, for every polynomial \( P \in \mathcal{P}_n(H) \) we define a non-negative vector \( \vec{l} = (l_2, \ldots, l_n) \). In [5, p99–100] it is shown that the number of such vectors is finite and depends on \( n, p \) and \( T \) only. All polynomials \( P \in \mathcal{P}_n(H) \) corresponding to the same vector \( \vec{l} \) are grouped together into a class \( \mathcal{P}_n(H, \vec{l}) \). We define

\[
\mathcal{P}_n(\vec{l}) = \bigcup_{H=1}^{\infty} \mathcal{P}_n(H, \vec{l}).
\]

(16)

Let \( K_0 = \{ w \in \mathbb{Q}_p : |w|_p < p^n \} \) be the disc of radius \( p^n \) centered at 0. Define

\[
p_j = p_j(P) = \frac{l_{j+1} + \cdots + l_n}{T} \quad (1 \leq j \leq n - 1).
\]

Lemma 5 Let \( w \in S(\alpha_1) \) and \( P \in \mathcal{P}_n(H) \). Then

\[
H^{-p_1} \ll |P'(\alpha_1)|_p \ll H^{-p_1 + (m-1)\varepsilon_1},
\]

\[
|P^{(j)}(\alpha_1)|_p \ll H^{-p_j + (m-j)\varepsilon_1} \quad \text{for} \quad 2 \leq j \leq m,
\]

and

\[
|P^{(j)}(\alpha_1)|_p \ll 1 \quad \text{for} \quad m < j \leq n.
\]

Proof. From Lemma 4 we have \( p^{-n} < |H|_p \leq 1 \). Then, after differentiating \( j \) times \( P(w) = H(w - \alpha_1) \ldots (w - \alpha_n) \) \( (1 \leq j \leq n) \) and using (14) and (15) the lemma is proved.

Lemma 6 Let \( \delta, \sigma \in \mathbb{R}_+ \). Further, let \( P, Q \in \mathbb{Z}[x] \) be two relatively prime polynomials of degree at most \( n \) with \( \max(H(P), H(Q)) \leq H \) and \( H > H_0(\delta) \). Let \( K(\alpha, H^{-\eta}) \) be a disc of radius \( H^{-\eta} \) centred at \( \alpha \). If there exists a number \( \tau > 0 \) such that for all \( w \in K(\alpha, H^{-\eta}) \),

\[
\max(|P(w)|_p, |Q(w)|_p) < H^{-\tau},
\]

then

\[
\tau + 2\max(\tau - \eta, 0) < 2n + \delta.
\]

For the proof see [4, Lemma 5].

Lemma 7 (Hensel) Let \( P \in \mathbb{Z}_p[x] \), \( \xi = \xi_0 \in \mathbb{Z}_p \) and \( |P(\xi)|_p \leq |P'(\xi)|_p^2 \). Then, for \( n \to \infty \) the sequence

\[
\xi_{n+1} = \xi_n - \frac{P(\xi_n)}{P'(\xi_n)}
\]

tends to some root \( \alpha \in \mathbb{Z}_p \) of the polynomial \( P \) and

\[
|\alpha - \xi_n|_p \leq \frac{|P(\xi_n)|_p}{|P'(\xi_n)|_p^{2}}.
\]

6
For the proof see [3, p134].

**Lemma 8** For any $Q > 1$, $w \in \mathbb{Q}_p$, $v \in [0, \frac{1}{2}]$ and $c_1 c_2 > p^3$ the system of inequalities

\[
\begin{align*}
|P(w)|_p &< c_1 Q^{-n-1+v}, \\
|P'(w)|_p &< c_2 Q^{-v},
\end{align*}
\]

has solutions in integer polynomials $P$ with $H(P) < Q$.

**Proof.** We will use Dirichlet’s pigeonhole principle. First, choose positive integer $l_1$ and $l_2$ such that

\[
p^{l_1 + l_2} < Q^{n+1} < p^{l_1 + l_2 + 1}.
\]

The number of possible values of $P(w) = a_1 + a_2 p + \cdots + a_k p^k + \cdots$ which have different vectors $\bar{a} = (a_0, a_1, \ldots, a_{l_1-1})$ is $p^{l_1}$. Similarly, the number of possible values of $P'(w) = b_1 + b_2 p + \cdots + b_k p^k + \cdots$ with different vectors $\bar{b} = (b_0, b_1, \ldots, b_{l_2-1})$ is $p^{l_2}$. The number of polynomials $P(x) = d_n x^n + d_{n-1} x^{n-1} + \cdots + d_1 x + d_0$ for $0 \leq d_i \leq Q$ is at least $Q^{n+1}$. Therefore, for $Q^{n+1} > p^{l_1 + l_2}$ there are at least two polynomials $P_1$ and $P_2$ which have the same vectors $\bar{a}$ and $\bar{b}$. Let $R(x) = P_1(x) - P_2(x)$. It is clear that $H(R) \leq Q$. Now, for fixed $c_1$, $c_2$ and $v$ choose $l_1$ and $l_2$ such that

\[
\begin{align*}
|P^{l_1 - 1} &< c_1^{-1} Q^{n+1-v} < p^{l_1}, \\
|P^{l_2 - 1} &< c_2^{-1} Q^v < p^{l_2}.
\end{align*}
\]

Since $|R(w)|_p \leq p^{-l_1}$ and $|R'(w)|_p \leq p^{-l_2}$ these inequalities imply that

\[
|R(w)|_p \leq p^{-l_1} < c_1 Q^{-n-1+v},
\]

and

\[
|R'(w)|_p \leq p^{-l_2} < c_1 Q^{-v}.
\]

Inequalities (18) together with (17) hold for $c_1 c_2 > p^3$ which completes the proof.

**Proof of Theorem 1 modulo Theorem 2.**

We will now show that Theorem 1 follows from Theorem 2. Let $w_1 \in B_1 = K \setminus M_{n,Q}(c_1, c_2)$ such that $\mu(B_1) \geq \frac{\mu(K)}{2}$. Assume that $\gamma = p^{-6} 2^{-n-9}$. Then, there exists a polynomial $P_1 \in (P)_n(Q)$ such that the following system of inequalities

\[
\begin{align*}
\gamma Q^{-n-1+v} &\leq |P_1(w_1)|_p < Q^{-n-1+v}, \\
\gamma Q^{-v} &\leq |P'_1(w_1)|_p < p^3 Q^{-v}
\end{align*}
\]

holds. The upper bounds follow from Lemma 8. If the first condition of (19) does not hold then $|P_1(w_1)|_p > \gamma Q^{-n-1+v}$ and $|P'_1(w_1)|_p < p^3 Q^{-v}$. This implies that $w_1$ belongs to a set with measure at most $\frac{\mu(K)}{4}$. If the second condition of (19) does not hold then $w_1$ also belongs to a set with measure at most $\frac{\mu(K)}{4}$. This contradicts the choice of $w_1$.  

7
Let \( w \in K_1 \) where \( K_1 \) is the disc \( \{ w \in \mathbb{Q}_p : |w - w_1|_p < Q^{-\frac{3}{2}} \} \) and obtain the Taylor representation of \( P_1'(w) \) on this disc so that

\[
P_1'(w) = P_1'(w_1) + \sum_{i=2}^{n} ((i - 1)!)^{-1} P_1^{(i)}(w_1)(w - w_1)^{i-1}.
\]

Since

\[
|(i - 1)!|_p^{-1} |P_1^{(i)}(w_1)|_p |w - w_1|_p^{i-1} \ll Q^{-\frac{3}{2}},
\]

and

\[
|P_1'(w)|_p \geq \gamma Q^{-v} \gg Q^{-\frac{3}{2}},
\]

for all \( w \in K_1 \) we obtain that \( |P_1'(w)|_p = |P_1'(w_1)|_p \). Let \( \alpha_1 \) be the closest root of \( P_1(w) \) to the point \( w_1 \). From Lemma 1

\[
|w_1 - \alpha|_p \leq |P_1(w_1)|_p |P_1'(w_1)|_p^{-1} \leq \gamma^{-1}Q^{-n-1+2v}.
\]  

(20)

To estimate the distance between \( w_1 \) and the root of the polynomial we can also apply Lemma 7. Since \( |P_1(w_1)|_p < |P_1'(w_1)|_p^2 \) we obtain that the sequence in Lemma 7 converges to the root \( \alpha_1 \in \mathbb{Q}_p \) and

\[
|w_1 - \alpha_1|_p \leq |P_1(w_1)|_p |P_1'(w_1)|_p^{-2} \leq \gamma^{-2}Q^{-n-1+3v}.
\]  

(21)

Since \( 0 < \gamma < 1 \) and \( v > 0 \) the right part of (20) is greater than the right part of (21). That imply the root \( \alpha \) belongs to the disc with center \( w_1 \) of radius less than the radius for the disc defined for the root \( \alpha_1 \). From Lemma 7 we find that \( \alpha_1 \in \mathbb{Q}_p \) but Lemma 1 does not guarantee that \( \alpha \in \mathbb{Q}_p \). Suppose that \( \alpha \neq \alpha_1 \) and consider the discriminant of the polynomial \( P \in \mathcal{P}_n(Q) \)

\[
D(P) = a_n^{2n-2} \prod_{1 \leq i < j \leq n} (\alpha_i - \alpha_j)^2.
\]  

(22)

Lemma 4 implies that \( |\alpha_i - \alpha_j|_p \ll 1 \). The product in (22) contains the difference \( (\alpha - \alpha_1) \) for some \( i \neq j \). We have \( D(P) \in \mathbb{Z} \) and \( |D(P)| \ll Q^{2n-2} \); therefore \( |D(P)|_p \gg Q^{-2n+2} \). From (20) and (21) we further obtain that

\[
|\alpha_1 - \alpha|_p = |\alpha_1 - w_1 + w_1 - \alpha|_p \leq \max\{|w_1 - \alpha_1|_p, |w_1 - \alpha|_p\} \leq \gamma^{-2}Q^{-n-1+3v}.
\]

Therefore

\[
Q^{-2n+2} \ll |D(P)|_p \ll |\alpha_1 - \alpha|_p^2 < \gamma^{-4}Q^{-2n-2+6v}.
\]  

(23)

For \( v \leq \frac{1}{2} \) and \( Q > Q_0 \) the inequality \( Q^{-2n+2} \ll \gamma^{-4}Q^{-2n-2+6v} \) is a contradiction. Hence, \( \alpha_1 = \alpha \). Thus, \( \alpha \in \mathbb{Q}_p \) and \( |w_1 - \alpha|_p \) satisfies condition (20).

From the representation of \( P'(\alpha) \) as a Taylor series

\[
P_1'(\alpha) = P_1'(w_1) + \sum_{i=2}^{n} ((i - 1)!)^{-1} P_1^{(i)}(w_1)(\alpha - w_1)^{i-1},
\]  

(24)
the estimate
\[ |(i - 1)!|^{-1}|P^{(i)}(w_1)|_p|α - w_1|_p^{i-1} \ll Q^{-n-1+2v} \]
and condition (20) we obtain
\[ |P'_1(α)|_p = |P'_1(w_1)|_p < p^3 Q^{-v}. \]
Therefore
\[ |D(P)|_p = |a_n^2 \prod_{k=2}^n (α_k - α) \prod_{i<j} (α_i - α_j)|_p \ll |P'(α)|_p^2 \ll Q^{-2v}. \]  
Inequality (20) implies that in the neighbourhood of the point \( w_1 \in B_1 \) there exists a root \( α \) of the polynomial \( P_1 \) with discriminant satisfying (25).

Further, let \( w_2 \in B_1 \) and construct a polynomial \( P_2 \) such that its discriminant satisfies (25). If \( |w_1 - w_2|_p > γ^{-1} Q^{-n-1+2v} \) then the polynomials \( P_1(w) \) and \( P_2(w) \) are different. From the Taylor expansion of \( P_1 \) at the point \( w = w_2 \) we obtain
\[ P_1(w_2) = P_1(w_1) + P'_1(w_1)(w_2 - w_1) + \sum_{i=2}^n (i!|^{-1}P^{(i)}(w_1)(w_2 - w_1)^i. \]  
Since
\[ |i!|^{-1}|P^{(i)}(w_1)|_p|w_2 - w_1|^i \ll Q^{-2n} \quad (i \geq 2), \]
\[ |P'_1(w_1)|_p|w_2 - w_1|_p > γQ^{-v}γ^{-1}Q^{-n-1+2v} = Q^{-n+1+v}, \]
and
\[ |P_1(w_1)|_p < Q^{-n-1+v}, \]
we can obtain from (26) that
\[ |P_1(w_2)|_p > Q^{-n-1+v}. \]
By construction \( |P_1(w_2)|_p < Q^{-n-1+v} \) which proves that \( P_2(w) \neq P_1(w_1) \). Repeating the described procedure several times, excluding at each step the disk \( K(w_i, c(n)Q^{-n-1+2v}) \), we can construct \( k \geq Q^{n+1-2v} \) points \( w_i \) and polynomials \( P_i \) corresponding to them, such that
\[ |D(P_i)|_p < Q^{-2v}. \]

Proof of Theorem 2.

From Lemma 8 for \( c_1 = 1, c_2 = p^3, Q > 1 \) and arbitrary \( w \in \mathbb{Q}_p \) the system
\[ \begin{cases} |P(w)|_p < Q^{-n-1+v}, \\ |P'(w)|_p < p^3 Q^{-v}, \end{cases} \]  
holds for some polynomial \( P \) with \( H(P) \leq Q \).
Denote by $M_1(c_3, c_4)$ the set of $w$ for which the system of inequalities
\begin{align}
&\begin{cases}
|P(w)|_p < c_3Q^{-n-1+v}, \\
Q^{-\frac{3}{4}} \leq |P'(w)|_p < c_4p^3Q^{-v},
\end{cases} 
\tag{28}
\end{align}
always has solutions in polynomials $P \in \mathcal{P}_n(Q)$. We again use the Taylor expansion of $P'(w)$ in the disc $\{w \in \mathbb{Q}_p : |w - \alpha|_p < Q^{-\frac{3}{4}}\}$, where $\alpha$ is the closest root to $w$. We have
\begin{align}
P'(w) = P'(\alpha) + \sum_{j=2}^{n} ((j - 1)!)^{-1}P^{(j)}(\alpha)(w - \alpha)^{j-1},
\tag{29}
\end{align}
For any $j, 2 \leq j \leq n$, we have
\begin{align}
|(j - 1)!|_p^{-1}|P^{(j)}(\alpha)|_p|w - \alpha|_p^{j-1} \ll Q^{-3/4},
\tag{30}
\end{align}
therefore from (29) and (30) it follows that $|P'(\alpha)|_p = |P'(w)|_p$ and the system of inequalities (27) can be rewritten as
\begin{align}
&\begin{cases}
|P(w)|_p < c_3Q^{-n-1+v}, \\
Q^{-\frac{3}{4}} \leq |P'(\alpha)|_p < c_4p^3Q^{-v}.
\end{cases} 
\tag{31}
\end{align}
Now we prove that $\mu(M_1(c_3, c_4)) \leq \frac{\mu(K)}{8}$. By Lemma 1 and (31) we have
\begin{align}
|w - \alpha|_p < c_3Q^{-n-1+v}|P'(\alpha)|_p^{-1}.
\tag{32}
\end{align}
Let $\sigma_1(P)$ denote the set of solutions of inequality (32). We introduce the disc $\sigma_2(P)$
\begin{align}
\sigma_2(P) := \{w \in \mathbb{Q}_p : |w - \alpha|_p < c_3Q^{-2+v}|P'(\alpha)|_p^{-1}\},
\tag{33}
\end{align}
where the value $c_5$ will be specified later. Obviously
\begin{align}
\mu(\sigma_1(P)) < c_3c_5^{-1}Q^{-n+1}\mu(\sigma_2(P)).
\tag{34}
\end{align}
Fix the vector $\bar{b} = (a_n, \ldots, a_2)$ of coefficients of $P$. All polynomials with the same vector $\bar{b}$ form the class $\mathcal{P}(\bar{b})$. Thus we divide $\mathcal{P}_n(Q)$ into classes according to the vector $\bar{b}$.
The disc $\sigma_2(P_1), P_1 \in \mathcal{P}(\bar{b})$, is called essential if for any $P_2 \in \mathcal{P}(\bar{b})$
\begin{align}
\sigma_2(P_1) \cap \sigma_2(P_2) = \emptyset
\tag{35}
\end{align}
holds. The disc $\sigma_2(P_1), P_1 \in \mathcal{P}(\bar{b})$, is called inessential otherwise; i.e. if it intersects another disc $\sigma_2(P_2)$ with $P_2 \in \mathcal{P}(\bar{b})$.

**a) The case of essential discs.** In this case every point $w \in K$ belongs to at most one essential disc $\sigma_2(P)$. Therefore for any vector $\bar{b}$
\begin{align}
\sum_{P \in \mathcal{P}_n(\bar{b})} \mu(\sigma_2(P)) \leq \mu(K).
\tag{36}
\end{align}
The number of different vectors in the class \( \mathcal{P}(\mathbf{b}) \) is at most \((2Q + 1)^{n-1} < 2^n Q^{n-1}\) for \(Q > Q_0\). Therefore
\[
\sum_{\mathbf{b}} \sum_{P \in \mathcal{P}(\mathbf{b})} \mu_s(\sigma_1(P)) < c_5 \mu_s(\sigma_2(P)) < c_5 \mu_s(\sigma_2(P)) < c_5 \mu_s(\sigma_2(P)) < c_5 \mu_s(\sigma_2(P)).
\] (37)

For \(c_5 = 2^{n+3}c_3\) the estimate in (37) is at most \(\mu_s(\sigma_2(P)) / 8\).

**b) The case of inessential discs.** We will first estimate the values of \(|P_1(w)|_p, j = 1, 2, \) on the intersection \(\sigma_2(P_1, P_2)\) of the discs \(\sigma_2(P_1)\) and \(\sigma_2(P_2)\). From the Taylor expansion of \(P_j\) on the disc \(\sigma_2(P_j)\) in the neighborhood of the root \(\alpha = \alpha(P_j)\) we obtain
\[
P_j(w) = P_j(\alpha)(w - \alpha) + \sum_{i=2}^{n} (i!)^{-1} P_j^{(i)}(\alpha)(w - \alpha)^i.
\] (38)

From (33) we further obtain that
\[
|P_j'(\alpha)|_p |w - \alpha|_p < c_5 Q^{-2+v},
\] (39)
and for \(i \geq 2\) from the lower bound for \(|P'(\alpha)|_p\) in (31) we have
\[
|\alpha|_p \leq Q^{-4+2v+\frac{4}{3}} < c_6 Q^{-2+v}
\] (40)
for \(v \in [0, \frac{1}{2}]\) and \(Q > Q_0\). The expansion (38), together with (39) and (40) imply that
\[
|P_j(w)|_p < c_5 Q^{-2+v}
\] (41)

Similarly, by the Taylor representation of \(P_j'\) on the disc \(\sigma_2(P_j)\) in the neighborhood of the root \(\alpha = \alpha(P_j)\) we have
\[
P_j'(w) = P_j'(\alpha) + \sum_{i=2}^{n} (i!)^{-1} P_j^{(i)}(\alpha)(w - \alpha)^{i-1}.
\]

The second inequality in (31) and the estimate
\[
|\alpha|_p \leq Q^{(i-1)(-2+v+\frac{2}{3})}, \quad i \geq 2
\]

imply that
\[
|P_j'(w)|_p < c_6 p^\delta Q^{-v}.
\]

Denote the difference between \(P_1\) and \(P_2\) by \(R(x) = P_2(x) - P_1(x)\). Since \(P_1\) and \(P_2\) belong to the same class \(\mathcal{P}(\mathbf{b})\) we know that \(R\) is linear, so that \(R(x) = ax + b\) for some \(a, b\). Also, (41) and the estimate for the derivative imply that
\[
\begin{cases}
|aw + b|_p < c_5 Q^{-2+v}, \\
|R'(w)|_p = |a|_p < c_6 p^\delta Q^{-v}, \\
\max\{|a|, |b|\} \leq Q.
\end{cases}
\] (42)
From the first two of these inequalities it follows that $|b|_p < c_4 p^3 Q^{-v}$ which further implies that for fixed $a$ the number $b$ can have at most $2a \mu(K)$ values. The value $a$ has at most $2c_4 p^3 Q^{1-v}$ values. As a result, the measure of the set of $w$ satisfying (42) does not exceed $4c_4 c_3 p^3 \mu(K) = 2^{n+5} c_4 p^3 \mu(K)$. Then, for $c_3 c_4 < p^{-3 - 2 - n - \delta}$, this estimate is not greater than $\frac{\mu(K)}{8}$.

Thus, the total measure for the essential and inessential discs is at most $\frac{\mu(K)}{4}$.

Our next task is to estimate the measure of $M_n^u(v)$ which we define to be the set of $w$ for which the system

$$
\begin{cases}
|P(w)|_p < Q^{-n-1+v} \\
|P'(w)|_p < Q^{-v_1} \ \\
|a_j| \leq Q, \ 0 \leq j \leq n,
\end{cases}
$$

(43)

has solutions. We will show that $\mu(M_n^u(v)) < \frac{\mu(K)}{4}$.

Four different cases will be considered according to the value of $l_2 T^{-1} + p_1$.

**Case 1.**

$$l_2 T^{-1} + p_1 \geq n + 1 - \varepsilon. \quad (44)$$

Let $\mathcal{P}_i(l)$ be the set $\{P \in \mathcal{P}_n(H, l) \mid 2^l \leq H < 2^{l+1}\}$. Since $Q$ is sufficiently large and $H \leq Q$, we may assume that $t_0 < t \ll \log Q$. Let $w \in S(\alpha_1)$, then

$$|w - \alpha_1|_p \leq 2^{l(n-1+v+p_1+(n-1)\varepsilon)} \quad (45)$$

or

$$|w - \alpha_1|_p \leq 2^{l(n-1+v+p_2+(n-2)\varepsilon)} \quad (46)$$

Simple calculations show that for this case the second inequality is better. For all the other cases we shall use the first one. Divide the disc $K$ into smaller discs $K_j$ of radius $2^{-(\frac{n+1-v-p_1}{2}-\gamma)}$, where $\gamma$ is a positive constant and will be specified below.

For an integer polynomial $P$ and a disk $K_j$ we shall use the definition that "$P(w)$ belongs to $K_j$" or "$K_j$ contains $P(w)$", if there is a point $w \in K_j$, that satisfies the system (43). Let $\sigma(P)$ denote the set of $w$ that satisfy (43).

a) Assume that at most one polynomial $P \in \mathcal{P}_i(l)$ belongs to every $K_j$. Then for every polynomial the measure of those $w$ which satisfy (43) is at most $c(n) 2^{-\varepsilon \frac{n+1-v-p_1}{2} - (n-2)\varepsilon} \mu(K)$. The number of $K_j$ is less than $2^{l(n+1-v-p_1-\gamma) \mu(K)}$. Therefore

$$\sum_{P \in \mathcal{P}_i(l)} \mu(\sigma(P)) \ll \sum_{P \in \mathcal{P}_i(l)} 2^{l(n+1-v-p_1-\gamma) \mu(K)c(n) 2^{-\varepsilon \frac{n+1-v-p_1}{2} - (n-2)\varepsilon}} \ll 2^{-t \gamma_1}, \quad (47)$$

where $\gamma_1 = \gamma - \frac{n-2}{2} \varepsilon$.

Sum (47) over all $t \geq t_0$. Since $\sum_{t \geq t_0} 2^{-t \gamma_1} \ll 2^{-t \gamma_1}$ for sufficiently large $t_0$ the measure of those $w$ such that (43) holds with polynomials $P$ satisfying case 1 a), does not exceed $\frac{\mu(K)}{32}$. 

12
b) Now assume the contrary: that there are discs $K_j$, that contain at least two polynomials, i.e. we can find polynomials $P_1$ and $P_2$ in $\mathcal{P}_1(\bar{1})$, points $w_1$ and $w_2$ in $K_j$, such that the system of inequalities

$$\begin{cases}
|P_1(w_1)|_p \ll 2^{t(-n-1+v)}, \\
|P_1'(x_1)|_p \ll 2^{-te_1}, \\
|P_2(x_2)|_p \ll 2^{t(-n-1+v)}, \\
|P_2'(x_2)|_p \ll 2^{-te_1}.
\end{cases}$$

holds. We will now estimate the values of $|P_1(w)|_p$ and $|P_2(w)|_p$ for $w \in K_j$. Using Taylor’s expansion for $P_i$ at the point $\alpha_1$ we have

$$P_i(w) = \sum_{j=1}^{n} (j!)^{-1} P^{(j)}(\alpha_1) \cdot (w - \alpha_1)^j, \quad i = 1, 2.$$  

Using estimates for $|P^{(j)}(\alpha_1)|_p$ from Lemmas 1 and 5 we therefore have

$$|P_i(x)|_p = \left| \sum_{j=1}^{n} (j!)^{-1} P^{(j)}(\alpha_1) \cdot (w - \alpha_1)^j \right|_p \ll$$

$$\ll \max_{1 \leq j \leq n} 2^{-tj} 2^{j(-n+v-1+e_{j+\eta})^\gamma} \ll 2^{t(-n-1+v+\gamma_1)}.$$

For polynomials $P_1$ and $P_2$ with no common roots we can apply Lemma 6 with $\tau = 1 + n - v - n\gamma_1$ and $\eta = n + 1 - v - p_2 - \gamma$. It is necessary to check whether $\tau + 2 \max(\tau - \sigma, 0) < 2n + \delta$ for any $\delta > 0$. We have

$$3 + 3n - 3v - 3n\gamma_1 - n - 1 + v + p_2 + 2\gamma < 2n + \delta.$$  

Hence, it follows that

$$2 - 2v + p_2 < \delta + (3n - 2)\gamma_1.$$  

However, $2 - 2v > \delta + (3n - 2)\gamma_1$ for $v \lesssim \frac{1}{2}$ and sufficiently small $\delta$ which leads to a contradiction.

Case 2.

$$n + 0.1 \leq l_2T^{-1} + p_1 < n + 1 - v.$$  

(48)

Again, divide the disc $K$ into smaller discs $K_j$, with radius $2^{t(-\frac{\gamma_2}{2} + \gamma)}.$

a) Assume that at most one polynomial $P \in \mathcal{P}_1(\bar{1})$ belongs to every $K_j$. From (45) for every polynomial the measure of those $w$ which satisfy (43), does not exceed $c(n)2^{-t(n+1-v-p_1-\gamma)e}$. The number of $K_j$ is less than $2^{t(n+1-v-p_1-\gamma)}\mu(K)$. Therefore

$$\sum_{P \in \mathcal{P}_1(\bar{1})} \mu(\sigma(P)) \ll \sum_{P \in \mathcal{P}_1(\bar{1})} 2^{t(\frac{\gamma_2}{2} - \gamma)} 2^{-t(n+1-v-p_1-(n-1)e)} \ll 2^{-t\gamma_2},$$  

(49)

where $\gamma_2 = \gamma - (n - 1)e$. Again we sum the estimate (49) over all $t > t_0$. And similarly as the argument following (47) we obtain an estimate less than $\frac{\mu(K)}{32}$. 

13
b) As in Case 1, now assume the existence of a disc $K_j$ that contains at least two different polynomials $P_1$ and $P_2$. For any $w \in K_j$ by Taylor’s expansion we get

$$|P_i(w)|_p \ll 2^{-t(l_2T^{-1}+p_1-n\gamma)}, \quad i = 1, 2.$$  

For $P_1$ and $P_2$ having no common roots in $K_j$ we apply Lemma 6 with $\tau = l_2T^{-1} + p_1 - n\gamma$ and $\eta = l_2T^{-1} - \gamma$. Note that $l_2T^{-1} \ll p_1$. Then

$$3l_2T^{-1} + 3p_1 - 3n\gamma - 2l_2T^{-1} + 2\gamma < 2n + \delta$$

and

$$3p_1 + l_2T^{-1} - (3n - 2)\gamma < 2n + \delta.$$

However,

$$3p_1 + l_2T^{-1} - (3n - 2)\gamma \geq 2p_1 + 2l_2T^{-1} - (3n - 2)\gamma > 2n + 0.2 - (3n - 2)\gamma > 2n + \delta$$

for sufficiently small $\delta$ and $\gamma$. Again there is a contradiction.

**Case 3.**

$$\frac{7}{4} \leq l_2T^{-1} + p_1 < n + 0.1. \quad (50)$$

This case covers the widest range of values for $l_2T^{-1} + p_1$ and is the most difficult one. We again divide the disc $K$ into smaller discs $K_j$ of radius $2^{-l_2T^{-1}}$. We need to estimate the value of a polynomial $P \in \mathcal{P}_n$ and its derivative for $w \in K_j$. For this purpose we rewrite $P(w)$ using the Taylor formula in the neighbourhood of the point $\alpha_1$

$$P(w) = P'(\alpha_1)(w - \alpha_1) + \sum_{i=2}^{n} (i!)^{-1}P^{(i)}(\alpha_1) \cdot (x - \alpha_1)^i,$$

$$|P'(\alpha_1)|_p \ll 2^{(-l_2T^{-1} - p_1)},$$

$$|2^{|p^{-1}}P''(\alpha_1)|_p \ll 2^{(-p_2 - 2l_2T^{-1})} \ll 2^{-t(l_2T^{-1} + p_1)},$$

$$|i!|_p^{-1}|P^{(i)}(\alpha_1)|_p \ll 2^{(-p_2 - 2l_2T^{-1})} \ll 2^{-t(l_2T^{-1} + p_1)}, \quad i \geq 3,$$

$$|P(w)|_p \ll 2^{-t(l_2T^{-1} + p_1)}. \quad (51)$$

Similarly from the expansion of $P'(w)$ on the disc $K_j$ we obtain

$$P'(w) = P'(\alpha_1) + \sum_{i=2}^{n} ((i - 1)!)^{-1}P^{(i)}(\alpha_1) \cdot (x - \alpha_1)^{i-1},$$

$$|P'(\alpha_1)|_p \ll 2^{-tp_1},$$

$$|(i - 1)!|_p^{-1}|P^{(i)}(\alpha_1)|_p \ll 2^{(-p_2 - (i - 1)l_2T^{-1})} \ll 2^{-tp_1}, \quad i \geq 2,$$

$$|P'(w)|_p \ll 2^{-tp_1}. \quad (52)$$
Thus if \( P \) belongs to the disc \( K_j \) then for \( w \in K_j \) the inequalities

\[
\begin{aligned}
|P(w)|_p &\ll 2^{-t(p_1 + l_2 T^{-1})} , \\
|P'(w)|_p &\ll 2^{-tp_1} 
\end{aligned}
\]  \tag{53}

hold.

For some \( \theta \geq 0 \) we assume that the disc \( K_j \) contains \( c(n)2^{t\theta} \) polynomials. Then by Lemma 1 the measure of \( w \in K \), that satisfy (43) does not exceed

\[
c(n)2^{t(n+v-1+p_1)}2^{\theta p_2}2^{l_2 T^{-1}} = m(t).
\]  \tag{54}

For \( \theta < n + 1 - v - l_2 T^{-1} - p_1 \) the series \( \sum_{t=1}^{\infty} m(t) \) converges. Therefore for \( t > t_0 \) we obtain the estimate \( \sum_{t>t_0} m(t) < \frac{\mu(K)}{2^r} \).

We introduce

\[
u = n + 1 - v - p_1 - l_2 T^{-1}.
\]

From (50) and \( v \leq 0.5 \) it follows that \( u \geq 0.4 \). In what follows let \( u_1 = u - 0.2 \geq 0.2 \) and write \( u_1 \) as \( u_1 = [u_1] + \{u_1\} \), where \([\ ]\) and \( \{\} \) denote the integer and fractional parts respectively.

Let \( \theta \geq u \). By Dirichlet’s principle we have that there are at least \( c(n)2^{t([u_1]+0.2)} \) polynomials \( P_1, P_2, \ldots, P_k \), with \( k \geq 2^{t([u_1]+0.2)} \), which have the same first \([u_1]\) coefficients.

Consider the polynomials \( R_j(w) = P_{j+1}(w) - P_j(w), j \geq 1 \), which satisfy:

\[
\begin{aligned}
\deg R_j(w) &\leq n - [u_1] = l_2 T^{-1} + p_1 + v + \{u_1\} - 0.8, \\
H(R_j) &\leq 2^{t+2}.
\end{aligned}
\]  \tag{55}

From (53) we get

\[
\begin{aligned}
|R_j(w)|_p &\ll 2^{-t(p_1 + l_2 T^{-1})} , \\
|R_j'(w)|_p &\ll 2^{-tp_1} , i = 1, \ldots, k.
\end{aligned}
\]  \tag{56}

Every coefficient \( a_j(R) \), \( 1 \leq j \leq n - [u_1] \) of the polynomial \( R_j \) lies in \([-2^{t+1}; 2^{t+1}] \). For each coefficient divide this interval into smaller intervals of length \( 2^{t\mu} \), where \( \mu = 1 - \frac{\{u_1\}}{n - [u_1]} \).

Then, at least \( c(n)2^{0.2t} \) polynomials have their lying in the same intervals. Hence the first \( u - [u_1] \) coefficients of the polynomials \( S_j(w) = R_{j+1}(w) - R_j(w) \) are at most \( c(n)2^{t\mu} \). For these polynomials \( S_j \) we obtain a similar system of inequalities to (56); namely

\[
\begin{aligned}
|S_i(w)|_p &\ll 2^{-t(l_2 T^{-1} + p_1)} , \\
|S_i'(w)|_p &\ll 2^{-tp_1} , \\
\deg S_i(w) &\leq n - [u_1] , \\
H(S_i) &< 2^{t\mu} , 1 \leq i \leq k - 1.
\end{aligned}
\]  \tag{57}

a) Suppose that among the polynomials \( S_j \) there are two which are coprime. Then we apply Lemma 6 on the disc \( K_j \) with \( \tau = p_1 + l_2 T^{-1} \) and \( \eta = l_2 T^{-1} \) to get

\[
\begin{aligned}
l_2 T^{-1} + 3p_1 &< 2(n - [u_1])(1 - \frac{\{u_1\}}{n - [u_1]}) + \delta_1 \\
 &= 2(n - u_1) + \delta_1 = 2p_1 + 2l_2 T^{-1} + 2v - 1.6 + \delta_1, \delta_1 < \delta.
\end{aligned}
\]  \tag{58}
Since $l_2T^{-1} \leq p_1$ and $v \leq \frac{1}{2}$ we have a contradiction for $\delta < 0.6$.

b) If all polynomials $S_j$ can be written $lS_0$ for some polynomial $S_0$ then there exists $l > \frac{k}{3}$ such that $(l, p) = 1$. Obviously the height of the polynomial $H(S_0) \ll 2^{-0.2t}H(S_1) \ll 2^{l(0.8 - \frac{\{u_1\}}{\pi \mu})}$. In the first inequality of (57) assume that $i = 0$, and on the RHS pass to the height $H(S_0)$ to obtain the inequality

$$|S_0(w)|_p \ll 2^{-l(l_2T^{-1} + p_1)(0.8 - \frac{\{u_1\}}{\pi \mu})^{-1}}. \tag{59}$$

If now the inequality

$$p_1 + l_2 \cdot T^{-1} > (n - [u_1] + 1)(0.8 - \frac{\{u_1\}}{n - [u_1]}) \tag{60}$$

holds, then by Sprindzuk’s theorem, see [11], the inequality (59) holds for a set of $w \in \mathbb{Q}_p$ with measure tending to zero for $t > t_0$ and $t_0 \to \infty$. Condition (50) implies that (60) is true. Therefore, for appropriate $t_0$ this measure can be bounded above by $\frac{\mu(K)}{32}$.

c) If there are reducible polynomials among the $S_i$ then we use Lemma 2. We again check the inequality

$$l_2 \cdot T^{-1} + p_1 > (n - [u_1])(1 - \frac{\{u_1\}}{n - [u_1]}) = n - u_1 = l_2 \cdot T^{-1} + p_1 + v - 0.8.$$

It obviously holds for $v \leq 0.5$. Thus the measure of those $w \in \mathbb{Q}_p$ for which the system (57) holds for these polynomials does not exceed $\frac{\mu(K)}{32}$.

**Case 4.**

$$l_2T^{-1} + p_1 < \frac{7}{4}. \tag{61}$$

We will estimate the expression $l_2T^{-1} + p_1$ from below. To do this we have to prove that $|P'(w)|_p = |P'\alpha)|_p \gg 2^{-m}$. From (43) we have that $|P'(w)|_p \ll 2^{-\frac{2t}{3}}$, therefore $p_1 \geq \frac{2}{3}$.

Thus we need to consider the system

$$\begin{cases} 
|P(w)|_p < 2^{l(-n-1+v)}, \\
|P'(w)|_p < 2^{-2t/3}, \\
\frac{2}{3} < l_2T^{-1} + p_1 < \frac{7}{4}. \tag{62}
\end{cases}$$

Let $\alpha_1$ be the closest root to $w$ and introduce the discs $\sigma(P)$ and $\sigma_1(P)$ given by

$$\sigma(P) = \{w \in K : |w - \alpha_1|_p < 2^{l(-n-1+v)}|P'(\alpha_1)|_p^{-1}\}. \tag{63}$$

$$\sigma_1(P) = \{w \in K : |w - \alpha_1|_p < 2^{-t}|P'(\alpha_1)|_p^{-1}\}. \tag{64}$$

From (63) and (64) we get

$$\mu(\sigma(P)) \ll 2^{-l(n-v)}\mu(\sigma_1(P)). \tag{65}$$
Fix a vector \( \bar{b} = (a_n, a_{n-1}, \ldots, a_2) \), where \( a_j \), \( 0 \leq j \leq n \), are the coefficients of the polynomial \( P \). The set of polynomials with vector \( \bar{b} \) is denoted by \( \mathcal{P}_b \). Obviously \( \#\mathcal{P}_b \leq 2^{(n-1)} \).

a) First, we consider polynomials \( P_1 \) such that \( \sigma_1(P_1) \cap \sigma_1(P_2) = \emptyset \) for any \( P_2 \in \mathcal{P}_b \). Then \( \sum_{P_1 \in \mathcal{P}_b} \mu(\sigma_1(P_1)) < \mu(K) \). Inequality (65) implies that

\[
\sum_{\bar{b}} \sum_{P \in \mathcal{P}_b} \mu(\sigma P) \ll 2^{(v-1)}.
\]

Since \( v-1 < 0 \) the series \( \sum_{t=1}^{\infty} 2^{(v-1)} \) converges and its sum \( \sum_{t>t_0} 2^{(v-1)} \) for appropriate \( t_0 \) will be less than \( \frac{\mu(K)}{32} \).

b) If there exists \( P_2 \in \mathcal{P}_b \) such that \( \sigma_1(P_1, P_2) = \sigma_1(P_1) \cap \sigma_1(P_2) \neq \emptyset \), then, using the Taylor expansion on this intersection for these polynomials and their derivatives we obtain

\[
\begin{cases}
|P_j(w)|_p \ll 2^{-t}, \\
|P'_j(w)|_p \ll 2^{-2t/3}, & j = 1, 2.
\end{cases}
\]

The polynomial \( R(w) = P_1(w) - P_2(w) \) is linear. Thus we can write \( R(w) = aw + b \). Then the system of inequalities (66) for \( R \) will be

\[
\begin{cases}
|aw + b|_p \ll 2^{-t}, \\
|a|_p \ll 2^{2t}, \\
\max(|a|, |b|) \leq 2Q.
\end{cases}
\]

The system of inequalities (67) is analogous to the system (42). We estimate the measure of the set of \( w \in \mathbb{Q}_p \), such that the system (67) has solutions in polynomials in the same way as we did when analyzing (42). We obtain that the measure is \( \ll 2^{-t/3} \). Again, \( \sum_{t>t_0} 2^{-t/3} < \frac{\mu(K)}{32} \).

Summing all the estimates for cases 1–4 together we finally obtain that \( \mu(M_{\alpha}^n(v)) < \frac{\mu(K)}{4} \) as required.

References


Institute of Mathematics
Surganova str. 11
220072, Minsk,
Belarus
E-mail: bernik@im.bas-net.by
olga_kukso@tut.by

Universität Bielefeld
Fakultät für Mathematik
Postfach 10 01 31
33501 Bielefeld
Germany
E-mail: goetze@math.uni-bielefeld.de

Received