UNIFORM RATES OF CONVERGENCE IN THE CLT FOR QUADRATIC FORMS

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Abstract. Let $X, X_1, X_2, \dots$ be a sequence of i.i.d. random vectors taking values in a d-dimensional real linear space $\mathbb{R}^d$. Assume that $\mathbf{E}\, X = 0$ and that X is not concentrated in a proper subspace of $\mathbb{R}^d$. Let G be a mean zero Gaussian random vector with the same covariance operator as that of X. We investigate the distributions of non-degenerate quadratic forms $\mathbb{Q}\,[S_N]$ of the normalized sums $S_N = N^{-1/2}\,(X_1 + \dots + X_N)$ and show that

$$\Delta_N \overset{\text{def}}{=} \sup_x \Big| \mathbf{P}\big\{ \mathbb{Q}\,[S_N] \le x \big\} - \mathbf{P}\big\{ \mathbb{Q}\,[G] \le x \big\} \Big| = \mathcal{O}\big(N^{-1}\big),$$

provided that $d \ge 5$ and the fourth moment of X exists. The bound $\mathcal{O}\big(N^{-1}\big)$ is optimal and improves, e.g., the well-known bound $\mathcal{O}\big(N^{-d/(d+1)}\big)$ due to Esseen (1945). Furthermore, we provide explicit bounds of order $\mathcal{O}\big(N^{-1}\big)$ for Δ_N and for the concentration function of the random variable $\mathbb{Q}\,[S_N + a]$, $a \in \mathbb{R}^d$. Our results extend the corresponding results of Bentkus and Götze (1997a) ($d \ge 9$) to the case $d \ge 5$.

1. Introduction and results

Let $\mathbb{R}^d$ denote the d-dimensional space of real vectors $x = (x_1, \dots, x_d)$ with scalar product $\langle x, x\rangle = x_1^2 + \dots + x_d^2$ and norm $\|x\| = \langle x, x\rangle^{1/2}$. We also denote by $\mathbb{R}^\infty$ a real separable Hilbert space consisting of all real sequences $x = (x_1, x_2, \dots)$ such that $\|x\|^2 = x_1^2 + x_2^2 + \dots < \infty$.

Let $X, X_1, X_2, \dots$ be a sequence of i.i.d. random vectors taking values in $\mathbb{R}^d$. Assume that $\mathbf{E}\, X = 0$ and $\mathbf{E}\, \|X\|^2 < \infty$, Let G be a mean zero Gaussian random vector such that its covariance operator $\mathbb{C} = \operatorname{cov} G : \mathbb{R}^d \to \mathbb{R}^d$ is equal to $\operatorname{cov} X$. It is well-known that the distributions $\mathcal{L}(S_N)$ of sums

$$S_N \overset{\text{def}}{=} N^{-1/2}\,(X_1 + \dots + X_N) \tag{1.1}$$

converge weakly to $\mathcal{L}(G)$.

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Let $\mathbb{Q} : \mathbb{R}^d \to \mathbb{R}^d$ be a linear symmetric bounded operator and let $\mathbb{Q}[x] = \langle \mathbb{Q}x, x\rangle$ be the corresponding quadratic form. We shall say that $\mathbb{Q}$ is non-degenerate if $\ker \mathbb{Q} = \big\{ 0 \big\}$.

Denote

$$\beta_q \overset{\text{def}}{=} \mathbf{E}\, \|X\|^q, \qquad \text{for } q \geq 0.$$

Introduce the distribution functions

$$F(x) \overset{\text{def}}{=} \mathbf{P}\big\{ \mathbb{Q}[S_N] \leq x \big\}, \qquad F_0(x) \overset{\text{def}}{=} \mathbf{P}\big\{ \mathbb{Q}[G] \leq x \big\}. \tag{1.2}$$

Write

$$\Delta_N \overset{\text{def}}{=} \sup_x \big| F(x) - F_0(x) \big|. \tag{1.3}$$

We shall provide explicit bounds for Δ_N.

Theorem 1.1. *Let* $\mathbf{E}\, X = 0$. *Assume that* $\mathbb{Q}$ *and* $\mathbb{C}$ *are non-degenerate and that* $d \geq 5$ *or* $d = \infty$. *Then*

$$\Delta_N \leq c(\mathbb{Q}, \mathbb{C})\, \beta_4 / N.$$

The constant $c(\mathbb{Q}, \mathbb{C})$ *in this bound depends on* $\mathbb{Q}$ *and* $\mathbb{C}$ *only.*

Theorem 1.1 confirms a conjecture of Bentkus and Götze (1997a) (below BG (1997a)). It generalizes to the case $d \geq 5$ the corresponding result of BG (1997a). In their Theorem 1.1 it was assumed that $d \geq 9$, while our Theorem 1.1 is proved for $d \geq 5$.

The distribution function of $\|S_N\|^2$ (for bounded X with values in $\mathbb{R}^d$) may have jumps of order $\mathcal{O}\big(N^{-1}\big)$, for all $1 \leq d \leq \infty$. See, e.g., BG (1996). Therefore, the bound of Theorem 1.1 is optimal.

Theorem 1.1 and the method of its proof are closely related to the lattice point problem in number theory. Suppose that $d < \infty$ and that $\langle \mathbb{Q}x, x\rangle > 0$, for $x \neq 0$. Let $\operatorname{vol} E_s$ be the volume of the ellipsoid

$$E_s = \big\{ x \in \mathbb{R}^d : \ \mathbb{Q}[x] \leq s \big\}, \qquad s \geq 0.$$

Write $\operatorname{vol}_{\mathbb{Z}} E_s$ for the number of points in $E_s \cap \mathbb{Z}^d$, where $\mathbb{Z}^d \subset \mathbb{R}^d$ is the standard lattice of points with integer coordinates.

The following result of Götze (2004), is related to Theorem 1.1. There are related results of Götze and Margulis (2009) for positive (as well as indefeinite) forms as well.

Theorem 1.2. (Götze (2004)) *For all dimensions* $d \geq 5$,

$$\sup_{a \in \mathbb{R}^d} \left| \frac{vol_{\mathbb{Z}}\,(E_s + a) - vol\, E_s}{vol\, E_s} \right| = \mathcal{O}\left(\frac{1}{s}\right), \qquad \textit{for} \quad s \geq 1,$$

where the constant in $\mathcal{O}(s^{-1})$ *depends on the dimension* d *and on the lengths of axes of the ellipsoid* E_1 *only.*

Theorem 1.2 solves the lattice point problem for $d \geq 5$, and it improves the classical estimate $\mathcal{O}(r^{-2d/(d+1)})$ due to Landau (1915), just as Theorem 1.1 improves the bound $\mathcal{O}(N^{-d/(d+1)})$ by Esseen (1945) for the CLT for ellipsoids with axes parallel to coordinate axes. Corresponding results for indefinite forms with $d \geq 5$, are proved in Götze and Margulis (2009).

For Hilbert spaces the order of error under the conditions of Theorem 1.1 had been investigated intensively. See Zalesskiĭ, Sazonov and Ulyanov (1988) and Nagaev (1989) for the optimal (with respect to eigenvalues of $\mathbb{C}$) bound of order $\mathcal{O}(N^{-1/2})$ under the assumption of finiteness of the third moment. For a more detailed discussion see Bentkus and Götze (1995a, 1996, 1997b) and Senatov (1989).

Under more restrictive moment and dimension conditions the estimate $\mathcal{O}(N^{-1+\varepsilon})$, with $\varepsilon \downarrow 0$ as $d \uparrow \infty$, was obtained by Götze (1979), with the help of a result for bivariate U-statistics. The symmetrization inequality for characteristic functions introduced in Götze (1979) and its extensions play the crucial role in the proofs of bounds in the CLT on ellipsoids and hyperboloids in finite and infinite dimensional cases. This inequality is related to Weyl's (1915/16) inequality for trigonometric sums. Under some special smoothness assumptions, error bounds $\mathcal{O}(N^{-1})$ (and even Edgeworth type expansions) were obtained in Götze (1979), Bentkus (1984), Bentkus, Götze and Zitikis (1993). BG (1995a, 1996, 1997b) established the bound of order $\mathcal{O}(N^{-1})$ without smoothness-type conditions. Similar bounds for the rate of infinitely divisible approximations were obtained by Bentkus, Götze and Zaitsev (1997). Among recent publications, we should mention the papers of Nagaev and Chebotarev (1999), (2005) ($d \geq 9$, a more precise dependence of constants on the eigenvalues of $\mathbb{C}$) and Bogatyrev, Götze and Ulyanov (2006) (non-uniform bounds for $d \geq 12$), see also Götze and Ulyanov (2000). The proofs of bounds of order $\mathcal{O}(N^{-1})$ are based on discretization (i.e., a reduction to lattice valued random vectors) and the symmetrization techniques mentioned above.

Additional restrictions like the diagonality of $\mathbb{Q}$, $\mathbb{C}$ and the independence of first five coordinates allowed already to reduce the dimension requirement for the bound $\mathcal{O}(N^{-1})$ to $d \geq 5$, see Bentkus and Götze (1996). The independence assumption in BG (1996) allowed to apply an adaption of the Hardy–Littlewood circle method. For the general case described in Theorem 1.1, we have to develop a new tool. Some yet unpublished results of Götze (1994) provide the rate $\mathcal{O}(N^{-1})$ for sums of two independent *arbitrary* quadratic forms (each of rank $d \geq 3$). Götze and Ulyanov (2003) obtained bounds of order $\mathcal{O}(N^{-1})$ for some ellipsoids in $\mathbb{R}^d$ with $d \geq 5$ in the case of lattice distributions of X.

The optimal possible dimension condition for this rate is just $d \geq 5$, due to the lower bounds of order $\mathcal{O}(N^{-1}\log N)$ for dimension $d = 4$ in the corresponding lattice point problem. The question about precise convergence rates in dimensions $2 \leq d \leq 4$ still remains completely open (even in the simplest case where $\mathbb{Q}$ is the identity operator $\mathbb{I}_d$, and for random vectors with independent Rademacher coordinates). It should be mentioned that, in the case $d = 2$, a precise convergence rate would imply a solution of the famous circle problem. Known lower bounds in the circle problem correspond to the bound $\mathcal{O}(N^{-3/4}\log^{\delta} N)$ for Δ_N. Hardy (1916) conjectured that up to logarithmic factors this is the optimal order.

To formulate the results we need more notation repeating most part of the notation used in BG (1997a). Write $\sigma^2 \overset{\text{def}}{=} \beta_2$. Let $\sigma_1^2 \geq \sigma_2^2 \geq \dots$ be the eigenvalues of $\mathbb{C}$, counting their multiplicities. We have $\sigma^2 = \sigma_1^2 + \sigma_2^2 + \cdots$. Denote by $\theta_1^4 \geq \theta_2^4 \geq \dots$ the eigenvalues of $(\mathbb{C}\mathbb{Q})^2$.

Throughout $\mathcal{S} = \{e_1, \dots, e_s\} \subset \mathbb{R}^d$ denotes a finite set of cardinality s. We shall write $\mathcal{S}_o$ instead of $\mathcal{S}$ if the system $\{e_1, \dots, e_s\}$ is orthonormal.

Let $p > 0$ and $\delta \geq 0$. Following BG (1997a), we introduce the non-degeneracy condition for the distribution of a d-dimensional vector Y:

$$\mathcal{N}(p, \delta, \mathcal{S}, Y): \qquad \mathbf{P}\big\{ \|Y - e\| \leq \delta \big\} \geq p, \qquad \text{for all } e \in \mathcal{S} \cup \mathbb{Q}\mathcal{S}. \tag{1.4}$$

We shall refer to condition (1.4) as condition $\mathcal{N}(p, \delta, \mathcal{S}, Y) = \mathcal{N}(p, \delta, \mathcal{S}, Y; \mathbb{Q})$.

A particular case where explicit lower bounds for p in (1.4) can be given in terms of eigenvalues of $\mathbb{C}$ and $\mathbb{Q}$, is described in BG (1997a). Assume that there exists an orthonormal system $\mathcal{S}_o = \{e_1, \dots, e_s\}$ of eigenvectors of $\mathbb{C}$ such that $\mathbb{Q}\mathcal{S}_o$ is again a system of eigenvectors of $\mathbb{C}$. Then we shall say that condition $\mathcal{B}(\mathcal{S}_o, \mathbb{C}) = \mathcal{B}(\mathcal{S}_o, \mathbb{C}; \mathbb{Q})$ is fulfilled. In this case we shall write

$$\lambda_s^2 = \min_{e \in \mathcal{S}_o \cup \mathbb{Q}\mathcal{S}_o} \sigma_e^2, \tag{1.5}$$

where σ_e^2 is the eigenvalue of $\mathbb{C}$ which corresponds to the eigenvector e. In particular, such a system $\mathcal{S}_o$ exists if we assume that $\mathbb{Q}$ and $\mathbb{C}$ are diagonal in a common orthonormal basis. If, in addition, $\mathbb{Q}$ is isometric, then we can choose $\mathcal{S}_o$ such that $\lambda_s^2 = \sigma_s^2$.

Introduce truncated random vectors

$$X^\diamond = X\, \mathbf{I}\big\{ \|X\| \leq \sigma\sqrt{N} \big\}, \qquad X_\diamond = X\, \mathbf{I}\big\{ \|X\| > \sigma\sqrt{N} \big\}, \qquad X^\diamond + X_\diamond = X, \tag{1.6}$$

and their moments

$$\Lambda_4 = \frac{N}{(\sigma\sqrt{N})^4}\, \mathbf{E}\, \|X^\diamond\|^4, \qquad \Pi_q = \frac{N}{(\sigma\sqrt{N})^q}\, \mathbf{E}\, \|X_\diamond\|^q. \tag{1.7}$$

Here and below $\mathbf{I}\big\{ A \big\}$ denotes the indicator of an event A.

By $c, c_1, c_2, \dots$ we shall denote absolute positive constants. If a constant depends on, say, s, then we shall point out the dependence writing c_s or $c(s)$. We denote by c universal constants which might be different in different places of the text. Furthermore, in the conditions of Theorems and Lemmas (see, e.g., Theorems 1.3, 1.4, 1.5 and 2.1) we write c_0 for an *arbitrary* positive absolute constant, for example one may choose $c_0 = 1$. We shall write $A \ll B$, if there exists an absolute constant c such that $A \leq cB$. Similarly, $A \ll_s B$, if $A \leq c(s)\, B$. We shall also write $A \asymp_s B$ if $A \ll_s B \ll_s A$.

Furthermore, in Theorems 1.3, 1.4 and 1.5 we assume that the symmetric operator $\mathbb{Q}$ is isometric, that is, that $\mathbb{Q}^2$ is the identity operator $\mathbb{I}_d$. This does not restrict generality (see Remark 1.7 in BG (1997a)).

The following Theorem 1.3 solved the problem in the case $13 \leq d \leq \infty$.

Theorem 1.3. (BG (1997a, Theorem 1.3)) *Let* $\mathbf{E}\,X = 0$, $\delta = 1/300$, $\mathbb{Q}^2 = \mathbb{I}_d$, $s = 13$ *and* $13 \le d \le \infty$. *Then we have:*

(i) *Assume that condition* $\mathcal{N}(p, \delta, \mathcal{S}_o, c_0 G/\sigma)$ *holds. Then*

$$\Delta_N \le C \big(\Pi_3 + \Lambda_4\big) \tag{1.8}$$

with $C = c p^{-6} + c(\sigma/\theta_8)^8$*;*

(ii) *Assume that condition* $\mathcal{B}(\mathcal{S}_o, \mathbb{C})$ *is fulfilled. Then the constant in* (1.8) *satisfies* $C \le \exp\big\{ c\sigma^2 \lambda_{13}^{-2} \big\}$.

Note that in Theorem 1.3 of BG (1997a) the more general quantity

$$\sup_x \Big| \mathbf{P}\big\{ \mathbb{Q}[S_N - a] \le x \big\} - \mathbf{P}\big\{ \mathbb{Q}[G - a] \le x \big\} - F_1(x) \Big|, \qquad a \in \mathbb{R}^d, \tag{1.9}$$

was estimated, where $F_1(x) = F_1(a; x)$ is the Edgeworth correction. In our case, where $a = 0$, the Edgeworth correction $F_1(x)$ is equal to zero for all x (see Remark 4.3).

Unfortunately, we cannot apply Theorem 1.3 for $d = 5, 6, \dots, 12$. The following Theorem 1.4 is valid for $5 \le d < \infty$ in finite-dimensional spaces $\mathbb{R}^d$ only. However, the bounds of Theorem 1.4 depend on the smallest σ_d. This makes them unstable if one of coordinates of X degenerates.

Theorem 1.4. *Let* $\mathbf{E}\,X = 0$, $\delta = 1/300$, $\mathbb{Q}^2 = \mathbb{I}_d$, $s = 5$ *and* $5 \le d < \infty$. *Then we have:*

(i) *Assume that condition* $\mathcal{N}(p, \delta, \mathcal{S}_o, c_0 G/\sigma)$ *holds. Then*

$$\Delta_N \le C \big(\Pi_3 + \Lambda_4\big), \tag{1.10}$$

with $C = c_d\, p^{-3}\, (\sigma/\sigma_d)^4$*;*

(ii) *Assume that condition* $\mathcal{B}(\mathcal{S}_o, \mathbb{C})$ *is fulfilled. Then the constant in* (1.10) *may be estimated as* $C \le c_d \sigma^4 \sigma_d^{-4} \exp\big\{ c\sigma^2/\lambda_5^2 \big\}$.

Theorems 1.3 and 1.4 yield Theorem 1.1, using the bound $\Pi_3 + \Lambda_4 \le \beta_4/(\sigma^4 N)$.

Theorem 1.4 extends to the case $d \ge 5$ the particular case ($a = 0$) of Theorem 1.5 of BG (1997a) which contains the corresponding bounds for (1.9), where $d \ge 9$. Direct attempts to prove similar bounds for $a \ne 0$, assuming $d \ge 5$ instead of $d \ge 9$ in Theorems 1.3, 1.4 and 1.5 of BG (1997a) failed. Theorem 1.4 of BG (1997a) deals with symmetric distributions of X. In this case the Edgeworth correction $F_1(x)$ vanishes as well. The main problem is that Lemma 3.2 allows us to integrate the remainder terms of expansions for $\alpha < s/2$ only. It still works for $s = 5$ and $\alpha \le 2$ in the proof of our Theorem 1.4 (see (3.23) and (4.19)) and failed for $s = 5$ and $\alpha = 4$ or $\alpha = 6$ in the proofs of Theorems 1.4 and 1.5 of BG (1997a) (see inequalities (3.33) and (3.35) in BG (1997a)). See lower bounds for Δ_N under different conditions on a and distribution of X in Götze and Ulyanov (2000).

The bounds for constants in Theorems 1.3 and 1.4 are not optimal. See Nagaev and Chebotarev (1999), (2005), Götze and Ulyanov (2000), and Bogatyrev, Götze and Ulyanov (2006) for more precise estimates of constants in the case $d \ge 9$.

Introduce the concentration function

$$Q(X;\,\lambda) = Q(X;\,\lambda;\,\mathbb{Q}) = \sup_{a,\,x} \mathbf{P}\big\{x \le \mathbb{Q}[X-a] \le x+\lambda\big\}, \quad \text{for} \ \ \lambda \ge 0. \tag{1.11}$$

By $\mathcal{L}(X)$ we shall denote the distribution of X. Define the symmetrization $\widetilde{X}$ of a random vector X as a random vector with distribution $\mathcal{L}(\widetilde{X}) = \mathcal{L}(X_1 - X_2)$. Instead of normalized sums S_N, it is sometimes more convenient to consider the sums $Z_N = X_1 + \dots + X_N$. Then $S_N = N^{-1/2} Z_N$.

Theorem 1.5. *Let $\mathbb{Q}^2 = \mathbb{I}_d$, $5 \le s \le d \le \infty$ and $0 \le \delta \le 1/(5s)$. For any random vector X we have:*

(i) If condition $\mathcal{N}(p,\delta,\mathcal{S}_o,\widetilde{X})$ is fulfilled with some $p>0$, then

$$Q(Z_N;\,\lambda) \le c_s\,(pN)^{-1} \max\big\{1;\,\lambda\big\}, \qquad \lambda \ge 0; \tag{1.12}$$

(ii) If, for some m, condition $\mathcal{N}(p,\delta,\mathcal{S}_o, m^{-1/2}\widetilde{Z}_m)$ is fulfilled, then

$$Q(Z_N;\,\lambda) \le c_s\,(pN)^{-1} \max\big\{m;\,\lambda\big\}, \qquad \lambda \ge 0. \tag{1.13}$$

We say that a random vector Y is concentrated in $\mathbb{L} \subset \mathbb{R}^d$ if $\mathbf{P}\{Y \in \mathbb{L}\} = 1$. In BG (1997a, item (iii) of Theorem 1.6) it was shown that if $\widetilde{X}$ is not concentrated in a proper closed linear subspace of $\mathbb{R}^d$, $1 \le d \le \infty$, then, for any $\delta > 0$ and $\mathcal{S}$ there exists a natural number m such that the condition

$$\mathcal{N}(p,\delta,\mathcal{S}, m^{-1/2}\widetilde{Z}_m) \qquad \text{holds with some } \ p > 0. \tag{1.14}$$

Theorem 1.5 and more explicit Theorem 2.1 extend to the case $d \ge 5$ Theorems 1.6 and 2.1 of BG (1997a) which were proved for $d \ge 9$. It should be mentioned that the supremum in (1.11) is taken not only over all x, but over all x and $a \in \mathbb{R}^d$. Thus, in Theorems 1.5 and 2.1 we consider the general case.

Introduce the notation used in the proofs.

Throughout we assume that all random vectors and variables are independent in aggregate, if the contrary is not clear from the context. By $\lceil \alpha \rceil$ we shall denote the integer part of a number α.

By $\overline{X}$ and $X_1, X_2, \dots$ we shall denote independent copies of a random vector X. By $G_1, G_2, \dots$ we shall denote independent copies of G.

For the sake of brevity we shall write throughout

$$\beta = \beta_4, \qquad \Pi = \Pi_2, \qquad \Lambda = \Lambda_4.$$

We shall identify the linear operators and corresponding matrices. By $\mathbb{I}_d : \mathbb{R}^d \to \mathbb{R}^d$ we denote the identity operator and, simultaneously, the diagonal matrix with entries 1 on the diagonal.

The expectation $\mathbf{E}_Y$ with respect to a random vector Y we define as the conditional expectation

$$\mathbf{E}_Y\, f(X,Y,Z,\dots) = \mathbf{E}\,\big(f(X,Y,Z\dots)\,\big|\,X,Z,\dots\big)$$

given all random vectors but Y.

Introduce the distribution functions

$$\Phi_a(x) = \mathbf{P}\big\{\mathbb{Q}[Z_N - a] \le x\big\}, \quad a \in \mathbb{R}^d, \tag{1.15}$$

$$\Psi(x) = \Phi_0(x) = \mathbf{P}\big\{\mathbb{Q}[Z_N] \le x\big\}, \qquad \Psi_0(x) = \mathbf{P}\big\{\mathbb{Q}[\sqrt{N}\, G] \le x\big\}. \tag{1.16}$$

Recall that the truncated random vectors $X^\diamond$, $X_\diamond$ and their moments are defined in (1.7). In Sections 2 and 3 we shall denote

$$X' = X^\diamond - \mathbf{E}\, X^\diamond + W, \tag{1.17}$$

where W is a centered Gaussian random vector which is independent of all other random vectors and variables and is chosen so that $\operatorname{cov} X' = \operatorname{cov} G$. Such random vector W exists by Lemma 2.3. Finally by $Z_N^\diamond$ (resp. Z_N') we shall denote sums of N independent copies of $X^\diamond$ (resp. X').

By

$$\widehat{F}(t) = \int_{-\infty}^{\infty} \mathrm{e}\{tx\}\, dF(x), \qquad \mathrm{e}\{x\} \overset{\text{def}}{=} \exp\{ix\}, \tag{1.18}$$

we denote the Fourier–Stieltjes transform of a function F of bounded variation, or in other words, the Fourier transform of the measure which has the distribution function F.

We conclude the Introduction by a brief description of the basic elements of the proof. We have to mention that a big part of the proof repeats the arguments of BG (1997a), see BG (1997a) for the description and application of symmetrization inequality, discretization procedure and double large sieve. We do not use here multiplicative inequalities of BG (1997a). We replace here their application by some arguments coming from the number theory. The original part of our proof is concentrated at Sections 5–7. In Section 2, we prove bounds for concentration functions. The proofs, being technically simpler as those of Theorem 1.4, already contain all the principal ideas. These proofs repeats almost literally the corresponding proofs of BG (1997a). The only difference consists in the use of new Lemma 7.3 which allows us to estimate characteristic functions for large values of argument t. In Sections 3 and 4 Theorem 1.4 is proved. In Sections 5–7 we prove estimates for characteristic functions. Section 5 is started with results from BG (1997a) (Lemmas 5.2 and 5.4). Their proofs in BG (1997a) are based on conditioning, discretization, as well as on the double large sieve.

Let $\varepsilon_1, \varepsilon_2, \dots$ denote i.i.d. symmetric Rademacher random variables. Let $\delta > 0$ and $\mathcal{S} = \{e_1, \dots, e_s\} \subset \mathbb{R}^d$. We say that a discrete random vector $Y \in \mathbb{R}^d$ (or its distribution $\mathcal{L}(Y)$) belongs to the class $\mathbf{\Gamma}(\delta;\, \mathcal{S})$ (briefly $\mathcal{L}(Y) \in \mathbf{\Gamma}(\delta;\, \mathcal{S})$) if Y is distributed as $\varepsilon_1 z_1 + \dots + \varepsilon_s z_s$, with some (non-random) $z_j \in \mathbb{R}^d$ such that $\|z_j - e_j\| \le \delta$, for all $1 \le j \le s$.

Introduce the function

$$\mathcal{M}(t; N) = 1\big/\sqrt{|\,t\,|\, N}, \ \text{ for } |\,t\,| \le N^{-1/2}, \qquad \mathcal{M}(t; N) = \sqrt{|\,t\,|}, \ \text{ for } |\,t\,| \ge N^{-1/2}. \tag{1.19}$$

Notice that, for $s > 0$,

$$2^{-1}\big(|\,tN|^{-s/2} + |\,t\,|^{s/2}\big) \le \mathcal{M}^s(t; N) \le |\,tN|^{-s/2} + |\,t\,|^{s/2}. \tag{1.20}$$

Assuming the condition $\mathcal{N}(p,\delta,\mathcal{S}_o,\widetilde{X})$ with $0\le\delta\le 1/(5s)$ and an orthonormal system $\mathcal{S}_o$, we can use Lemma 5.2 which implies that, for any $a\in\mathbb{R}^d$ and $t\in\mathbb{R}$,

$$\big|\,\widehat{\Psi}_a(t)\,\big| = \big|\mathbf{E}\ \mathrm{e}\big\{t\,\mathbb{Q}[Z_N-a]\big\}\,\big| \ll_s \mathcal{M}^s(t;k), \quad k=pN. \tag{1.21}$$

Moreover, by Lemma 5.4, for any $0<A\le B$, $a\in\mathbb{R}^d$ and $\gamma>0$,

$$\int\limits_A^B \big|\,\widehat{\Psi}_a(t)\,\big|\,\frac{dt}{|t|} \le c_\gamma(s)\,(pN)^{-\gamma}\,\log\frac{B}{A} + \sup_{\Gamma}\int\limits_A^B \sqrt{\mathbf{E}\ \mathrm{e}\big\{t\,\langle\widetilde{W},\widetilde{W'}\rangle/2\big\}}\;\frac{dt}{|t|}\,, \tag{1.22}$$

where $W=V_1+\dots+V_n$ and $W'=V_1'+\dots+V_n'$ are independent sums of independent copies of random vectors V and V' respectively, and the supremum $\sup_\Gamma$ is taken over all $\mathcal{L}(V),\mathcal{L}(V')\in\mathbf{\Gamma}(\delta;\,\mathcal{S}_o)$.

Inequalities of type (1.21) allow to prove in Theorem 1.1 only error bounds $\mathcal{O}(N^{-\alpha})$, for some $\alpha<1$. This is due to possible oscillations of $\big|\,\widehat{\Phi}_a(t)\,\big|$ between 0 and 1, as $|\,t\,|\sim N^{1-\varepsilon}$ with small $\varepsilon\ge 0$. In Section 5, we reduce the estimation of $\mathbf{E}\ \mathrm{e}\big\{t\,\langle\widetilde{W},\widetilde{W'}\rangle/2\big\}$ to the estimation of a theta-series see Lemma 5.6 and inequalities (5.28) and (5.29). To this end, we write the expectation with respect to Rademacher random variables as a sum with binomial weights $p(m)$ and $p(\overline{m})$. Then we estimate $p(m)$ and $p(\overline{m})$ from above by discrete Gaussian exponential weights $c_s\,q(m)$ and $c_s\,q(\overline{m})$, see (5.13), (5.16), (5.18) and (5.19). Together with the non-negativity of some characteristic functions (see (5.17) and (5.21)), this allows us to apply then the Poisson summation formula from Lemma 5.5. This formula reduces the problem to an estimation of integrals of theta-series. Section 6 is devoted to some facts from Number Theory. We consider the lattices, their α-characteristics and Minkowski's successive minima. In Section 7 we reduce the estimation of integrals of theta-series to some integrals of α-characteristics. An application of a new Lemma 7.2 proved by Götze and Margulis (2009) ends the proof.

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2. Proofs of bounds for concentration functions; truncation

We start the Section with Theorem 2.1 which (under additional restrictions) provides more explicit bounds for the concentration than those of Theorem 1.5. In the next Theorem, c_0 is an arbitrary positive absolute constant. Recall as well that we write $\beta=\beta_4$, $\Pi=\Pi_2$ and $\Lambda=\Lambda_4$.

Theorem 2.1. *Assume that* $5\le d\le\infty$ *and that the operator* $\mathbb{Q}$ *is isometric. Then, for any random vector* X *such that* $\mathbf{E}\,X=0$ *and* $\sigma^2<\infty$, *we have:*

(i) *Assume condition* $\mathcal{N}(p,\delta,\mathcal{S}_o,c_0\,G/\sigma)$ *with* $s=5$ *and* $\delta=1/200$. *Then*

$$Q(Z_N;\,\lambda) \ll p^{-2}\max\big\{\Pi+\Lambda;\ \lambda\sigma^{-2}N^{-1}\big\}, \qquad \lambda\ge 0. \tag{2.1}$$

In particular, $Q(Z_N;\,\lambda)\ll p^{-2}N^{-1}\max\big\{\beta/\sigma^4;\ \lambda/\sigma^2\big\}$*;*

(ii) Let the condition $\mathcal{B}(\mathcal{S}_o, \mathbb{C})$ *(see* (1.5)*) be fulfilled with* $s = 5$. *Then*

$$Q(Z_N;\, \lambda) \le \exp\bigl\{ c\sigma^2/\lambda_5^2 \bigr\} \max\bigl\{ \Pi + \Lambda;\, \lambda\sigma^{-2} N^{-1} \bigr\}, \qquad \lambda \ge 0, \tag{2.2}$$

where c *denotes a sufficiently large absolute constant.*

Proof of Theorems 1.5 *and* 2.1. Below we shall prove the assertions (1.12); (1.12) $\Longrightarrow$ (1.13); (1.13) $\Longrightarrow$ (2.1) and (2.1) $\Longrightarrow$ (2.2). The proof repeats almost literally the corresponding proof of BG (1997a). It is given here for the sake of completeness. The only essential difference is in the use of Lemma 7.3 in the proof of Lemma 2.2. $\square$

For $T \ge t_0, t_1 \ge 0$ and $a \in \mathbb{R}^d$, define the integrals

$$I_0 = \int_{-t_1}^{t_1} \bigl| \widehat{\Phi}_a(t) \bigr| \, dt, \qquad I_1 = \int_{t_0 \le |t| \le T} \bigl| \widehat{\Phi}_a(t) \bigr| \, \frac{dt}{|t|}\,,$$

where

$$\widehat{\Phi}_a(t) = \mathbf{E}\; \mathrm{e}\bigl\{ t\, \mathbb{Q}[Z_N - a] \bigr\} \tag{2.3}$$

denotes the Fourier–Stieltjes transform of the distribution function Φ_a of $\mathbb{Q}[Z_N - a]$ (see (1.15) and (1.18)). Note that $\bigl|\widehat{\Phi}_a(-t)\bigr| = \bigl|\widehat{\Phi}_a(t)\bigr|$.

Lemma 2.2. *Assume the condition* $\mathcal{N}(p, \delta, \mathcal{S}_o, \widetilde{X})$ *with some* $0 \le \delta \le 1/(5s)$ *and* $s \ge 5$. *Let*

$$t_0 = c_1(s)\,(pN)^{-1+2/s}, \qquad t_1 = c_2(s)\,(pN)^{-1/2}, \qquad c_3(s) \le T \le c_4(s) \tag{2.4}$$

with some positive constants $c_j(s)$, $1 \le j \le 4$. *Then*

$$I_0 \ll_s (pN)^{-1}, \qquad I_1 \ll_s (pN)^{-1}. \tag{2.5}$$

Proof. Denote $k = pN$. Without loss of generality we assume that $k \ge c_s$, for a sufficiently large constant c_s. Indeed, if $k \le c_s$, then we can derive (2.5) using $|\widehat{\Phi}_a| \le 1$. Choosing c_s to be large enough, we ensure that $k \ge c_s$ implies $1/k \le t_0 \le t_1 \le T$.

Let us prove (2.5) for I_0. By Theorem 5.2 we have

$$|\widehat{\Phi}_a(t)| \ll_s \mathcal{M}^s(t;k), \qquad k = pN. \tag{2.6}$$

Since $|\widehat{\Phi}_a| \le 1$, we have $\bigl|\widehat{\Phi}_a(t)\bigr| \ll_s \min\bigl\{1;\, \mathcal{M}^s(t;k)\bigr\}$. Furthermore, denoting $t_2 = k^{-1/2} \max\{1; c_2(s)\}$ and using the definition of the function $\mathcal{M}$, we obtain

$$I_0 \ll_s \int_0^{1/k} dt + \int_{1/k}^{\infty} \frac{dt}{(tk)^{s/2}} + \int_0^{t_2} t^{s/2}\, dt = \frac{1}{k} + \frac{c_s}{k} + \frac{c_s}{k^{(s+2)/4}} \ll_s \frac{1}{k}\,,$$

thus proving (2.5) for I_0.

It remains to estimate I_1. Using (1.20), (2.4) and (2.6), it is easy to verify that

$$\int_{t_0}^{k^{-2/s}} \bigl|\widehat{\Phi}_a(t)\bigr| \frac{dt}{t} \ll_s \int_{t_0}^{\infty} \frac{dt}{t\,(tk)^{s/2+1}} + \int_0^{k^{-2/s}} t^{s/2-1}\, dt \ll_s \frac{1}{k}\,. \tag{2.7}$$

Furthermore, Lemma 7.3 implies that

$$\int_{c_4(s)k^{-1}}^{T} \bigl|\,\widehat{\Phi}_a(t)\bigr|\,\frac{dt}{t} \ll_s \frac{1}{k}\,, \tag{2.8}$$

with some $c_5(s)$, and $c_5(s)\,k^{-1} < k^{-2/s}$, if $k \geq c_s$ with sufficiently large c_s. The second inequality in (2.5) follows now from (2.7) and (2.8). $\square$

Proof of (1.12). Using a well-known inequality for concentration functions (see, for example, Petrov (1975, Lemma 3 of Ch. 3)), we have

$$Q(Z_N;\,\lambda) \leq 2 \sup_{a\in\mathbb{R}^d} \max\Bigl\{\lambda;\ \frac{1}{T}\Bigr\} \int_{-T}^{T} \bigl|\,\widehat{\Phi}_a(t)\bigr|\,dt, \tag{2.9}$$

for any $T > 0$. To estimate the integral in (2.9) we shall apply Lemma 2.2. Let us choose $T = 1$. Without loss of generality we assume that $pN \geq 1$. Then, using $1 \leq 1/|\,t\,|$, for $|\,t\,| \leq 1$, we have

$$\int_{-T}^{T} \bigl|\,\widehat{\Phi}_a(t)\bigr|\,dt \leq \int_{|t|\leq(pN)^{-1/2}} \bigl|\,\widehat{\Phi}_a(t)\bigr|\,dt + \int_{(pN)^{-1/2}\leq|t|\leq 1} \bigl|\,\widehat{\Phi}_a(t)\bigr|\,\frac{dt}{|\,t\,|} \overset{\mathrm{def}}{=} I_0 + I_2.$$

Lemma 2.2 implies $I_0 \ll_s 1/(pN)$ and $I_2 \ll_s 1/(pN)$. $\square$

Proof of (1.12) $\Longrightarrow$ (1.13). Without loss of generality we can assume that $N/m \geq 2$. Let $Y_1, Y_2, \ldots$ be independent copies of $m^{-1/2}\,Z_m$. Write $W_k = Y_1 + \cdots + Y_k$. Then $\mathcal{L}(Z_N) = \mathcal{L}(\sqrt{m}W_k + b)$ with $k = \lceil N/m \rceil$ and with some b independent of W_k. Consequently, we have $Q(Z_N;\lambda) \leq Q(W_k;\lambda/m)$. In order to estimate $Q(W_k;\lambda/m)$ we can apply (1.12) replacing Z_N by W_k. We obtain

$$Q(W_k;\lambda/m) \ll_s (pk)^{-1}\max\{1;\ \lambda/m\} \ll_s (pN)^{-1}\max\{m;\ \lambda\}. \qquad \square$$

Recall that truncated random vectors and moments were defined by (1.6), (1.7) and (1.17), and that $\mathbb{C} = \operatorname{cov} X = \operatorname{cov} G$.

Lemma 2.3. (BG (1997a, Lemma 2.4)) *The random vectors $X^{\diamond}$, $X_{\diamond}$ satisfy*

$$\langle \mathbb{C}x, x\rangle = \langle \operatorname{cov} X^{\diamond}\, x, x\rangle + \mathbf{E}\,\langle X_{\diamond}, x\rangle^2 + \langle \mathbf{E}\,X^{\diamond}, x\rangle^2.$$

There exist independent centered Gaussian vectors G_* and W such that $\mathcal{L}(G) = \mathcal{L}(G_* + W)$ and

$$2\operatorname{cov} G_* = 2\operatorname{cov} X^{\diamond} = \operatorname{cov} \widetilde{X^{\diamond}}, \qquad \langle \operatorname{cov} W\, x, x\rangle = \mathbf{E}\,\langle X_{\diamond}, x\rangle^2 + \langle \mathbf{E}\,X^{\diamond}, x\rangle^2.$$

Furthermore, $\mathbf{E}\,\|G\|^2 = \mathbf{E}\,\|G_*\|^2 + \mathbf{E}\,\|W\|^2$ *and* $\mathbf{E}\,\|W\|^2 \leq 2\sigma^2\Pi$.

Lemma 2.4. (BG (1997a, Lemma 5.4)) *Assume that $0 < 4\varepsilon \leq \delta \leq 1$. Let $e \in \mathbb{R}^d$, $\|e\| = 1$, be an eigenvector of the covariance operator $\mathbb{C} = \operatorname{cov} G$, so that $\mathbb{C}e = \sigma_e e$ with some $\sigma_e > 0$. Then the probability $p_e = \mathbf{P}\bigl\{\,\|\varepsilon\sigma^{-1}G - e\| \leq \delta\bigr\}$ satisfies the inequality $p_e \geq \exp\bigl\{-c\sigma^2\varepsilon^{-2}\sigma_e^{-2}\bigr\}$ with some positive absolute constant c.*

Recall, that $Z_N^\diamond$ denotes a sum of N independent copies of $X^\diamond$.

Lemma 2.5. (BG (1997a, Lemma 2.5)) *Let $\varepsilon > 0$. There exist absolute positive constants c and c_1 such that the condition $\Pi \le c_1 p\delta^2/(\varepsilon^2\sigma^2)$ implies that*

$$\mathcal{N}(p,\delta,\mathcal{S},\varepsilon G) \Longrightarrow \mathcal{N}(p/4, 4\delta, \mathcal{S}, \varepsilon(2m)^{-1/2}\widetilde{Z_m^\diamond}),$$

for $m \ge c\varepsilon^4\sigma^4 N\,\Lambda/(p\delta^4)$.

Proof (1.13) $\Longrightarrow$ (2.1). By a well known truncation argument, we have

$$\bigl|\mathbf{P}\bigl\{\, Z_N \in A \bigr\} - \mathbf{P}\bigl\{\, Z_N^\diamond \in A \bigr\}\bigr| \le N\mathbf{P}\bigl\{\, \|X\| > \sigma\sqrt{N}\,\bigr\} \le \Pi, \tag{2.10}$$

for any measurable set A, and

$$Q(Z_N,\lambda) \le \Pi + Q(Z_N^\diamond,\lambda). \tag{2.11}$$

Write $K = \varepsilon/\sqrt{2}$ with $\varepsilon = c_0/\sigma$. Then, by Lemma 2.5, we have

$$\mathcal{N}(p,\delta,\mathcal{S}_o,\varepsilon G) \Longrightarrow \mathcal{N}(p/4, 4\delta, \mathcal{S}_o, m^{-1/2}\,K\,\widetilde{Z_m^\diamond}), \tag{2.12}$$

provided that

$$\Pi \le c_1 p, \qquad\qquad m \ge cN\,\Lambda/p. \tag{2.13}$$

Without loss of generality we may assume that $\Pi/p \le c_1$, since otherwise the result easily follows from the trivial estimate $Q(Z_N;\lambda) \le 1$.

Recall that we are proving (2.1) assuming that $s = 5$ and $\delta = 1/200$. Hence, $4\delta = 1/50 < 1/(5s)$. The non-degeneracy condition (2.12) for $K\,\widetilde{Z_m^\diamond}$ allows to apply (1.13) of Theorem 1.5, and we obtain

$$Q(Z_N^\diamond,\lambda) = Q(K\,Z_N^\diamond, K^2\lambda) \ll (pN)^{-1}\max\{m;\, K^2\lambda\}, \tag{2.14}$$

for any m such that (2.13) is fulfilled. Choosing the minimal m in (2.13), we obtain

$$Q(Z_N^\diamond,\lambda) \ll p^{-2}\max\{\Lambda;\, \lambda/(\sigma^2 N)\}. \tag{2.15}$$

Combining the estimates (2.11) and (2.15), we conclude the proof. □

Proof (2.1) $\Longrightarrow$ (2.2). Note that the bound (2.1) holds with a probability p of condition $\mathcal{N}(p,\delta,\mathcal{S}_o, c_0 G/\sigma)$. Let us choose $4c_0 = \delta = 1/200$. Then, using Lemma 2.4 and the assumption $\mathcal{B}(\mathcal{S}_o,\mathbb{C})$, the effective lower bound $p \ge \exp\bigl\{\, c\sigma^2/\lambda_5^2 \bigr\}$ follows. □

3. Beginning of the proof of Theorem 1.4

The proof of Theorem 1.4 repeats almost literally the proof of Theorem 1.5 in BG (1997a). It starts with a truncation of random vectors and an application of the Fourier transform to the functions Ψ and Ψ_0. We shall estimate integrals over the Fourier transforms using results of Sections 2, 5–7 and some technical lemmas of BG (1997a). We shall apply as well some elements of the standard techniques used in the case of the CLT in multidimensional spaces (cf. e.g., Bhattacharya and Rao (1986)).

We shall use the following approximate and precise formulas for the Fourier inversion. A smoothing inequality of Prawitz (1972) implies (see Bentkus and Götze 1996, Section 4) that

$$F(x)=\frac{1}{2}+\frac{i}{2\pi}\ \mathrm{V.\,P.}\int_{|t|\le K}\mathrm{e}\big\{-xt\big\}\widehat{F}(t)\ \frac{dt}{t}\ +R,\tag{3.1}$$

for any $K>0$ and any distribution function F with characteristic function $\widehat{F}$, where

$$|R|\le\frac{1}{K}\int_{|t|\le K}|\widehat{F}(t)|\,dt.$$

Here $\mathrm{V.\,P.}\int f(t)\,dt=\lim_{\varepsilon\to0}\int_{|t|>\varepsilon}f(t)\,dt$ denotes the Principal Value of the integral.

For any function $F:\mathbb{R}\to\mathbb{R}$ of bounded variation such that $F(-\infty)=0$ and $2F(x)=F(x+)+F(x-)$, for all $x\in\mathbb{R}$, the following Fourier–Stieltjes inversion formula holds (see, e.g., Chung (1974))

$$F(x)=\frac{1}{2}\,F(\infty)+\frac{i}{2\pi}\lim_{M\to\infty}\mathrm{V.\,P.}\int_{|t|\le M}\mathrm{e}\{-xt\}\,\widehat{F}(t)\,\frac{dt}{t}\,.\tag{3.2}$$

The formula is well-known for distribution functions. For functions of bounded variation, it extends by linearity arguments.

In this section we shall assume that the following conditions are fulfilled

$$\mathbb{Q}^2=\mathbb{I}_d,\quad\sigma^2=1,\quad s=5,\quad\delta=1/300,\quad\mathcal{N}(p,\delta,\mathcal{S}_o,c_0G).\tag{3.3}$$

Notice that the assumption $\sigma^2=1$ does not restrict generality since from Theorem 1.4 with $\sigma=1$ we can derive the general result replacing X, G by X/σ, G/σ, etc. Other assumptions in (3.3) are included as conditions in Theorem 1.4. The assumption $\sigma^2=1$ yields (recall that we write $\Pi=\Pi_2$ and $\Lambda=\Lambda_4$)

$$N^{-1}\le2(\Pi+\Lambda),\qquad\Pi+\Lambda\le1,\qquad\sigma_j\le1,\qquad\lambda_j\le1.\tag{3.4}$$

Recall that Δ_N and functions Ψ and Ψ_0 are defined by (1.3) and (1.15). In that notation we have

$$\Delta_N=\sup_{x\in\mathbb{R}}\big|\Delta_N(x)\big|,\tag{3.5}$$

where

$$\Delta_N(x)=\Psi(x)-\Psi_0(x),\tag{3.6}$$

since the functions F and F_0 defined by (1.2) satisfy

$$F(x)=\Psi(xN),\quad F_0(x)=\Psi_0(xN),\quad\widehat{F}(tN)=\widehat{\Psi}(t),\quad\widehat{F}_0(tN)=\widehat{\Psi}_0(t).\tag{3.7}$$

To prove the statement (i) of Theorem 1.4, we have to derive that

$$\Delta_N\ll_d p^{-3}\sigma_d^{-4}\,(\Pi+\Lambda)\,,\qquad\qquad\text{for } s=5.\tag{3.8}$$

While proving (3.8) we can and shall assume that

$$\Pi\le cp\sigma_d^4,\qquad\qquad\Lambda\le cp\sigma_d^4,\tag{3.9}$$

with a sufficiently small positive absolute constant c. These assumptions do not restrict generality. Indeed, we have $\Delta_N \le 1$. If the assumption (3.9) does not hold, this immediately implies (3.8).

Recall that the random vectors $X^\diamond$, X' and sums $Z_N^\diamond$, Z_N' of their independent copies are defined in (1.6) and (1.17). Write $\Psi^\diamond$ and Ψ' for the distribution function of $\mathbb{Q}[Z_N^\diamond]$ and $\mathbb{Q}[Z_N']$ respectively. For $0 \le k \le N$ introduce the the distribution function

$$\Psi^{(k)}(x) = \mathbf{P}\bigl\{\mathbb{Q}\bigl[G_1 + \dots + G_k + X_{k+1}' + \dots + X_N'\bigr] \le x\bigr\}. \tag{3.10}$$

Notice that $\Psi^{(0)} = \Psi'$, $\Psi^{(N)} = \Psi_0$.

Lemma 3.1. (BG (1997a, Lemma 3.1)) *Assume that* $\Pi \le c_1 p$ *and that a number* $1 \le m \le N$ *satisfies* $m \ge c_2 N\Lambda/p$, *with some sufficiently small* (*resp. large*) *positive absolute constant* c_1 (*resp.* c_2). *Let* c_3 *be an absolute constant. Write*

$$K = c_0^2/(2m), \qquad\qquad t_1 = c_3\,(pN/m)^{-1/2}.$$

Let F *denote any of the functions* $\Psi^\diamond$, Ψ', $\Psi^{(k)}$ *or* Ψ_0. *Then we have*

$$F(x) = \frac{1}{2} + \frac{i}{2\pi}\,\mathrm{V.P.}\int_{|t|\le t_1} \mathrm{e}\{-xtK\}\,\widehat{F}(tK)\,\frac{dt}{t} + R_1, \tag{3.11}$$

with $|R_1| \ll (pN)^{-1} m$.

Proof. Let us prove (3.11). For the proof we shall combine (3.1) and Lemma 2.2. Changing the variable $t = \tau K$ in the approximate Fourier inversion formula (3.1), we obtain

$$F(x) = \frac{1}{2} + \frac{i}{2\pi}\,\mathrm{V.P.}\int_{|t|\le 1} \mathrm{e}\{-xtK\}\,\widehat{F}(tK)\,\frac{dt}{t} + R, \tag{3.12}$$

where

$$|R| \le \int_{|t|\le 1} \bigl|\widehat{F}(tK)\bigr|\,dt. \tag{3.13}$$

Notice all functions $\Psi^\diamond$, Ψ', $\Psi^{(k)}$, Ψ_0 are distribution functions of the following type of random variables:

$$\mathbb{Q}[U+T], \qquad\qquad U \overset{\mathrm{def}}{=} G_1 + \dots + G_k + X_{k+1}^\diamond + \dots + X_N^\diamond,$$

with some $0 \le k \le N$, where the random vector T is independent of $X_j^\diamond$ and G_j, for all j. Let us consider two alternative cases: $k \ge N/2$ and $k < N/2$.

The case $k < N/2$. Let Y denote a sum of m independent copies of $K^{1/2} X^\diamond$. Let $Y_1, Y_2, \dots$ be independent copies of Y. Then we can write

$$K^{1/2} U \overset{\mathcal{D}}{=} Y_1 + \dots + Y_l + T_1 \tag{3.14}$$

with $l = \lceil N/(2m) \rceil$ and some random T_1 independent of $Y_1, \dots, Y_l$. By Lemma 2.5, we have

$$\mathcal{N}(p, \delta, \mathcal{S}, c_0 G) \Longrightarrow \mathcal{N}(p/4, 4\delta, \mathcal{S}, \widetilde{Y}) \tag{3.15}$$

provided that

$$\Pi \le c_1 p \qquad \text{and} \qquad m \ge c_2 N\Lambda/p. \tag{3.16}$$

The inequalities in (3.16) are just conditions of the Lemma 3.1. Due to (3.14) and (3.15), we can use Lemma 2.2 in order to estimate the integrals in (3.12) and (3.13). Replacing in Lemma 2.2 X by Y and N by l, we obtain (3.11) in the case $k < N/2$.

The case $k \ge N/2$. We can proceed as in the previous case defining however Y as a sum of m independent copies of $K^{1/2}G$. The condition (3.21) is fulfilled since now $\mathcal{L}(\widetilde{Y}) = \mathcal{L}(c_0 G/\sigma)$, and (3.11) follows. □

Following BG (1997a), introduce the upper bound $\varkappa = \varkappa(t) = \varkappa\bigl(t; N, \mathcal{L}(X), \mathcal{L}(G)\bigr)$ for the characteristic function of quadratic forms (cf. Bentkus (1984) and Bentkus, Götze and Zitikis (1993)). We define $\varkappa = \varkappa\bigl(t; N, \mathcal{L}(X)\bigr) + \varkappa\bigl(t; N, \mathcal{L}(G)\bigr)$, where

$$\varkappa(t; N, \mathcal{L}(X)) = \sup_{a\in\mathbb{R}^d} \bigl|\mathbf{E}\ \mathrm{e}\bigl\{t\mathbb{Q}[Z_j] + \langle a, Z_j\rangle\bigr\}\bigr|, \qquad Z_j = X_1 + \cdots + X_j, \tag{3.17}$$

with $j = \bigl\lceil (N-2)/14 \bigr\rceil$.

Lemma 3.2. *Assume the conditions of Lemma* 3.1. *Then*

$$\int_{|t|\le t_1} \bigl(|t|K\bigr)^{\alpha} \varkappa\bigl(tK; N, \mathcal{L}(X^{\diamond}), \mathcal{L}(G)\bigr) \,\frac{dt}{|t|} \ll_\alpha \begin{cases} (Np)^{-\alpha}, & \textit{for } 0 \le \alpha < s/2, \\ (Np)^{-\alpha}\bigl(1 + |\log(Np/m)|\bigr), & \textit{for } \alpha = s/2, \\ (Np)^{-\alpha}\bigl(1 + (Np/m)^{(2\alpha - s)/4}\bigr), & \textit{for } \alpha > s/2. \end{cases} \tag{3.18}$$

Lemma 3.2 is a generalization of Lemma 3.2 from BG (1997a) which contains the same bound for $0 \le \alpha < s/2$. In this paper, we have to estimate the left hand side of (3.18) in the case $s/2 \le \alpha$ too.

Proof. By (3.15), the condition $\mathcal{N}(p/4, 4\delta, \mathcal{S}, K^{1/2}\,\widetilde{Z_m^{\diamond}})$ is fulfilled. Therefore, collecting independent copies of $K^{1/2}X^{\diamond}$ in groups as in (3.14), we can apply Theorem 5.2. We obtain

$$\varkappa(tK; N, \mathcal{L}(X^{\diamond})) \ll \mathcal{M}^s(t; pN/m).$$

A similar upper bound holds for $\varkappa(tK; N, \mathcal{L}(G))$ (cf. the proof of (3.11) in the case of $k > N/2$). Using the definition of the function $\mathcal{M}(\cdot,\cdot)$ and (1.19) in order to get rid of absolute constants, we get

$$\varkappa(tK; N, \mathcal{L}(X^{\diamond}), \mathcal{L}(G)) \ll_s \min\Bigl\{1;\ \bigl(m/(tpN)\bigr)^{s/2}\Bigr\}, \qquad \text{for} \quad |t| \le t_1.$$

Integrating that bound (cf. the estimation of I_1 in Lemma 2.2), we conclude the proof of the Lemma. □

Reduction of (3.8) *to an estimation of*

$$\Delta'_N = \sup_x \bigl|\Psi'(x) - \Psi_0(x)\bigr|, \tag{3.19}$$

where Ψ' *is the distribution function of* $\mathbb{Q}[Z_N']$. It suffices to prove that the quantity Δ_N' satisfies inequalities of type (3.8). Indeed, let us prove that

$$\sup_x \bigl|\Psi(x) - \Psi'(x)\bigr| \ll p^{-2}(\Pi + \Lambda). \tag{3.20}$$

Using truncation (cf. (2.10)), we have $|\Psi - \Psi^\diamond| \le \Pi$, and

$$\sup_x \bigl|\Psi(x) - \Psi'(x)\bigr| \le \Pi + \sup_x \bigl|\Psi^\diamond(x) - \Psi'(x)\bigr|. \tag{3.21}$$

In order to estimate $|\Psi^\diamond - \Psi'|$, we shall apply Lemmas 3.1 and 3.2. The number m in these Lemmas exists, as it follows from the second inequality in (3.9). Let us choose the minimal m, that is, $m \asymp N\Lambda/p$. Then $(pN)^{-1}m \ll \Lambda/p^2$ and $m/N \ll \Lambda/p$. Therefore, using Lemma 3.1, we have

$$\sup_x \bigl|\Psi^\diamond(x) - \Psi'(x)\bigr| \ll p^{-2}\Lambda + \int_{|t|\le t_1} \bigl|\widehat{\Psi}^\diamond(\tau) - \widehat{\Psi}'(\tau)\bigr| \,\frac{dt}{|t|}\,, \qquad \tau = tK. \tag{3.22}$$

We shall prove that

$$\bigl|\widehat{\Psi}^\diamond(\tau) - \widehat{\Psi}'(\tau)\bigr| \ll \varkappa\Pi\,|\tau|N(1 + |\tau|N), \tag{3.23}$$

with $\varkappa = \varkappa(\tau; N, \mathcal{L}(X^\diamond))$. Combining (3.21)–(3.23), using $\tau = tK$ and integrating inequality (3.23) with help of Lemma 3.2, we derive (3.20).

Let us prove (3.23). Recall that $X' = X^\diamond - \mathbf{E}\,X^\diamond + W$, where W denotes a centered Gaussian random vector which is independent of all other random vectors and such that $\operatorname{cov} X' = \operatorname{cov} G$ (see Lemma 2.3). Writing $D = Z_N^\diamond - \mathbf{E}\,Z_N^\diamond$, we have

$$Z_N^\diamond = D + \mathbf{E}\,Z_N^\diamond, \qquad Z_N' \overset{\mathcal{D}}{=} D + \sqrt{N}W,$$

and $|\widehat{\Psi}^\diamond(\tau) - \widehat{\Psi}'(\tau)| \le |f_1(\tau)| + |f_2(\tau)|$ with

$$\begin{aligned} f_1(\tau) &= \mathbf{E}\ \mathrm{e}\bigl\{\tau\mathbb{Q}[D + \sqrt{N}W]\bigr\} - \mathbf{E}\ \mathrm{e}\bigl\{\tau\mathbb{Q}[D]\bigr\}, \\ f_2(t) &= \mathbf{E}\ \mathrm{e}\bigl\{\tau\mathbb{Q}[D + \mathbf{E}\,Z_N^\diamond]\bigr\} - \mathbf{E}\ \mathrm{e}\bigl\{\tau\mathbb{Q}[D]\bigr\}. \end{aligned} \tag{3.24}$$

We have to show that both $|f_1(t)|$ and $|f_2(t)|$ are bounded from above by the right hand side of (3.23). Let us consider f_1. We can write $\mathbb{Q}[D + \sqrt{N}W] = \mathbb{Q}[D] + A + B$ with $A = 2\sqrt{N}\langle \mathbb{Q}D, W\rangle$ and $B = N\mathbb{Q}[W]\bigr\}$. Taylor expansions of the exponent in (3.24) in powers of $i\tau B$ and $i\tau A$ with remainders $\mathcal{O}(\tau B)$ and $\mathcal{O}(\tau^2 A^2)$ respectively imply (notice that $\mathbf{E}\,W = 0$)

$$|f_1(\tau)| \ll \varkappa|\tau|N\mathbf{E}\,\|W\|^2 + \varkappa\tau^2 N\mathbf{E}\,\|W\|^2\,\mathbf{E}\,\|D\|^2, \tag{3.25}$$

where $\varkappa = \varkappa(\tau; N, \mathcal{L}(X^\diamond))$. The estimation of the remainders of these expansions is based on the splitting and conditioning techniques described in Section 9 of BG (1997a), see also Bentkus, Götze and Zaitsev (1997). Using $\sigma = 1$, $\mathbf{E}\,\|W\|^2 \ll \Pi$ and $\mathbf{E}\,\|D\|^2 \ll N$, we derive from (3.25) that

$$\bigl|f_1(\tau)\bigr| \ll \varkappa\Pi|\tau|N\bigl(1 + |\tau|N)\bigr). \tag{3.26}$$

Expanding in powers of $\mathbf{E}\, Z_N^\diamond = N\,\mathbf{E}\, X^\diamond$ and proceeding similarly to the proof of (3.26), we obtain

$$|f_2(t)| \ll \varkappa \Pi |\tau| N,$$

which concludes the proof of (3.23). $\square$

4. End of the proof of Theorem 1.4

We start this Section with two Lemmas from BG (1997a). Throughout we shall assume that $\mathbf{E}\, X = \mathbf{E}\, G = 0$ and $\operatorname{cov} X = \operatorname{cov} G$. Recall that we write

$$\beta_s = \mathbf{E}\, \|X\|^s, \quad \beta = \beta_4, \quad \sigma^2 = \beta_2, \quad Z_N = X_1 + \cdots + X_N.$$

We shall denote as well $\mathbb{D} = t\mathbb{Q}$ and $U_l = G_1 + \cdots + G_l$. Since $\sqrt{N}\, G \overset{\mathcal{D}}{=} U_N$, we can write

$$\widehat{\Psi}(t) = \mathbf{E}\ \mathrm{e}\big\{\mathbb{D}[Z_N]\big\}, \qquad \widehat{\Psi}_0(t) = \mathbf{E}\ \mathrm{e}\big\{\mathbb{D}[U_N]\big\}.$$

We shall use the upper bound $\varkappa = \varkappa(t; N, \mathcal{L}(X)) + \varkappa(t; N, \mathcal{L}(G))$ (cf. (3.17)), where

$$\varkappa(t; N, \mathcal{L}(X)) = \sup_{a \in \mathbb{R}^d} \big|\mathbf{E}\ \mathrm{e}\big\{\mathbb{D}[Z_j] + \langle a, Z_j\rangle\big\}\big|, \qquad j = \big[(N-2)/14\big].$$

For $0 \le k \le N$ define

$$\widehat{\Phi}^{(k)}(t) = \mathbf{E}\ \mathrm{e}\big\{t\,\mathbb{Q}[G_1 + \cdots + G_k + X_{k+1} + \cdots + X_N]\big\}.$$

Notice that $\widehat{\Phi}^{(0)}(t) = \widehat{\Psi}(t)$ and $\widehat{\Phi}^{(N)}(t) = \widehat{\Psi}_0(t)$.

Lemma 4.1. (BG (1997a, Lemma 9.3)) *Assume that $\sigma = 1$ and $\mathbb{Q}^2 = \mathbb{I}_d$. Then we have*

$$\big|\widehat{\Psi}(t) - \widehat{\Phi}^{(k)}(t)\big| \ll \varkappa t^2 k\,\big(\beta + |t| N \beta + |t| N \sqrt{N\beta}\big).$$

Fix an integer k, $1 \le k \le N$. Define the distributions

$$\mu(A) = \mathbf{P}\Big\{U_k + \sum_{j=k+1}^{N} X_j \in \sqrt{N} A\Big\}, \qquad \mu_0(A) = \mathbf{P}\big\{U_N \in \sqrt{N} A\big\}.$$

For measurable sets $A \subset \mathbb{R}^d$ define the Edgeworth correction (to the distribution μ) as $\mu_1^{(k)}(A) = (N-k)\, N^{-3/2} \chi(A)/6$, where the signed measure χ is given by

$$\chi(A) = \int_A \mathbf{E}\, p'''(x) X^3\, dx, \tag{4.1}$$

and where

$$p'''(x)\, u^3 = p(x)\Big(3\,\langle \mathbb{C}^{-1} u, u\rangle \langle \mathbb{C}^{-1} x, u\rangle - \langle \mathbb{C}^{-1} x, u\rangle^3\Big) \tag{4.2}$$

denotes the third Frechet derivative of p in the direction u, and $p(x) = \phi(\mathbb{C}^{-1/2} x)/\sqrt{\det \mathbb{C}}$, where $\phi(\cdot)$ is the standard normal density in $\mathbb{R}^d$. Introduce the signed measure $\nu = \mu - \mu_0 - \mu_1^{(k)}$.

Lemma 4.2. *Assume that* $d < \infty$ *and* $1 \le k \le N$. *Then there exists* c_d *such that*

$$\delta_N \overset{\text{def}}{=} \sup_{A \subset \mathbb{R}^d} \bigl|\nu(A)\bigr| \ll_d \frac{\beta}{\sigma_d^4 N} + \frac{N^{d/2}}{k^{d/2}} \exp\{-c_d\, k\sigma_d^4/\beta\}. \tag{4.3}$$

An outline of the proof. We repeat and slightly improve the proof of Lemma 9.3 in BG (1997a) (cf. the proof of Lemma 2.5 in Bentkus and Götze (1996)). Assuming that $\operatorname{cov} X = \operatorname{cov} G = \mathbb{I}_d$, we shall prove that

$$\delta_N \ll_d \frac{\beta}{N} + \frac{N^{d/2}}{k^{d/2}} \exp\{-c_d\, k/\beta\}. \tag{4.4}$$

Applying (4.4) to $\mathbb{C}^{-1/2}X$ and $\mathbb{C}^{-1/2}G$ and estimating $\|\mathbb{C}^{-1/2}\| \le 1/\sigma_d$, we obtain (4.3).

While proving (4.4) we can assume that $\beta/N \le c_d$ and $N \ge 1/c_d$ with a sufficiently small positive constant c_d. Otherwise (4.4) follows from the trivial bound

$$\delta_N \ll_d 1 + (\beta/N)^{1/2} \int_{\mathbb{R}^d} \|x\|^3 p(x)\, dx \ll_d 1 + (\beta/N)^{1/2}.$$

Define truncated random vectors $X^* = X\,\mathbf{I}\bigl\{\|X\| \le \sqrt{N}\bigr\}$ and $X_* = X\,\mathbf{I}\bigl\{\|X\| > \sqrt{N}\bigr\} = X - X^*$. Similarly to Lemma 2.3, the random vectors X^*, X_* satisfy

$$\langle \operatorname{cov} X\, x, x\rangle = \|x\|^2 = \langle \operatorname{cov} X^*\, x, x\rangle + \mathbf{E}\,\langle X_*, x\rangle^2 + \langle \mathbf{E}\, X^*, x\rangle^2.$$

There exist independent centered Gaussian vectors G^* and W such that $\mathcal{L}(G) = \mathcal{L}(G_* + W)$ and

$$\operatorname{cov} G^* = \operatorname{cov} X^*, \qquad \langle \operatorname{cov} W x, x\rangle = \mathbf{E}\,\langle X_*, x\rangle^2 + \langle \mathbf{E}\, X^*, x\rangle^2.$$

Furthermore, $\mathbf{E}\,\|G\|^2 = \mathbf{E}\,\|G^*\|^2 + \mathbf{E}\,\|W\|^2$. Using standard methods, we can show that $\mathbf{E}\,\|W\|^2 \le 2d/N$, $\mathbf{P}\bigl\{X \ne X^*\bigr\} = \mathbf{P}\bigl\{\|X\| > \sqrt{N}\bigr\} \le \beta/N^2$, $\mathbf{E}\,\bigl\|X_*\bigr\| \le \beta/N^{3/2}$ and $\mathbf{E}\,\bigl\|X_*\bigr\|^3 \le \beta/N^{1/2}$.

To prove (4.4) we shall apply truncation of X_j, centering and a correction of the covariances of Gaussian summands G_j, for $k+1 \le j \le N$. Namely, in (4.4) we can replace X_j by $X_j^* = X_j \mathbf{I}\{\|X_j\| \le \sqrt{N}\}$ up to an error β/N. The centering, that is, a replacement of X_j^* by $X_j^\bullet \overset{\text{def}}{=} X_j^* - \mathbf{E}\, X^*$, produces an error bounded by β/N. A correction of the covariances of the Gaussian random vectors yields a similar error. We shall denote the corrected Gaussian random vectors (with $\operatorname{cov} G_j^\bullet = \operatorname{cov} X_j^\bullet$) by $G_j^\bullet$. Note that actually we truncate the vectors $\mathbb{C}^{-1/2}X$. Therefore, X^* and $X^\diamond$ are different truncated vectors. The use of X^* is important for the estimation of constants.

After such a replacement of X_j and G_j by $X_j^\bullet$ and $G_j^\bullet$, for $k+1 \le j \le N$, we can assume that all eigenvalues of $\operatorname{cov} X_j^\bullet$ belong to the interval $[1/2, 3/2]$. Otherwise a trivial bound $\beta \ge cN$ implies the result. As a consequence of the truncation we have

$$\mathbf{E}\,\|S_l\|^\gamma \ll_{\gamma,d} 1, \qquad \gamma > 0, \quad 1 \le l \le N. \tag{4.5}$$

Denoting by $Z_j^\bullet$ and $U_j^\bullet$ sums of j independent copies of $X^\bullet$ and $G^\bullet$ respectively, introduce the multidimensional characteristic functions $g(t) = \mathbf{E}\,\mathrm{e}\big\{\,\langle t, G\rangle\,\big\}$,

$$f(t) = \mathbf{E}\,\mathrm{e}\big\{\,\langle N^{-1/2}t, Z_{N-k}^\bullet\rangle\,\big\}, \qquad f_0(t) = \mathbf{E}\,\mathrm{e}\big\{\,\langle N^{-1/2}t, U_{N-k}^\bullet\rangle\,\big\}, \tag{4.6}$$

$$f_1(t) = \frac{N-k}{6N^{3/2}}\,\mathbf{E}\,\langle it, X^\bullet\rangle^3 f_0(t), \qquad \widehat{\nu}(t) = \big(f(t) - f_0(t) - f_1(t)\big)\,g(\varepsilon t), \quad \varepsilon^2 = k/N. \tag{4.7}$$

By a slight extension of the proof of Lemma 11.6 in Bhattacharya and Rao (1986), see as well the proof of Lemma 2.5 in Bentkus and Götze (1996), we obtain

$$\delta_N \ll_d \frac{\beta}{N} + \max_{|\alpha|\le 2d} \int_{t\in\mathbb{R}^d} \big|\,\partial^\alpha \widehat{\nu}(t)\,\big|\,dt. \tag{4.8}$$

Here $|\alpha| = |\alpha_1| + \dots + |\alpha_d|$, $\alpha = (\alpha_1, \dots, \alpha_d)$, $\alpha_j \in \mathbb{Z}$, $\alpha_j \ge 0$. In order to derive (4.4) from (4.8), it suffices to prove that, for $|\alpha| \le 2d$,

$$\big|\,\partial^\alpha \widehat{\nu}(t)\,\big| \ll_d g(c_1\,\varepsilon t), \qquad \varepsilon^2 = k/N, \tag{4.9}$$

$$\big|\,\partial^\alpha \widehat{\nu}(t)\,\big| \ll_d \beta N^{-1}\,(1 + \|t\|^6)\,\exp\{-c_2(d)\,\|t\|^2\}, \quad \text{for } \|t\|^2 \le c_3(d)\,N/\beta. \tag{4.10}$$

Indeed, using (4.9) and denoting $T = \sqrt{c_3(d)\,N/\beta}$, we obtain

$$\int\limits_{\|t\|\ge T} \big|\,\partial^\alpha \widehat{\nu}(t)\,\big|\,dt \ll_d \int\limits_{\|t\|\ge T} g(c_1\,\varepsilon t)\,dt \ll_d \varepsilon^{-d}\,\exp\{-c_1\,\varepsilon^2 T^2/8\} \int\limits_{\mathbb{R}^d} \exp\{-c_1\,\|t\|^2/8\}\,dt, \tag{4.11}$$

and it is easy to see that the right hand side of (4.11) is bounded from above by the second summand in the right hand side of (4.4). In the proof of (4.9) and (4.11) we used relations (4.5)–(4.7), $1 \le k \le N$, $\sqrt{N}/\beta^{1/2} \gg_d 1$, $\big|\,\partial^\alpha g(t)\,\big| \ll_\alpha \exp\{-c\,\|t\|^2\}$ and $y^{c_d}\exp\{-y\} \ll_d 1$, for $y > 0$. Similarly, using (4.10), we can integrate $\big|\,\partial^\alpha \widehat{\nu}(t)\,\big|$ over $\|t\| \le T$, and the integral is bounded from above by $c_d\beta/N$.

One can prove (4.10) using a Bergström type identity, the estimates (4.5) and a version of the standard techniques provided in Bhattacharya and Rao (1986). □

Remark 4.3. Let $\mathfrak{S}$ be the set of all symmetric Borel sets $A \subset \mathbb{R}^d$. This means that $A \in \mathfrak{S}$ iff $A = -A$. Using (4.1) and (4.2), we see that $\chi(A) = 0$ (and, hence, $\mu_1^{(k)}(A) = 0$), for any $A \in \mathfrak{S}$, since the function $p'''(x)\,u^3$ is an odd function in x. In particular, we can consider the set $A = \big\{\,y \in \mathbb{R}^d : \mathbb{Q}[y] \le x\,\big\} \in \mathfrak{S}$.

Proof of Theorem 1.4. Recall that we have to verify that Δ_N' is bounded by the right hand side of (3.8), that is, that

$$\Delta_N' \ll p^{-3}\sigma_d^{-4}\big(\Pi + \Lambda\big). \tag{4.12}$$

Recall that the distribution function $\Psi^{(k)}$ is defined in (3.10). For any $1 \le k \le N$, we have

$$\Delta'_N \le I_1 + I_2, \qquad I_1 = \sup_x \bigl|\Psi'(x) - \Psi^{(k)}(x)\bigr|, \tag{4.13}$$

$$I_2 = \sup_x \bigl|\Psi^{(k)}(x) - \Psi_0(x)\bigr|. \tag{4.14}$$

Below we shall prove that

$$I_1 \ll p^{-2}(\Pi+\Lambda) + p^{-3}kN^{-2}\bigl(\beta' + \sqrt{N\beta'}\bigr)\bigl(1 + p^{1/4}/(\Pi+\Lambda)^{1/4}\bigr), \tag{4.15}$$

$$I_2 \ll_d \frac{\beta'}{\sigma_d^4 N} + \frac{N^{d/2}}{k^{d/2}}\exp\{-c_d\, k\sigma_d^4/\beta'\} \tag{4.16}$$

with $\beta' = \mathbf{E}\,\|X'\|^4$. Let us choose $k \asymp_d \sigma_d^{-3}N^{1/4}\beta'^{3/4}$. Such $k \le N$ exists since we assumed (3.9) and $\beta' \ll N(\Pi+\Lambda)$. Then (4.15) and (4.16) turn into

$$I_1 \ll_d p^{-2}(\Pi+\Lambda) + \frac{1}{\sigma_d^3 p^3}\,\Bigl(\Bigl(\frac{\beta'}{N}\Bigr)^{5/4} + \Bigl(\frac{\beta'}{N}\Bigr)^{7/4}\Bigr)\bigl(1 + p^{1/2}/(\Pi+\Lambda)^{1/4}\bigr) \tag{4.17}$$

and

$$I_2 \ll_d \frac{\beta'}{\sigma_d^4 N} + \Bigl(\frac{\sigma_d^4 N}{\beta'}\Bigr)^{3d/8}\exp\Bigl\{-c_d\Bigl(\frac{\sigma_d^4 N}{\beta'}\Bigr)^{1/4}\Bigr\} \ll_d \frac{\beta'}{\sigma_d^4 N}\,. \tag{4.18}$$

Thus, (4.13), (4.14), (4.17) and (4.18) yield (4.12) by an application of $\sigma_d \le 1$, $\beta' \ll N(\Pi+\Lambda)$ and $\Pi+\Lambda \le 1$.

Let us prove (4.15). As in the proof of (3.22), applying Lemma 3.1 (choosing $m \asymp N(\Lambda+\Pi)/p$), we obtain

$$I_1 \ll p^{-2}(\Lambda+\Pi) + \int_{|t|\le t_1}\bigl|\widehat{\Psi}'(\tau) - \widehat{\Psi}^{(k)}(\tau)\bigr|\,dt/|t|, \qquad \tau = tK.$$

Applying Lemma 4.1 and replacing in that Lemma X by X', t by τ, β by $\beta' = \mathbf{E}\,\|X'\|^4$, $\Phi^{(k)}$ by $\Psi^{(k)}$ and Ψ by Ψ', we have

$$\bigl|\widehat{\Psi}'(\tau) - \widehat{\Psi}^{(k)}(\tau)\bigr| \ll \varkappa\tau^2 k\,\bigl(\beta' + |\tau|N\beta' + |\tau|N\sqrt{N\beta'}\bigr). \tag{4.19}$$

Integrating with the help of Lemma 3.2 we obtain (4.15).

Let us prove (4.16). Define the measure χ' by replacing X by X' in (4.1). Write $Z'_{kN} = \sum_{j=1}^k G_j + \sum_{j=k+1}^N X'_j$. A re-normalization of random vectors implies

$$I_2 \le \delta'_N \overset{\text{def}}{=} \sup_{A\subset\mathfrak{S}}\Bigl|\mathbf{P}\bigl\{Z'_{kN} \in \sqrt{N}\,A\bigr\} - \mathbf{P}\bigl\{Z'_{NN} \in \sqrt{N}\,A\bigr\} - \frac{N-k}{6N^{3/2}}\,\chi'(A)\Bigr|. \tag{4.20}$$

Recall that $\chi'(A) = 0$, for any $A \subset \mathfrak{S}$ (see Remark 4.3). To estimate δ'_N we can apply Lemma 4.2 with X_j replaced by X'_j. We get

$$\delta'_N \ll_d \frac{\beta'}{\sigma_d^4 N} + \frac{N^{d/2}}{k^{d/2}}\exp\{-c_d\, k\sigma_d^4/\beta'\}. \tag{4.21}$$

The statement (i) of Theorem 1.4 is proved.

Let us prove $(i) \Longrightarrow (ii)$. To obtain $p \geq \exp\{c\lambda_5^{-2}\}$ we can use (i) in the case when the condition $\mathcal{N}(p, \delta, \mathcal{S}_o, c_0 G)$ is fulfilled with $c_0 = \delta/4 = 1/1200$. Indeed, the condition $\mathcal{B}(\mathcal{S}_o, \mathbb{C})$ guarantees that $e \in \mathcal{S}_o \cup \mathbb{Q}\mathcal{S}_o$ are eigenvectors of the covariance operator $\mathbb{C}$, and we can get the lower bound for p by an application of Lemma 2.4 using $c_0 = \delta/4$. □

5. From Probability to Number Theory

In Section 5 we shall reduce the estimation of the integrals of the modulus of characteristic functions $\widehat{\Psi}_a(t)$ to the estimation the integrals of some theta-series. We shall use the following lemmas.

Lemma 5.1. (BG (1997a, Lemma 5.1)) *Let $L, C \in \mathbb{R}^d$. Let Z, U, V and W denote independent random vectors taking values in $\mathbb{R}^d$. Denote by*

$$P(x) = \langle \mathbb{Q}x, x\rangle + \langle L, x\rangle + C, \qquad\qquad \textit{for } x \in \mathbb{R}^d,$$

a real-valued polynomial of second order. Then

$$2\Big|\mathbf{E}\,\mathrm{e}\big\{t\,P(Z+U+V+W)\big\}\Big|^2 \leq \mathbf{E}\,\mathrm{e}\big\{2\,t\,\langle\mathbb{Q}\widetilde{Z}, \widetilde{U}\rangle\big\} + \mathbf{E}\,\mathrm{e}\big\{2\,t\,\langle\mathbb{Q}\widetilde{Z}, \widetilde{V}\rangle\big\}.$$

Lemma 5.2. (BG (1997a, Theorem 7.1)) *Assume that $\mathbb{Q}^2 = \mathbb{I}_d$ and that the condition $\mathcal{N}(p, \delta, \mathcal{S}_o, \widetilde{X})$ holds with some $0 < p \leq 1$ and $0 \leq \delta \leq 1/(5s)$. Then, for any $a \in \mathbb{R}^d$ and $t \in \mathbb{R}$,*

$$\big|\widehat{\Psi}_a(t)\big| \ll_s \mathcal{M}^s\big(t; pN\big),$$

where the function $\mathcal{M}$ and $\widehat{\Psi}_a(t)$ are defined by (1.19) *and* (2.3) *respectively.*

Let $\varepsilon_1, \varepsilon_2, \dots$ denote i.i.d. symmetric Rademacher random variables. Let $\delta > 0$ and $\mathcal{S} = \{e_1, \dots, e_s\} \subset \mathbb{R}^d$. We shall write $\mathcal{L}(Y) \in \mathbf{\Gamma}(\delta;\, \mathcal{S})$ if a discrete random vector Y is distributed as $\varepsilon_1 z_1 + \dots + \varepsilon_s z_s$, with some (non-random) $z_j \in \mathbb{R}^d$ such that $\|z_j - e_j\| \leq \delta$, for all $1 \leq j \leq s$. Recall that $\mathcal{S}_o = \{e_1, \dots, e_s\} \subset \mathbb{R}^d$ denotes an orthonormal system.

Lemma 5.3. (BG (1997a, Corollary 6.3)) *Let $\mathbb{Q}^2 = \mathbb{I}_d$. Assume $\mathcal{N}(p, \delta, \mathcal{S}, \widetilde{X})$. Write $n = \lceil pN/(5s)\rceil$. Then, for any $0 < A \leq B$ and $\gamma \geq 0$, we have*

$$\int_A^B \big|\widehat{\Psi}_a(t)\big|\,\frac{dt}{|t|} \leq I + c_\gamma(s)\,(pN)^{-\gamma}\,\log\frac{B}{A}\,,$$

with

$$I = \sup_{\Gamma}\,\sup_{b\in\mathbb{R}^d}\int_A^B \sqrt{\varphi(t/4)}\,\frac{dt}{|t|}\,, \qquad \varphi(t) \overset{\mathrm{def}}{=} \Big|\mathbf{E}\,\mathrm{e}\big\{t\,\mathbb{Q}[Y + \mathbb{Q}Y' + b]\big\}\Big|^2, \tag{5.1}$$

where $Y = U_1 + \dots + U_n$ and $Y' = U_1' + \dots + U_n'$ denote sums of independent (*non-i.i.d.*) *vectors, and $\sup_\Gamma$ is taken over all $\big\{\mathcal{L}(U_j), \mathcal{L}(U_j') :\ 1 \leq j \leq n\big\} \subset \mathbf{\Gamma}(\delta;\, \mathcal{S})$.*

Lemma 5.4. *Assume that* $\mathbb{Q}^2 = \mathbb{I}_d$ *and that the condition* $\mathcal{N}(p, \delta, \mathcal{S}, \widetilde{X})$ *holds with some* $0 < p \le 1$ *and* $\delta > 0$. *Let*

$$n \overset{\mathrm{def}}{=} \lceil pN/(11s) \rceil \ge 1. \tag{5.2}$$

Then, for any $0 < A \le B$, $a \in \mathbb{R}^d$ *and* $\gamma > 0$,

$$\int_A^B \bigl| \widehat{\Psi}_a(t) \bigr| \, \frac{dt}{|t|} \le c_\gamma(s) \, (pN)^{-\gamma} \log \frac{B}{A} + \sup_\Gamma \int_A^B \sqrt{\mathbf{E}\ \mathrm{e}\bigl\{ t \, \langle \widetilde{W}, \widetilde{W}' \rangle /2 \bigr\}} \, \frac{dt}{|t|}, \tag{5.3}$$

where $W = V_1 + \dots + V_n$ *and* $W' = V_1' + \dots + V_n'$ *are independent sums of independent copies of random vectors* V *and* V' *respectively, and the supremum* $\sup_\Gamma$ *is taken over all* $\mathcal{L}(V), \mathcal{L}(V') \in \mathbf{\Gamma}(\delta; \mathcal{S})$.

Note that this lemma was proved for general $\mathcal{S}$, but in this paper we need $\mathcal{S} = \mathcal{S}_o$ only.

Proof of Lemma 5.4. Let us show that

$$\int_A^B \bigl| \widehat{\Psi}_a(t) \bigr| \, \frac{dt}{|t|} \le c_\gamma(s) \, (pN)^{-\gamma} \log \frac{B}{A} + \sup_\Gamma \int_A^B \sqrt{\mathbf{E}\ \mathrm{e}\bigl\{ t \, \langle \widetilde{W}, \widetilde{W}' \rangle /2 \bigr\}} \, \frac{dt}{|t|}, \tag{5.4}$$

where $W = V_1 + \dots + V_n$ and $W' = V_1' + \dots + V_n'$ are independent sums of of independent (non-i.i.d.) vectors, and $\sup\limits_\Gamma$ is taken over all $\bigl\{ \mathcal{L}(V_j), \mathcal{L}(V_j') : \ 1 \le j \le n \bigr\} \subset \mathbf{\Gamma}(\delta; \mathcal{S})$.

While proving (5.4) we can assume that $pN \ge c_s$ with a sufficiently large constant c_s, since otherwise (5.4) is trivially fulfilled.

Let $\varphi(t)$ be defined in (5.1), where $Y = U_1 + \dots + U_n$ and $Y' = U_1' + \dots + U_n'$ denote sums of independent (non-i.i.d.) vectors with $\bigl\{ \mathcal{L}(U_j), \mathcal{L}(U_j') : \ 1 \le j \le n \bigr\} \subset \mathbf{\Gamma}(\delta; \mathcal{S})$.

We shall apply the symmetrization Lemma 5.1. Split $Y = T + T_1$ and $Y' + \mathbb{Q}b = R + R_1 + R_2$ into sums of independent sums of independent summands so that each of the sums T, R and R_1 contains $n = \lceil pN/(11s) \rceil$ independent summands U_j and U_j' respectively. Such an n exists since $pN \ge c_s$ with a sufficiently large c_s. Lemma 5.1 and symmetry of $\mathbb{Q}$ imply

$$2 |\varphi(t)|^2 \le \mathbf{E}\ \mathrm{e}\bigl\{ 2\, t \, \langle \widetilde{T}, \mathbb{Q}^2 \widetilde{R} \rangle \bigr\} + \mathbf{E}\ \mathrm{e}\bigl\{ 2\, t \, \langle \widetilde{T}, \mathbb{Q}^2 \widetilde{R}_1 \rangle \bigr\}. \tag{5.5}$$

Recall that $\mathbb{Q}^2 = \mathbb{I}$. Inequality (5.4) follows now from (5.5) and Lemma 5.3.

Let now $W = V_1 + \dots + V_n$ and $W' = V_1' + \dots + V_n'$ be independent sums of of independent (non-i.i.d.) vectors with $\bigl\{ \mathcal{L}(V_j), \mathcal{L}(V_j') : \ 1 \le j \le n \bigr\} \subset \mathbf{\Gamma}(\delta; \mathcal{S})$. Using that all $\mathcal{L}(\widetilde{V}_j)$ are symmetrized and have non-negative characteristic functions and Hölder's inequality, we obtain, for each t,

$$\begin{aligned}
\mathbf{E}\,\mathrm{e}\big\{t\,\langle\widetilde{W},\widetilde{W'}\rangle\big\} &= \mathbf{E}_{\widetilde{W'}}\Big(\prod_{j=1}^{n}\mathbf{E}_{\widetilde{V}_j}\,\mathrm{e}\big\{t\,\langle\widetilde{V}_j,\widetilde{W'}\rangle\big\}\Big) && (5.6)\\
&\le \Big(\prod_{j=1}^{n}\mathbf{E}_{\widetilde{W'}}\big(\mathbf{E}_{\widetilde{V}_j}\,\mathrm{e}\big\{t\,\langle\widetilde{V}_j,\widetilde{W'}\rangle\big\}\big)^{n}\Big)^{1/n} && (5.7)\\
&= \Big(\prod_{j=1}^{n}\mathbf{E}_{\widetilde{W'}}\big(\mathbf{E}_{\widetilde{T}_j}\,\mathrm{e}\big\{t\,\langle\widetilde{T}_j,\widetilde{W'}\rangle\big\}\big)\Big)^{1/n} && (5.8)\\
&= \Big(\prod_{j=1}^{n}\mathbf{E}\,\mathrm{e}\big\{t\,\langle\widetilde{T}_j,\widetilde{W'}\rangle\big\}\Big)^{1/n}, && (5.9)
\end{aligned}$$

where $\widetilde{T}_j \overset{\mathrm{def}}{=} \sum_{l=1}^{n}\widetilde{V}_{jl}$ denotes a sum of i.i.d. copies $\widetilde{V}_{jl}$ of $\widetilde{V}_j$ which are independent of all other random vectors and variables.

Repeating the steps (5.6)–(5.9) for each factor $\mathbf{E}\,\mathrm{e}\big\{t\,\langle\widetilde{T}_j,\widetilde{W'}\rangle\big\}$ instead of the expectation $\mathbf{E}\,\mathrm{e}\big\{t\,\langle\widetilde{W},\widetilde{W'}\rangle\big\}$ on the right hand side separately, we get (with $\widetilde{T}'_k \overset{\mathrm{def}}{=} \sum_{l=1}^{n}\widetilde{V}'_{kl}$, where $\widetilde{V}'_{kl}$ are i.i.d. copies of $\widetilde{V}'_k$ independent of all other random vectors)

$$\mathbf{E}\,\mathrm{e}\big\{t\,\langle\widetilde{W},\widetilde{W'}\rangle\big\} \le \Big(\prod_{j=1}^{n}\prod_{k=1}^{n}\mathbf{E}\,\mathrm{e}\big\{t\,\langle\widetilde{T}_j,\widetilde{T}'_k\rangle\big\}\Big)^{1/n^2}. \tag{5.10}$$

Thus, using (5.10) and the arithmetic-geometric mean inequality, we have

$$\begin{aligned}
\int_A^B \sqrt{\mathbf{E}\,\mathrm{e}\big\{t\,\langle\widetilde{W},\widetilde{W'}\rangle/2\big\}}\,\frac{dt}{|t|} &\le \int_A^B \Big(\prod_{j=1}^{n}\prod_{k=1}^{n}\mathbf{E}\,\mathrm{e}\big\{t\,\langle\widetilde{T}_j,\widetilde{T}'_k\rangle/2\big\}\Big)^{1/2n^2}\,\frac{dt}{|t|}\\
&\le \frac{1}{n^2}\sum_{j=1}^{n}\sum_{k=1}^{n}\int_A^B \Big(\mathbf{E}\,\mathrm{e}\big\{t\,\langle\widetilde{T}_j,\widetilde{T}'_k\rangle/2\big\}\Big)^{1/2}\,\frac{dt}{|t|}\\
&\le \sup_{\Gamma}\int_A^B \sqrt{\mathbf{E}\,\mathrm{e}\big\{t\,\langle\widetilde{T},\widetilde{T}'\rangle/2\big\}}\,\frac{dt}{|t|}\,, && (5.11)
\end{aligned}$$

where $T = V_1+\cdots+V_n$ and $T' = U'_1+\cdots+U'_n$ are independent sums of independent copies of random vectors U and U' respectively, and the supremum $\sup_\Gamma$ is taken over all $\mathcal{L}(U), \mathcal{L}(U') \in \mathbf{\Gamma}(\delta;\,\mathcal{S})$. Inequalities (5.4) and (5.11) imply now the statement of the lemma. $\square$

The following Lemma 5.5 provides a Poisson summation formula.

Lemma 5.5. *Let* $\operatorname{Re} z > 0$, $a, b \in \mathbb{R}^s$ *and* $\mathbb{S} : \mathbb{R}^s \to \mathbb{R}^s$ *be a positive definite symmetric non-degenerate linear operator. Then*

$$\begin{aligned}&\sum_{m\in\mathbb{Z}^s} \exp\bigl\{-z\,\mathbb{S}[m+a] + 2\pi i\,\langle m, b\rangle\bigr\}\\ &= \bigl(\det(\mathbb{S}/\pi)\bigr)^{-1/2} z^{-s/2} \exp\bigl\{-2\pi i\,\langle a, b\rangle\bigr\} \sum_{l\in\mathbb{Z}^s} \exp\Bigl\{-\frac{\pi^2}{z}\,\mathbb{S}^{-1}[l+b] - 2\pi i\,\langle a, l\rangle\Bigr\},\end{aligned}$$

where $\mathbb{S}^{-1} : \mathbb{R}^s \to \mathbb{R}^s$ *denotes the inverse positive definite operator for* $\mathbb{S}$.

Proof. See, for example, Fricker (1982), p. 116, or Mumford (1983), p. 189, formula (5.1); and p. 197, formula (5.9). □

Let the conditions of Lemma 5.4 be satisfied. Introduce one-dimensional lattice probability distributions $H_n = \mathcal{L}(\xi_n)$ with integer valued ξ_n setting

$$\mathbf{P}\bigl\{\xi_n = k\bigr\} = A_n\, n^{-1/2} \exp\bigl\{-k^2/2n\bigr\}, \quad \text{for } k \in \mathbb{Z}.$$

It is easy to see that $1 \ll A_n \ll 1$. Moreover, by Lemma 5.5,

$$\widehat{H}_n(t) \geq 0, \qquad \text{for all } t \in \mathbb{R}. \tag{5.12}$$

Introduce the s-dimensional random vector ζ_n having as coordinates independent copies of ξ_n. Then, for $m = (m_1, \dots, m_s) \in \mathbb{Z}^s$, we have

$$q(m) \overset{\text{def}}{=} \mathbf{P}\bigl\{\zeta_n = m\bigr\} = A_n^s\, n^{-s/2} \exp\bigl\{-\|m\|^2/2n\bigr\}. \tag{5.13}$$

Lemma 5.6. *Let* $W = V_1 + \dots + V_n$ *and* $W' = V_1' + \dots + V_n'$ *denote independent sums of independent copies of random vectors* V *and* V' *such that*

$$V = \varepsilon_1 z_1 + \dots + \varepsilon_s z_s, \qquad V' = \varepsilon_{s+1} z_1' + \dots + \varepsilon_{2s} z_s',$$

with some $z_j, z_j' \in \mathbb{R}^d$. *Introduce the matrix* $\mathbb{B}_t = \{b_{ij}(t) : 1 \leq i, j \leq s\}$ *with* $b_{ij}(t) = t\,\langle z_i, z_j'\rangle$. *Then*

$$\mathbf{E}\ \mathrm{e}\bigl\{t\,\langle \widetilde{W}, \widetilde{W}'\rangle/4\bigr\} \ll_s \mathbf{E}\ \mathrm{e}\bigl\{\langle \mathbb{B}_t \zeta_n, \zeta_n'\rangle\bigr\} + \exp\{-cn\}, \qquad \textit{for all } t \in \mathbb{R},$$

where ζ_n' *are independent copies of* ζ_n *and* c *is an absolute constant.*

Proof. Without loss of generality, we shall assume that $n \geq c$, where c is an absolute constant which is so large as it will be needed for the validity of arguments below. Consider the random vector $Y = (\widetilde{\varepsilon}_1, \dots, \widetilde{\varepsilon}_s) \in \mathbb{R}^s$ with coordinates which are symmetrizations of i.i.d. Rademacher random variables. Let $R = (R_1, \dots, R_s)$ and T denote independent sums of n independent copies of $Y/2$. Then we can write

$$\mathbf{E}\ \mathrm{e}\bigl\{t\,\langle \widetilde{W}, \widetilde{W}'\rangle/4\bigr\} = \mathbf{E}\ \mathrm{e}\bigl\{\langle \mathbb{B}_t R, T\rangle\bigr\}, \qquad \text{for all } t \in \mathbb{R}, \tag{5.14}$$

Note that the scalar product $\langle\cdot,\cdot\rangle$ in $\mathbf{E}\,\mathrm{e}\big\{\langle\mathbb{B}_t R,T\rangle\big\}$ means the scalar product of vectors in $\mathbb{R}^s$. In order to estimate this expectation, we write it in the form

$$\begin{aligned}\mathbf{E}\,\mathrm{e}\big\{\langle\mathbb{B}_t R,T\rangle\big\} &= \mathbf{E}\,\mathbf{E}_R\,\mathrm{e}\big\{\langle\mathbb{B}_t R,T\rangle\big\}\\ &= \sum_{\overline{m}\in\mathbb{Z}^s} p(\overline{m})\sum_{m\in\mathbb{Z}^s} p(m)\,\mathrm{e}\big\{\langle\mathbb{B}_t m,\overline{m}\rangle\big\},\end{aligned}\tag{5.15}$$

with summing over $m=(m_1,\dots,m_s)\in\mathbb{Z}^s$, $\overline{m}=(\overline{m}_1,\dots,\overline{m}_s)\in\mathbb{Z}^s$ and

$$p(m)=\mathbf{P}\big\{R=m\big\}=\prod_{j=1}^{s}\mathbf{P}\big\{R_j=m_j\big\}=\prod_{j=1}^{s}2^{-2n}\binom{2n}{m_j+n},\tag{5.16}$$

if $\max\limits_{1\le j\le s}|m_j|\le n$ and $p(m)=0$ otherwise. Clearly, for fixed $T=\overline{m}$,

$$\mathbf{E}_R\,\mathrm{e}\big\{\langle\mathbb{B}_t R,T\rangle\big\}=\sum_{m\in\mathbb{Z}^s}p(m)\,\mathrm{e}\big\{\langle\mathbb{B}_t m,\overline{m}\rangle\big\}\ge 0\tag{5.17}$$

is a value of the characteristic function of symmetrized random vector $\mathbb{B}_t R$. Using Stirling's formula, it is easy to show that there exist absolute constants c_1 and c_2 such that

$$\mathbf{P}\big\{R_j=m_j\big\}\ll n^{-1/2}\exp\big\{-m_j^2/2n\big\},\qquad\text{for } |m_j|\le c_1 n,\tag{5.18}$$

and

$$\mathbf{P}\big\{|R_j|\ge c_1 n\big\}\ll\exp\{-c_2 n\}.\tag{5.19}$$

Using (5.15)–(5.19), we obtain

$$\begin{aligned}\mathbf{E}\,\mathrm{e}\big\{\langle\mathbb{B}_t R,T\rangle\big\} &\ll_s \sum_{\overline{m}\in\mathbb{Z}^s} q(\overline{m})\sum_{m\in\mathbb{Z}^s} p(m)\,\mathrm{e}\big\{\langle\mathbb{B}_t m,\overline{m}\rangle\big\}+\exp\{-c_2 n\}\\ &= \sum_{m\in\mathbb{Z}^s} p(m)\sum_{\overline{m}\in\mathbb{Z}^s} q(\overline{m})\,\mathrm{e}\big\{\langle\mathbb{B}_t m,\overline{m}\rangle\big\}+\exp\{-c_2 n\}\\ &= \mathbf{E}\,\mathbf{E}_{\zeta_n}\,\mathrm{e}\big\{\langle\mathbb{B}_t R,\zeta_n\rangle\big\}+\exp\{-c_2 n\}\\ &= \mathbf{E}\,\mathrm{e}\big\{\langle\mathbb{B}_t R,\zeta_n\rangle\big\}+\exp\{-c_2 n\}.\end{aligned}\tag{5.20}$$

Now we repeat our previous arguments, noting that

$$\mathbf{E}_{\zeta_n}\,\mathrm{e}\big\{\langle\mathbb{B}_t R,\zeta_n\rangle\big\}=\sum_{\overline{m}\in\mathbb{Z}^s}q(\overline{m})\;\mathrm{e}\big\{\langle\mathbb{B}_t R,\overline{m}\rangle\big\}\ge 0\tag{5.21}$$

is a value of the non-negative characteristic function of the random vector ζ_n (see (5.12)). Using again (5.18) and (5.19), we obtain

$$\mathbf{E}\,\mathrm{e}\big\{\langle\mathbb{B}_t R,\zeta_n\rangle\big\}\ll_s\mathbf{E}\,\mathrm{e}\big\{\langle\mathbb{B}_t\zeta_n,\zeta_n'\rangle\big\}+\exp\{-c_2 n\}.\tag{5.22}$$

Relations (5.14), (5.20) and (5.22) imply the statement of the lemma. □

Let us estimate the expectation $\mathbf{E}\ \mathrm{e}\big\{ \langle \mathbb{B}_t \zeta_n, \zeta_n' \rangle \big\}$ under the conditions of Lemmas 5.4 and 5.6, assuming that $\delta \le 1/(5s)$, $n \ge c$, where c is a sufficiently large absolute constant, and

$$\|z_j - e_j\| \le \delta, \qquad \|z_j' - e_j\| \le \delta, \qquad \text{for } 1 \le j \le s, \tag{5.23}$$

with an orthonormal system $\mathcal{S}_o = \big\{ e_1, e_2, \dots, e_s \big\}$ involved in the conditions of Lemma 5.4. We can rewrite $\mathbf{E}\ \mathrm{e}\big\{ \langle \mathbb{B}_t \zeta_n, \zeta_n' \rangle \big\}$ as

$$\mathbf{E}\ \mathrm{e}\big\{ \langle \mathbb{B}_t \zeta_n, \zeta_n' \rangle \big\} = \sum_{\overline{m} \in \mathbb{Z}^s} q(\overline{m}) \sum_{m \in \mathbb{Z}^s} q(m)\, \mathrm{e}\big\{ \langle \mathbb{B}_t \overline{m}, m \rangle \big\}.$$

Thus, by (5.13),

$$\mathbf{E}\ \mathrm{e}\big\{ \langle \mathbb{B}_t \zeta_n, \zeta_n' \rangle \big\} = A_n^{2s}\, n^{-s} \sum_{\overline{m} \in \mathbb{Z}^s} \sum_{m \in \mathbb{Z}^s} \exp\big\{ i\, \langle \mathbb{B}_t \overline{m}, m \rangle - \|m\|^2/2n - \|\overline{m}\|^2/2n \big\}.$$

Applying Lemma 5.5 with $\mathbb{S} = \mathbb{I}_s$, $z = 1/2n$, $a = 0$, $b = (2\pi)^{-1}\, \mathbb{B}_t\, \overline{m}$, we obtain

$$\mathbf{E}\ \mathrm{e}\big\{ \langle \mathbb{B}_t \zeta_n, \zeta_n' \rangle \big\} \ll_s n^{-s/2} \sum_{l, m \in \mathbb{Z}^s} \exp\big\{ -2\pi^2 n\, \|\, l + (2\pi)^{-1} \mathbb{B}_t m \|^2 - \|m\|^2/2n \big\}. \tag{5.24}$$

Note that the vectors ζ_n and ζ_n' are i.i.d. Hence,

$$\mathbf{E}\ \mathrm{e}\big\{ \langle \mathbb{B}_t \zeta_n, \zeta_n' \rangle \big\} = \mathbf{E}\ \mathrm{e}\big\{ \langle \mathbb{B}_t^* \zeta_n, \zeta_n' \rangle \big\}, \tag{5.25}$$

where $\mathbb{B}_t^*$ denotes the adjoint operator for $\mathbb{B}_t$. Similarly to (5.24), we could derive the inequality

$$\mathbf{E}\ \mathrm{e}\big\{ \langle \mathbb{B}_t^* \zeta_n, \zeta_n' \rangle \big\} \ll_s n^{-s/2} \sum_{l, m \in \mathbb{Z}^s} \exp\big\{ -2\pi^2 n\, \|\, l + (2\pi)^{-1} \mathbb{B}_t^* m \|^2 - \|m\|^2/2n \big\}. \tag{5.26}$$

Denote

$$r = \sqrt{2\pi^2 n}. \tag{5.27}$$

By (5.24)–(5.27), we have

$$\begin{aligned}
\mathbf{E}\ \mathrm{e}\big\{ \langle \mathbb{B}_t \zeta_n, \zeta_n' \rangle \big\} &= \Big(\mathbf{E}\ \mathrm{e}\big\{ \langle \mathbb{B}_t \zeta_n, \zeta_n' \rangle \big\}\, \mathbf{E}\ \mathrm{e}\big\{ \langle \mathbb{B}_t^* \zeta_n, \zeta_n' \rangle \big\} \Big)^{1/2} \\
&\ll_s n^{-s/2} \Big(\sum_{l, m \in \mathbb{Z}^s} \exp\big\{ -2\pi^2 n\, \|\, l + (2\pi)^{-1} \mathbb{B}_t m \|^2 - \|m\|^2/2n \big\} \\
&\qquad \times \sum_{\bar{l}, \overline{m} \in \mathbb{Z}^s} \exp\big\{ -2\pi^2 n\, \|\, \bar{l} + (2\pi)^{-1} \mathbb{B}_t^* \overline{m} \|^2 - \|\overline{m}\|^2/2n \big\} \Big)^{1/2} \\
&\ll_s r^{-s} \Big(\sum_{l, m, \bar{l}, \overline{m} \in \mathbb{Z}^s} \exp\big\{ -r^2\, \|\, l + (2\pi)^{-1} \mathbb{B}_t m \|^2 - \|m\|^2/r^2 \\
&\qquad - r^2\, \|\, \bar{l} + (2\pi)^{-1} \mathbb{B}_t^* \overline{m} \|^2 - \|\overline{m}\|^2/r^2 \big\} \Big)^{1/2}.
\end{aligned} \tag{5.28}$$

Now we rewrite the last sum as

$$\sum_{l,m,\bar{l},\overline{m}\in\mathbb{Z}^s} \exp\big\{-r^2\,\|l+(2\pi)^{-1}\mathbb{B}_t m\|^2-\|m\|^2/r^2-r^2\,\|\bar{l}+(2\pi)^{-1}\mathbb{B}_t^*\overline{m}\|^2-\|\overline{m}\|^2/r^2\big\}$$
$$=\sum_{m,\overline{m}\in\mathbb{Z}^{2s}} \exp\big\{-r^2\,\|m-t\,\mathbb{V}\overline{m}\|^2-\|\overline{m}\|^2/r^2\big\}, \tag{5.29}$$

where $\mathbb{V}:\mathbb{R}^{2s}\to\mathbb{R}^{2s}$ is the operator with matrix

$$\mathbb{V}=\begin{pmatrix} \mathbb{O}_s & (2\pi)^{-1}\mathbb{B}_1 \\ (2\pi)^{-1}\mathbb{B}_1^* & \mathbb{O}_s \end{pmatrix}, \tag{5.30}$$

where $\mathbb{O}_s$ denotes the $(s\times s)$ matrix with zero entries. It is important that the matrix $\mathbb{V}$ is symmetric. This will allow us to apply below Lemma 6.2 (which is related to Number Theory) to derive inequality (7.14).

Let us show that (cf. BG (1997a, proof of Lemma 7.4))

$$\|\mathbb{B}_1\|\le 3/2 \quad\text{and}\quad \|\mathbb{B}_1^{-1}\|\le 2. \tag{5.31}$$

Indeed, the entries of the matrix $\mathbb{B}_1$ are $b_{ij}(1)=\langle z_i,z_j'\rangle$ with some $z_j,z_j'\in\mathbb{R}^d$ satisfying (5.23). Since $\mathcal{S}_o=\big\{e_1,e_2,\dots,e_s\big\}$ is an orthonormal system, inequalities (5.23) imply that $\mathbb{B}_1=\mathbb{I}_s+\mathbb{A}$ with some matrix $\mathbb{A}=\{a_{ij}\}$ such that $|a_{ij}|\le 2\delta+\delta^2$. Thus, we have $\|\mathbb{A}\|\le\|\mathbb{A}\|_2\le 2s\delta+s\delta^2$, where $\|\mathbb{A}\|_2$ denotes the Hilbert–Schmidt norm of the matrix $\mathbb{A}$. Therefore the condition $\delta\le 1/(5s)$ implies $\|\mathbb{A}\|\le 1/2$ and inequalities (5.31).

By (5.30) and (5.31), for any $x\in\mathbb{R}^{2s}$, we have

$$\|x\|\ll\|\mathbb{V}x\|+\|\mathbb{V}^{-1}x\|\ll\|x\|. \tag{5.32}$$

6. Some facts from Number Theory

In Section 6, we consider some facts of the geometry of numbers (see Davenport (1958) or Cassels (1959)). They will help us to estimate the right-hand side of inequality (5.29).

Let $e_1,e_2,\dots,e_d$ be linearly independent vectors in $\mathbb{R}^d$. The set

$$\Lambda=\Big\{\sum_{j=1}^d n_je_j:n_j\in\mathbb{Z},j=1,2,\dots,d\Big\} \tag{6.1}$$

is called the lattice with basis $e_1,e_2,\dots,e_d$. The determinant $\det(\Lambda)$ of a lattice Λ is the modulus of the determinant of the matrix formed from the vectors $e_1,e_2,\dots,e_d$:

$$\det(\Lambda)\overset{\text{def}}{=}\big|\det(e_1,e_2,\dots,e_d)\big|. \tag{6.2}$$

The determinant of a lattice does not depend on the choice of basis. Any lattice $\Lambda\subset\mathbb{R}^d$ can be represented as $\Lambda=\mathbb{A}\,\mathbb{Z}^d$, where $\mathbb{A}$ is a non-degenerate linear operator. Clearly, $\det(\Lambda)=|\det\mathbb{A}|$.

Let $m_1, \dots, m_l \in \Lambda$ be linearly independent vectors belonging to a lattice Λ. Then the set

$$\Lambda' = \Big\{ \sum_{j=1}^{l} n_j m_j : n_j \in \mathbb{Z}, j = 1, 2, \dots, l \Big\} \tag{6.3}$$

is an l-dimensional sublattice of the lattice Λ. Its determinant $\det(\Lambda')$ is the determinant of the matrix formed from the coordinates of the vectors $m_1, m_2, \dots, m_l$ with respect to an orthonormal basis of the linear span of the vectors $m_1, m_2, \dots, m_l$.

Let $F : \mathbb{R}^d \to [0, \infty]$ denote a norm on $\mathbb{R}^d$, that is $F(\alpha x) = |\alpha| \, F(x)$, for $\alpha \in \mathbb{R}$, and $F(x + y) \le F(x) + F(y)$. The successive minima $M_1 \le \dots \le M_d$ of F with respect to a lattice Λ are defined as follows: Let $M_1 = \inf\big\{ F(m) : m \ne 0, m \in \Lambda \big\}$ and define M_j as the infimum of $\lambda > 0$ such that the set $\big\{ m \in \Lambda : F(m) < \lambda \big\}$ contains j linearly independent vectors. It is easy to see that these infima are attained, that is there exist linearly independent vectors $b_1, \dots, b_d \in \Lambda$ such that $F(b_j) = M_j$, $j = 1, \dots, d$.

Lemma 6.1. (Davenport (1958, Lemma 1)) *Let $M_1 \le \dots \le M_d$ be the successive minima of a norm F with respect to the lattice $\mathbb{Z}^d$. Suppose that $1 \le j \le d$ and $M_j \le b \le M_{j+1}$, for some $b > 0$. Then the number of $m = (m_1, \dots, m_d) \in \mathbb{Z}^d$ such that $F(m) < b$ is bounded from above by $4^j \, b^j (M_1 \cdot M_2 \cdots M_j)^{-1}$.*

Representing $\Lambda = \mathbb{A}\,\mathbb{Z}^d$, we see that the lattice $\mathbb{Z}^d$ may be replaced in Lemma 6.1 by any lattice $\Lambda \subset \mathbb{R}^d$. It suffices to apply this lemma to the norm $G(m) = F(\mathbb{A} m)$, $m \in \mathbb{Z}^d$.

Let $\|x\|_\infty = \max_{1 \le j \le d} |x_j|$, for $x = (x_1, \dots, x_d) \in \mathbb{R}^d$, and let $z(v)$ denote the distance of the number v to the nearest integer.

Lemma 6.2. *Let $L_j(x) = \sum_{k=1}^{d} q_{jk} x_k$, $1 \le j \le d$, denote linear forms on $\mathbb{R}^d$ such that $q_{jk} = q_{kj}$, $j, k = 1, \dots, d$. Assume that $r \ge 1$. Let μ be the number of $m = (m_1, \dots, m_d) \in \mathbb{Z}^d$ such that*

$$z(L_j(m)) < r^{-1}, \quad |m_j| < r, \qquad \text{for all } \ 1 \le j \le d. \tag{6.4}$$

Then

$$\mu \ll_d (M_1 \cdot M_2 \cdots M_d)^{-1}, \tag{6.5}$$

where $M_1 \le \dots \le M_d$ are the first d of the $2d$ successive minima $M_1 \le \dots \le M_{2d}$ (with respect to $\mathbb{Z}^d$) of the norm $F : \mathbb{R}^{2d} \to [0, \infty)$ defined for vectors $y = (x, \overline{x}) \in \mathbb{R}^{2d}$, $x, \overline{x} \in \mathbb{R}^d$, $\overline{x} = (\overline{x}_1, \dots, \overline{x}_d)$, as

$$F(y) \overset{\text{def}}{=} \max\big\{ r\,|\,L_1(x) - \overline{x}_1\,|, \dots, r\,|\,L_d(x) - \overline{x}_d\,|,\; r^{-1} \|x\|_\infty \big\}. \tag{6.6}$$

Moreover,

$$\frac{1}{2d} \le M_k M_{2d+1-k} \le (2d)^{2d-1}, \qquad 1 \le k \le 2d. \tag{6.7}$$

Here and below writing (a, b), for $a \in \mathbb{R}^{d_1}$, $b \in \mathbb{R}^{d_2}$, means that $(a, b) \in \mathbb{R}^{d_1 + d_2}$ and the coordinates of (a, b) are the coordinates of the vectors a and b in the corresponding order, that is, $(a, b) = (a_1, a_2, \dots, a_{d_1}, b_1, b_2, \dots, b_{d_2})$. A similar notation will be used as well for more vectors $(a, b, c, \dots, z)$.

Lemma 6.2 is proved in Lemma 3 of Davenport (1958), see also Davenport (1958, formula (20), p. 113). Note that inequality (6.5) of Lemma 6.2 is, in a sense, a particular case of Lemma 6.1.

Note that

$$r^{-1} \le M_1 \le \cdots \le M_d \ll_d 1, \tag{6.8}$$

where the first inequality is obvious by $F(y) \ge r^{-1}\|x\|_\infty$ (if $\|x\|_\infty = 0$ then $F(y) \ge r\,\|\overline{x}\|_\infty \ge r^{-1}\,\|\overline{x}\|_\infty$) and $M_d \ll_d 1$ follows from (6.7) for $k = d$.

Lemma 6.3. *Let $F_j(m)$, $j = 1, 2$, be some norms in $\mathbb{R}^d$ and $M_1 \le \cdots \le M_d$ and $N_1 \le \cdots \le N_d$ be the successive minima of F_1 with respect to the lattice Λ_1 and of F_2 with respect to the lattice Λ_2 respectively. Assume that $M_k \gg_d F_2(n_k)$, $k = 1, 2, \dots, d$, for some linearly independent vectors $n_1, n_2, \dots, n_d \in \Lambda_2$. Then*

$$M_k \gg_d N_k, \quad k = 1, \dots, d. \tag{6.9}$$

The proof of this lemma is elementary and therefore omitted.

Lemma 6.4. *Let Λ be a lattice in $\mathbb{R}^d$ and let $c_1(d)$ and $c_2(d)$ be positive quantities depending on d only. Then*

$$\sum_{v\in\Lambda} \exp\bigl\{-c_1(d)\,\|v\|^2\bigr\} \asymp_d \#H, \tag{6.10}$$

where $H \overset{\text{def}}{=} \bigl\{v \in \Lambda : \|v\|_\infty < c_2(d)\bigr\}$.

Proof. Introduce for $\mu = (\mu_1, \dots, \mu_d) \in \Xi \overset{\text{def}}{=} c_2(d)\,\mathbb{Z}^d$ the sets

$$B_\mu \overset{\text{def}}{=} \Bigl[\mu_1 - \frac{c_2(d)}{2}\,,\, \mu_1 + \frac{c_2(d)}{2}\Bigr) \times \cdots \times \Bigl[\mu_d - \frac{c_2(d)}{2}\,,\, \mu_d + \frac{c_2(d)}{2}\Bigr)$$

such that $\mathbb{R}^d = \bigcup_\mu B_\mu$. For any fixed $w^* \in H_\mu \overset{\text{def}}{=} \bigl\{w \in \Lambda \cap B_\mu\bigr\}$ we have

$$w - w^* \in H, \qquad \text{for any } w \in H_\mu.$$

Hence we conclude for any $\mu \in \Xi$

$$\#H_\mu \le \#H. \tag{6.11}$$

Since $x \in B_\mu$ implies $\|x\|_\infty \ge \|\mu\|_\infty/2$, we obtain by (6.11)

$$\begin{aligned}
\sum_{v\in\Lambda} \exp\bigl\{-c_1(d)\,\|v\|^2\bigr\} &\le \sum_{v\in\Lambda} \exp\bigl\{-c_d\,\|v\|_\infty^2\bigr\} \\
&\ll_d \#H_0 + \sum_{\mu\in\Xi\setminus 0}\ \sum_{v\in\Lambda} \mathbf{I}\bigl\{\,v \in B_\mu\bigr\}\exp\bigl\{-c(d)\,\|\mu\|_\infty^2\bigr\} \\
&\ll_d \#H \cdot \sum_{\mu\in\Xi} \exp\bigl\{-c(d)\,\|\mu\|_\infty^2\bigr\} \\
&\ll_d \#H.
\end{aligned} \tag{6.12}$$

On the other hand,

$$\sum_{v\in\Lambda}\exp\bigl\{-c_1(d)\,\|v\|^2\bigr\}\geq\sum_{v\in\Lambda}\exp\bigl\{-c_d\,\|v\|_\infty^2\bigr\}\gg_d\#H.\tag{6.13}$$

This conclude the proof of Lemma 6.4. □

For a lattice $\Lambda\subset\mathbb{R}^d$, $\dim\Lambda=d$ and $1\leq l\leq d$ we define its α_l-characteristics by

$$\alpha_l(\Lambda)\overset{\text{def}}{=}\sup\Bigl\{\bigl|\det(\Lambda')\bigr|^{-1}:\Lambda'\subset\Lambda,\ \ l\text{-dimensional sublattice of }\Lambda.\Bigr\}\tag{6.14}$$

Lemma 6.5. *Let $F(\cdot)$ be a norm in $\mathbb{R}^d$ such that $F(\cdot)\asymp_d\|\cdot\|$. Let $M_1\leq\cdots\leq M_d$ be the successive minima of F with respect to a lattice $\Lambda\subset\mathbb{R}^d$. Then*

$$\alpha_l(\Lambda)\asymp_d(M_1\cdot M_2\cdots M_l)^{-1},\quad l=1,\dots,d.\tag{6.15}$$

For the proof of Lemma 6.5 we shall use the following lemma formulated in Proposition (p. 517) and Remark (p. 518) in A.K. Lenstra, H.W. Lenstra and Lovász (1982).

Lemma 6.6. *Let $M_1\leq\cdots\leq M_d$ be the successive minima of the standard Euclidean norm with respect to a lattice $\Lambda\subset\mathbb{R}^d$. Then there exists a basis $e_1,e_2,\dots,e_d$ of Λ such that*

$$M_l\asymp_d\|e_l\|,\quad l=1,\dots,d.\tag{6.16}$$

Moreover,

$$\det(\Lambda)\asymp_d\prod_{l=1}^d\|e_l\|.\tag{6.17}$$

Proof of Lemma 6.5. According to Lemma 6.3, we can replace the Euclidean norm $\|\cdot\|$ by the norm $F(\cdot)$, in the formulation of Lemma 6.6. Let $\Lambda'\subset\Lambda$ be an arbitrary l-dimensional sublattice of Λ and $N_1\leq\cdots\leq N_l$ be the successive minima of the norm $F(\cdot)$ with respect to Λ'. It is clear that $M_j\leq N_j$, $j=1,2,\dots,l$. If $M_j=F(m_j)$ for some linearly independent vectors $m_1,m_2,\dots,m_l\in\Lambda$ and

$$\Lambda'=\Bigl\{\sum_{j=1}^l n_jm_j:n_j\in\mathbb{Z},j=1,2,\dots,l\Bigr\},\tag{6.18}$$

then $M_j=N_j$, $j=1,2,\dots,l$. It remains to apply Lemma 6.6. □

7. From Number Theory to Probability

In Section 7, we shall use these number theory results to estimate the right-hand side of (5.29), which may be considered as a theta-series. Recall that we have assumed the conditions of Lemmas 5.4 and 5.6, as well as (5.23), $\delta\leq 1/(5s)$ and $n\geq c$. The notation $\mathrm{SL}(d,\mathbb{R})$ is used below for the set of all $(d\times d)$-matrices with real entries and determinant 1.

Introduce the matrices

$$\mathbb{D}_r \overset{\text{def}}{=} \begin{pmatrix} r\,\mathbb{I}_{2s} & \mathbb{O}_{2s} \\ \mathbb{O}_{2s} & r^{-1}\,\mathbb{I}_{2s} \end{pmatrix} \in \mathrm{SL}(4s,\mathbb{R}), \quad r>0, \tag{7.1}$$

$$\mathbb{K}_t \overset{\text{def}}{=} \begin{pmatrix} \mathbb{I}_{2s} & -t\,\mathbb{I}_{2s} \\ t\,\mathbb{I}_{2s} & \mathbb{I}_{2s} \end{pmatrix}, \quad t\in\mathbb{R}, \tag{7.2}$$

$$\mathbb{U}_t \overset{\text{def}}{=} \begin{pmatrix} \mathbb{I}_{2s} & -t\,\mathbb{I}_{2s} \\ \mathbb{O}_{2s} & \mathbb{I}_{2s} \end{pmatrix} \in \mathrm{SL}(4s,\mathbb{R}), \quad t\in\mathbb{R}, \tag{7.3}$$

and the lattices

$$\Lambda = \Lambda_{\mathbb{V}} \overset{\text{def}}{=} \begin{pmatrix} \mathbb{I}_{2s} & \mathbb{O}_{2s} \\ \mathbb{O}_{2s} & \mathbb{V} \end{pmatrix} \mathbb{Z}^{4s} \subset \mathbb{R}^{4s}, \tag{7.4}$$

$$\Lambda_j = \mathbb{D}_j\,\mathbb{U}_{j^{-1}}\,\Lambda = \begin{pmatrix} j\,\mathbb{I}_{2s} & -\mathbb{V} \\ \mathbb{O}_{2s} & j^{-1}\,\mathbb{V} \end{pmatrix} \mathbb{Z}^{4s}, \quad j=1,2,\ldots. \tag{7.5}$$

Below we shall use the following simplest properties of these matrices:

$$\mathbb{D}_a\mathbb{D}_b=\mathbb{D}_{ab}, \quad \mathbb{U}_a\mathbb{U}_b=\mathbb{U}_{a+b} \quad \text{and} \quad \mathbb{D}_a\mathbb{U}_b=\mathbb{U}_{a^2b}\,\mathbb{D}_a, \quad \text{for } a,b>0. \tag{7.6}$$

In the sequel we shall apply Lemma 6.2 to linear forms

$$L_j(x)=\sum_{k=1}^{s} t\,a_{jk}\,x_k, \qquad 1\le j\le 2s, \tag{7.7}$$

where $t\in\mathbb{R}$ is arbitrary and a_{jk} are the elements of the symmetric matrix $\mathbb{V}$. For fixed t, we denote the corresponding successive minima of the norm $F(\cdot)$ (defined by (6.6) and (7.7)) by $M_{j,t},\ j=1,\ldots,4s$. Thus, we can write

$$M_{j,t}=\bigl|L(m,\overline{m},t)\bigr|_\infty, \tag{7.8}$$

for some $m,\overline{m}\in\mathbb{Z}^{2s}$, where

$$L(m,\overline{m},t)=\bigl(r\,(m_1-t\,(\mathbb{V}\overline{m})_1),\ldots,r\,(m_{2s}-t\,(\mathbb{V}\overline{m})_{2s}),\ r^{-1}\overline{m}_1,\ldots,r^{-1}\overline{m}_{2s}\bigr). \tag{7.9}$$

It is easy to see from the definition that $M_{j,t}$ are the successive minima of the norm $\|\cdot\|_\infty$ with respect to the lattice

$$\Xi_t \overset{\text{def}}{=} \begin{pmatrix} r\,\mathbb{I}_{2s} & -r\,t\,\mathbb{V} \\ \mathbb{O}_{2s} & r^{-1}\,\mathbb{I}_{2s} \end{pmatrix} \mathbb{Z}^{4s}. \tag{7.10}$$

Moreover, simultaneously, $M_{j,t}$ are the successive minima of the norm $F^*(\cdot)$ defined for $(m,\overline{m})\in\mathbb{R}^{4s}$, $m,\overline{m}\in\mathbb{R}^{2s}$ by

$$F^*((m,\overline{m})) \overset{\text{def}}{=} \max\bigl\{\|m\|_\infty, \|\mathbb{V}^{-1}\overline{m}\|_\infty\bigr\} \tag{7.11}$$

with respect to the lattice

$$\Omega_t \overset{\text{def}}{=} \begin{pmatrix} r\,\mathbb{I}_{2s} & -r\,t\,\mathbb{V} \\ \mathbb{O}_{2s} & r^{-1}\,\mathbb{V} \end{pmatrix} \mathbb{Z}^{4s} = \mathbb{D}_r\,\mathbb{U}_t\,\Lambda. \tag{7.12}$$

Note that, according to (5.32),

$$F^*(\cdot) \asymp_s \|\cdot\|. \tag{7.13}$$

Using (5.32), (7.10)–(7.13) and Lemmas 6.2, 6.4 and 6.5, we obtain

$$\begin{aligned}\sum_{m,\overline{m}\in\mathbb{Z}^{2s}} \exp\bigl\{-r^2\,\|m-t\,\mathbb{V}\overline{m}\|^2-\|\overline{m}\|^2/r^2\bigr\} \quad &= \quad \sum_{v\in\Xi_t}\exp\bigl\{-\|v\|^2\bigr\}\\ &\ll_s \quad \#\bigl\{v\in\Xi_t\,:\,\|v\|_\infty<1\bigr\}\\ &\ll_s \quad (M_{1,t}\cdot M_{2,t}\cdots M_{2s,t})^{-1}\\ &\asymp_s \quad \alpha_{2s}(\Xi_t)\asymp_s\alpha_{2s}(\Omega_t). \end{aligned}\tag{7.14}$$

Here we have used essentially that the matrix $\mathbb{V}$ is symmetric. Now, by (5.28), (5.29), (7.12) and (7.14), we have

$$\mathbf{E}\ \mathrm{e}\bigl\{\langle\mathbb{B}_t\zeta_n,\zeta_n'\rangle\bigr\}\ll_s r^{-s}\bigl(\alpha_{2s}(\Omega_t)\bigr)^{1/2}=r^{-s}\bigl(\alpha_{2s}(\mathbb{D}_r\,\mathbb{U}_t\,\Lambda)\bigr)^{1/2}. \tag{7.15}$$

Using (7.15) and Lemmas 5.4 and 5.6, we derive the following lemma.

Lemma 7.1. *Let the conditions of Lemma* 5.4 *be satisfied with* $\delta\le 1/(5s)$. *Then*

$$\int_{r^{-1}}^1\bigl|\,\widehat{\Psi}_a(t/2)\bigr|\,\frac{dt}{t}\ \ll_s\ (pN)^{-1}+r^{-s/2}\,\sup_\Gamma\,\int_{r^{-1}}^1(\alpha_{2s}(\mathbb{D}_r\,\mathbb{U}_t\,\Lambda))^{1/4}\,\frac{dt}{t}\,,\qquad \textit{for all }\ t\in\mathbb{R}, \tag{7.16}$$

where r, $\alpha_{2s}(\cdot)$ *and the lattice* Λ *are defined in relations* (5.2), (5.27), (5.30), (6.14) *and* (7.4) *and in Lemma* 5.6. *The* $\sup_\Gamma$ *means here the supremum over all possible values of* $z_j,z_j'\in\mathbb{R}^d$ *such that*

$$\|z_j-e_j\|\le\delta,\qquad \|z_j'-e_j\|\le\delta,\qquad \textit{for }\ 1\le j\le s, \tag{7.17}$$

where $\mathcal{S}_o=\{e_1,\dots,e_s\}\subset\mathbb{R}^d$ *is an orthonormal system involved in the conditions of Lemma* 5.4.

Let $v=(m,\overline{m})\in\mathbb{R}^{4s}$, $m,\overline{m}\in\mathbb{R}^{2s}$. Then

$$\overline{m}+t\,m=(1+t^2)\,\overline{m}+t\,(m-t\overline{m}). \tag{7.18}$$

Equality (7.18) implies that

$$\|\overline{m}+t\,m\|\ll_s\|\overline{m}\|+\|m-t\overline{m}\|,\quad \text{for } |t|\ll_s 1. \tag{7.19}$$

Hence,

$$r\,\|m-t\overline{m}\|+r^{-1}\,\|\overline{m}+t\,m\|\ll_s r\,\|m-t\overline{m}\|+r^{-1}\,\|\overline{m}\|,\quad \text{for } r\ge 1,\ |t|\ll_s 1. \tag{7.20}$$

According to (7.1)–(7.3), we have

$$\mathbb{D}_r\mathbb{U}_t\,v=(r(m-t\overline{m}),r^{-1}\overline{m})\quad\text{and}\quad \mathbb{D}_r\mathbb{K}_t\,v=(r(m-t\overline{m}),r^{-1}(\overline{m}+t\,m)). \tag{7.21}$$

It is clear that the operators $\mathbb{D}_r\mathbb{U}_t$ and $\mathbb{D}_r\mathbb{K}_t$ are invertible. Therefore, using (7.20) and (7.21) and applying Lemmas 6.3 and 6.5, we derive the inequality

$$\alpha_{2s}(\mathbb{D}_r\mathbb{U}_t\Omega)\ll_s\alpha_{2s}(\mathbb{D}_r\mathbb{K}_t\Omega),\quad \text{for } r\ge 1,\ |t|\ll_s 1, \tag{7.22}$$

which is valid for any lattice $\Omega\subset\mathbb{R}^{4s}$.

Let $\mathbb{T}$ be the $(4s\times 4s)$ permutation matrix which permutes the rows of a $(4s\times 4s)$ matrix $\mathbb{A}$ so that the new order (corresponding to the matrix $\mathbb{T}\mathbb{A}$) is: $1,2s,2,2s+1,\dots,2s-1,4s$. Note that the operator $\mathbb{T}$ is isometric and $\mathbb{A}\mapsto\mathbb{A}\,\mathbb{T}^{-1}$ rearrange the columns of $\mathbb{A}$ in the order mentioned above. It is easy to see that

$$\alpha_{2s}(\mathbb{T}\Omega)=\alpha_{2s}(\Omega), \tag{7.23}$$

for any lattice $\Omega\subset\mathbb{R}^{4s}$.

Note now that

$$\mathbb{T}\mathbb{D}_r\mathbb{K}_t\Lambda_j=\mathbb{T}\mathbb{D}_r\mathbb{K}_t\mathbb{T}^{-1}\mathbb{T}\Lambda_j=\mathbb{W}_t\Delta_j, \tag{7.24}$$

where

$$\Delta_j=\mathbb{T}\Lambda_j \tag{7.25}$$

and where $\mathbb{W}_t$ is $(4s\times 4s)$-matrix

$$\mathbb{W}_t=\begin{pmatrix}\mathbb{G}_{r,t} & \mathbb{O}_2 & : & \mathbb{O}_2\\ \mathbb{O}_2 & \mathbb{G}_{r,t} & : & \mathbb{O}_2\\ \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot\\ \mathbb{O}_2 & \mathbb{O}_2 & : & \mathbb{G}_{r,t}\end{pmatrix} \tag{7.26}$$

constructed of (2×2)-matrices $\mathbb{O}_2$ and

$$\mathbb{G}_{r,t}\overset{\text{def}}{=}\begin{pmatrix} r & -rt\\ r^{-1}t & r^{-1}\end{pmatrix}. \tag{7.27}$$

Let $|t|\le 2$ and

$$\theta=\arcsin\bigl(t\,(1+t^2)^{-1/2}\bigr)\quad\text{or, equivalently,}\quad t=\tan\theta. \tag{7.28}$$

Then we have

$$|\theta|\le c^*\overset{\text{def}}{=}\arcsin(2/\sqrt5),\quad \cos\theta=(1+t^2)^{-1/2},\quad \sin\theta=t\,(1+t^2)^{-1/2}. \tag{7.29}$$

It is easy to see that

$$\mathbb{G}_{r,t}=(1+t^2)^{1/2}\,\overline{\mathbb{D}}_r\,\overline{\mathbb{K}}_\theta \tag{7.30}$$

and

$$\mathbb{W}_t=(1+t^2)^{1/2}\,\widetilde{\mathbb{D}}_r\,\widetilde{\mathbb{K}}_\theta, \tag{7.31}$$

where

$$\widetilde{\mathbb{D}}_r=\begin{pmatrix}\overline{\mathbb{D}}_r & \mathbb{O}_2 & : & \mathbb{O}_2\\ \mathbb{O}_2 & \overline{\mathbb{D}}_r & : & \mathbb{O}_2\\ \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot\\ \mathbb{O}_2 & \mathbb{O}_2 & : & \overline{\mathbb{D}}_r\end{pmatrix}\quad\text{and}\quad \widetilde{\mathbb{K}}_\theta=\begin{pmatrix}\overline{\mathbb{K}}_\theta & \mathbb{O}_2 & : & \mathbb{O}_2\\ \mathbb{O}_2 & \overline{\mathbb{K}}_\theta & : & \mathbb{O}_2\\ \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot\\ \mathbb{O}_2 & \mathbb{O}_2 & : & \overline{\mathbb{K}}_\theta\end{pmatrix} \tag{7.32}$$

are $(4s\times 4s)$ matrices with

$$\overline{\mathbb{D}}_r\overset{\text{def}}{=}\begin{pmatrix} r & 0\\ 0 & r^{-1}\end{pmatrix}\quad\text{and}\quad \overline{\mathbb{K}}_\theta\overset{\text{def}}{=}\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}\in\mathrm{SL}(2,\mathbb{R}). \tag{7.33}$$

Substituting (7.31) into equality (7.24), we obtain

$$\mathbb{T}\mathbb{D}_r\mathbb{K}_t\Lambda_j = (1+t^2)^{1/2}\,\widetilde{\mathbb{D}}_r\,\widetilde{\mathbb{K}}_\theta\,\Delta_j. \tag{7.34}$$

Below we shall also use the following crucial lemma of Götze and Margulis (2009).

Lemma 7.2. *Let* $\widetilde{\mathbb{K}}_\theta$ *and*

$$\widetilde{\mathbb{H}} = \begin{pmatrix} \overline{\mathbb{H}} & \mathbb{O}_2 & : & \mathbb{O}_2 \\ \mathbb{O}_2 & \overline{\mathbb{H}} & : & \mathbb{O}_2 \\ \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot \\ \mathbb{O}_2 & \mathbb{O}_2 & : & \overline{\mathbb{H}} \end{pmatrix} \tag{7.35}$$

be $(2d\times 2d)$*-matrices such that* $\overline{\mathbb{H}}\in G=\mathrm{SL}(2,\mathbb{R})$ *and* $\widetilde{\mathbb{K}}_\theta$ *is defined in* (7.32) *and* (7.33). *Let* β *is a positive number such that* $\beta d>2$. *Then, for any* $\overline{\mathbb{H}}\in G$ *and any lattice* $\Delta\subset\mathbb{R}^{2d}$,

$$\int_0^{2\pi}(\alpha_d(\widetilde{\mathbb{H}}\,\widetilde{\mathbb{K}}_\theta\,\Delta))^\beta d\theta \ll_{\beta,d} (\alpha_d(\Delta))^\beta\,\|\overline{\mathbb{H}}\|^{\beta d-2}. \tag{7.36}$$

Here $\|\overline{\mathbb{H}}\|$ *is the standard norm of the linear operator* $\mathbb{H}:\mathbb{R}^2\to\mathbb{R}^2$.

Consider, under the conditions of Lemma 7.1,

$$I_0 \overset{\text{def}}{=} \int_{r^{-1}/2}^{1/2}\big|\,\widehat{\Psi}_a(t)\big|\,\frac{dt}{t} = \int_{r^{-1}}^{1}\big|\,\widehat{\Psi}_a(t/2)\big|\,\frac{dt}{t}\,. \tag{7.37}$$

By Lemma 7.1, we have

$$I_0 \ll_s (pN)^{-1} + r^{-s/2}\sup_{\Gamma}\ J, \tag{7.38}$$

where

$$J = \int_{r^{-1}}^{1}(\alpha_{2s}(\mathbb{D}_r\,\mathbb{U}_t\,\Lambda))^{1/4}\,\frac{dt}{t} \le \sum_{j=2}^{\rho} I_j, \tag{7.39}$$

with

$$I_j \overset{\text{def}}{=} \int_{j^{-1}}^{(j-1)^{-1}}(\alpha_{2s}(\mathbb{D}_r\,\mathbb{U}_t\,\Lambda))^{1/4}\,\frac{dt}{t}\,,\quad j=2,3,\dots,\rho\overset{\text{def}}{=}\lceil r\rceil+1. \tag{7.40}$$

Changing variable $t=vj^{-2}$ and $v=w+j$ in I_j and using the properties of matrices $\mathbb{D}_r$ and $\mathbb{U}_t$, we have

$$\begin{aligned} I_j &= \int_j^{j^2(j-1)^{-1}}(\alpha_{2s}(\mathbb{D}_r\,\mathbb{U}_{vj^{-2}}\,\Lambda))^{1/4}\,\frac{dv}{v} \\ &\le \int_j^{j+2}(\alpha_{2s}(\mathbb{D}_r\,\mathbb{U}_{vj^{-2}}\,\Lambda))^{1/4}\,\frac{dv}{v} \\ &= \int_0^{2}(\alpha_{2s}(\mathbb{D}_r\,\mathbb{U}_{wj^{-2}}\,\mathbb{U}_{j^{-1}}\,\Lambda))^{1/4}\,\frac{dw}{w+j}\,. \end{aligned} \tag{7.41}$$

By (7.6),

$$\mathbb{D}_r\,\mathbb{U}_{wj^{-2}} = \mathbb{D}_{rj^{-1}}\,\mathbb{D}_j\,\mathbb{U}_{wj^{-2}} = \mathbb{D}_{rj^{-1}}\,\mathbb{U}_w\,\mathbb{D}_j. \tag{7.42}$$

According to (7.41) and (7.42),

$$I_j \ll \frac{1}{j}\int_0^2 \big(\alpha_{2s}(\mathbb{D}_{rj^{-1}}\mathbb{U}_t\Lambda_j)\big)^{1/4}\,dt, \tag{7.43}$$

where the lattices Λ_j are defined in (7.5) (see also (7.1), (7.3) and (7.4)). Let $N_1^{(j)} \le \cdots \le N_{4s}^{(j)}$ be the successive minima of the Euclidean norm with respect to the lattice Λ_j. Using (5.32) and (7.5), it is easy to show that

$$\det(\Lambda_j) = \big|\det\mathbb{V}\big| \asymp_s 1 \quad\text{and}\quad N_1^{(j)} \gg_s 1. \tag{7.44}$$

Therefore, by Lemmas 6.5 and 6.6, $N_k^{(j)} \asymp_s 1$, $k = 1, 2, \dots, 4s$, and

$$\alpha_{2s}(\Lambda_j) \ll_s 1, \tag{7.45}$$

By (7.22), (7.23) and (7.34), we have

$$\begin{aligned}\alpha_{2s}(\mathbb{D}_{rj^{-1}}\mathbb{U}_t\Lambda_j) &\ll_s \alpha_{2s}(\mathbb{D}_{rj^{-1}}\mathbb{K}_t\Lambda_j) = \alpha_{2s}(\mathbb{T}\mathbb{D}_{rj^{-1}}\mathbb{K}_t\Lambda_j)\\ &\ll_s \alpha_{2s}(\widetilde{\mathbb{D}}_{rj^{-1}}\,\widetilde{\mathbb{K}}_\theta\,\Delta_j),\end{aligned} \tag{7.46}$$

for $|t| \ll_s 1$, $r \ge 1$, where θ is defined in (7.28). Using Lemma 7.2 (with $d = 2s$) coupled with (7.29) and (7.46), we obtain

$$\begin{aligned}\int_0^2 \big(\alpha_{2s}(\mathbb{D}_{rj^{-1}}\mathbb{U}_t\Lambda_j)\big)^{1/4}\,dt &\ll_s \int_0^{c^*}\Big(\alpha_{2s}(\widetilde{\mathbb{D}}_{rj^{-1}}\,\widetilde{\mathbb{K}}_\theta\,\Delta_j)\Big)^{1/4}\,\frac{d\theta}{\cos^2\theta}\\ &\ll \int_0^{2\pi}\Big(\alpha_{2s}(\widetilde{\mathbb{D}}_{rj^{-1}}\,\widetilde{\mathbb{K}}_\theta\,\Delta_j)\Big)^{1/4}\,d\theta\\ &\ll_s \|\overline{\mathbb{D}}_{rj^{-1}}\|^{s/2-2}\,\big(\alpha_{2s}(\Delta_j)\big)^{1/4},\end{aligned} \tag{7.47}$$

if $s \ge 5$. It is clear that $\|\overline{\mathbb{D}}_{rj^{-1}}\| = rj^{-1}$. Therefore, according to (7.23), (7.25), (7.43) and (7.47),

$$I_j \ll_s \frac{1}{j}\,(rj^{-1})^{s/2-2}\,\big(\alpha_{2s}(\Lambda_j)\big)^{1/4}. \tag{7.48}$$

By (7.39), (7.45) and (7.48), we obtain, for $s \ge 5$,

$$J \ll_s \sum_{j=2}^{\rho}\frac{1}{j}\,(rj^{-1})^{s/2-2} \ll_s r^{s/2-2}. \tag{7.49}$$

By (5.2), (5.27), (7.38) and (7.49), we have $r \asymp_s (Np)^{1/2}$ and

$$I_0 \ll_s r^{-2} \ll_s (Np)^{-1}. \tag{7.50}$$

It is clear that in a similar way we can establish that

$$\int_1^{c(s)}\big|\widehat{\Psi}_a(t/2)\big|\,\frac{dt}{t} \ll_s r^{-2} \ll_s (Np)^{-1}, \tag{7.51}$$

for any quantity $c(s)$ depending on s only. The proof will be easier due to the fact that t cannot be small in this integral.

Thus, we have proved the following lemma.

Lemma 7.3. *Let the conditions of Lemma* 5.4 *be satisfied with* $s \geq 5$ *and* $c(s)$ *be a quantity depending on* s *only. Then there exists a* c_s *such that*

$$\int_{r^{-1}}^{c(s)} \bigl|\,\widehat{\Psi}_a(t)\bigr|\,\frac{dt}{t} \ll_s (Np)^{-1}, \tag{7.52}$$

if $Np \gg_s c_s$*, where* r *is defined in* (5.2) *and* (5.27).

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