

ON THE DISTRIBUTION OF THE NODAL SETS OF RANDOM SPHERICAL HARMONICS

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ABSTRACT. We study the volume of the nodal set of eigenfunctions of the Laplacian on the m -dimensional sphere. It is well known that the eigenspaces corresponding to $E_n = n(n + m - 1)$ are the spaces \mathcal{E}_n of spherical harmonics of degree n , of dimension \mathcal{N} . We use the multiplicity of the eigenvalues to endow \mathcal{E}_n with the Gaussian probability measure and study the distribution of the m -dimensional volume of the nodal sets of a randomly chosen function. The expected volume is proportional to $\sqrt{E_n}$. One of our main results is bounding the variance of the volume to be $O(\frac{E_n}{\sqrt{\mathcal{N}}})$.

In addition to the volume of the nodal set, we study its Leray measure. We find that its expected value is n independent. We are able to determine that the asymptotic form of the variance is $\frac{const}{\mathcal{N}}$.

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1. INTRODUCTION

Let M be a smooth compact manifold and f a real valued function on M . We define its nodal set to be the subset of M , where f vanishes and we are interested mainly in the nodal sets of eigenfunctions of the Laplacian on M . It is known [6], that generically, the nodal sets are smooth submanifolds of M with codimension 1. For example, if M is a surface, the nodal sets are *lines*. One is interested in studying their volume (i.e. the length of the nodal line for the 2-dimensional case), and other nodal properties for highly excited eigenstates. It was conjectured by Yau that the volume of the nodal set is bounded from above and below by a multiple of the square root of the Laplace eigenvalue. The lower bound was proven by Bruning and Gromes [5] and Bruning [4] for the planar case. Donnelly-Fefferman’s celebrated result [7] resolved Yau’s conjecture for real analytic metrics. However the general case of a smooth manifold is still open.

1.1. Spherical Harmonics. In this paper, we study the nodal sets for the eigenfunctions of the Laplacian Δ on the m -dimensional unit sphere \mathcal{S}^m . It is well known that the eigenvalues E of the Laplace equation

$$\Delta f + Ef = 0$$

on \mathcal{S}^m are all the numbers of the form

$$(1) \quad E = E_n = n(n + m - 1),$$

where n is an integer. Given a number E_n , the corresponding eigenspace is the space \mathcal{E}_n of the spherical harmonics of degree n . Its dimension is given by

$$(2) \quad \mathcal{N} = \mathcal{N}_n = \frac{2n + m - 1}{n + m - 1} \binom{n + m - 1}{m - 1} \sim \frac{2}{(m - 1)!} n^{m-1}.$$

Given an integral number n , we fix an $L^2(\mathcal{S}^m)$ orthonormal basis of \mathcal{E}_n

$$\eta_1(x) = \eta_1^n(x), \eta_2(x) = \eta_2^n(x), \dots, \eta_{\mathcal{N}}(x) = \eta_{\mathcal{N}}^n(x),$$

giving an identification $\mathcal{E}_n \cong \mathbb{R}^{\mathcal{N}}$. For further reading on the spherical harmonics we refer the reader to [1], chapter 9.

1.2. **Random model.** We consider a *random eigenfunction*

$$(3) \quad f(x) = \sqrt{\frac{|\mathcal{S}^m|}{\mathcal{N}}} \sum_{k=1}^{\mathcal{N}} a_k \eta_k(x),$$

where a_k are Gaussian $N(0, 1)$ i.i.d. which we assume to be defined on the same sample space Ω . That is, we use the identification $\mathcal{E}_n \cong \mathbb{R}^{\mathcal{N}}$ to induce the Gaussian probability measure ν on \mathcal{E}_n as

$$d\nu(f) = e^{-\frac{1}{2}\|\vec{a}\|^2} \frac{da_1 \cdot \dots \cdot da_{\mathcal{N}}}{(2\pi)^{\mathcal{N}/2}},$$

where $\vec{a} = (a_i) \in \mathbb{R}^{\mathcal{N}}$ are as in (3).

Note that ν is invariant with respect to the orthonormal basis for \mathcal{E}_n . As usual, for any random variable X on Ω , we denote its expectation $\mathbb{E}X$. For example, with the normalization factor in (3), for every *fixed* point $x \in \mathcal{S}^m$, one has

$$(4) \quad \mathbb{E}[f(x)^2] = \frac{|\mathcal{S}^m|}{\mathcal{N}} \sum_{i=1}^{\mathcal{N}} \eta_i(x)^2 = 1,$$

a simple corollary from the addition theorem (see (28)).

Any characteristic $X(L)$ of the nodal set

$$L = L_f = \{x \in \mathcal{S}^m : f(x) = 0\}$$

is a random variable defined on the same sample space Ω . We are interested in the distribution of two different characteristics. The most natural characteristic of the nodal set L_f of f is, of course, its $(m-1)$ -dimensional volume $\mathcal{Z} = \mathcal{Z}(f)$. The study of the distribution of the random variable \mathcal{Z} for a random $f \in \mathcal{E}_n$ is one of the goals of the present paper.

Berard [2] showed that the expected volume $\mathbb{E}\mathcal{Z}$ is

$$\mathbb{E}\mathcal{Z}(f) = \text{const} \cdot \sqrt{E_n}$$

(see proposition 1.4) and Neuheisel [8] proved that as $n \rightarrow \infty$,

$$(5) \quad \text{Var}(\mathcal{Z}) = O\left(\frac{E_n}{n^{\frac{(m-1)^2}{3m+1}}}\right) = O\left(\frac{E_n}{\mathcal{N}^{\frac{m-1}{3m+1}}}\right).$$

Remark 1.1. Rather than taking a_k standard Gaussian i.i.d., Neuheisel assumes that the vector $\vec{a} = (a_k) \in \mathbb{R}^{\mathcal{N}}$ is chosen uniformly on the unit sphere $\mathcal{S}^{\mathcal{N}-1}$. However, it is easy to see, that, since $\mathcal{Z}(f) = \mathcal{Z}(b \cdot f)$ for any constant $b \in \mathbb{R}$, both of those models are equivalent.

The volume of the nodal line of a random eigenfunction on the torus

$$\mathcal{T}^m = \mathbb{R}^m / \mathbb{Z}^m$$

was studied by Rudnick and Wigman [11]. In this case, it is not difficult to see that the expectation is given by $\mathbb{E}\mathcal{Z}(f^{\mathcal{T}^m}) = \text{const} \cdot \sqrt{E}$. Moreover, they prove that as the eigenspace dimension \mathcal{N} grows to infinity, the variance is bounded by

$$\text{Var} \mathcal{Z}(f^{\mathcal{T}^m}) = O\left(\frac{E}{\sqrt{\mathcal{N}}}\right),$$

which, in particular, implies that the tails of the distribution of the normalized random variable $\frac{\mathcal{Z}}{\mathbb{E}\mathcal{Z}}$ die.

More generally, one may also consider a random model of eigenfunctions for a generic compact manifold M . Of course, for generic manifolds, one does not expect the Laplacian to have any multiplicities, so that we cannot introduce a Gaussian ensemble on the eigenspace. Let E_j be the eigenvalues and ϕ_j the corresponding eigenfunctions. It is well known that the E_j are discrete, $E_j \rightarrow \infty$ and $L^2(M) = \text{span}\{\phi_j\}$.

In this case, rather than considering random eigenfunctions, one considers random *combinations* of eigenfunctions with growing energy window of either type

$$f^L(x) = \sum_{E_j \in [0, E]} a_j \phi_j(x)$$

(called the long range), or

$$f^S(x) = \sum_{\sqrt{E_j} \in [\sqrt{E}, \sqrt{E+1}]} a_j \phi_j(x),$$

(called the short range), as $E \rightarrow \infty$. Berard [2] found that

$$\mathbb{E}\mathcal{Z}(f^L) \sim c_M \cdot \sqrt{E}$$

and recently Zelditch [13] proved that

$$\mathbb{E}\mathcal{Z}(f^S) \sim c_M \cdot \sqrt{E},$$

notably with the same constant c_M for both the long and the short ranges.

Berry [3] computed the expected length of nodal lines for isotropic, monochromatic random waves in the plane, which are eigenfunctions of the Laplacian with eigenvalue E_n . He found that the expected length (per unit area) is again of size approximately $\sqrt{E_n}$ and argued that the variance should be of order $\log E_n$.

1.3. Leray nodal measure. Another property of the nodal line we consider is its *Leray measure* (also called *the microcanonical measure*). Given a function f on \mathcal{S}^m , we define the Leray nodal measure to be

$$(6) \quad \mathcal{L}(f) := \lim_{\epsilon \rightarrow 0} \frac{1}{2\epsilon} \text{meas}\{x \in \mathcal{S}^m : |f(x)| < \epsilon\},$$

provided that the last limit exists. One may write the definition (6) of the Leray nodal measure formally as

$$(7) \quad \mathcal{L}(f) := \int_{\mathcal{S}^m} \delta(f(x)) dx,$$

where δ is the Dirac delta function.

As is well known, the limit (6) exists when $\nabla f \neq 0$ on the nodal set in which case

$$\mathcal{L}(f) = \int_{\{x: f(x)=0\}} \frac{d\nu'(x)}{|\nabla f(x)|},$$

where ν' is the Riemannian hypersurface measure on the nodal set. This holds almost always on \mathcal{E}_n (see section 2.1).

The distribution of the Leray nodal measure on the sphere was also considered by Neuheisel. As in case of the volume, one may compute the expected value

$$\mathbb{E}\mathcal{L} = \frac{|\mathcal{S}^m|}{\sqrt{2\pi}}$$

using a rather standard computation (see proposition 1.3) and Neuheisel proved that the variance is bounded by

$$(8) \quad \text{Var } \mathcal{L} = O\left(\frac{1}{n^{\frac{m-1}{2}}}\right) = O\left(\frac{1}{\sqrt{\mathcal{N}}}\right)$$

Remark 1.2. Here, as in the case for the volume, Neuheisel considered a slightly different variation of the random model (see remark 1.1). Even though the Leray nodal measure is not invariant under dilations, i.e.

$$\mathcal{L}(b \cdot f) = \frac{1}{b}\mathcal{L}(f),$$

those models are still equivalent *asymptotically*, as $\mathcal{N} \rightarrow \infty$.

The Leray measure $\mathcal{L}(f^{\mathcal{T}^m})$ for the random eigenfunctions on the torus \mathcal{T}^n was considered by Oravecz, Rudnick and Wigman [10]. The expectation is given by

$$\mathbb{E}\mathcal{L}(f^{\mathcal{T}^m}) = \frac{1}{\sqrt{2\pi}}.$$

These authors were able to establish the variance to be asymptotic to

$$\text{Var } \mathcal{L}(f^{\mathcal{T}^m}) \sim c \cdot \frac{1}{\mathcal{N}}$$

for some $c > 0$, for $m = 2$ and $m \geq 5$.

1.4. The expectation.

Proposition 1.3. *For n sufficiently large, the expectation of the Leray nodal measure of the random eigenfunction is given by*

$$\mathbb{E}\mathcal{L}(f) = \frac{|\mathcal{S}^m|}{\sqrt{2\pi}}.$$

Proposition 1.4. *One has*

$$(9) \quad \mathbb{E}\mathcal{Z}(f) = c_m \cdot \sqrt{E_n},$$

with the constant c_m defined by

$$(10) \quad c_m = \frac{2\pi^{m/2}}{\sqrt{m}\Gamma(\frac{m}{2})}.$$

1.5. Statement of the main results. Our main results concern the variance of the Leray nodal measure \mathcal{L} and the volume \mathcal{Z} of the nodal set. We improve on Neuheisel's results (5) and (8), and need to use some of the steps in his work; however because some of the arguments in Neuheisel contain gaps, we need to redo them, partially accounting for the length of this paper.

For \mathcal{L} we were able to determine its asymptotics precisely.

Theorem 1.5. *As $n \rightarrow \infty$, the variance of the Leray nodal measure is asymptotic to*

$$(11) \quad \text{Var } \mathcal{L}(f) \sim \frac{2^{m-2} \pi^{\frac{m-2}{2}} \Gamma(\frac{m}{2}) |\mathcal{S}^m|}{(m-1)!} \cdot \frac{1}{\mathcal{N}}.$$

One should compare the asymptotic result (11) to Neuheisel's bound (8).

Remark 1.6. Note that unlike the torus, our proof here works for any dimension $m \geq 2$, including $m = 3, 4$. The reason is that for the sphere, the so-called two point function u (to be defined, see (19)) is related to the ultraspherical polynomials, a standard family of orthogonal polynomials [12]. In particular, using Hilb's asymptotics for the ultraspherical polynomials, it is easy to show that the 4th moment of u is dominated by its second moment (see lemmas A.1 and A.4).

Unlike the spherical case, the two point function for the d -dimensional torus is related to the distribution of points

$$\vec{n} = (n_1, \dots, n_d) \in \mathbb{Z}^d$$

so that

$$\|\vec{n}\|^2 = n_1^2 + \dots + n_d^2 = \frac{E}{4\pi^2}.$$

For $d \geq 5$ a strong equidistribution result for \vec{n} implies in particular that the 4th moment of u is dominated by its second moment. For the two-dimensional case we used a special result due to Zygmund. The remaining cases $d = 3, 4$ are, to our best knowledge, open.

Concerning the volume, we have the following result:

Theorem 1.7. *One has*

$$(12) \quad \text{Var } \mathcal{Z}(f) = O\left(\frac{E_n}{\sqrt{\mathcal{N}}}\right),$$

asymptotically as $n \rightarrow \infty$.

Note that theorem 1.7 implies that the variance of the *normalized* random variable $\tilde{\mathcal{Z}} := \frac{\mathcal{Z}}{\mathbb{E}\mathcal{Z}}$ with expectation 1, vanishes as $n \rightarrow \infty$. Thus, in particular, the tails of the distribution of $\tilde{\mathcal{Z}}$ “die”, that is, for every $\epsilon > 0$, most of the mass of $\tilde{\mathcal{Z}}$ is concentrated in $[1 - \epsilon, 1 + \epsilon]$. In addition, theorem 1.7 bounds the “typical” size of the tail of the distribution of $\tilde{\mathcal{Z}}$ (and thus of \mathcal{Z}). One should compare (12) to (5), obtained by Neuheisel. Partially motivated by the recent result [GW] for the analogous ensemble of random one dimensional trigonometric polynomials, it may be possible to improve the bound to E_n/\mathcal{N} .

1.6. On the proofs of the main results. The spherical case offers some marked differences from that of the torus [10] and [11]. Unlike the torus, which is identified with the unit square with its sides pairwise glued, the sphere possesses a nontrivial geometry. In the course of the proof of the main results, one has to study the joint distribution of the gradients $\nabla f(x)$ and $\nabla f(y)$ as random vectors, where $x, y \in \mathcal{S}^m$ are *fixed*, and $f \in \mathcal{E}_n$ is randomly chosen. The main obstacle here is that for different points

$x \in \mathcal{S}^m$, the gradients live in different spaces, namely, the tangent spaces $T_x(\mathcal{S}^m)$ which are, in general, different.

One then has to canonically identify the spaces $T_x(\mathcal{S}^m)$ via a family of isometries ϕ_x smooth w.r.t. x . In reality, such a choice is not possible for every x , and we treat this complication in section 2.4.

Once the geometric problems are resolved, the treatment of the so-called *two-point* (or, alternatively, *covariance*) function and its derivatives is more standard, related to the well-known ultraspherical polynomials (see [12] or appendix A). In particular, we find that the geometrical structure of the so-called *singular sets* on \mathcal{S}^m is less complicated than the singular set on the torus (see section 4.2).

1.7. Some Conventions. Throughout the paper, the letters x, y and z will denote either points on the sphere \mathcal{S}^m or spherical variables and t will denote a real variable. For $x, y \in \mathcal{S}^m$, $d(x, y)$ will stand for the spherical distance between x and y . The letters μ, ν, ν will be reserved for measures, where the measure ν will stand for the uniform measure on \mathcal{S}^m so that $d\nu(x) = dx$.

Finally, given a set S , we denote its volume by $|S|$. For example,

$$(13) \quad |\mathcal{S}^m| = \frac{2\pi^{\frac{m+1}{2}}}{\Gamma(\frac{m+1}{2})}.$$

In this manuscript we will use the notations $A \ll B$ and $A = O(B)$ interchangeably.

1.8. Plan of the paper. This paper is organized as follows. Section 2 is devoted to the computation of the expected value of the Leray nodal measure and the volume, that is, proving propositions 1.3 and 1.4, where the rest of the paper focuses on the variance of those characteristics, i.e. proving theorems 1.5 and 1.7. The treatment of the variance in both cases will be divided into two steps. First, we express it in an integral form in section 3. We treat the integrals obtained in section 3 throughout section 4. In case of the Leray measure, we will be able to give a precise asymptotic expression. In the case of volume, we give an upper bound.

Appendix A will introduce the reader to the ultraspherical polynomials and will also provide all the necessary background we will need in this paper. The goal of appendix B is to prove that the set of “bad” (singular) eigenfunctions in the space of all the eigenfunctions, is “rare” in some strong sense. Finally, appendix C will prove a particular nondegeneracy result for the distribution of the eigenfunctions and its gradients, needed to give meaning to the integral formula obtained for the variance of the volume given in section 3.

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2. EXPECTATION

In this section we prove propositions 1.3 and 1.4. As a start, we wish to stay away from the set of the singular functions discussed in section 2.1.

2.1. The singular functions. In this section we define the notion of the singular functions and formulate the intuitive statement that they are “rare”. The proofs are given in appendix B.

Definition 2.1. An eigenfunction $f \in \mathcal{E}_n$ is *singular* if $\exists x \in \mathcal{S}^m$ with $f(x) = 0$ and $\nabla f(x) = \vec{0}$. An eigenfunction $f \in \mathcal{E}_n$ is *nonsingular* if $\nabla f \neq \vec{0}$ on the nodal set.

A nonsingular eigenfunction has no self-intersections. We denote $Sing \subseteq \mathcal{E}_n$ to be the set of all the singular eigenfunctions. First, we claim that as a set, $Sing$ is “small”.

Lemma 2.2. *The set $Sing$ has codimension 1 in \mathcal{E}_n .*

Now, given $x \in \mathcal{S}^m$ and $b \in \mathbb{R}$, we denote \mathcal{P}_b^x to be the set of all the eigenfunctions which attain the value b at the point x . That is,

$$(14) \quad \mathcal{P}_b^x = \{f \in \mathcal{E}_n : f(x) = b\}.$$

The set \mathcal{P}_b^x is a hyperplane in \mathcal{E}_n .

Moreover, given $(x, y) \in \mathcal{S}^m \times \mathcal{S}^m$ and $b = (b_1, b_2) \in \mathbb{R}^2$ we denote

$$(15) \quad \mathcal{P}_b^{x,y} = \{f \in \mathcal{E}_n : f(x) = b_1, f(y) = b_2\}.$$

For $x \neq \pm y$, $\mathcal{P}_b^{x,y}$ is an affine subspace of \mathcal{E}_n of codimension 2, as it is easy to see from the addition theorem (see section 2.2).

The next couple of lemmas establish the fact that the intersections of $Sing$ with \mathcal{P}_b^x and $\mathcal{P}_b^{x,y}$ for $x \neq \pm y$, are of codimension 1. Lemma 2.4 is essential while treating the variance of the Leray nodal measure (section 3.2).

Lemma 2.3. *For every $x \in \mathcal{S}^m$ and $b \in \mathbb{R}$, the set*

$$Sing_b^x := Sing \cap \mathcal{P}_b^x$$

has codimension 1 in \mathcal{P}_b^x .

Lemma 2.4. *If $x, y \in \mathcal{S}^m$ and $x \neq \pm y$, then for every $b = (b_1, b_2) \in \mathbb{R}^2$, the set $Sing_b^{x,y} := Sing \cap \mathcal{P}_b^{x,y}$ has codimension 1 in $\mathcal{P}_b^{x,y}$.*

The proofs of all the lemmas of this section are given in appendix B.

2.2. Two-point function. We define the so called *two-point* function, also referred in the literature as the *covariance* function

$$(16) \quad u(x, y) = u_n^m(x, y) = \mathbb{E}[f(x)f(y)] = \frac{|\mathcal{S}^m|}{\mathcal{N}} \sum_{k=1}^{\mathcal{N}} \eta_k(x)\eta_k(y).$$

The addition theorem [1], page 456, theorem 9.6.3 implies that

$$(17) \quad u(x, y) = Q_n^m(\cos d(x, y)),$$

where

$$Q_n^m : [-1, 1] \rightarrow \mathbb{R}$$

are the *normalized* ultraspherical polynomials defined and studied in appendix A. Recall that $d(x, y)$ is the spherical distance so that

$$\cos d(x, y) = \langle x, y \rangle,$$

thinking of \mathcal{S}^m as being embedded into \mathbb{R}^{m+1} .

It is immediate that u is rotationally invariant, i.e.

$$(18) \quad u(Rx, Ry) = u(x, y),$$

where R is any rotation on \mathcal{S}^m . In case y is not specified, it is taken to be the northern pole $N \in \mathcal{S}^m$, that is

$$(19) \quad u(x) := u(x, N).$$

For every $t \in [-1, 1]$, $|Q_n^m(t)| \leq 1$ and $|Q_n^m(t)| = 1$, if and only if $t = \pm 1$. Therefore

$$(20) \quad (u(x, y) = \pm 1) \Leftrightarrow (x = \pm y),$$

and

$$(21) \quad (u(x) = \pm 1) \Leftrightarrow (x \in \{N, S\}),$$

where N and S are the northern and the southern poles respectively.

2.3. Leray nodal measure. We will need the following definitions from [10], section 3.

For $\epsilon > 0$, set

$$\mathcal{L}_\epsilon(f) := \frac{1}{2\epsilon} \text{meas}\{x : |f(x)| < \epsilon\}.$$

so that $\mathcal{L}(f) = \lim_{\epsilon \rightarrow 0} \mathcal{L}_\epsilon(f)$.

For $\alpha > 0$, $\beta > 0$ let

$$\mathcal{E}_n(\alpha, \beta) = \{f \in \mathcal{E}_n : |f(x)| \leq \alpha \Rightarrow |\nabla f(x)| > \beta\}.$$

The sets $\mathcal{E}_n(\alpha, \beta)$ are open, and have the monotonicity property

$$\alpha_1 > \alpha_2 \Rightarrow \mathcal{E}_n(\alpha_1, \beta) \subseteq \mathcal{E}_n(\alpha_2, \beta)$$

and

$$\beta_1 > \beta_2 \Rightarrow \mathcal{E}_n(\alpha, \beta_1) \subseteq \mathcal{E}_n(\alpha, \beta_2).$$

Moreover, for any sequence $\alpha_n, \beta_n \rightarrow 0$ we have

$$\mathcal{E}_n \setminus \text{Sing} = \bigcup_n \mathcal{E}_n(\alpha_n, \beta_n).$$

We have (cf. [10], lemma 3.1)

Lemma 2.5. *For $f \in \mathcal{E}_n(\alpha, \beta)$ and $0 < \epsilon < \alpha$, we have*

$$\mathcal{L}_\epsilon(f) \ll \sqrt{E_n}$$

where the constant involved in the ' \ll '-notation depends only on α and β .

To prove lemma 2.5, we will need lemma 3.2 from [10].

Lemma 2.6 (Lemma 3.2 from [10]). *Let $g(t)$ be a trigonometric polynomial on $[0, 2\pi]$ of degree at most M so that there are $\alpha > 0$, $\beta > 0$ such that $|g'(t)| > \beta$ whenever $|g(t)| < \alpha$. Then for all $0 < \epsilon < \alpha$ we have*

$$\frac{1}{2\epsilon} \text{meas}\{t \in [0, 2\pi] : |g(t)| < \epsilon\} \ll \frac{M}{\beta},$$

where the constant in the ' \ll '-notation may depend on m only.

Proof of lemma 2.5. Let (ϕ_1, \dots, ϕ_m) be the standard multi-dimensional spherical coordinates so that $x \in \mathcal{S}^m$ is parameterized by

$$x = (\cos \phi_1, \sin \phi_1 \cos \phi_2, \dots, \sin \phi_1 \dots \sin \phi_{m-1})$$

for $(\phi_1, \dots, \phi_m) \in R := [0, \pi] \times \dots \times [0, \pi] \times [0, 2\pi]$. It is well-known that for $\phi_i \neq 0, \pi, 2\pi$, $\{\frac{\partial}{\partial \phi_k}\}$ is an orthogonal basis of $T_x(\mathcal{S}^m)$ and we have

$$\left\| \frac{\partial}{\partial \phi_k} \right\| = \sin \phi_1 \cdot \dots \cdot \sin \phi_{k-1},$$

so that the Jacobian

$$J = J(\phi_1, \dots, \phi_m) = \frac{Dx}{D(\phi_1, \dots, \phi_m)}$$

satisfies

$$J = \sin \phi_1^{m-1} \cdot \sin \phi_2^{m-2} \cdot \dots \cdot \sin \phi_{m-1}.$$

Let $0 < \epsilon < \alpha$. We write

$$\text{meas}\{x \in \mathcal{S}^m : |f(x)| < \epsilon\}$$

as an integral

(22)

$$\text{meas}\{x \in \mathcal{S}^m : |f(x)| < \epsilon\} = \int_{\mathcal{S}^m} \chi\left(\frac{f(x)}{\epsilon}\right) dx = \int_{A_\epsilon} |J(\phi_1, \dots, \phi_m)| d\phi_1 \cdot \dots \cdot d\phi_m$$

in the spherical coordinates, where we denoted

$$A_\epsilon := \{P \in R : |f(P)| < \epsilon\}.$$

For $P \in R$, $1 \leq k \leq m$ we define $p_k(P) = \frac{1}{\left\| \frac{\partial}{\partial \phi_k} \right\|} \frac{\partial f}{\partial \phi_k}(P)$, so that

$$(23) \quad \|\nabla f(x)\|^2 = p_1^2 + p_2^2 + \dots + p_m^2.$$

We decompose

$$A_\epsilon = W_1 \cup W_2 \cup \dots \cup W_m$$

with

$$W_k := \{P \in A_\epsilon : |p_k(P)| = \max_j |p_j(P)|\}.$$

Note that on W_k ,

$$|p_i(P)| > \frac{\beta}{\sqrt{m}},$$

by (23) and $\|\nabla f(x)\| > \beta$ on A_ϵ .

Note that for $\phi_k, k \neq k_0$ fixed, $g(\phi_{k_0}) := f(\phi_1, \dots, \phi_m)$ is a trigonometric polynomial in ϕ_{k_0} on either $[0, \pi]$ or $[0, 2\pi]$ of degree $\leq n \leq \sqrt{E_n}$ with derivative

$$g'(\phi) = \left\| \frac{\partial}{\partial \phi_{k_0}} \right\| \cdot p_1(P)$$

so that on W_{k_0} , $|g'(\phi)| > \left\| \frac{\partial}{\partial \phi_{k_0}} \right\| \frac{\beta}{\sqrt{m}}$. Thus lemma 2.6 implies

$$\text{meas}\{\theta : |g(\phi_{k_0})| < \epsilon\} \ll \frac{\sqrt{E_n}}{\left\| \frac{\partial}{\partial \phi_{k_0}} \right\|} \cdot \epsilon.$$

Therefore the contribution of W_{k_0} to the integral (22) is

$$\begin{aligned} \int_{W_1} |J| d\phi_1 \cdots d\phi_m &\leq \int \text{meas}\{\phi : |g(\phi_{k_0})| < \epsilon\} d\phi_1 \cdots d\hat{\phi}_{k_0} \cdots d\phi_m \\ &\ll \int \frac{|J|}{\left\| \frac{\partial}{\partial \phi_{k_0}} \right\|} \cdot \sqrt{E_n} \epsilon d\phi_1 \cdots d\hat{\phi}_{k_0} \cdots d\phi_m \ll \epsilon \sqrt{E_n}. \end{aligned}$$

which concludes the proof of the lemma. \square

We conclude the section with a formal derivation of proposition 1.3. A rigorous proof proceeds along the same lines as the proof of theorem 4.1 in [10] (see section 4.2), using lemmas 2.5, 2.2 and 2.3. We omit it here.

Formal derivation of proposition 1.3. Given a function $f \in \mathcal{E}_n$, we write its Leray nodal measure formally as

$$\mathcal{L}(f) = \int_{S^m} \delta(f(x)) dx,$$

see (7).

Then, taking the expected value of both sides and changing the order of the expectation and the limit, we obtain

$$(24) \quad \mathbb{E}\mathcal{L}(f) = \int_{S^m} \mathbb{E}\delta(f(x)) dx.$$

Now, for each *fixed* $x \in S^m$, the random variable $v = f(x)$ is a linear combination of Gaussian random variables, and therefore, Gaussian itself. Its mean is zero and its variance is 1 by (4). Writing the Gaussian probability density function explicitly, we have

$$\mathbb{E}\delta(f(x)) = \mathbb{E}\delta(v) = \int_{-\infty}^{\infty} \delta(a) \frac{e^{-\frac{1}{2}a^2}}{\sqrt{2\pi}} da = \frac{1}{\sqrt{2\pi}}.$$

To finish the proof of this proposition we integrate the last equality on the sphere and substitute it into (24). \square

2.4. Choice of orthonormal bases for $T_x(\mathcal{S}^m)$. For every $x \in \mathcal{S}^m$ we will need to identify

$$(25) \quad \phi_x : T_x(\mathcal{S}^m) \cong \mathbb{R}^m,$$

so that given a smooth function f on \mathcal{S}^m , the function

$$\nabla f(x) \in \mathbb{R}^m,$$

is, under the identification (25), *almost everywhere smooth* (i.e, C^k , if f is C^{k+1}) of argument x .

Since we will be typically interested in the length of the gradient, we will require the identifications (25) to be length preserving, namely, isometries. This is naturally accomplished, given a choice

$$B_x = \{e_1^x, \dots, e_m^x\}$$

of an *orthonormal* basis of $T_x(\mathcal{S}^m)$ for every $x \in \mathcal{S}^m$, so that for every vector e_i , all of its coordinates satisfy the appropriate smoothness condition. To do so, we consider the sphere without its southern pole S

$$R := \mathcal{S}^m \setminus \{S\}.$$

Choosing an *arbitrary* orthonormal basis $B = B_N$ corresponding to the northern pole provides such a basis B_x for every $x \in R$ by means of the parallel transport of B along the unique geodesic linking N and x on R . We choose an arbitrary orthonormal basis B_S of the tangent plane $T_S(\mathcal{S}^m)$ of \mathcal{S}^m at the southern pole. It doesn't affect any of the computations below, and we will neglect it from this point on.

Let $g(x) : \mathcal{S}^m \rightarrow \mathbb{R}$ be any smooth function. We will use the notation

$$\frac{\partial}{\partial e_i} g(x) = \frac{\partial}{\partial e_i^x} g|_x$$

for the directional derivative of $g(x)$ at x along e_i^x , i.e.

$$\frac{\partial}{\partial e_i} g(x) = \langle \nabla g(x), e_i^x \rangle.$$

In case of ambiguity, i.e. if we deal with a two variable function

$$h(x, y) : \mathcal{S}^m \times \mathcal{S}^m \rightarrow \mathbb{R},$$

we write $\frac{\partial}{\partial e_i^x} h$ or $\frac{\partial}{\partial e_i^y} h$ for the derivative of g as a function of x with y constant, or vice versa respectively. Similarly, we will use the notation $\nabla_x g(x, y) \in T_x(\mathcal{S}^m)$ and $\nabla_y g(x, y) \in T_y(\mathcal{S}^m)$ to denote the gradient of $g(x, y)$ as a function of x or y respectively.

Remark 2.7. Note that with the choice of the identifications (25) as above, we have

$$(26) \quad \nabla_x d(x, y)|_{(x, N)} = -\nabla_y d(x, y)|_{(x, N)},$$

which is going to be useful in simplifying the covariance matrix Σ (see section 3.1).

Remark 2.8. In fact, for all our purposes, it is sufficient to make the choice of the orthonormal bases *locally*. Such a choice is possible for any manifold.

2.5. The covariance matrix, expectation. Given a point $x \in \mathcal{S}^m$ we consider the random vector $(v, w) \in \mathbb{R} \times \mathbb{R}^m$

$$(v, w) = (f(x), \nabla f(x)),$$

where we use the identification (25). It is easy to see that being a linear transformation of a mean zero Gaussians, its distribution is a mean zero Gaussian as well.

We claim that the covariance matrix of (v, w) is

$$(27) \quad \tilde{\Sigma}_{(m+1) \times (m+1)} := \begin{pmatrix} \mathbb{E}f(x)^2 & \mathbb{E}[f(x)\nabla f(x)] \\ \mathbb{E}[f(x)\nabla f(x)]^t & \mathbb{E}[\nabla f(x)^t \nabla f(x)] \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{E_n}{m} I_m \end{pmatrix}.$$

First,

$$(28) \quad \mathbb{E}f(x)^2 = u(x, x) = 1,$$

by the definition (16), (17) of $u(x, y)$ and (20).

Next, we have

$$\mathbb{E}(f(x)\nabla f(x)) = \frac{1}{2} \nabla \mathbb{E}(f(x)^2) = \nabla 1/2 = \vec{0}$$

by (4).

Finally, we compute $\mathbb{E}[\nabla f(x)^t \nabla f(x)]$. For $i \neq j$, we have

$$\mathbb{E} \left[\frac{\partial}{\partial e_i^x} f(x) \frac{\partial}{\partial e_j^x} f(x) \right] = \left[\frac{\partial}{\partial e_i^x \partial e_j^y} u(x, y) \right] \Big|_{x=y} = 0,$$

computing the second partial derivative explicitly in local coordinates (see section 2.4 for an explanation of the partial derivatives notations).

For $i = j$, we have by the rotational symmetry on \mathcal{S}^m ,

$$\begin{aligned} \mathbb{E} \left(\left(\frac{\partial}{\partial e_i^x} f(x) \right)^2 \right) &= \frac{1}{m|\mathcal{S}^m|} \int_{\mathcal{S}^m} \mathbb{E} \left(\nabla f(x) \cdot \nabla f(x) \right) dx \\ &= \frac{1}{m|\mathcal{S}^m|} \mathbb{E} \left[\int_{\mathcal{S}^m} \nabla f(x) \cdot \nabla f(x) dx \right] = -\frac{1}{m|\mathcal{S}^m|} \mathbb{E} \left[\int_{\mathcal{S}^m} f(x) \cdot \Delta f(x) dx \right] \\ &= \frac{E_n}{m|\mathcal{S}^m|} \mathbb{E} \left[\int_{\mathcal{S}^m} f(x)^2 dx \right] = \frac{E_n}{m} \cdot \frac{1}{|\mathcal{S}^m|} \int_{\mathcal{S}^m} \mathbb{E}[f(x)^2] dx = \frac{E_n}{m}, \end{aligned}$$

by the divergence theorem and (4). Thus

$$(29) \quad \mathbb{E}(\nabla f(x)^t \nabla f(x)) = \mathbb{E}(\nabla f(y)^t \nabla f(y)) = \frac{E_n}{m} I_m.$$

2.6. Riemannian volume. Let χ be the indicator function of the interval $[-1, 1]$. For $\epsilon > 0$, we define the random variable

$$\mathcal{Z}_\epsilon(f) := \frac{1}{2\epsilon} \int_{\mathcal{S}^m} \chi \left(\frac{f(x)}{\epsilon} \right) |\nabla f(x)| dx.$$

Lemma 2.9 (Lemma 3.1 from [11]). *Suppose that $f \in \mathcal{E}_n$ is nonsingular. Then*

$$\text{vol}(f^{-1}(0)) = \lim_{\epsilon \rightarrow 0} \mathcal{Z}_\epsilon(f).$$

Lemma 2.9 implies that the expectation of the volume and its second moments are given by the following.

Corollary 2.10 (Corollary 3.4 from [11]). *The first and second moments of the volume $\mathcal{Z}(f)$ of the nodal set of f are given by*

$$\mathbb{E}(\mathcal{Z}) = \mathbb{E}(\lim_{\epsilon \rightarrow 0} \mathcal{Z}_\epsilon), \quad \mathbb{E}(\mathcal{Z}^2) = \mathbb{E}(\lim_{\epsilon_1, \epsilon_2 \rightarrow 0} \mathcal{Z}_{\epsilon_1} \mathcal{Z}_{\epsilon_2}).$$

Lemma 2.11. *For every $f \in \mathcal{E}_n$ and $\epsilon > 0$, one has*

$$\mathcal{Z}_\epsilon(f) = O(\sqrt{E_n}),$$

where the constant involved in the 'O' notation depends only on m .

To prove lemma 2.11, we use lemma 3.3 from [11].

Lemma 2.12 (Lemma 3.3 from [11]). *Let $g(t)$ be a trigonometric polynomial on $[0, 2\pi]$ of degree at most M . Then for all $\epsilon > 0$ we have*

$$\frac{1}{2\epsilon} \int_{\{t: |g(t)| \leq \epsilon\}} |g'(t)| dt \leq 6M.$$

Proof of lemma 2.11. We write \mathcal{Z}_ϵ in the multi-dimensional spherical coordinates (see the proof of lemma 2.11) as

$$\mathcal{Z}_\epsilon(f) := \frac{1}{2\epsilon} \int_R \chi\left(\frac{f(\phi_1, \dots, \phi_m)}{\epsilon}\right) \|\nabla f(\phi_1, \dots, \phi_m)\| \cdot |J| d\phi_1 \dots d\phi_m.$$

Note that in the spherical coordinates, for $\phi_k \neq 0, \pi, 2\pi$

$$\nabla f(\phi_1, \dots, \phi_m) = \left(\frac{1}{\|\frac{\partial}{\partial \phi_k}\|} \frac{\partial f}{\partial \phi_k} \right)_{1 \leq k \leq m},$$

in the orthonormal basis associated to $\{\frac{\partial}{\partial \phi_k}\}$. Thus

$$\|\nabla f\| \cdot |J| \ll \sum_{k=1}^m \left| \frac{\partial f}{\partial \phi_k} \right|.$$

Note that for $\phi_k, k \neq k_0$ fixed, $f(\phi_1, \dots, \phi_m)$ is a 1-variable trigonometric polynomial in ϕ_{k_0} of degree $\leq n \ll \sqrt{E_n}$. Therefore,

$$\begin{aligned} \mathcal{Z}_\epsilon(f) &\ll \sum_{k_0=1}^m \int \frac{1}{2\epsilon} \left[\int_{\{\theta: |f(\phi_1, \dots, \phi_m)| < \epsilon\}} \left| \frac{\partial f(\phi_1, \dots, \phi_m)}{\partial \phi_{k_0}} \right| d\phi_{k_0} \right] d\phi_1 \dots d\hat{\phi}_{k_0} \dots d\phi_m \\ &\ll \sqrt{E_n}, \end{aligned}$$

by lemma 2.12. □

Now we are in a position to prove the main result of this section, namely proposition 1.4.

Proof of proposition 1.4. We saw that

$$\mathbb{E}\mathcal{Z}(f) = \mathbb{E} \lim_{\epsilon \rightarrow 0} \mathcal{Z}_\epsilon(f)$$

by corollary 2.10. Lemma 2.11 and the dominated convergence theorem allow us to exchange the order of taking expectation and the limit to obtain

$$(30) \quad \mathbb{E}\mathcal{Z}(f) = \lim_{\epsilon \rightarrow 0} \mathbb{E}\mathcal{Z}_\epsilon(f).$$

By Fubini's theorem,

$$(31) \quad \mathbb{E}\mathcal{Z}_\epsilon(f) = \mathbb{E} \left[\frac{1}{2\epsilon} \int_{\mathcal{S}^m} \chi \left(\frac{f(x)}{\epsilon} \right) |\nabla f(x)| dx \right] = \int_{\mathcal{S}^m} K_\epsilon^1(x) dx,$$

where $K_\epsilon^1(x)$ is defined by

$$(32) \quad K_\epsilon^1(x) := \mathbb{E} \left[\frac{1}{2\epsilon} \chi \left(\frac{f(x)}{\epsilon} \right) |\nabla f(x)| \right] = \frac{1}{2\epsilon} \int_{\mathcal{E}_n} \chi \left(\frac{f(x)}{\epsilon} \right) |\nabla f(x)| dv(f).$$

We write K_ϵ^1 in terms of the random vector (v, w) , introduced in section 2.5¹ as

$$K_\epsilon^1(x) = \frac{1}{2\epsilon} \int_{\mathbb{R} \times \mathbb{R}^m} \chi \left(\frac{v}{\epsilon} \right) \|w\| d\mu(v, w),$$

where $d\mu(v, w)$ is the joint probability density function of (v, w) , namely mean zero Gaussian with covariance $\tilde{\Sigma}$ given by (27). Writing the Gaussian probability explicitly, we have

$$\begin{aligned} K_\epsilon^1(x) &= \frac{1}{2\epsilon} \int_{\mathbb{R} \times \mathbb{R}^m} \chi \left(\frac{v}{\epsilon} \right) \|w\| \exp \left(-\frac{1}{2} (v, w) \tilde{\Sigma}^{-1} (v, w)^t \right) \frac{dv dw}{(2\pi)^{(m+1)/2} \sqrt{\det \tilde{\Sigma}}} \\ &= \frac{1}{2\epsilon} \int_{-\epsilon}^{\epsilon} \exp \left(-\frac{1}{2} v^2 \right) dv \int_{\mathbb{R}^m} \|w\| \exp \left(-\frac{1}{2} \frac{\|w\|^2 m}{E_n} \right) \frac{m^{m/2} dw}{(2\pi)^{(m+1)/2} E_n^{m/2}} \\ &= \frac{1}{2\epsilon} \int_{-\epsilon}^{\epsilon} \exp \left(-\frac{1}{2} v^2 \right) dv \cdot \frac{\sqrt{E_n}}{\sqrt{m}} \int_{\mathbb{R}^m} \|w'\| \exp \left(-\frac{1}{2} \|w'\|^2 \right) \frac{dw'}{(2\pi)^{(m+1)/2}}, \end{aligned}$$

changing the variables

$$w = \sqrt{\frac{E_n}{m}} w'.$$

Following (31) and (30), we integrate the last expression and take the limit $\epsilon \rightarrow 0$ to obtain

$$\mathbb{E}\mathcal{Z}(f) = c_m \sqrt{E_n},$$

where

$$(33) \quad c_m = \frac{|\mathcal{S}^m|}{\sqrt{m} (2\pi)^{(m+1)/2}} \int_{\mathbb{R}^m} \|w'\| \exp \left(-\frac{1}{2} \|w'\|^2 \right) dw'.$$

Finally, substituting

$$\int_{\mathbb{R}^m} \|w'\| \exp \left(-\frac{1}{2} \|w'\|^2 \right) dw' = \sqrt{2} (2\pi)^{m/2} \frac{\Gamma \left(\frac{m+1}{2} \right)}{\Gamma \left(\frac{m}{2} \right)}$$

¹We use this opportunity to note that in [11], page 7, the boundedness of $\frac{1}{2\epsilon} \chi \left(\frac{f(x)}{\epsilon} \right) |\nabla f(x)|$ is *unnecessary*, and, in fact, wrong.

(see e.g. [11], page 7) and (13) into the last expression yields (10). \square

3. AN INTEGRAL FORMULA FOR THE SECOND MOMENT

3.1. Covariance matrices, second moment. Similarly to the computation of the expected volume, we will naturally encounter a random vector on

$$\mathbb{R} \times \mathbb{R} \times \mathbb{R}^m \times \mathbb{R}^m,$$

defined as

$$(f(x), f(y), \nabla f(x), \nabla f(y)),$$

for some *fixed* $x, y \in \mathcal{S}^m$, where we again use the identification (25). We will use the rotational symmetry of the sphere to reduce the discussion to the case $y = N$ is the northern pole. Thus we consider

$$(34) \quad Z := (v_1, v_2, w_1, w_2) = (f(x), f(N), \nabla f(x), \nabla f(N))$$

for some $y \in \mathcal{S}^m$.

It is obvious that the joint distribution of this vector, is mean zero Gaussian. It remains, therefore, to compute the covariance matrix. We need the following notations.

Let $D = D(x)$ be the vector in \mathbb{R}^m defined by

$$D(x) = \nabla_x u(x, y)|_{(x, N)} \in T_x(\mathcal{S}^m) \cong \mathbb{R}^m.$$

Note that for $x \neq \pm N$, we may use (17) to obtain

$$D(x) = Q_n^{m'}(d(x, N)) \sin(d(x, N)) \nabla_x d(x, y)|_{(x, N)}.$$

It is then clear from (26) that we then have

$$(35) \quad \nabla_x u(x, y)|_{(x, N)} = -\nabla_y u(x, y)|_{(x, N)}.$$

Finally, let

$$H = H(x) = (h_{ij})$$

be the $m \times m$ matrix defined as

$$(36) \quad H = \nabla_x \nabla_y u(x, y)|_{(x, N)},$$

i.e. $H = (h_{jk})$ with entries given by

$$h_{jk} = \frac{\partial^2}{\partial e_j^x \partial e_k^y} u(x, y)|_{(x, N)}.$$

We will be in particular interested in the conditional distribution of

$$Z_1 = (w_1, w_2) = (\nabla f(x), \nabla f(N)),$$

conditioned upon $f(x) = f(N) = 0$.

For the variance computation of the Leray nodal measure, we will need the distribution of the random vector

$$\tilde{Z} := (v_1, v_2) = (f(x), f(N)).$$

It is distributed mean zero Gaussian as well.

The covariance matrices of the random vectors above are given in the following lemma.

Lemma 3.1. *Let $x \in \mathcal{S}$. Then*

- (1) The distribution of the random vector $\tilde{Z} = (v_1, v_2)$ is mean zero Gaussian with covariance matrix given by

$$(37) \quad A = \begin{pmatrix} 1 & u(x, N) \\ u(x, N) & 1 \end{pmatrix}.$$

- (2) The covariance matrix of the random vector Z is the $(2m + 2) \times (2m + 2)$ matrix

$$\Sigma = \begin{pmatrix} A & B \\ B^t & C \end{pmatrix},$$

where $A \in M_{2 \times 2}$ is given by (37), $B \in M_{2 \times 2m}$ is given by

$$(38) \quad B = \begin{pmatrix} \vec{0} & -D(x) \\ D(x) & \vec{0} \end{pmatrix},$$

and $C \in M_{2m \times 2m}$ is given by

$$(39) \quad C = \begin{pmatrix} \frac{E_n}{m} I_m & H \\ H^t & \frac{E_n}{m} I_m \end{pmatrix}$$

with the “pseudo-Hessian” matrix $H = (h_{jk})$ of u given by (36). The distribution of Z is nondegenerate for $x \neq \pm N$ (this is equivalent to Σ being invertible).

- (3) The covariance matrix of the conditional distribution of Z_1 , conditioned upon $v_1 = v_2 = 0$ is given by

$$(40) \quad \Omega = \left[\begin{pmatrix} \frac{E_n}{m} I & H \\ H^t & \frac{E_n}{m} I \end{pmatrix} - \frac{1}{1 - u^2} \begin{pmatrix} D^t D & -u D^t D \\ -u D^t D & D^t D \end{pmatrix} \right].$$

We call the matrix Ω the “reduced covariance matrix” of Z_1 , and one has

$$(41) \quad \det \Sigma = \det A \det \Omega = (1 - u^2) \det \Omega.$$

Proof. Part (1) of the lemma is evident from the definition of the two-point function. It is also clear that part (2) of the lemma implies part (3), since one computes the covariance matrix Ω of the conditional distribution from Σ employing

$$\Sigma^{-1} = \begin{pmatrix} * & * \\ * & \Omega^{-1} \end{pmatrix}.$$

The nondegeneracy of the distribution of the random vector Z for $x \neq \pm y$ follows directly from appendix C. The matrix Σ is then invertible, being the covariance of a nonsingular joint Gaussian distribution.

It remains, therefore, to prove part (2) of the lemma. It is clear that the block A is the same as the covariance matrix in part (1), i.e. given by (37).

Now by the definition,

$$B = \begin{pmatrix} \mathbb{E}(f(x) \nabla f(x)) & \mathbb{E}(f(x) \nabla f(N)) \\ \mathbb{E}(f(N) \nabla f(x)) & \mathbb{E}(f(N) \nabla f(N)) \end{pmatrix},$$

and we have already seen that

$$\mathbb{E}(f(x) \nabla f(x)) = \vec{0}$$

in section 2.5 as well as

$$\mathbb{E}(f(N) \nabla f(N)) = \vec{0}.$$

Also

$$\mathbb{E}(f(N)\nabla f(x)) = \nabla_x \mathbb{E}(f(x)f(N)) = \nabla_x u(x, y)|_{(x, N)} = D(x),$$

and similarly

$$\mathbb{E}(f(x)\nabla f(N)) = -D(x),$$

which finishes the proof of (38).

Finally, we compute C . By the definition,

$$C = \begin{pmatrix} \mathbb{E}(\nabla f(x)^t \nabla f(x)) & \mathbb{E}(\nabla f(x)^t \nabla f(N)) \\ \mathbb{E}(\nabla f(N)^t \nabla f(x)) & \mathbb{E}(\nabla f(N)^t \nabla f(N)) \end{pmatrix}.$$

We have already computed that $\mathbb{E}(\nabla f(x)^t \nabla f(x))$ and $\mathbb{E}(\nabla f(N)^t \nabla f(N))$ are given by (29). Finally,

$$\mathbb{E}(\nabla f(x)^t \nabla f(N)) = \nabla_x \nabla_y \mathbb{E}[f(x)f(N)] = \nabla_x \nabla_y u(x, y)|_{(x, N)} = H,$$

and similarly

$$\mathbb{E}(\nabla f(N)^t \nabla f(x)) = H^t.$$

This implies (39) and finishes the proof of the lemma. \square

3.2. Leray nodal measure.

Proposition 3.2. *The second moment of the Leray nodal measure is given by*

$$(42) \quad \mathbb{E}\mathcal{L}(f)^2 = \frac{|\mathcal{S}^m|}{2\pi} \int_{\mathcal{S}^m} \frac{dx}{\sqrt{1-u(x)^2}},$$

where $u(x)$ is the two-point function given by (19) and (17).

As in the case of expectation, we give a formal derivation of proposition 3.2, omitting a rigorous treatment. A rigorous proof is obtained following the lines of the proof of theorem 5.1 in [10] (see section 5.3), using lemma 2.4 in our case. The convergence of the integral on the RHS of (42), necessary to the proof, follows from (50) and lemma 4.1.

Formal derivation of proposition 3.2. We write the Leray measure as (7) again, so that

$$(43) \quad \begin{aligned} \mathbb{E}\mathcal{L}(f)^2 &= \mathbb{E} \left[\int_{\mathcal{S}^m} \int_{\mathcal{S}^m} \delta(f(x))\delta(f(y)) dx dy \right] = \int_{\mathcal{S}^m \times \mathcal{S}^m} \mathbb{E}[\delta(f(x))\delta(f(y))] dx dy \\ &= |\mathcal{S}^m| \int_{\mathcal{S}^m} \mathbb{E}[\delta(f(x))\delta(f(N))] dx, \end{aligned}$$

by the rotational symmetry of the sphere. Now, for a fixed $x \in \mathcal{S}^m$ with $x \neq \pm N$, the random variables $v_1 := f(x)$ and $v_2 := f(N)$ are multivariate mean zero Gaussian with covariance matrix A given by (37).

Thus, writing the Gaussian measure explicitly, we obtain

$$\begin{aligned} \mathbb{E}(\delta(f(x))\delta(f(y))) &= E[\delta(v_1)\delta(v_2)] \\ &= \int_{\mathbb{R}^2} \delta(a_1)\delta(a_2) \exp(-\frac{1}{2}aA^{-1}a^t) \frac{da}{2\pi\sqrt{\det A}} \\ &= \frac{1}{2\pi\sqrt{\det A}} = \frac{1}{2\pi\sqrt{1-u(x,y)^2}}. \end{aligned}$$

Plugging this into (43) yields (42). □

3.3. Riemannian volume.

Proposition 3.3. *The second moment of $\mathcal{Z}(f)$ is given by*

$$(44) \quad \mathbb{E}(\mathcal{Z}^2) = |\mathcal{S}^m| \int_{\mathcal{S}^m} K(x) dx$$

where

$$(45) \quad K(x) = \frac{1}{\sqrt{1-u^2}} \int_{\mathbb{R}^m \times \mathbb{R}^m} \|w_1\| \|w_2\| \frac{\exp(-\frac{1}{2}(w_1, w_2)\Omega^{-1}(w_1, w_2)^t)}{\sqrt{\det \Omega}} \frac{dw_1 dw_2}{(2\pi)^{m+1}},$$

where $\Omega = \Omega(x)$ is defined by (40).

Denote

$$K_{\epsilon_1, \epsilon_2}(x, y) := \frac{1}{4\epsilon_1\epsilon_2} \int_{\mathcal{E}_n} \|\nabla f(x)\| \|\nabla f(y)\| \chi\left(\frac{f(x)}{\epsilon_1}\right) \chi\left(\frac{f(y)}{\epsilon_2}\right) dv(f).$$

To prove the proposition we will need the following lemma (cf. lemma 5.3 in [11]).

Lemma 3.4. *For $(x, y) \in \mathcal{S}^m \times \mathcal{S}^m$ with $x \neq y$, one has the inequality*

$$(46) \quad K_{\epsilon_1, \epsilon_2}(x, y) \ll_m \frac{E_n}{\sqrt{1-u(x, y)^2}},$$

where the implied constant depends only on the dimension m .

The proof is almost identical to the proof of lemma 5.3 of [11]².

Proof. Write $f(x) = \langle f, U(x) \rangle$, where $U(x)$ is the unit vector

$$U(x) = \sqrt{\frac{|\mathcal{S}^m|}{\mathcal{N}}} (\eta_i(x))_i \in S^{\mathcal{N}-1},$$

where $\{\eta_i(x)\}_{i=1}^{\mathcal{N}}$ is the L_2 orthonormal basis of \mathcal{E}_n chosen, and where we identify the function f with a vector in $\mathbb{R}^{\mathcal{N}}$ via (3). Note that

$$\langle U(x), U(y) \rangle = u(x, y)$$

is the cosine of the angle between $U(x)$ and $U(y)$.

We have

$$\nabla f(x) = DU \cdot f$$

²Note that in [11], there is a misprint in the course of the proof of lemma 5.3.

where the derivative DU is a $m \times \mathcal{N}$ matrix. Equivalently,

$$(\nabla f(x))_i = \left\langle f, \left(\frac{\partial}{\partial e_i} U(x) \right) \right\rangle, \quad 1 \leq i \leq m.$$

By the triangle and Cauchy-Schwartz inequalities,

$$\|\nabla f(x)\| \leq \sum_{i=1}^m \|f\| \cdot \left\| \left(\frac{\partial}{\partial e_i} U(x) \right) \right\| \ll \sqrt{E_n} \|f\|,$$

due to

$$\left\| \left(\frac{\partial}{\partial e_i} U(x) \right) \right\|^2 = \mathbb{E} \left[\left(\frac{\partial f}{\partial e_i} \right)^2 \right] = \frac{E_n}{m},$$

by (29).

Therefore

$$(47) \quad K_{\epsilon_1, \epsilon_2}(x, y) \ll \frac{E_n}{4\epsilon_1\epsilon_2} \int_{\substack{|f(x)| < \epsilon_1 \\ |f(y)| < \epsilon_2}} \|f\|^2 e^{-\|f\|^2/2} df.$$

Consider the plane $\pi \subset \mathbb{R}^{\mathcal{N}}$ spanned by $U(x)$ and $U(y)$. The domain of the integration is all the vectors $f \in \mathbb{R}^{\mathcal{N}}$ so that the projection of f on π falls into the parallelogram P defined by the perpendiculars l_x^\pm and l_y^\pm to the endpoints of $\pm U(x)$ and $\pm U(y)$. Denote the angle α between the sides of P , computed as

$$\cos \alpha = \langle U(x), U(y) \rangle = u(x, y).$$

We claim that the area of P is

$$\text{area}(P) = 4\epsilon_1\epsilon_2 \frac{1}{\sqrt{1 - u(x, y)^2}}.$$

To see that, we assume, with no loss of generality that $\epsilon_2 \cos \alpha \leq \epsilon_1$ (otherwise exchange between x and y) and $\alpha \in (0, \frac{\pi}{2})$. Now if furthermore,

$$\epsilon_2 \leq \epsilon_1 \cos \alpha,$$

then the line l_y^+ does not intersect the interval $[0, \epsilon_1 U(y)]$, and the sides of P are easily seen to have lengths $\frac{2\epsilon_1}{\sin \alpha}$ and $\frac{2\epsilon_2}{\sin \alpha}$, and the angle between the sides of P is α , so that our claim follows. Otherwise (namely if $\epsilon_2 > \epsilon_1 \cos \alpha$), a little trigonometric computation shows that the lengths of the sides of P are again $\frac{2\epsilon_1}{\sin \alpha}$ and $\frac{2\epsilon_2}{\sin \alpha}$ and the angle between the sides of P is α .

Write the multiple integral in (47) as the iterated integral

$$(48) \quad \int_P \left(\int_{p+\pi^\perp} \|f\|^2 e^{-\|f\|^2/2} df \right) dp,$$

where the variable p runs over all the points of the parallelepiped P . The inner integral in (48) is $O(1)$.

Indeed, note that for every $f_1 \in \pi^\perp$,

$$\begin{aligned} \|p + f_1\|^2 e^{-\|p+f_1\|^2/2} &= (\|p\|^2 + \|f_1\|^2) e^{-(\|p\|^2 + \|f_1\|^2)/2} \\ &\ll (1 + \|f_1\|^2) \cdot e^{-\|f_1\|^2/2}, \end{aligned}$$

since $\|p\|^2 e^{-\|p\|^2/2}$ is bounded. Our claim follows from convergence of the integral $\int_{\mathbb{R}^{\mathcal{N}-2}} (1 + \|w\|^2) e^{-\|w\|^2/2} dw$. Therefore

$$\int_{\substack{|f(x)| < \epsilon_1 \\ |f(y)| < \epsilon_2}} \|f\|^2 e^{-\|f\|^2/2} df \ll \text{area}(P) \ll \epsilon_1 \epsilon_2 \frac{1}{\sqrt{1 - u(x, y)^2}}.$$

Substituting the last estimate into (47) proves (46). \square

We give a formal derivation of proposition 3.3. Having lemma 3.4 in our hands, a rigorous proof of proposition 3.3 is identical to the proof of proposition 5.2 of [11] and we omit it here. In the course of the proof one shows that

$$K(x) = \lim_{\epsilon_1, \epsilon_2 \rightarrow 0} K_{\epsilon_1, \epsilon_2}(x, N).$$

Therefore, taking the limit $\epsilon_1, \epsilon_2 \rightarrow 0$ in (46), we obtain

Corollary 3.5. *If $u(x)^2 \neq 1$ then*

$$K(x) \ll \frac{E_n}{\sqrt{1 - u(x)^2}}.$$

Formal derivation of proposition 3.3. Corollary 2.10 allows us to write an expression for the second moment formally as

$$\mathbb{E} \mathcal{Z}(f)^2 = \mathbb{E} \left[\int_{\mathcal{S}^m \times \mathcal{S}^m} \delta(f(x)) \|\nabla f(x)\| \delta(f(y)) \|\nabla f(y)\| dx dy \right],$$

and changing the order of taking the integration, we obtain

$$\begin{aligned} \mathbb{E} \mathcal{Z}(f)^2 &= \int_{\mathcal{S}^m \times \mathcal{S}^m} \mathbb{E} \left[\delta(f(x)) \cdot \|\nabla f(x)\| \cdot \delta(f(y)) \cdot \|\nabla f(y)\| \right] dx dy \\ (49) \quad &= |\mathcal{S}^m| \int_{\mathcal{S}^m} \mathbb{E} \left[\delta(f(N)) \cdot \|\nabla f(N)\| \cdot \delta(f(x)) \cdot \|\nabla f(x)\| \right] dx, \end{aligned}$$

by the rotational symmetry of the sphere. In fact, the integrand

$$\mathbb{E} \left[\delta(f(x)) \cdot \|\nabla f(x)\| \cdot \delta(f(y)) \cdot \|\nabla f(y)\| \right]$$

depends on $d(x, y)$ only (this is the isotropic property of the random ensemble \mathcal{E}_n).

Now for a fixed $x \in \mathcal{S}^m$ with $x \neq \pm N$, the joint distribution of the random vector Z defined as in (34) is Gaussian with mean zero and covariance $\Sigma = \Sigma(x)$ as in lemma 3.1. Thus we may write

$$\begin{aligned} &\mathbb{E} \left[\delta(f(x)) \cdot \|\nabla f(x)\| \cdot \delta(f(N)) \cdot \|\nabla f(N)\| \right] \\ &= \int_{\mathbb{R}^2 \times \mathbb{R}^{2m}} \delta(v_1) \cdot \|w_1\| \cdot \delta(v_2) \cdot \|w_2\| \exp\left(-\frac{1}{2}(v, w) \Sigma^{-1}(v, w)^t\right) \frac{dv dw}{(2\pi)^{m+1} \sqrt{1 - u^2} \sqrt{\det \Omega}}, \end{aligned}$$

substituting the explicit expression for the Gaussian measure and using (41) (recall that $\Omega = \Omega(x)$ is defined by (40)).

Substituting $v_1 = v_2 = 0$, we have

$$\begin{aligned} & \mathbb{E} \left[\delta(f(x)) \cdot \|\nabla f(x)\| \cdot \delta(f(N)) \cdot \|\nabla f(N)\| \right] \\ &= \int_{\mathbb{R}^{2m}} \|w_1\| \|w_2\| \exp\left(-\frac{1}{2} w \Sigma^{-1} w^t\right) \frac{dw}{(2\pi)^{m+1} \sqrt{1-u^2} \sqrt{\det \Omega}}. \end{aligned}$$

To obtain the statement of the proposition, we integrate the last expression over \mathcal{S}^m and plug it into (49). \square

4. ASYMOTOTICS OF THE VARIANCE

In this section we prove theorems 1.5 and 1.7.

4.1. Leray nodal measure. Here we use the ultraspherical or Gegenbauer polynomials (see appendix A for details).

Concluding the proof of theorem 1.5. Using proposition 3.2, (19) and proposition 1.3, we obtain

$$(50) \quad \text{Var}(\mathcal{Z}) = \frac{|\mathcal{S}^m|}{2\pi} \int_{\mathcal{S}^m} \frac{dx}{\sqrt{1-u(x)^2}} - \frac{|\mathcal{S}^m|^2}{2\pi} = \frac{|\mathcal{S}^m|}{2\pi} \int_{-1}^1 \left(\frac{1}{\sqrt{1-Q_n^m(t)^2}} - 1 \right) d\mu(t),$$

where $\mu = \mu_m$ is the measure on $I := [-1, 1]$ defined by

$$(51) \quad d\mu(t) = \frac{2\pi^{m/2}}{\Gamma(\frac{m}{2})} \cdot (1-t^2)^{\frac{m-2}{2}} dt.$$

It is easy to check that $\mu = g_*\nu$, where $g : \mathcal{S}^m \rightarrow I$ is the function

$$(52) \quad g(x) := \cos d(x, N),$$

and d is the spherical distance (recall that ν is the uniform measure on \mathcal{S}^m).

Lemma 4.1 together with (50) conclude the proof of the theorem once noting (2). \square

Lemma 4.1. *One has the following asymptotics*

$$\int_{-1}^1 \left[\frac{1}{\sqrt{1-Q_n^m(t)^2}} - 1 \right] d\mu(t) = 2^{m-2} \pi^{m/2} \Gamma\left(\frac{m}{2}\right) \frac{1}{n^{m-1}} + O(\epsilon(m; n)),$$

where $\epsilon(m; n)$ is given by

$$(53) \quad \epsilon(m; n) := \begin{cases} \frac{\log n}{n^2}, & m = 2 \\ n^{-m}, & m \geq 3 \end{cases},$$

and μ is the measure defined by (51).

To prove lemma 4.1, we will divide the domain of the integral (i.e. the interval $I := [-1, 1]$) into two subintervals: $B := [-1 + \frac{c_0}{n^2}, 1 - \frac{c_0}{n^2}]$ with c_0 constant, and $B^c := I \setminus B$. We will show that the main contribution to the integral in (50) comes from B , *bounding* the contribution of B^c to that integral.

We will reuse this partition while proving theorem 1.7 (see section 4.4). This justifies devoting a separate section (namely, section 4.2) to the treatment of B^c . In analogy to the situation of [10] (cf. section 6.1) and [11] (cf. section 6.2), we will call B and B^c the *nonsingular* and the *singular* intervals respectively. The proof of lemma 4.1 will be finally given in section 4.3.

The singular and nonsingular intervals, as well as some of their properties will be given in section 4.2. The proof of lemma 4.1 will be finally given in section 4.3.

4.2. The singular interval. In the course of the proofs of theorems 1.5 and 1.7, we are going to deal with the function

$$h(t) = \frac{1}{\sqrt{1 - (Q_n^m(t))^2}}$$

defined on $[-1, 1]$. We wish to expand it into the Taylor polynomial of $f(s) = \frac{1}{\sqrt{1-s^2}}$ around $s = 0$ as

$$(54) \quad \frac{1}{\sqrt{1 - (Q_n^m(t))^2}} = 1 + \frac{(Q_n^m(t))^2}{2} + O((Q_n^m(t))^4).$$

To be able to justify the expansion above, we will have to bound $Q_n^m(t)$ away from ± 1 , as in corollary A.3. This corollary provides us with a subinterval $B \subseteq [-1, 1]$ (which will be referred as the *nonsingular* interval) of large measure μ , such that $Q_n^m(t)$ is bounded away from ± 1 for all $t \in B$. Giving a special treatment to its complement (referred as the *singular* interval, even though it is in fact a union of two disjoint intervals), we will show that its contribution is negligible (see sections 4.3.1 and 4.4.2). We give a rigorous treatment below.

Let I be the interval $I = [-1, 1]$. Choose any $0 < \epsilon_0 < 1$ and the constant $c_0 > 0$ guaranteed by corollary A.3, corresponding to ϵ_0 , assuming that n is large enough in the sense of corollary A.3. We fix ϵ_0 and c_0 throughout the rest of the paper and define the nonsingular interval

$$B = B_n := \left[-1 + \frac{c_0}{n^2}, 1 - \frac{c_0}{n^2} \right].$$

Corollary A.3 implies that the expansion (54) holds on B with the constant involved in the ' O '-notation dependent only on ϵ_0 .

By an explicit computation, it is clear that

$$(55) \quad \mu(B^c) \ll n^{-m},$$

where μ is the measure on I defined by (51).

Recall that μ is the measure on $[-1, 1]$ induced from the uniform measure ν on \mathcal{S}^m by $g : \mathcal{S}^m \rightarrow [-1, 1]$ defined by (52). We also define the *spherical* nonsingular set

$$SB := g^{-1}(B),$$

and the *spherical singular set*

$$SB^c := \mathcal{S}^m \setminus SB.$$

Since, as it was mentioned earlier, $\mu = g_*\nu$, it is evident that

$$(56) \quad \nu(SB^c) = \mu(B^c) = O(n^{-m}).$$

The set SB is analogous to the *nonsingular* set in the sense of [10] (cf. section 6.1) and [11] (cf. section 6.2). The structure of SB on the sphere (i.e., its projection B into $[-1, 1]$ by g) is by far simpler than that of the singular set on the torus, due to the lack of problems of arithmetic nature.

4.3. Proof of lemma 4.1.

Proof. We write

$$\int_{-1}^1 \frac{d\mu(t)}{\sqrt{1 - (Q_n^m(t))^2}} dt = \int_B + \int_{B^c}.$$

This, together with lemmas 4.2 and 4.3 imply the result. \square

4.3.1. The contribution of the singular interval B^c .

Lemma 4.2. *One has*

$$(57) \quad \int_{B^c} \frac{d\mu(t)}{\sqrt{1 - (Q_n^m(t))^2}} \ll n^{-m}.$$

Proof. We will bound the contribution of the integral on

$$B^c \cap [0, 1] = [1 - \frac{c_0}{n^2}, 1],$$

the rest being similar. Furthermore, we may assume by symmetry, that $Q_n^m(t) \geq 0$ so that

$$\frac{1}{\sqrt{1 - Q_n^m(t)^2}} \ll \frac{1}{\sqrt{1 - Q_n^m(t)}}.$$

In what follows we will, consistently with appendix A, adapt the notation

$$\alpha := \frac{m-2}{2}.$$

Writing $t = \cos \psi$, we have $\phi \in [0, \frac{c_1}{n}]$ for some constant $c_1 > 0$. Substituting into Hilb's generalized asymptotic formula (see lemma A.2), we have

$$Q_n^m(\cos \psi) = C \cdot \sqrt{\frac{\psi}{\sin \psi} \frac{J_\alpha(n\psi)}{(\sin \psi)^\alpha}} + O(\psi^2),$$

for some constant $C = C_n^m$, using the normalization defined by (73). Taking the limit $\phi \rightarrow 0$, the value of the constant C is easily seen to be

$$(58) \quad C = \left[\lim_{\phi \rightarrow 0} \frac{J_\alpha(n\phi)}{\phi^\alpha} \right]^{-1} = n^{-\alpha} \tilde{C},$$

where

$$(59) \quad \tilde{C} = \tilde{C}^m := \left[\lim_{\phi \rightarrow 0} \frac{J_\alpha(\phi)}{\phi^\alpha} \right]^{-1} \neq 0,$$

since $Q_n^m(1) \neq 0$ (one can obtain an explicit expression for this constant using the expansion of the Bessel function into power series, see e.g. [9], page 57).

Thus, the contribution of the singular interval to the integral, is, for n large enough

$$\begin{aligned}
(60) \quad & \int_{1-\frac{c_0}{n^2}}^1 \ll \int_0^{c_1/n} \frac{(\sin \phi)^{m-1}}{\sqrt{1-Q_n^m(\cos \phi)}} d\phi \ll \int_0^{c_1/n} \frac{\phi^{m-1}}{\sqrt{1-C \cdot \sqrt{\frac{\phi}{\sin \phi} \frac{J_\alpha(n\phi)}{(\sin \phi)^\alpha} + O(\phi^2)}}} d\phi \\
& = n^{-m} \int_0^{c_1} \frac{\psi^{m-1}}{\sqrt{1-C \cdot (1+O(\frac{\psi}{n})^2) \frac{J_\alpha(\psi)}{(\frac{\psi}{n})^\alpha} + O((\frac{\psi}{n})^2)}} d\psi \\
& = n^{-m} \int_0^{c_1} \frac{\psi^{m-1}}{\sqrt{1-\tilde{C} \frac{J_\alpha(\psi)}{\psi^\alpha} + O((\frac{\psi}{n})^2)}} d\psi,
\end{aligned}$$

by (58).

We claim that

$$(61) \quad 1 - \tilde{C} \frac{J_\alpha(\psi)}{\psi^\alpha} \gg_{c_1} \psi^2.$$

Having (61) proved would imply that

$$\int_{1-\frac{c_0}{n^2}}^1 \ll n^{-m} \int_0^{c_1} \psi^{m-2} d\psi \ll n^{-m},$$

which is the statement of the lemma.

To see (61), it is sufficient to show that

$$\lim_{\psi \rightarrow 0} \frac{1 - \tilde{C} \frac{J_\alpha(\psi)}{\psi^\alpha}}{\psi^2} > 0$$

and

$$(62) \quad \left| \tilde{C} \frac{J_\alpha(\psi)}{\psi^\alpha} \right| < 1$$

for every $\psi \in (0, c_1]$. However the former inequality follows from the Bessel function expansion into power series around $\psi = 0$ (see [9], page 57, (9.09))

$$1 - \tilde{C} \frac{J_\alpha(\psi)}{\psi^\alpha} = a_0 \psi^2 + O(\psi^4),$$

for some constant $a_0 > 0$, so that the limit is positive.

To see (62), we note that in the course of establishing (60), we showed

$$Q_n^m(\cos \frac{\psi}{n}) = \tilde{C} \frac{J_\alpha(\psi)}{\psi^\alpha} + O\left(\left(\frac{\psi}{n}\right)^2\right).$$

Therefore, if (62) is not satisfied, taking n large enough would contradict $|Q_n^m(t)| \leq 1$.

□

4.3.2. *The contribution of the nonsingular interval B .*

Lemma 4.3.

$$(63) \quad \int_B \left[\frac{1}{\sqrt{1 - (Q_n^m(t))^2}} - 1 \right] d\mu(t) = 2^{m-2} \pi^{m/2} \Gamma\left(\frac{m}{2}\right) \cdot \frac{1}{n^{m-1}} + O(\epsilon(m; n)),$$

where $\epsilon(m; n)$ is given by (53).

Proof. On B we may write

$$\frac{1}{\sqrt{1 - (Q_n^m(t))^2}} = 1 + \frac{(Q_n^m(t))^2}{2} + O(Q_n^m(t)^4)$$

(see section 4.2). Integrating, we obtain

$$\begin{aligned} \int_B \left[\frac{1}{\sqrt{1 - (Q_n^m(t))^2}} - 1 \right] d\mu(t) &= \frac{1}{2} \int_B (Q_n^m(t))^2 d\mu(t) + O\left(\int_B (Q_n^m(t))^4 d\mu(t) \right) \\ &= O(\mu(B^c)) + \left(\frac{1}{2} \int_{-1}^1 (Q_n^m(t))^2 d\mu(t) + O(\mu(B^c)) \right) + O\left(\int_{-1}^1 (Q_n^m(t))^4 d\mu(t) \right) \\ &= \frac{1}{2} (2^{m-1} \pi^{m/2} \Gamma\left(\frac{m}{2}\right) \frac{1}{n^{m-1}} + O(n^{-m})) + O(\epsilon(m; n)) \\ &= 2^{m-2} \pi^{m/2} \Gamma\left(\frac{m}{2}\right) \cdot \frac{1}{n^{m-1}} + O(\epsilon(m; n)), \end{aligned}$$

as stated, by (55) and lemmas A.1 and A.4. \square

4.4. **Riemannian volume.** The goal of this section is to prove theorem 1.7.

4.4.1. *Plan of the proof of theorem 1.7.* We have by proposition 3.3,

$$(64) \quad \text{Var}(\mathcal{Z}(f)) = |\mathcal{S}^m| \int_{\mathcal{S}^m} K(x) dx - c_m E_n,$$

where

$$K(x) = \frac{1}{\sqrt{1 - u^2}} \int_{\mathbb{R}^m \times \mathbb{R}^m} \|w_1\| \|w_2\| \frac{\exp(-\frac{1}{2}(w_1, w_2)\Omega^{-1}(w_1, w_2)^t)}{\sqrt{\det \Omega}} \frac{dw_1 dw_2}{(2\pi)^{m+1}},$$

and c_m is a constant given by (10).

As in case of the Leray nodal measure, we divide the integration range into the nonsingular set SB and its complement SB^c (see section 4.2). We bound the corresponding contributions to the integral separately (see lemmas 4.4 and 4.5). Using corollary 3.5, it is easy to relate the contribution of SB^c to the last integral in (50), which we already bounded while treating the variance of the Leray nodal measure (lemma 4.2).

It then remains to bound the contribution of the integral on SB . Here we may write $\frac{1}{\sqrt{1 - u^2}} = 1 + O(u^2)$ and one may show that, up to an admissible error, we may replace it by 1. We will define a new matrix S by

$$\Omega = \frac{E_n}{m} (I - S),$$

and notice that substituting $S = 0$ into the integral, the identity matrix I recovers the square of the expected volume $(\mathbb{E}\mathcal{Z})^2$. Bounding the variance is then equivalent to “bounding” the matrix S in some *average* sense.

To quantify the last statement we set $\sigma(x)$ to be the spectral norm of the matrix $S(x)$. We will show that the variance is bounded by

$$E_n \cdot \left(\int_{S^m} \sigma(x) dx + O\left(\frac{1}{N}\right) \right).$$

To bound $\int \sigma(x)$, we use the trivial inequality $\sigma(x) \leq \sqrt{\text{tr } S^2}$. We will prove that $\int \text{tr}(S(x)^2) \ll \frac{1}{N}$, and together with the Cauchy-Schwartz inequality this implies the statement of the theorem.

4.4.2. *A bound for the contribution on the singular interval SB^c .*

Lemma 4.4. *One has*

$$\int_{SB^c} K(x) dx \ll E_n \epsilon(m; n),$$

where $\epsilon(m; n)$ is defined by (53).

Proof. We use corollary 3.5 to write

$$\int_{SB^c} K(x) dx \ll_m E_n \int_{SB^c} \frac{dx}{\sqrt{1-u(x)^2}} = E_n \int_{B^c} \frac{d\mu(t)}{\sqrt{1-Q_n^m(t)}} \ll E_n \epsilon(m; n),$$

obtaining the last inequality by lemma 4.2. \square

4.4.3. *A bound for the contribution on the nonsingular interval SB .*

Lemma 4.5.

$$\int_{SB} K(x) dx = \frac{1}{|S^m|} (\mathbb{E}(\mathcal{Z}))^2 + O\left(\frac{E_n}{\sqrt{N}}\right).$$

Proof. Define $\Omega_1 = \Omega_1(x)$ by $\Omega = \frac{E_n}{m} \cdot \Omega_1$. The matrix Ω_1 is symmetric, and positive for a set of $x \neq \pm N$, since Ω is such. Therefore it has a positive definite square root $P_1^2 = \Omega_1$.

Intuitively, Ω_1 approximates the identity matrix I . To quantify this intuitive statement, we introduce the matrix

$$(65) \quad S = I - \Omega_1 = \frac{m}{E_n} \frac{1}{1-u^2} \begin{pmatrix} D^t D & -uD^t D - (1-u^2)H \\ -uD^t D - (1-u^2)H^t & D^t D \end{pmatrix},$$

and its spectral norm $\sigma = \sigma(x)$, i.e

$$\sigma = \max_{1 \leq i \leq 2m} |\alpha_i|,$$

where α_i are the eigenvalues of S . Note that, since Ω_1 is positive definite, $S \ll I$ in the sense that all its eigenvalues are in $(-\infty, 1)$.

Changing the coordinates

$$w = \sqrt{\frac{E_n}{m}} z P_1,$$

we write the definition (45) of $K(x)$ as

$$(66) \quad K(x) = \frac{E_n}{m\sqrt{1-u^2}} \int_{\mathbb{R}^{2m}} \|(zP_1)_1\| \cdot \|(zP_1)_2\| e^{-\frac{1}{2}\|z\|^2} \frac{dz}{(2\pi)^{m+1}},$$

where for $a \in \mathbb{R}^{2m}$ we write $(a)_1 \in \mathbb{R}^m$ and $(a)_2 \in \mathbb{R}^m$ to denote either the first or the last m coordinates.

We claim that

$$(67) \quad P_1 = I(1 + O(\sigma)).$$

This follows from that fact that if $S \sim \text{diag}(\alpha_i)$ then $P_1 \sim \text{diag}(\sqrt{1-\alpha_i})$ so that

$$P_1 - I \sim \text{diag}(\sqrt{1-\alpha_i} - 1) \ll \text{diag}(|\alpha_i|) \ll \sigma I,$$

since $\sqrt{1-y} - 1 < |y|$ on $(-\infty, 1)$.

Moreover, by the definition of the spherical nonsingular set, on SB , $u(x)$ is bounded away from 1, so that one may expand

$$(68) \quad \frac{1}{\sqrt{1-u^2}} = 1 + O(u^2),$$

where the constant involved in the ' O ' notation is absolute.

Substituting (67) and (68) into (66), we obtain

$$K(x) = \frac{E_n}{m(2\pi)^{m+1}} \int_{\mathbb{R}^{2m}} \|(z)_1\| \cdot \|(z)_2\| e^{-\frac{1}{2}\|z\|^2} (1 + O(u^2))(1 + O(\sigma))^2 dz.$$

Continuing, we have

$$\begin{aligned} K(x) &= \frac{E_n}{m(2\pi)^{m+1}} \int_{\mathbb{R}^m \times \mathbb{R}^m} \|(z)_1\| \cdot \|(z)_2\| e^{-\frac{1}{2}(\|z_1\|^2 + \|z_2\|^2)} dz_1 dz_2 (1 + O(u^2) + O(\sigma) + O(\sigma^2)) \\ &= \left(\frac{\sqrt{E_n}}{\sqrt{m}(2\pi)^{\frac{m+1}{2}}} \int_{\mathbb{R}^m} \|z'\| e^{-\frac{1}{2}\|z'\|^2} dz' \right)^2 (1 + O(u^2) + O(\sigma) + O(\sigma^2)) \\ &= \frac{1}{|\mathcal{S}^m|^2} (\mathbb{E}\mathcal{Z})^2 (1 + O(u^2) + O(\sigma) + O(\sigma^2)), \end{aligned}$$

by (9) and (33).

Integrating on SB , we obtain

$$\int_{SB} K(x) dx - \frac{1}{|\mathcal{S}^m|} (\mathbb{E}\mathcal{Z})^2 \ll E_n \left(\frac{1}{\sqrt{N}} + \int_{SB} u^2 dx + O(\nu(SB^c)) \right),$$

by (9) and lemma 4.6.

To bound the last expression, we use (56), as well as, by the definition (19) of the two-point function, we have

$$\int_{SB} u^2 dx \leq \int_{S^m} (Q_n^m(\cos d(x, N)))^2 dx = \int_{-1}^1 (Q_n^m(t))^2 d\mu(t) \ll \frac{1}{N},$$

by lemma A.1 and (2). This concludes the proof of the lemma. \square

Lemma 4.6. *For a fixed m , as $n \rightarrow \infty$, one has*

$$(1) \quad \int_{SB} \sigma(x)^2 dx \ll \frac{1}{\mathcal{N}}.$$

$$(2) \quad \int_{SB} \sigma(x) dx \ll \frac{1}{\sqrt{\mathcal{N}}}.$$

Proof. Part 2 of the lemma clearly follows from part 1 by the Cauchy-Schwartz inequality. Thus we are only to prove part 1.

To prove the statement, we recall that σ is by the definition the spectral norm of S , defined by (65). To bound $\int \sigma^2$, we use the trivial inequality $\sigma \leq \text{tr}(S^2)$.

Since, by the definition of the nonsingular set SB , the two-point function $u(x)$ is bounded away from 1, we may disregard the $1 - u^2$ altogether. We define the matrix

$$S_1 := \frac{E_n}{m}(1 - u^2)S = \begin{pmatrix} D^t D & -uD^t D - (1 - u^2)H \\ -uD^t D - (1 - u^2)H^t & D^t D \end{pmatrix}.$$

We claim that

$$\int_{S^m} \text{tr} S_1^2 dx = O(n^{5-m}).$$

This is sufficient for the statement of the present lemma, since then

$$\int_{SB} \sigma^2 dx \ll \frac{1}{E_n^2} \int_{SB} \text{tr} S_1^2 dx \leq \frac{1}{E_n^2} \int_{S^m} \text{tr} S_1^2 dx \ll \frac{1}{n^4} \cdot n^{5-m} \ll \frac{1}{\mathcal{N}}.$$

Now the elements of the matrix S^2 are bounded by elements either of the form

$$\frac{\partial u}{\partial e_{i_1}^z} \Big|_{(x,N)} \cdot \frac{\partial u}{\partial e_{i_2}^z} \Big|_{(x,N)} \cdot \frac{\partial u}{\partial e_{i_3}^z} \Big|_{(x,N)} \cdot \frac{\partial u}{\partial e_{i_4}^z} \Big|_{(x,N)},$$

the form

$$\frac{\partial u}{\partial e_{i_1}^z} \Big|_{(x,N)} \cdot \frac{\partial u}{\partial e_{i_2}^z} \Big|_{(x,N)} \cdot \frac{\partial^2 u}{\partial e_{i_3}^z \partial e_{i_4}^z} \Big|_{(x,N)},$$

or the form

$$\frac{\partial^2 u}{\partial e_{i_1}^z \partial e_{i_2}^z} \Big|_{(x,N)} \cdot \frac{\partial^2 u}{\partial e_{i_3}^z \partial e_{i_4}^z} \Big|_{(x,N)},$$

where in all the expressions above z may be either x or y (see section 2.4 for an explanation of the partial derivatives notations).

Using the Cauchy-Schwartz inequality again and the symmetry with respect to the variables, it suffices to prove the inequalities

$$(69) \quad \int_{S^m} \left(\frac{\partial u}{\partial e_1^x}(x) \right)^4 dx \ll n^{5-m}$$

and

$$(70) \quad \int_{S^m} \left(\frac{\partial^2 u}{\partial e_1^x \partial e_2^x}(x) \right)^2 dx \ll n^{5-m}.$$

We may compute the partial derivative in (69) (assuming $x \neq \pm N$) as

$$\frac{\partial}{\partial e_1^x} Q_n^m(\cos d(x, N)) = -Q_n^{m'}(\cos d(x, N)) \sin d(x, N) \frac{\partial}{\partial e_1^x} d(x, N),$$

so that, since $\frac{\partial}{\partial e_1^x} d(x, y)$ is obviously bounded on \mathcal{S}^m , it is sufficient to bound

$$\int_{\mathcal{S}^m} (Q_n^{m'}(\cos d(x, N)) \sin d(x, y))^4 dx = \int_{-1}^1 (Q_n^{m'}(t))^4 (1-t^2)^2 d\mu(t),$$

and thus (69) follows from lemma A.6, recalling the definition (51) of the measure μ .

As for (70), we write the second partial derivative in the integrand as

$$\begin{aligned} \frac{\partial^2 u}{\partial e_1^x \partial e_2^x}(x) &= Q_n^{m''}(\cos d(x, N)) \sin^2 d(x, N) \frac{\partial}{\partial e_1^x} d(x, N) \frac{\partial}{\partial e_2^x} d(x, N) \\ &\quad - Q_n^{m'}(\cos d(x, N)) \cdot \frac{\partial}{\partial e_2^x} \left[\sin d(x, N) \frac{\partial}{\partial e_1^x} d(x, N) \right], \end{aligned}$$

so that, using a similar argumentation, we conclude that the integral in (70) is bounded by

$$\begin{aligned} &\ll \int_{\mathcal{S}^m} (Q_n^{m''}(\cos d(x, N)))^2 (\sin d(x, N))^4 dx + \int_{\mathcal{S}^m} (Q_n^{m'}(\cos d(x, N)))^2 dx \\ &= \int_{-1}^1 (Q_n^{m''}(t))^2 (1-t^2)^2 d\mu(t) + \int_{-1}^1 (Q_n^{m'}(t))^2 d\mu(t). \end{aligned}$$

Therefore, (70) follows from lemmas A.5 and A.7. \square

4.4.4. Concluding the proof of theorem 1.7.

Proof of theorem 1.7. We write (64) as,

$$\text{Var}(\mathcal{Z}) = \int_{SB^c} K(x) dx + \left(\int_{SB} K(x) dx - \mathbb{E}(\mathcal{Z})^2 \right) \ll E_n \epsilon(m; n) + \frac{E_n}{\sqrt{\mathcal{N}}} \ll \frac{E_n}{\sqrt{\mathcal{N}}},$$

by lemmas 4.4 and 4.5, where we use

$$\epsilon(m; n) \ll \frac{E_n}{\sqrt{\mathcal{N}}},$$

due to (53) and (2). \square

APPENDIX A. LEGENDRE AND ULTRASPHERICAL POLYNOMIALS

The ultraspherical (or Gegenbauer) polynomials $P_n^\alpha(t) : [-1, 1] \rightarrow \mathbb{R}$ of degree n generalize the Legendre polynomials $P_n(t) = P_n^0(t)$. We use the corresponding *normalized* polynomials $Q_n^m(t)$ for an integral $m \geq 0$, which differ from P_n^α (for a suitably chosen α), by a constant, defined by

$$Q_n^m(1) = 1.$$

A.1. Definition and basic facts. The Legendre polynomials P_n are the unique polynomials of degree n , orthogonal on $[-1, 1]$ (w.r.t. the trivial weight function), normalized by $P_n(t) = 1$. More generally, for a real

$$\alpha > -1,$$

we define the ultraspherical polynomials $P_n^\alpha(t)$, being, up to a constants, the unique sequence polynomials of degree n , pairwise orthogonal w.r.t. the weight function on $[-1, 1]$ defined by

$$(71) \quad \omega(t) = (1 - t^2)^\alpha.$$

It is defined uniquely by the normalizing condition

$$(72) \quad P_n^\alpha(1) = \frac{\Gamma(n + \alpha + 1)}{\Gamma(n + 1) \cdot \Gamma(\alpha + 1)},$$

once we know that $t = 1$ is not a zero of P_n^α , see [12], chapter 3.3. The ultraspherical polynomials is a particular case $\alpha = \beta$ of a more general class of polynomials, usually referred to as the Jacobi polynomials $P_n^{\alpha, \beta}$ (see e.g. [12] for more information).

While studying the spherical harmonics on the m -dimensional sphere, we are interested in the ultraspherical polynomials with $\alpha = \frac{m-2}{2}$, and moreover, we would like to normalize it by setting its value at 1 to be 1. That is, we define

$$(73) \quad Q_n^m(t) := \frac{P_n^\alpha(t)}{P_n^\alpha(1)},$$

where

$$(74) \quad \alpha := \frac{m-2}{2}.$$

For example,

$$Q_n^2(t) = P_n(t) = P_n^0(t)$$

are the usual Legendre polynomials.

Throughout the section, we fix an integral number $m \geq 2$, and use the associated value of α , defined by (74). It is well known that Q_n^m is either even or odd, for the even and odd values of n respectively, and $|Q_n^m(t)|$ has a maximum at $t = \pm 1$.

The function $v = P_n^\alpha(t)$ satisfies the differential equation ([12], page 60, (4.2.1))

$$(75) \quad (1 - t^2)v'' - mtv' + n(n + m - 1)v = 0.$$

Due to its linear nature, it is also satisfied by $v = Q_n^m(t)$. The following recurrence relation ([12], page 83, (4.7.27)) will prove itself as very useful

$$(76) \quad (1 - t^2)P_n^{\alpha'}(t) + ntP_n^\alpha(t) - (n + \alpha)P_{n-1}^\alpha(t) = 0.$$

Note that this recurrence relation is not satisfied by $Q_n^m(t)$ due to the different normalization constants for P_n and P_{n-1} .

A.2. Some basic results. Recall the definition (51) and (71) of the measure $\mu = \mu_m$ and the weight function ω_m respectively. We note that

$$d\mu(t) = \frac{2\pi^{m/2}}{\Gamma(\frac{m}{2})} \cdot \omega(t)dt,$$

so for purposes of giving an upper bound only, we may disregard the difference between $d\mu$ and ωdt .

Concerning the 2nd and the 4th moments of the ultraspherical polynomials, we have the following:

Lemma A.1. *For m fixed, the second moment of the normalized ultraspherical polynomials is*

$$\int_{-1}^1 Q_n^m(t)^2 d\mu(t) = 2^{m-1} \pi^{m/2} \Gamma(\frac{m}{2}) \cdot \frac{1}{n^{m-1}} + O(n^{-m}),$$

as $n \rightarrow \infty$.

Proof. One has ([12], (4.3.3))

$$(77) \quad \int_{-1}^1 P_n^\alpha(t)^2 \omega(t) dt = \frac{2^{m-1}}{2n+m-1} \frac{\Gamma(n+\frac{m}{2})^2}{\Gamma(n+1)\Gamma(n+m-1)},$$

and (72) implies that

$$(78) \quad P_\alpha^m(1) = \frac{\Gamma(n+\frac{m}{2})}{\Gamma(n+1)\Gamma(\frac{m}{2})} \sim c \cdot n^\alpha.$$

Thus, using the definition (73) of the normalized ultraspherical polynomials, we obtain

$$\begin{aligned} \int_{-1}^1 Q_n^m(t)^2 d\mu(t) &= \frac{2^{m-1}}{2n+m-1} \frac{\Gamma(n+1)\Gamma(\frac{m}{2})^2}{\Gamma(n+m-1)} \cdot \frac{2\pi^{m/2}}{\Gamma(\frac{m}{2})} \\ &= \frac{2^m \pi^{m/2} \Gamma(\frac{m}{2})}{2n+m-1} \frac{n!}{(n+m-2)!} \\ &= 2^{m-1} \pi^{m/2} \Gamma(\frac{m}{2}) \cdot \frac{1}{n} \cdot \frac{1}{n^{m-2}} (1 + O(\frac{1}{n})) \\ &= 2^{m-1} \pi^{m/2} \Gamma(\frac{m}{2}) \frac{1}{n^{m-1}} + O(n^{-m}), \end{aligned}$$

as stated. \square

Lemma A.2 (Hilb Asymptotics (formula (8.21.17) on page 197 of Szego)).

$$(79) \quad \left(\frac{1}{2} \sin \theta\right)^\alpha P_n^\alpha(\cos \theta) = N^{-\alpha} \frac{\Gamma(n+\alpha+1)}{n!} \left(\frac{\theta}{\sin \theta}\right)^{1/2} J_\alpha(N\theta) + \delta(\theta),$$

uniformly for $0 \leq \theta \leq \pi/2$, where $N = n + \frac{m-1}{2}$, J_α is the Bessel J function of order α and the error term is

$$\delta(\theta) \ll \begin{cases} \theta^{1/2} O(n^{-3/2}), & cn^{-1} < \theta < \pi/2 \\ \theta^{\alpha+2} O(n^\alpha), & 0 < \theta < cn^{-1}. \end{cases}$$

Remark: It is clear, that

$$(80) \quad n^{-\alpha} \frac{\Gamma(n + \alpha + 1)}{n!} = 1 + O\left(\frac{1}{n}\right),$$

so we will usually omit this factor.

Corollary A.3. *For every $\epsilon_0 > 0$, there exists a constant $c_0 > 0$ depending on m only, such that if $c \geq c_0$ and $t \in [0, 1 - \frac{c}{n^2}]$, where n is large enough so that the interval above is not empty, one has*

$$|Q_n^m(t)| < \epsilon_0.$$

Proof. Let $t = \cos \theta$. Then if $0 \leq t < 1 - \frac{c_0}{n^2}$, $\theta > C_0 \cdot \frac{\sqrt{c_0}}{n}$ for some absolute constant $C_0 > 0$. Lemma A.2 implies that one has

$$|P_n^\alpha(\cos \theta)| \leq C \frac{1}{\sin^\alpha \theta} |J_\alpha(N\theta)|$$

for some absolute constant $C > 0$. We bound it by

$$|P_n^\alpha(\cos \theta)| \leq C_1 \frac{n^\alpha}{c_0^{\alpha/2}} |J_\alpha(N\theta)|,$$

so that (78) implies that

$$|Q_n^m(\cos \theta)| \leq \frac{C_2}{c_0^{\alpha/2}} |J_\alpha(N\theta)| < \epsilon_0,$$

provided that we choose c_0 large enough, since J_α is bounded. \square

Lemma A.4. *The 4th moment of the ultraspherical polynomials satisfies*

$$\int_{-1}^1 Q_n^m(t)^4 d\mu(t) \ll \epsilon(m; n),$$

where $\epsilon(m; n)$ is defined by (53).

Proof. We will limit ourselves to the interval $[0, 1]$. To prove the statement there, we invoke the generalized Hilb's asymptotics (lemma A.2).

We have, using (80), that

$$(\sin \theta)^{m-2} (P_n^\alpha(\cos \theta))^4 \ll J_\alpha^4(N\theta) \frac{\theta^2}{(\sin \theta)^m} + \frac{\delta^4(\theta)}{\sin^{m-2} \theta}$$

and claim that

$$(81) \quad \int_{-1}^1 P_n^\alpha(t)^4 d\mu(t) \ll \begin{cases} \frac{\log n}{n^2}, & m = 2 \\ n^{m-4}, & m \geq 3 \end{cases}.$$

We have

$$(82) \quad \int_{-1}^1 P_n^\alpha(t)^4 d\mu(t) \ll \int_0^{\pi/2} J_\alpha^4(N\theta) \frac{\theta^2}{(\sin \theta)^{m-1}} d\theta + \int_0^{\pi/2} \frac{\delta^4(\theta)}{\sin^{m-3} \theta} d\theta$$

The contribution of the main term in (82) to the integral in (81) is

$$\begin{aligned} &\ll \int_0^{\pi/2} J_\alpha^4(N\theta) \frac{1}{\theta^{m-3}} d\theta = N^{m-4} \int_0^{\frac{\pi N}{2}} \frac{J_\alpha^4(\phi)}{\phi^{m-3}} d\phi \\ &= N^{m-4} \left[\int_0^1 \frac{J_\alpha^4(\phi)}{\phi^{m-3}} d\phi + \int_1^{\frac{\pi N}{2}} \frac{d\phi}{\phi^{m-1}} \right], \end{aligned}$$

using the well known decay

$$J_\alpha(y) \ll \frac{1}{\sqrt{y}}$$

of the Bessel J functions at infinity.

The first integral involved in the expression above is $O(1)$, since J_α vanishes with multiplicity (at least) α at zero (it follows, for example, from Hilb's formula). The second one is bounded by

$$\ll \begin{cases} \log n, & m = 2 \\ 1, & m \geq 3 \end{cases}.$$

Therefore the contribution of the main term in (82) to the LHS of (81) is dominated by the RHS of (81).

The contribution of the error term in (82) is at most

$$n^{2m-4} \int_0^{1/n} \theta^{m+7} d\theta + n^{-6} \int_{1/n}^{\pi/2} \theta^{5-m} d\theta = O(n^{m-12}).$$

We obtain the statement of the lemma by using (81) and (73) with (78). \square

A.3. Moments of the derivatives of the ultraspherical polynomials.

Lemma A.5.

$$\int_{-1}^1 Q_n^{m'}(t)^2 d\mu(t) \ll \frac{\log n}{n^{m-4}}.$$

Proof. We will bound the integral on $[0, 1]$ only, having a similar bound on $[-1, 0]$. By (78), the statement of the lemma is equivalent to

$$\int_0^1 P_n^{\alpha'}(t)^2 d\mu(t) \ll n^2 \log n.$$

We rewrite the last integral using (76) as

$$\int_0^1 \frac{\left((n + m/2 - 1)P_{n-1}^\alpha(t) - ntP_n^\alpha(t) \right)^2}{(1-t^2)^2} d\mu(t).$$

To give a bound, we partition the range of the integration into 2 subranges:

$$(83) \quad \int_0^1 = \int_0^{1-1/n^2} + \int_{1-1/n^2}^1 .$$

To bound the second integral in (83), we define

$$(84) \quad f(t) := (n + m/2 - 1)P_{n-1}^\alpha(t) - ntP_n^\alpha(t) = (1 - t^2)P_n^{\alpha'}(t).$$

Computing the derivative $f'(t)$ and using (75) again we obtain

$$(85) \quad f'(t) = (m - 2)tP_n^{\alpha'}(t) - n(n + m - 1)P_n^\alpha(t).$$

We claim that this implies

$$(86) \quad f'(t) \ll n^2 P_n^\alpha(1) \ll n^{\frac{m+2}{2}},$$

the second inequality being a consequence of (78). To see the first inequality of (86), we note that it is sufficient to show that

$$(87) \quad tP_n^{\alpha'}(t) \ll n^{\frac{m+2}{2}},$$

by (85). With no loss of generality we may assume that $t = 1$ or $P_n^{\alpha'}$ has a local extremum, i.e. $P_n^{\alpha''}(t) = 0$. In both cases the equation (75) implies

$$tP_n^{\alpha'}(t) = P_n^\alpha(t)O(n^2),$$

which implies (87).

Now using the linear Taylor approximation of $f(t)$ around $t = 1$ with (86), the second integral in (83) is, since $f(1) = 0$,

$$\ll n^{m+2} \int_{1-1/n^2}^1 \frac{(t-1)^2}{(1-t^2)^2} \cdot (1-t^2)^{\frac{m-2}{2}} dt \ll n^{m+2} \int_{1-1/n^2}^1 (1-t^2)^{\frac{m-2}{2}} dt \ll n^2.$$

In order to bound the first integral in (83), we employ the generalized Hilb's asymptotics (79). The integrand is (taking the change of variables $t = \cos \theta$ and (51) into the account),

$$(88) \quad \begin{aligned} & n^2 \frac{\left((1 + O(\frac{1}{n}))P_{n-1}^\alpha(\cos \theta) - \cos \theta P_n^\alpha(\cos \theta) \right)^2}{(\sin \theta)^3} \cdot (\sin \theta)^{2\alpha} \\ & \ll n^2 \cdot \frac{\frac{\sin \theta}{\theta} \cdot \left((1 + O(\frac{1}{n}))J_\alpha((N-1)\theta) - (1 + O(\theta^2) + O(\frac{1}{n}))J_\alpha(N\theta) \right)^2}{(\sin \theta)^3} + n^2 \frac{\delta^2(\theta)}{(\sin \theta)^3} \\ & \ll n^2 \frac{\left(J_\alpha(N\theta) - J_\alpha((N-1)\theta) \right)^2}{\theta^3} + \frac{1}{\theta^3} + O(n^2\theta) + n^2 \frac{\delta^2(\theta)}{\theta^3}, \end{aligned}$$

and the integration range is essentially $[\frac{1}{n}, \frac{\pi}{2}]$.

The contribution of the last error term in (88) is

$$\ll n^2 \cdot n^{-3} \int_{1/n}^{\pi/2} \frac{\theta}{\theta^3} d\theta \ll 1,$$

the other ones being trivially bounded by $O(n^2)$.

The contribution of the main term in (88) is

$$\begin{aligned} n^2 \int_{1/n}^{\pi/2} \frac{(J_\alpha(N\theta) - J_\alpha((N-1)\theta))^2}{\theta^3} d\theta &\ll n^4 \int_1^{\frac{n\pi}{2}} \frac{\left(J_\alpha(\phi) - J_\alpha(\phi(1 - \frac{1}{N}))\right)^2}{\phi^3} d\phi \\ &\ll n^4 \cdot \frac{1}{n^2} \int_1^{\frac{n\pi}{2}} \frac{\phi^2}{\phi^3} d\phi \ll n^2 \log n, \end{aligned}$$

due to the boundness of the derivative $J'_\alpha(t)$. As it was stated, this is equivalent to the statement of the lemma. \square

Lemma A.6. *One has*

$$\int_{-1}^1 Q_n^{m'}(t)^4 (1-t^2)^2 d\mu(t) \ll \begin{cases} n^2 \log n, & m = 2 \\ \frac{1}{n^{m-4}}, & m \geq 3 \end{cases}.$$

Proof. The proof of the lemma is similar to the one of lemma A.5.

We will bound the integral only on $[0, 1]$, having a similar bound on $[-1, 0]$. The statement of the lemma is equivalent to

$$\int_0^1 \frac{\left((n + m/2 - 1)P_{n-1}^\alpha(t) - nxP_n^\alpha(t)\right)^4}{(1-t^2)^2} d\mu(t) \ll \begin{cases} n^2 \log n, & m = 2 \\ n^m, & m \geq 3 \end{cases},$$

using (78) and (76).

We partition the range of the integration into 2 subranges:

$$(89) \quad \int_0^1 = \int_0^{1-\frac{1}{n^2}} + \int_{1-\frac{1}{n^2}}^1.$$

To bound the second integral in (89) we use the definition (84) of the function $f(t)$, as well as the inequality (86), as in the course of proof of lemma A.5. Thus the integral is

$$\ll n^{2(m+2)} \int_{1-1/n^2}^1 \frac{(1-t)^4}{(1-t)^2} (1-t)^{\frac{m-2}{2}} dt \ll n^{2(m+2)} n^{-(m+4)} = n^m.$$

To bound the first integral in (89), we employ the generalized Hilb's asymptotics (79). The integrand is (taking into consideration the change of

variables $t = \cos \theta$,

(90)

$$\begin{aligned} & n^4 \frac{\left((1 + O(\frac{1}{n})) P_{n-1}^\alpha(\cos \theta) - \cos \theta P_n^\alpha(\cos \theta) \right)^4}{\sin \theta^3 \cdot \sin \theta^{2\alpha}} \cdot (\sin \theta)^{4\alpha} \\ & \ll n^4 \cdot \frac{\frac{\sin \theta}{\theta} \cdot \left((1 + O(\frac{1}{n})) J_\alpha((N-1)\theta) - (1 + O(\theta^2) + O(\frac{1}{n})) J_\alpha(N\theta) \right)^4}{(\sin \theta)^{m+1}} + n^4 \frac{\delta^4(\theta)}{(\sin \theta)^{m+1}} \\ & \ll n^4 \frac{\left(J_\alpha(N\theta) - J_\alpha((N-1)\theta) \right)^4}{\theta^{m+1}} + \frac{1}{\theta^{m+1}} + O\left(n^4 \frac{1}{\theta^{m-7}}\right) + n^4 \frac{\delta^4(\theta)}{\theta^{m+1}}, \end{aligned}$$

and the integration range is (up to a constant) $[\frac{1}{n}, \frac{\pi}{2}]$.

The contribution of the last error term in (90) is

$$\ll \frac{n^4}{n^6} \int_{1/n}^{\pi/2} \frac{\theta^2}{\theta^{m+1}} d\theta = n^{-2} \int_{1/n}^{\pi/2} \frac{d\theta}{\theta^{m-1}} \ll \max(n^{m-4} \log n, 1),$$

the other ones being trivially bounded by $O(n^m)$.

The contribution of the main term in (88) is

(91)

$$n^4 \int_{1/n}^{\pi/2} \frac{\left(J_\alpha(N\theta) - J_\alpha((N-1)\theta) \right)^4}{\theta^{m+1}} d\theta \ll n^{m+4} \int_1^{\frac{\pi N}{2}} \frac{\left(J_\alpha(\phi) - J_\alpha(\phi(1 - \frac{1}{N})) \right)^4}{\phi^{m+1}} d\phi.$$

Let $g(\phi)$ be the function

$$g(t) := J_\alpha(\phi) - J_\alpha(\phi(1 - \frac{1}{N})).$$

Then by the mean value theorem,

$$g(\phi) = \frac{\phi}{N} J'_\alpha(s),$$

where $\phi(1 - \frac{1}{N}) < s < \phi$, and using the decay

$$|J'_\alpha(s)| \ll \frac{1}{\sqrt{s}},$$

we obtain

$$(92) \quad |g(\phi)| \ll \frac{\sqrt{\phi}}{N}.$$

Substituting (92) into (91), we have that the contribution is

$$\ll n^{m+4} \cdot \frac{1}{n^4} \int_1^{\frac{\pi n}{2}} \frac{\phi^2}{\phi^{m+1}} d\phi \ll n^m \int_1^{\frac{\pi n}{2}} \frac{d\phi}{\phi^{m-1}} \ll \begin{cases} n^2 \log n, & m = 2 \\ n^m, & m \geq 3 \end{cases},$$

which concludes the proof of the lemma. \square

Lemma A.7.

$$\int_{-1}^1 Q_n^{m''}(t)^2 (1-t^2)^2 d\mu(t) \ll \frac{1}{n^{m-5}}$$

Proof. We use the differential equation (75) to write the integral as

$$\begin{aligned} & \int_{-1}^1 (mtQ_n^{m'}(t) - n(n+m-1)Q_n^m(t))^2 d\mu(t) \\ & \ll \int_{-1}^1 ((Q_n^{m'}(t))^2 d\mu(t) + n^4 \int_{-1}^1 Q_n^m(t)^2 d\mu(t) \ll \frac{1}{n^{m-5}}, \end{aligned}$$

by lemmas A.1 and A.5. □

APPENDIX B. THE SINGULAR FUNCTIONS ARE “RARE”

In this section we give the proofs of lemmas 2.2, 2.3 and 2.4 (see section 2.1).

Notation B.1. Here and in appendix C we adapt the following notations. Let x and y on the sphere \mathcal{S}^m such that $x \neq \pm y$.

- (1) Denote $\bar{x}y$ the (unique) big circle through x and y .
- (2) The smaller arc of $\bar{x}y$ connecting x to y will be denoted by $\check{x}y$.
- (3) Let $z \in \mathcal{S}^m$ be a point not lying on the plane $\Pi = \Pi(x, y)$ defined by O, x and y . We denote $\mathcal{S}^2 = \mathcal{S}^2(x, y, z)$ the (unique) 2-dimensional big sphere containing O, x, y and z , i.e.

$$\mathcal{S}^2 := \mathcal{S}^m \cap \Pi(x, y, z).$$

(Note that there is no ambiguity in notations for $m = 2$).

We also recall the fact that if $\mathcal{S}^2 \subseteq \mathcal{S}^m$ is any big sphere, then for any two points $x, y \in \mathcal{S}^2$,

$$\bar{x}y_{\mathcal{S}^2} = \bar{x}y_{\mathcal{S}^m}.$$

In particular, the shortest path between x and y on \mathcal{S}^m passes inside \mathcal{S}^2 and

$$\nabla_x d_{\mathcal{S}^m}(x, y) = \nabla_x d_{\mathcal{S}^2}(x, y) \in T_x(\mathcal{S}^2)$$

under the natural embedding

$$T_x(\mathcal{S}^2) \subseteq T_x(\mathcal{S}^m).$$

The following simple geometric lemma will prove itself as quite useful.

Lemma B.2. *Let $x, y \in \mathcal{S}^m$ such that $x \neq y$ and $\xi \neq \xi' \in \mathcal{S}^m$ such that $d(x, \xi) = d(x, \xi')$ and $d(y, \xi) = d(y, \xi')$. Denote $v := \nabla_x d(x, y)$ and*

$$v_1 = v_1(\xi, \xi') = \nabla_x d(x, \xi) - \nabla_x d(x, \xi').$$

Then for all ξ and ξ' , $v \perp v_1$, and moreover the vectors v_1 span v^\perp in $T_x(\mathcal{S}^m)$.

Proof. To see the claim of the lemma, we first note that it is obvious for $m = 2$. For higher dimensions, it follows from the fact that any 2-dimensional big sphere is given by $\mathcal{S}^2 = \mathcal{S}^m \cap \Pi$, where Π is a 3-dimensional linear subspace of \mathbb{R}^{m+1} , i.e. one direction vector orthogonal to the plane containing $\bar{x}z$. \square

Recall that the set $Sing \subseteq \mathcal{E}_n$ is the set of singular functions (see definition 2.1).

Proof of lemma 2.2. We define the map

$$\Psi : \mathcal{E}_n \times \mathcal{S}^m \rightarrow \mathbb{R} \times \mathbb{R}^m$$

by

$$(f, x) \mapsto (f(x), \nabla f(x)),$$

using the isometry $T_x(\mathcal{S}^m) \cong \mathbb{R}^m$ again, so that

$$Sing = \pi_{\mathcal{E}_n}(\Psi^{-1}(0, \vec{0})).$$

We claim that Ψ is submersion. Having this claim in our hands would imply

$$\Psi^{-1}(0, \vec{0}) \leq \mathcal{N} - 1$$

by the submersion theorem. Therefore

$$\dim(Sing) \leq \mathcal{N} - 1$$

as well.

To see that Ψ is indeed a submersion, we compute its differential to be

$$d\Psi = \begin{pmatrix} \eta_1(x) & \eta_2(x) & \dots & \eta_{\mathcal{N}}(x) & * \\ \nabla\eta_1(x) & \nabla\eta_2(x) & \dots & \nabla\eta_{\mathcal{N}}(x) & * \end{pmatrix},$$

where $\{\eta_k\}$ is the orthonormal basis of \mathcal{E}_n , which appears in the definition (3) of f . Denote the matrix $A_{(m+1) \times \mathcal{N}}$ with the first \mathcal{N} columns of $d\Psi$. We claim that A is of full rank, i.e. $rk(A) = m + 1$. To see that we compute the Gram matrix of its rows to be

$$A \cdot A^t = \begin{pmatrix} 1 & 0 \\ 0 & \frac{E_n}{m} I_m \end{pmatrix},$$

see section 2.5. Since it is clearly invertible, we conclude that $rk(A) = m + 1$. \square

Recall that we defined \mathcal{P}_b^x and $\mathcal{P}_b^{x,y}$ in section 2.1 (see (14) and (15)).

Proof of lemma 2.3. Define $B_b^x \subseteq Sing \cap \mathcal{P}_b^x$ to be the set of function having $\pm x$ as their singular point, that is

$$B_b^x = \{f \in \mathcal{P}_b^x : f(x) = 0, \nabla f(x) = 0\} \cup \{f \in \mathcal{E}_n : f(-x) = 0, \nabla f(-x) = 0\}.$$

It is obvious that B_b^x is nonempty only if $b = 0$. Since $f(-y) = \pm f(y)$ for every $y \in \mathcal{S}^m$, $f \in B_b^x$ implies that f is singular at x . The set B_b^x is of codimension m , since the covariance matrix (27) is invertible, so that the Gaussian distribution of $\nabla f(x)$ conditioned upon $f(x) = 0$, is nonsingular.

Next, we define

$$\bar{B}_b^x := Sing \cap \mathcal{P}_b^x \setminus B_b^x.$$

To prove the statement of the lemma, we need to prove that \bar{B}_b^x is of codimension 1 in $Sing \cap \mathcal{P}_b^x$. To do so, we follow closely the proof of lemma 2.2. This time we define

$$\Psi_x : \mathcal{E}_n \times \mathcal{S}^m \setminus \{\pm x\} \rightarrow \mathbb{R}^2 \times \mathbb{R}^m$$

by

$$(f, y) \mapsto (f(x), f(y), \nabla f(y)),$$

satisfying

$$\bar{B}_b^x = \pi_{\mathcal{E}_n}(\Psi^{-1}(b, 0, \vec{0})).$$

Using a similar dimensional approach, it is sufficient to prove that Ψ_x is a submersion. The differential of Ψ_x is

$$d\Psi_x = \begin{pmatrix} \eta_1(x) & \eta_2(x) & \dots & \eta_{\mathcal{N}}(x) & * \\ \eta_1(y) & \eta_2(y) & \dots & \eta_{\mathcal{N}}(y) & * \\ \nabla\eta_1(y) & \nabla\eta_2(y) & \dots & \nabla\eta_{\mathcal{N}}(y) & * \end{pmatrix}.$$

Assume by contradiction that the vectors $(\eta_k(x), \eta_k(y), \nabla\eta_k(y))$ satisfy a nontrivial linear functional. Since η_k span the whole space \mathcal{E}_n , that functional is satisfied by

$$(Q_n^m(\cos d(x, \xi)), Q_n^m(\cos d(y, \xi)), \nabla Q_n^m(\cos d(y, \xi))),$$

for every $\xi \in \mathcal{S}^m$. The surjectivity of $d\Psi_x$ then follows from lemma B.3.

Finally, we note that

$$Sing \cap \mathcal{P}_b^x = B_b^x \cup \bar{B}_b^x,$$

which concludes the proof of this lemma. \square

Lemma B.3. *For every $x \in \mathcal{S}^m$, $y \neq \pm x$, the only solutions in $\alpha, \beta \in \mathbb{R}$, $C \in \mathbb{R}^m$ for*

$$(93) \quad \alpha Q_n^m(\cos d(x, \xi)) + \beta Q_n^m(\cos d(y, \xi)) - Q_n^{m'}(\cos d(y, \xi)) \sin d(y, \xi) \langle C, \nabla_y d(y, \xi) \rangle = 0$$

are $\alpha = \beta = 0$, $C = \vec{0}$.

Proof. It is obvious that either $\alpha = 0$ or $\beta = 0$, imply that $\alpha = \beta = 0$, $C = \vec{0}$. Therefore we may assume that $\alpha = -1$, $\beta \neq 0$.

Substituting ξ in (93) and, in addition, any $\xi' \neq \xi$ not lying on $\bar{x}y$ with $d(\xi', x) = d(\xi, x)$ and $d(\xi', y) = d(\xi, y)$, we obtain that C is collinear to any $v_1 = v_1(\xi, \xi') \in T_y(\mathcal{S}^m)$ of the form

$$v_1 = \nabla_y d(y, \xi) - \nabla_y d(y, \xi').$$

Lemma B.2 implies that C is collinear to $v := \nabla_y d(x, y)$.

We restrict ourselves to any two-dimensional big sphere $\mathcal{S}^2 \subseteq \mathcal{S}^m$ containing x and y . Knowing that $C \parallel v$, for $\xi \in \mathcal{S}^2$ on the big circle perpendicular to $\nabla_y d(x, y)$, (93) is

$$Q_n^m(bt) = \frac{1}{\beta} Q_n^m(t),$$

by the spherical cosine theorem, where we denote $t := \cos d(y, \xi)$ and $b := \cos d(x, y)$. It is clear that it is only possible if $b = 1$, that is $x = \pm y$, which is a contradiction. \square

Proof of lemma 2.4. To prove the statement of the lemma, we partition the set $Sing \cap \mathcal{P}_b^{x,y}$ into

$$Sing \cap \mathcal{P}_b^{x,y} = B_b^x \cup B_b^{x,y} \cup \bar{B}_b^{x,y},$$

defining appropriately each of the sets above and proving the statements regarding each of them separately.

First, similarly to the proof of lemma 2.3, we treat the set

$$B_b^x \subseteq Sing \cap \mathcal{P}_b^{x,y}$$

of function having $\pm x$ as their singular point, that is

$$B_b^x = \{f \in \mathcal{P}_b^{x,y} : f(x) = 0, \nabla f(x) = 0\} \cup \{f \in \mathcal{E}_n : f(-x) = 0, \nabla f(-x) = 0\}.$$

It is easy to see (exactly as in case of lemma 2.3) that B_b^x has codimension ≥ 1 in $\mathcal{P}_b^{x,y}$.

Next, we treat the case when the function f has a singular point on $D \subseteq \mathcal{S}^m$, a distinguished codimension 1 set of points on the sphere we are about to define. Let $A_1 := \bar{x}y$ be the big circle linking x to y and $A_2 \subseteq \mathcal{S}^m$ be the set of all the points z such that the spherical angle $\angle xzy = \frac{\pi}{2}$ is right angle. Define

$$D = D_{x,y} = A_1 \cup A_2 \setminus \{\pm x\}.$$

It is clear that D is a codimension one set on the sphere satisfying $\pm x \notin D$, $\pm y \in D$.

Define $B_b^{x,y} \subseteq Sing \cap \mathcal{P}_b^{x,y}$ to be the set of singular functions having a D -point as their singular point, that is

$$B = B_b^{x,y} = \{f \in \mathcal{P}_b^{x,y} : \exists z \in D : f(z) = 0, \nabla f(z) = 0\}.$$

We claim that B has codimension at least 1 in $\mathcal{P}_b^{x,y}$. To see that we define the map

$$\tilde{\Psi}_{b_1}^{x,y} : \mathcal{E}_n \times D \rightarrow \mathbb{R}^2 \times \mathbb{R}^m$$

by

$$(f, z) \mapsto (f(x), f(z), \nabla f(z)).$$

It is clear that

$$B \subseteq \pi_{\mathcal{E}_n}((\tilde{\Psi}_{b_1}^{x,y})^{-1}(b_1, 0, \vec{0})).$$

Moreover, $\tilde{\Psi}_{b_1}^{x,y}$ is a submersion (see the proof of lemma 2.3).

Therefore, $(\tilde{\Psi}_{b_1}^{x,y})^{-1}(b_1, 0, \vec{0})$ is of codimension $m + 2$ in $\mathcal{E}_n \times D$, i.e. of dimension $\mathcal{N} - 3$, so that B is of codimension ≥ 1 in $\mathcal{P}_b^{x,y}$.

Finally, we treat the “generic” case. We define the set

$$\bar{B} = \bar{B}_b^{x,y} := Sing \cap \mathcal{P}_b^{x,y} \setminus (B_b^x \cup B_b^{x,y})$$

of functions in $Sing \cap \mathcal{P}_b^{x,y}$ having the set of their singular points outside of $\{\pm x\} \cup D$ (i.e. having at least one singular point there).

We define

$$\Psi_b^{x,y} : \mathcal{E}_n \times \mathcal{S}^m \setminus (D \cup \{\pm x\}) \rightarrow \mathbb{R}^3 \times \mathbb{R}^m$$

by

$$(f, z) \mapsto (f(x), f(y), f(z), \nabla f(z)).$$

It is obvious that

$$\bar{B} \subseteq \pi_{\mathcal{E}_n}((\Psi_{b_1}^{x,y})^{-1}(b_1, b_2, 0, \vec{0})).$$

As before, to prove that \bar{B} is of codimension 1, it is sufficient to prove that $\Psi_b^{x,y}$ is a submersion. To see that $\Psi_b^{x,y}$ is a submersion, we compute its differential to be

$$d\Psi = \begin{pmatrix} \eta_1(x) & \eta_2(x) & \dots & \eta_{\mathcal{N}}(x) & * \\ \eta_1(y) & \eta_2(y) & \dots & \eta_{\mathcal{N}}(y) & * \\ \eta_1(z) & \eta_2(z) & \dots & \eta_{\mathcal{N}}(z) & * \\ \nabla\eta_1(z) & \nabla\eta_2(z) & \dots & \nabla\eta_{\mathcal{N}}(z) & * \end{pmatrix}.$$

Its surjectivity follows from lemma B.4.

This concludes the proof of this lemma. \square

Lemma B.4. *Let x, y and z be points on the sphere \mathcal{S}^m . Suppose that $x \neq y$, $z \notin \bar{xy}$, and $\angle xzy \neq \frac{\pi}{2}$. Then the only solution to*

$$(94) \quad \alpha f(x) + \beta f(y) + \gamma f(z) + \langle C, \nabla_z f(z) \rangle = 0$$

for every $f \in \mathcal{E}_n$ is $\alpha = \beta = \gamma = 0$, $C = \vec{0}$.

Proof. Substituting

$$f(z) = Q_n^m(\cos d(z, \xi))$$

with $\xi \in \mathcal{S}^m$, (94) is

$$(95) \quad \begin{aligned} & \alpha Q_n^m(\cos d(x, \xi)) + \beta Q_n^m(\cos d(y, \xi)) + Q_n^m(\cos d(z, \xi)) \\ & - Q_n^{m'}(\cos d(z, \xi)) \sin d(z, \xi) \langle C, \nabla_z d(z, \xi) \rangle = 0. \end{aligned}$$

for $\xi \neq \pm z$.

Comparing the equality (95) for ξ not lying on \bar{yz} and any ξ' with $d(x, \xi) = d(x, \xi')$ and $d(z, \xi) = d(z, \xi')$, we obtain

$$(96) \quad \begin{aligned} & Q_n^{m'}(\cos d(z, \xi)) \sin(d(z, \xi)) \langle C, \nabla_z d(z, \xi) - \nabla_z d(z, \xi') \rangle \\ & = \beta \left[Q_n^m(\cos d(y, \xi')) - Q_n^m(\cos d(y, \xi)) \right]. \end{aligned}$$

Let

$$\xi'' \neq \xi''' \in \mathcal{S}^m$$

with $\xi'' \neq \xi$ be the unique pair of points with

$$d(z, \xi'') = d(z, \xi''') = d(z, \xi)$$

and

$$\nabla_z d(z, \xi'') - \nabla_z d(z, \xi''') = \nabla_z d(z, \xi) - \nabla_z d(z, \xi').$$

In particular, we have

$$d(x, \xi'') = d(x, \xi''').$$

Substituting ξ'' and ξ''' into (96), as we may do, yields

$$(97) \quad \begin{aligned} & Q_n^{m'}(\cos d(z, \xi)) \sin(d(z, \xi)) \langle C, \nabla_z d(z, \xi'') - \nabla_z d(z, \xi''') \rangle \\ & = \beta \left[Q_n^m(\cos d(y, \xi''')) - Q_n^m(\cos d(y, \xi'')) \right], \end{aligned}$$

and comparing (96) to (97) we see that either $\beta = 0$ or

$$(98) \quad Q_n^m(\cos d(y, \xi)) - Q_n^m(\cos d(y, \xi')) - Q_n^m(\cos d(y, \xi'')) + Q_n^m(\cos d(y, \xi''')) = 0.$$

Suppose by contradiction that the latter holds. We restrict ourselves to any big two-dimensional sphere $\mathcal{S}^2(x, y, z) \subseteq \mathcal{S}^m$ (recall notation B.1). Let $d > 0$ be a small number and $\phi \neq \phi' \in \mathcal{S}^2$ be the (unique) points which satisfy

$$d(z, \phi) = d(z, \phi') = d$$

and $\bar{\phi}\bar{\phi}' \perp \bar{x}z$. We may approach to ϕ by \mathcal{S}^2 -points ξ and ξ'' and to ϕ' by ξ' and ξ''' of the form as above with the additional requirement

$$d(\xi, \bar{x}z) = d(\xi', \bar{x}z) = d(\xi'', \bar{x}z) = d(\xi''', \bar{x}z) = d.$$

Dividing (98) by $d(\xi, \xi'') = d(\xi', \xi''')$, and taking the limit as $\xi \rightarrow \phi$, we obtain

(99)

$$Q_n^m(\cos d(y, \phi)) \sin d(y, \phi) \frac{\partial}{\partial e^\phi} d(y, \phi) = Q_n^m(\cos d(y, \phi')) \sin d(y, \phi') \frac{\partial}{\partial e^{\phi'}} d(y, \phi'),$$

where e^ϕ and $e^{\phi'}$ are the unit tangent vectors in the directions $\check{\phi}\check{\xi}''$ and $\check{\phi}'\check{\xi}'''$ respectively.

Denote $d_0 := d(y, z)$, $d_1 := d(y, \phi)$ and $d_2 := d(y, \phi')$. Let δ be the angle $\delta = \angle xzy$. We have $\delta \neq 0, \frac{\pi}{2}$ by the assumptions of the lemma. We compute

$$c_1 = c_1(d) := \cos d_1 = \cos d_0 \cos d + \cos \delta \sin d_0 \sin d$$

and

$$c_2 = c_2(d) := \cos d_2 = \cos d_0 \cos d - \cos \delta \sin d_0 \sin d,$$

by the spherical cosine theorem. It is obvious that the LHS of (99) is an analytic function of c_1 , and the RHS is the same function evaluated at $c_2 = g(c_1)$ for some analytic function g . The function g is defined on an neighbourhood of $\cos d_0$ satisfying $g(\cos d_0) = \cos d_0$. Therefore, lemma B.5 implies that

$$g'(\cos d_0) = \pm 1.$$

On the other hand, computing the derivative explicitly, we have

$$g'(\cos d_0) = \frac{-\cos d_0 \sin d_0 - \cos \delta \cos d_0 \sin d_0}{-\cos d_0 \sin d_0 + \cos \delta \cos d_0 \sin d_0} = \frac{1 + \cos \delta}{1 - \cos \delta},$$

which, clearly, under the assumptions of the lemma, cannot be equal to ± 1 , and therefore we obtain the necessary contradiction. This proves that $\beta = 0$. By the symmetry, we have $\alpha = 0$ as well.

Thus (96) implies that

$$C \perp v_1(\xi) := \nabla_z d(z, \xi) - \nabla_z d(z, \xi'),$$

for every ξ, ξ' of the form above. However, for every ξ , the vectors

$$v_1(\xi, \xi') \in T_z(\mathcal{S}^m)$$

are all orthogonal to $v := \nabla_z d(x, z)$, the vector in the direction of $\bar{x}z$, and moreover, they span the orthogonal complement v^\perp , by lemma B.2.

Therefore C must be collinear to v . Similarly we may argue that C is collinear to $v' := \nabla_z d(z, y)$. However, v and v' are not collinear by the assumptions of the present lemma, so that $C = 0$. Knowing that, $\gamma = 0$ is easy to obtain. \square

Lemma B.5. *Let $f(t)$ an analytic, not identically vanishing function, and $g(t)$ a differentiable function defined on an neighbourhood I of $t_0 \in I$ such that $g(t_0) = t_0$. Suppose that we have on I*

$$f(g(t)) = f(t).$$

Then $g'(t_0) = \pm 1$.

Proof. We have by the chain rule,

$$f'(g(t))g'(t) = f'(t).$$

Therefore, if $f'(t_0) \neq 0$ then $g'(t_0) = 1$ and we are done. Otherwise, we continue differentiating to obtain

$$f''(g(t))g'^2(t) + f'(g(t))g''(t) = f''(t)$$

so that if $f''(t_0) \neq 0$, we have $g'^2(t_0) = 1$ and we are done again. Otherwise we continue differentiating until we encounter the first derivative $f^{(k)}(t_0) \neq 0$ implying $g^{(k)}(t) = 1$. Such a number k exists, since f is *analytic*. \square

APPENDIX C. NON DEGENERACY OF POINT VALUE AND GRADIENT DISTRIBUTION

In this section we prove that for $\pm N \neq x \in \mathcal{S}^m$, the distribution of the random vector Z defined in section 3.1, is *nonsingular* Gaussian.

Lemma C.1. *Let $x \neq \pm N \in \mathcal{S}^m$ and $V = V_x$ be vector space*

$$V = \mathbb{R}^2 \times T_x(\mathcal{S}^m) \times T_N(\mathcal{S}^m).$$

Define the subspace

$$U = U_{x,n} \subseteq V$$

by

$$U = \{(f(x), f(N), \nabla f(x), \nabla f(y)) : f \in \mathcal{E}_n\}.$$

Then one has

$$U = V,$$

provided that n large enough. That is, the distribution of the random vector

$$V = (f(x), f(N), \nabla f(x), \nabla f(N))$$

is Gaussian nondegenerate and one may identify

$$U \cong \mathbb{R}^{2m+2},$$

as in section 2.4.

Proof. Let $x \neq \pm N$. We assume by contradiction, that U is a proper subspace of V , i.e. there is a nontrivial functional $h : V \rightarrow \mathbb{R}$ vanishing on U .

We wish to work with coordinates and employ the orthonormal bases for $T_x(\mathcal{S}^m)$ and $T_N(\mathcal{S}^m)$ chosen in section 2.4, so that under the corresponding identification, one has (26).

By our assumption, there exist numbers $\alpha, \beta \in \mathbb{R}$ and vectors $C, D \in \mathbb{R}^2$ so that

$$(100) \quad \alpha f(x) + \beta f(N) + \langle C, \nabla f(x) \rangle + \langle D, \nabla f(N) \rangle = 0.$$

We know that for every $\eta \in \mathcal{S}^m$, the function

$$f(x) := Q_n^m(\cos d(x, \eta)),$$

is a spherical harmonic lying in \mathcal{E}_n . For this particular function (100) is for $\eta \neq \pm x, \pm N$,

$$(101) \quad \begin{aligned} & \alpha Q_n^m(\cos d(x, \eta)) + \beta Q_n^m(\cos d(N, \eta)) \\ & - Q_n^{m'}(\cos d(x, \eta)) \sin(d(x, \eta)) \cdot \langle C, \nabla_x d(x, \eta) \rangle \\ & - Q_n^{m'}(\cos d(N, \eta)) \sin(d(N, \eta)) \cdot \langle D, \nabla_N d(N, \eta) \rangle = 0. \end{aligned}$$

First choose $\eta \in \mathcal{S}^m$ not lying on $x\bar{N}$ and compare (101) for η and any $\eta' \neq \eta$ satisfying $d(x, \eta') = d(x, \eta)$ and $d(N, \eta') = d(N, \eta)$. We obtain

$$\begin{aligned} & Q_n^{m'}(\cos d(x, \eta)) \sin(d(x, \eta)) \cdot \langle C, \nabla_x d(x, \eta) \rangle \\ & + Q_n^{m'}(\cos d(N, \eta)) \sin(d(N, \eta)) \cdot \langle D, \nabla_N d(N, \eta) \rangle \\ & = Q_n^{m'}(\cos d(x, \eta)) \sin(d(x, \eta)) \cdot \langle C, \nabla_x d(x, \eta') \rangle \\ & + Q_n^{m'}(\cos d(N, \eta)) \sin(d(N, \eta)) \cdot \langle D, \nabla_N d(N, \eta') \rangle. \end{aligned}$$

Equivalently,

$$(102) \quad \begin{aligned} & Q_n^{m'}(\cos d(x, \eta)) \sin(d(x, \eta)) \cdot \langle C, \nabla_x d(x, \eta) - \nabla_x d(x, \eta') \rangle \\ & = -Q_n^{m'}(\cos d(N, \eta)) \sin(d(N, \eta)) \cdot \langle D, \nabla_N d(N, \eta) - \nabla_N d(N, \eta') \rangle. \end{aligned}$$

For every η , the vectors

$$v_1(\eta) = \nabla_x d(x, \eta) - \nabla_x d(x, \eta') \in T_x(\mathcal{S}^m)$$

are all orthogonal to $v := \nabla_x d(x, N)$, the vector in the direction of $x\bar{N}$, and moreover, they span the orthogonal complement v^\perp by lemma B.2. We claim that the equality (102) implies that

$$C \perp \text{sp}\{v_1(\eta)\}$$

and thus C and v are collinear. Similarly, D and $v' := \nabla_N d(x, N)$ are collinear, and since we identify v with $-v'$, that implies $C = \lambda D$ are collinear.

Suppose otherwise. Let $v_0 = v_1(\eta_0)$ such that $\langle C, v_0 \rangle \neq 0$, and consider the two-dimensional sphere $\mathcal{S}^2 \subseteq \mathcal{S}^m$ defined by $x\bar{N}$ and v_0 . For $\eta \in \mathcal{S}^2$, one has

$$v_1(\eta) \parallel v_0.$$

We fix $d = d(N, \eta)$ so that $\cos d$ is a zero of $Q_n^{m'}$. Then the RHS of (102) vanishes and our assumptions imply that $\cos d(N, \eta)$ is a zero of $Q_n^{m'}$.

However the function $\cos d(x, \eta)$ is a continuous nonconstant function of η on the arc

$$A := \{\eta : d(N, \eta) = d\} \subseteq \mathcal{S}^2,$$

and therefore its image contains an interval, contradicting the finiteness of number of zeros of $Q_n^{m'}$. Therefore

$$C \parallel v,$$

which proves our claim, i.e. $C = \lambda D$ for some $\lambda \in \mathbb{R}$.

Substituting the last equality into (102) with $\eta \in \mathcal{S}^m$ such that

$$d(x, \eta) = d(N, \eta),$$

implies $\lambda = -1$, i.e.

$$(103) \quad C = -D.$$

Now substitute $\eta \rightarrow x$ in (101) to obtain

$$(104) \quad \alpha + \beta Q_n^m(\cos d) + Q_n^{m'}(\cos d) \sin(d) \cdot \langle C, \nabla_N d(N, x) \rangle = 0,$$

where $d = d(x, N)$. We obtain similarly

$$(105) \quad \alpha Q_n^m(\cos d) + \beta - Q_n^{m'}(\cos d) \sin(d) \cdot \langle C, \nabla_x d(x, N) \rangle = 0,$$

upon substitution $\eta \rightarrow N$. Since in our identification, we have $\nabla_x d(x, N) = -\nabla_N d(x, N)$, (104) together with (105) imply

$$(106) \quad \alpha = \beta,$$

since

$$(Q_n^m(\cos d) \neq 1) \Leftrightarrow (d \neq 0, \pi) \Leftrightarrow (x \neq \pm N).$$

We claim that $\alpha = 0$ and $C = 0$. Assume otherwise. Consider any two-dimensional sphere \mathcal{S}^2 containing $x\bar{N}$, and the big circle $E \subseteq \mathcal{S}^2$ defined by

$$E = \{\eta \in \mathcal{S}^2 : d(x, \eta) = d(N, \eta)\}.$$

On E , (101) is, substituting (103) and (106)

$$(107) \quad 2\alpha Q_n^m(\cos d(x, \eta)) + Q_n^{m'}(\cos d(x, \eta)) \sin d(x, \eta) \langle C, \nabla_N d(N, \eta) - \nabla_x d(x, \eta) \rangle = 0.$$

It is clear that the vector

$$v = \nabla_N d(N, \eta) - \nabla_x d(x, \eta)$$

is collinear to $\nabla_x d(x, N)$, which, as we have seen, is collinear to C . In particular, $\alpha = 0$ if and only if $C = 0$ and thus we may assume by contradiction $\alpha \neq 0$ and $C \neq 0$.

Since $x \neq \pm N$, the point $\eta \in E$ lying on $x\bar{N}$, satisfies

$$d(x, \eta) < \frac{\pi}{2}$$

and the point $\eta' = -\eta \in E$ satisfies

$$d(x, \eta) > \frac{\pi}{2}.$$

Therefore there exists a point $\eta_0 \in E$ with

$$d(x, \eta_0) = \frac{\pi}{2}.$$

Then either $Q_n^m(\cos d(x, \eta_0)) = 0$ or $Q_n^{m'}(\cos d(x, \eta_0)) = 0$, depending on whether n is even or odd. However, the equality (107) implies then

$$Q_n^m(\cos d(x, \eta_0)) = Q_n^{m'}(\cos d(x, \eta_0)) = 0.$$

This contradicts the fact that Q_n^m does not have any double zeros, since then the differential equation (75) satisfied by Q_n^m would imply $Q_n^m \equiv 0$. \square

REFERENCES

- [1] Andrews, George E.; Askey, Richard; Roy, Ranjan *Special functions Encyclopedia of Mathematics and its Applications* 71. Cambridge University Press, Cambridge, 1999.
- [2] Bérard, P. *Volume des ensembles nodaux des fonctions propres du laplacien*. Bony-Sjostrand-Meyer seminar, 1984–1985, Exp. No. 14 , 10 pp., École Polytech., Palaiseau, 1985.
- [3] Berry, M. V. *Statistics of nodal lines and points in chaotic quantum billiards: perimeter corrections, fluctuations, curvature* J.Phys.A **35** (2002), 3025-3038.
- [4] J. Brüning *Über Knoten Eigenfunktionen des Laplace-Beltrami Operators*, Math. Z. 158 (1978), 15–21.
- [5] J. Brüning and D. Gromes *Über die Länge der Knotenlinien schwingender Membranen*, Math. Z. 124 (1972), 79–82.
- [6] S. Y. Cheng, *Eigenfunctions and nodal sets*, Comm. Math. Helv. **51** (1976), 43–55.
- [7] H. Donnelly, and C. Fefferman *Nodal sets of eigenfunctions on Riemannian manifolds*, Invent. Math. **93** (1988), 161–183.
- [GW] Granville, A.; Wigman, I. The distribution of the zeroes of random stationary trigonometric polynomials, available online <http://arxiv.org/abs/0809.1848>.
- [8] J. Neuheisel, *The asymptotic distribution of nodal sets on spheres*, Johns Hopkins Ph.D. thesis (2000).
- [9] Olver, F. W. J. *Asymptotics and special functions* Computer Science and Applied Mathematics. Academic Press [A subsidiary of Harcourt Brace Jovanovich, Publishers], New York-London, 1974.
- [10] F. Oravecz, Z. Rudnick and I. Wigman, *The Leray measure of nodal sets for random eigenfunctions on the torus*, Annales de l'institut Fourier, 58 no. 1 (2008), p. 299-335.
- [11] Z. Rudnick and I. Wigman *On the volume of nodal sets for eigenfunctions of the Laplacian on the torus*, Annales Henri Poincare, Vol. 9 (2008), No. 1, 109–130
- [12] Szego, Gabor *Orthogonal polynomials*. Fourth edition. American Mathematical Society, Colloquium Publications, Vol. XXIII. American Mathematical Society, Providence, R.I., 1975.
- [13] S. Zelditch, *Real and complex zeros of Riemannian random waves*, (2007) preprint.

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