

ON BILINEAR FORMS OF HEIGHT 2 AND DEGREE 1 OR 2 IN CHARACTERISTIC 2

AHMED LAGHRIBI AND ULF REHMANN

ABSTRACT. Our aim in this paper is to complete some results given in [15] on the classification of symmetric bilinear forms of height 2 and degree $d = 1$ or 2 over fields of characteristic 2, i.e., those whose anisotropic parts over their own function fields are similar to d -fold bilinear Pfister forms.

1. INTRODUCTION AND MAIN RESULTS

Let F be a field of characteristic 2. Throughout this paper, the expression “bilinear form” means “finite dimensional regular symmetric bilinear form”.

For a field extension L/F and a bilinear (or quadratic) form B over L , we say that B is *definable over F* if B is isometric to C_L for some bilinear (or quadratic) form C over F . If moreover, C is unique, then we say that B is *defined (by C) over F* .

To a bilinear form B with underlying vector space V , we associate a unique quadratic form \tilde{B} given on V by: $\tilde{B}(v) = B(v, v)$ for $v \in V$. The function field of B , denoted by $F(B)$, is by definition the function field of \tilde{B} . The standard splitting tower of a nonzero bilinear form B is the sequence of forms and fields defined as follows:

$$\begin{cases} F_0 = F & \text{and } B_0 = B_{\text{an}} \\ \text{For } n \geq 1 : & F_n = F_{n-1}(B_{n-1}) \quad \text{and } B_n = ((B_{n-1})_{F_n})_{\text{an}}, \end{cases}$$

where C_{an} denotes the anisotropic part of a bilinear form C . The height $h(B)$ of B is the smallest integer h such that $\dim B_h \leq 1$, where $\dim C$ denotes the dimension of a bilinear form C . As was done by the first author in [15], we associate to the form B another numerical invariant $\deg(B)$, called the degree of B , as follows: If $h = h(B)$ and $(B_i, F_i)_{0 \leq i \leq h}$ is the standard splitting tower of B , then the form $B_{h(B)-1}$ is of height 1. By the classification of height 1 bilinear forms [15, Th. 4.1], there

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exists a unique bilinear Pfister form π over $F_{h(B)-1}$ such that $B_{h(B)-1}$ is similar to π or to the pure part of π according as $\dim B$ is even or odd. If $\dim B$ is even, then we put $\deg(B) = d$ where $\dim \pi = 2^d$. Otherwise, we put $\deg(B) = 0$.

We call π the leading form of B . The form B is called good if π is definable over F , and in this case, we know from [15, Prop. 5.3] that π is defined over F by a d -fold bilinear Pfister form. For example, if B is of even dimension and nontrivial determinant, then it is good of degree 1 and leading form $(\langle 1, \det B \rangle_b)_{F_{h-1}}$.

An important problem considered in [15] is the classification of bilinear forms by height and degree. Bilinear forms of height 1 are completely classified as we said before in the definition of the degree. For good bilinear forms of height 2, the first author gave in [15] a complete classification of those of degree 0, and a partial classification of those of degree ≥ 1 . In this paper, we complete the classification of bilinear forms which are good of height 2 and degree 1 or 2, and with [15, Th. 5.10] we get the following theorem:

Theorem 1.1. *Let B be an anisotropic bilinear form over F which is good of height 2 and degree $d = 1$ or 2 . Let λ be the unique d -fold bilinear Pfister form over F such that $\lambda_{F(B)}$ is the leading form of B . Then, we are in one of the following cases:*

(1) $\dim B = 2^n$ with $n \geq d + 1$: *In this case, there exists $\alpha \in F^* := F \setminus \{0\}$ and π similar to an n -fold bilinear Pfister form such that $B \perp \alpha \lambda \perp \pi$ is metabolic.*

(2) $\dim B = 2^m - 2^d$ with $m \geq d + 2$: *In this case, $B \simeq \rho \otimes \lambda$ such that $\dim \rho$ is odd and $B \perp \langle \det \rho \rangle_b \otimes \lambda$ is similar to an m -fold bilinear Pfister form.*

Conversely, any anisotropic bilinear form satisfying the conditions described in (1) or (2) is good of height 2 and degree d .

Moreover, the first author gave a formula on the possible dimensions of bilinear forms of height 2 (good or not) [15, Cor 5.20]. As a consequence of it, we get that the dimension of any anisotropic bilinear form of height and degree 2 (good or not) can be 2^n , $2^n - 2$, or $2^n - 4$ for some $n \geq 3$ [15, Comment after Remark 5.21]. Note that Theorem 1.1 shows that the integers 2^n for $n \geq 3$, and $2^n - 4$ for $n \geq 4$ occur as dimensions of good anisotropic bilinear forms of height and degree 2. We know by [15] that any Albert bilinear form, i.e., a 6-dimensional bilinear form of trivial determinant, is of height and degree 2 but not good. Before this work and except for the integer 6, we did not know

other integers which really do occur as dimensions of anisotropic nongood bilinear forms of height and degree 2. Here, we clarify this point by proving the following theorem:

Theorem 1.2. (1) *There are 8-dimensional anisotropic nongood bilinear forms of height and degree 2.*
 (2) *An anisotropic bilinear form B over F is nongood of height and degree 2 iff one of the following conditions holds:*
 (i) *B is an Albert form.*
 (ii) *$\dim B = 8$ and there exists an anisotropic Albert bilinear form θ , unique up to similarity, that becomes isotropic over $F(B)$ and satisfies $B \perp \theta \in I^3 F$.*

The existence of 8-dimensional anisotropic bilinear forms of height and degree 2 which are not good is new in comparison with what is known in characteristic $\neq 2$. In fact, in this case, Kahn proved that a nongood anisotropic quadratic form of height and degree 2 is necessarily of dimension 6 and trivial discriminant, i.e., an Albert quadratic form [7]. Kahn's proof is based on the index reduction theorem of Merkurjev [18], [21]. But we do not have such a theorem for bilinear forms in characteristic 2. In our case, we will be inspired from a descent method due to Kahn [8], and we will use a result of Aravire and Baeza [1] to get the following theorem which is essential for the proofs of Theorems 1.1 and 1.2:

Theorem 1.3. *Let B be an anisotropic bilinear form over F of dimension ≥ 3 , and τ an anisotropic bilinear form similar to a d -fold bilinear Pfister form over $F(B)$, with $d = 1$ or 2 . Let C be a bilinear form over F .*

(1) *Suppose that $d = 2$ and $\tau \perp C_{F(B)} \in I^3 F(B)$. Then, there exists an Albert bilinear form θ over F such that $\tau \perp \theta_{F(B)} \in I^3 F(B)$. Furthermore, if $\dim B > 8$, then there exists a unique 2-fold bilinear Pfister form λ over F such that τ is similar to $\lambda_{F(B)}$.*

(2) *Suppose that $\dim B > 2^{d+1}$ and $\tau \perp C_{F(B)} \in I^{d+2} F(B)$. Let λ be as in (1) if $d = 2$, or $\lambda = \langle 1, \det C \rangle_b$ if $d = 1$. Moreover, suppose that $B_{F(\lambda)}$ is anisotropic or $C_{F(\lambda)}$ is metabolic. Then, τ is defined over F by a form similar to λ .*

Obviously, statement (1) of Theorem 1.3 implies that an anisotropic nongood bilinear form of height and degree 2 is of dimension 6 or 8. Moreover, as we see in Theorem 1.2, a complete classification of such bilinear forms consists in studying the isotropy of Albert bilinear forms over function fields of quadrics. This is an affair of norm field and norm degree (see subsection 2.3 for the definitions). More precisely, for θ an