LONG TIME BEHAVIOUR OF RANDOM WALKS ON ABELIAN GROUPS.

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Abstract. We show that on any locally compact non-compact second - countable Abelian group one can construct symmetric random walks with return probabilities decaying as close as possible to that of the exponential function $n \to \exp(-n)$ at infinity.

Key words: random walk, locally compact Abelian group, infinite divisible distribution, Laplace transform, Köhlbecker transform, Legendre transform, return probability, heat kernel.

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1. Introduction.

Let $\{X_k\}$ be a sequence of independent identically distributed real valued random variables with common distribution $\mathbb{P}_{X_1} := \mu$. We assume that:

- X_k is symmetric,
- μ is absolutely continuous with respect to the Lebesgues measure and has density $x \to \mu(x)$,
- $\sigma^2 = \int x^2 \mu(x) dx < \infty$.

Let us denote by $S_n = \sum_{k \leq n} X_k$ and let $I = (-\epsilon, \epsilon)$ be a finite interval, $|I| = 2\epsilon < \infty$, then:

$$\mathbb{P}(S_n \in I) \sim \frac{1}{\sigma\sqrt{2\pi}} |I| n^{-\frac{1}{2}} \qquad \text{as} \qquad n \to \infty.$$

This fact is a consequence of Local Limit Theorem [11, Ch.4, Thm.4.3.1]

$$\mathbb{P}_{\frac{S_n}{\sigma\sqrt{n}}} \to \phi \qquad \text{uniformly as } n \to \infty,$$

where $\phi(x) = \frac{1}{\sqrt{2\pi}} \exp(-x^2/2)$ is the normal density.

Our aim is to investigate the decay of the function $n \to \mathbb{P}(S_n \in I)$ at infinity in the case where $\int x^2 \mu(x) dx = \infty$.

Important observation: Let X_k belongs to the domain of attraction of

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a stable law with exponent $0 < \alpha < 2$, that is $\mathbb{P}(|X_k| > t) \asymp t^{-\alpha}$ as $t \to \infty^{(*)}$. Then we have (see, for instance [13]):

$$\mathbb{P}(S_n \in I) \sim c_\alpha |I| n^{-\frac{1}{\alpha}} \qquad \text{as} \quad n \to \infty.$$

Hence as $\alpha \downarrow 0$ the decay of the function $n \to \mathbb{P}(S_n \in I)$ becomes faster than that of $n \to n^{-\frac{1}{2}}$. Very generally: Let μ be a symmetric probability measure on a locally compact non-compact second-countable Abelian group $\mathbb{G}$. Assume that it has density $x \to \mu(x)$ with respect to the Haar measure ν on $\mathbb{G}$. We assume also that this density is symmetric and is in $L_2(\nu)$. Then according to [14] and [3] we have the following properties, for any relatively compact neighbourhood $I \subset \mathbb{G}$ of the neutral element $e \in \mathbb{G}$,

- $\mathbb{P}(S_{2n} \in I) \asymp \mu^{*2n}(0)$ as $n \to \infty$,
- $\lim_{n\to\infty} \frac{1}{2n} \log \mathbb{P}(S_{2n} \in I) = 0$.

These two facts imply that for any probability measure μ on $\mathbb{G}$ as above we must have:

$$\mathbb{P}(S_{2n} \in I) = \exp\{-2n \cdot o(1)\} \qquad \text{as} \quad n \to \infty.$$

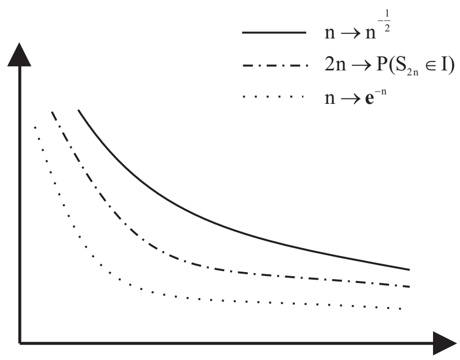

FIG. 1. *The decays of the functions* $n \to n^{-\frac{1}{2}}$, $n \to \mathbb{P}(S_{2n} \in I)$ *and* $n \to \exp(-n)$.

The main aim of this paper is to show that the decay of the function

$$2n \to \mathbb{P}(S_{2n} \in I) \qquad \text{as} \quad n \to \infty.$$

can be as close as possible to the decay of the exponential function $n \to e^{-n}$ as $n \to \infty$. In showing that, first, we consider the case when $\mathbb{G} = \mathbb{R}^1$ or $\mathbb{Z}^1$. Then using the structure theory of locally compact Abelian groups [8], [9] and [7], our knowledge of the result for the groups $\mathbb{R}^1$ and $\mathbb{Z}^1$ and the Fourier analysis on locally compact Abelian groups we build random walks on arbitrary group $\mathbb{G}$ with desired properties.

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* $f(x) \asymp g(x)$ as $x \to a$ means that $\exists c > 0$ such that $c \le f(x)/g(x) \le \frac{1}{c}$ when $|x - a| < \epsilon$, for some $\epsilon > 0$.

2. Symmetric random walks on the group $\mathbb{R}^1$.

In this case we choose $\mu = \mathbb{P}_{X_1}$ to be symmetric and infinitely divisible, that is there exists a one-parametric convolution semigroup of probability measures $(\mu_t)_{t>0}$ such that:

- $\mu = \mu_t$ when $t = 1$. In particular, $\mu^{*n} = \mu_n$.
- $\mu_t \to \varepsilon_0$ weakly as $t \to \infty$.

In particular, we have:

- $\hat{\mu}_t = \exp(-t\Psi)$,

where $\hat{\mu}_t$ is the Fourier transform of the measure μ_t and Ψ is *even non-negative definite* function on $\mathbb{R}^1$ (the symbol).

Assumption 1. We assume that *the function* $\exp(-t\Psi)$ *is summarable, for any* $t > 0$. This implies that

$$\mu_t(0) = \int_{\mathbb{R}^1} e^{-t\Psi(\xi)} d\xi = 2\int_0^\infty e^{-ts} dF(s),$$

where $F(s) = \text{Lebesgues measure}\{\tau > 0 : \quad \Psi(\tau) \le s\}$.

Assumption 2. We assume that *there exists a function* $f : \mathbb{R}^1_+ \to \mathbb{R}^1_+$, *such that* f *is increasing,* $\log f(t) = o(t)$ *as* $t \to \infty$ *and*

$$F(s) = \int_0^s f(t)dt, \quad s > 0. \tag{2.1}$$

It follows that for all $n \ge 1$,

$$\mu^{*n}(0) = \mu_n(0) = 2\int_0^\infty e^{-ns} f(s)ds. \tag{2.2}$$

Therefore we have the following comparison

$$\mathbb{P}(S_n \in I) \asymp \int_0^\infty e^{-ns} f(s)ds \qquad \text{as} \quad n \to \infty. \tag{2.3}$$

Because of the equation (2.3) we are left to investigate the asymptotic behaviour of the Laplace integral of the function f.

Remark 2.1. *That the representation (2.1) is possible follows from the celebrated Polya's theorem: Let* $\Psi \ge 0$ *be even continuous function,* $\Psi(0) = 0$. *Assume that* Ψ *being restricted to* $\mathbb{R}^1_+$ *is increasing and concave. Then* Ψ *is a negative definite function. In particular by the celebrated Bochner's theorem there exists a continuous convolution semigroup of probability measures* $(\mu_t)_{t>0}$ *such that* $\hat{\mu}_t = \exp(-t\Psi)$, *for all* $t > 0$. *See* [6], [12]. *In our case, for* $s \ge 0$, *the function* $s \to \Psi(s)$ *is the inverse to the function* $s \to \int_0^s f(t)dt$, $s \ge 0$.

Thanks to our choice (the assumptions 1 and 2) the semigroup $(\mu_t)_{t>0}$ has the following important properties:

(i) For each $t > 0$, μ_t admits an even, C^∞-density $x \to \mu_t(x)$ with respect to the Lebesgues measure.
(ii) $\max_{x\in\mathbb{R}^1} \mu_t(x) = \mu_t(0) = 2\int_0^\infty e^{-ts} f(s)ds$.
(iii) If $\frac{1}{f^2}$ is convex, then $x \to \mu_t(x)$ is unimodal.

The first statement is a consequence of the fact that $\Psi(s)/\log s \to \infty$ at ∞. The second one follows from the fact that $s \to \Psi(s)$ is real and even. The third one is an application of the non-trivial criteria of unimodality which is due to Askey [1].

To investigate the Laplace transform in (ii) we introduce two auxiliary transforms. Let $M : \mathbb{R}^1_+ \to \mathbb{R}^1_+$ be a decreasing function such that $M(0) = +\infty$. We define two transforms:

- The Köhlbecker transform of M:

$$\mathcal{K}(M)(x) := -\log\left(\int_0^\infty e^{-xt} de^{-M(t)}\right) , \ x > 0.$$

- The Legendre transform of M:

$$\mathcal{L}(M)(x) := \inf_{\tau>0}\{x\tau + M(\tau)\} \ , \ x > 0.$$

Theorem 2.1. *([2]) In the notations from above,*

$$\mathcal{K}(M)(x) \sim \mathcal{L}(M)(x) \qquad as \quad x \to \infty.$$

Thanks to the theorem (2.1), we have

$$\begin{aligned}\mu_t(0) &= 2\int_0^\infty e^{-st} dF(s) = 2\int_0^\epsilon e^{-st} dF(s) + O(e^{-\epsilon t}) = \\ &= 2\int_0^\infty e^{-st} de^{-M(s)} + O(e^{-\epsilon t}) = 2(1+o(1))e^{-\mathcal{K}(M)(t)},\end{aligned}$$

where $M(s) = \log\frac{1}{F(s)}$ at zero and $F(s) = \int_0^s f(\tau)d\tau$. This implies the following relation

$$-\log \mu_t(0) \sim \mathcal{K}(M)(t) \sim \mathcal{L}(M)(t) \qquad \text{at} \quad \infty. \tag{2.4}$$

The behaviour of the function $t \to \mathcal{L}(M)(t)$ at infinity depends on the behaviour of the function $t \to M(t)$ (equivalently of $t \to f(t)$) at zero. Then our strategy is to do computations, when f is very flat at zero.

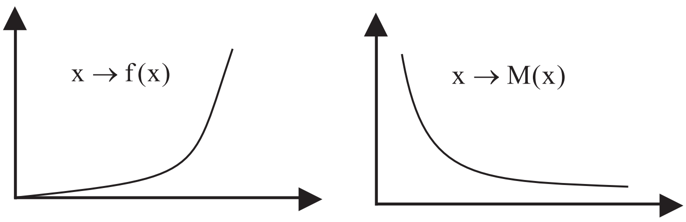

FIG. 2. *The flatter f at 0, the faster M at 0.*

Some particular results based on the direct computation of $\mathcal{L}(M)$ are presented in the table bellow. Assume that

$$f \;=\; e^{-g} \text{ at zero, } g(0) = +\infty$$

and write down $\mu_t(0)$ in the following form

$$\mu_t(0) = \exp\left\{-t\left[\frac{-\log\mu_t(0)}{t}\right]\right\} = \exp\{-t\cdot o(t)\}.$$

With these notations our computations are presented in the table:

	$g(s) \asymp$ at zero	$-\log\mu_t(0) \asymp$ at infinity	$o(t) \asymp$ at infinity
1	$(\log\frac{1}{s})^\alpha,\ \alpha > 1$	$(\log t)^\alpha$	$\frac{(\log t)^\alpha}{t}$
2	$s^{-\beta},\ \beta > 0,\ \beta_0 := \frac{\beta}{\beta+1}$	t^{β_0}	$\left(\frac{1}{t}\right)^{1-\beta_0}$
3	$\exp\{s^{-\gamma}\},\ \gamma > 0$	$\frac{t}{(\log t)^{\frac{1}{\gamma}}}$	$\frac{1}{(\log t)^{\frac{1}{\gamma}}}$
4	$\exp_{(k)}\{s^{-\nu}\},\ \nu > 0$ $(*)$	$\frac{t}{(\log_{(k)} t)^{\frac{1}{\nu}}}$ $(**)$	$\frac{1}{(\log_{(k)} t)^{\frac{1}{\nu}}}$

$(*)$ $\exp_{(k)}(t) = \exp(\exp(\ldots\exp(t)))$,
$(**)$ $\log_{(k)}(t) = \log(\log(\ldots\log(t)))$ at ∞.

TABLE. 1. *Some examples of fast decaying function $t \to \mu_t(0)$.*

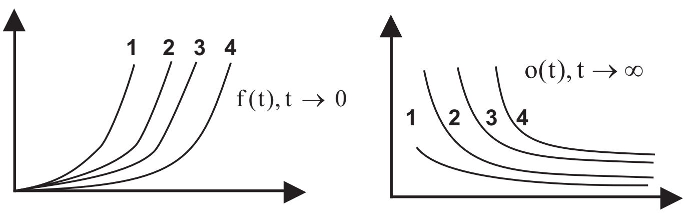

FIG. 3. *The flatter f at 0, the "weaker" $o(t)$ at ∞.*

In what follows we use the notation $f_1(x) \preceq f_2(x)$ at $x = a$ as an abbreviation of the following property: $\liminf_{x\to a} f_2(x)/f_1(x) > 0$. *Our main result for the group $\mathbb{R}^1$ is following theorem.*

Theorem 2.2. *For any non-decreasing function* $G : \mathbb{R}^1_+ \to \mathbb{R}^1_+$, *which is* $o(t)$ *at* ∞ *there exists a symmetric probability measure* μ *on* $\mathbb{R}^1$ *such that*

$$\mu^{*n}(e) \preceq e^{-G(n)} \qquad at \quad \infty.$$

In fact, μ *can be chosen such that*

$$-\log \mu^{*n}(e)/G(n) \to \infty \qquad at \quad \infty.$$

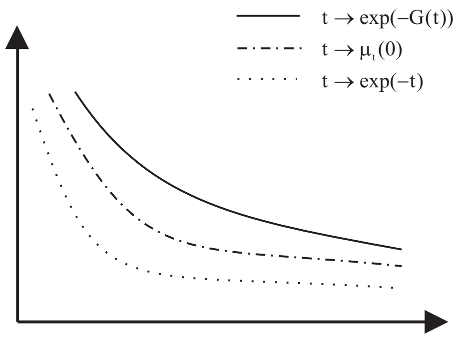

FIG. 4. *The decays of the functions* $t \to \exp(-G(t))$, $t \to \mu_t(0)$ *and* $t \to \exp(-t)$.

Proof. Choose the function $x \to \widetilde{G}(x)$ concave, $o(x)$ and $\widetilde{G}/G \to \infty$ at ∞ (FIG. 5.). Put $f = e^{-\mathcal{L}^*(\widetilde{G})}$, where we define the conjugate Legendre transform $\mathcal{L}^*(\widetilde{G})$ as follows

$$\mathcal{L}^*(\widetilde{G})(x) = \sup_{t>0}\{-tx + \widetilde{G}(t)\}, \quad x > 0.$$

Let $F(t) = \int_0^t f(x)dx$, $\Psi = F^{-1}$ and $\hat{\mu}_t = \exp(-t\Psi)$. Then we have

$$-\log \mu_t(0) = \mathcal{K}(\mathcal{L}^*(\widetilde{G}))(t) \sim \mathcal{L}(\mathcal{L}^*(\widetilde{G}))(t) \qquad \text{at} \quad \infty.$$

Since $\widetilde{G}$ is concave, $\mathcal{L}\left(\mathcal{L}^*(\widetilde{G})\right) = \widetilde{G}$, hence

$$-\log \mu_t(0)/G(t) \sim \widetilde{G}(t)/G(t) \to +\infty \qquad \text{at} \quad \infty.$$

Construction of the function $\widetilde{G}$: We can choose a decreasing sequence of reals ε_k such that $0 < \varepsilon_k \leq \varepsilon_0 = 1$, for all $k = 0, 1, 2, \ldots$ and $\lim_{n\to\infty} \varepsilon_n = 0$ and then we choose $t_0 < t_1 < \ldots$, $\lim_{n\to\infty} t_n = \infty$, such that

$$G(t) < \varepsilon_0 t \ \text{ for } t \in [t_0, t_1],$$

and

$$G(t) < \varepsilon_k t + \sum_{i=1}^{k} t_i(\varepsilon_{i-1} - \varepsilon_i) \ \text{ for } t \in [t_k, t_{k+1}],\ k = 1, 2, \ldots.$$

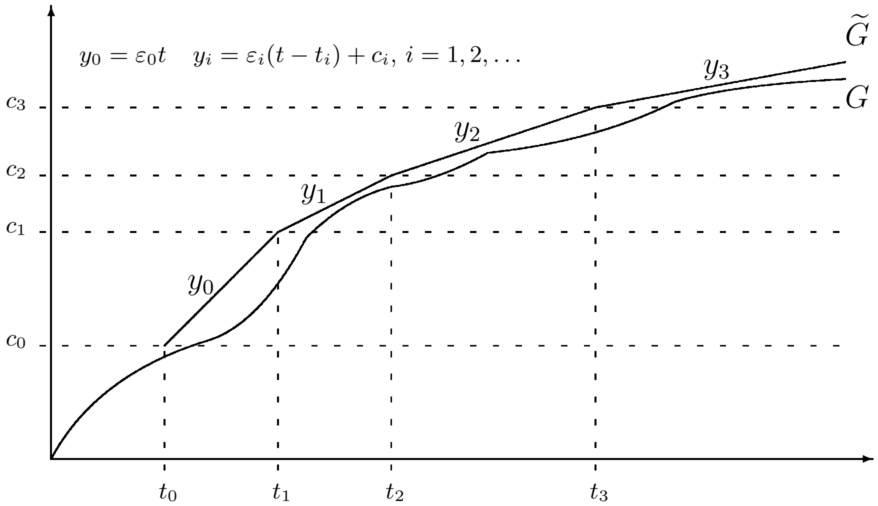

FIG. 5. *Construction of the function* $\widetilde{G}$.

This construction finish the proof of the theorem 2.2.

□

3. Symmetric random walks on the group $\mathbb{Z}^1$.

As in the case of the group $\mathbb{R}^1$ for any symmetric probability measure μ on $\mathbb{Z}$ which has finite second moment and any finite and symmetric interval $I \subset \mathbb{Z}$ we have

- $\mathbb{P}(S_n \in I) \asymp n^{-\frac{1}{2}} \quad$ as $\quad n \to \infty.$

Also in general (without assumption that the second moment is finite) we have

- $\mathbb{P}(S_{2n} \in I) = \exp\{-2n \cdot o(1)\} \quad$ as $\quad n \to \infty.$

We aim to show that, as in the case of the group $\mathbb{R}^1$, the decay of the function

$$2n \to \mathbb{P}(S_{2n} \in I), \qquad n \to \infty$$

can be as close as possible to the exponential one. This problem can be reduced to the similar problem on the group $\mathbb{R}^1$.

Reduction to the group $\mathbb{R}^1$. Let μ be a symmetric probability measure on $\mathbb{Z}^1$ and $\Phi = \hat{\mu}$ be its characteristic function. Then we have

$$\text{(3.1)} \quad \mathbb{P}(S_{2n} \in I) \asymp \mu^{*2n}(0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} [\Phi(x)]^{2n}\, dx = \frac{1}{\pi} \int_{0}^{\pi} [\Phi(x)]^{2n}\, dx.$$

We are looking for Φ supported by $[-\epsilon, \epsilon] \subset [-\pi, \pi]$ and having around zero the form $\Phi = e^{-g}$. Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be an increasing function such that $f(0) = 0$. Define g and Φ_0 by the equalities

$$g = \left(\lambda \to \int_0^\lambda f(\tau)d\tau\right)^{-1}, \ \Phi_0 = e^{-g}.$$

Then, by the Polya's theorem, Φ_0 is a characteristic function of some probability measure μ_0 on $\mathbb{R}^1$, that is $\Phi_0 = \hat{\mu}_0$. Now define Φ as on the figure below.

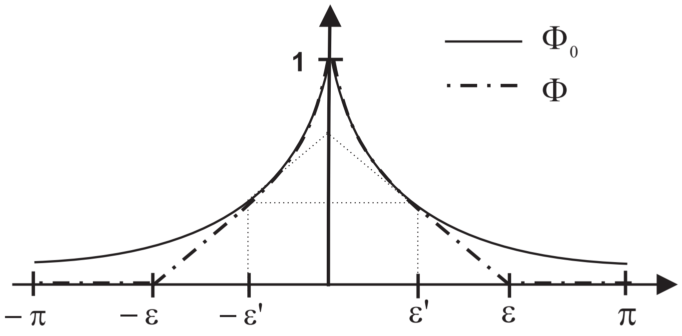

FIG. 6. *Construction of the function* Φ.

By construction, Φ being restricted to $\mathbb{R}^1_+$ is continuous, decreasing and convex function. **Polya's theorem** implies that there exists a probability measure μ_1 on $\mathbb{R}^1$ such that $\hat{\mu}_1 = \Phi$. Since $\Phi \in L_1$ with respect to the Lebesgue measure, μ_1 is absolutely continuous with respect to the Lebesgues measure and its density $x \to \mu_1(x)$, $x \in \mathbb{R}^1$ can be expressed as the inverse Fourier transform of the function Φ. Next we apply the Poisson summation formula to (Φ, μ_1). We have:

$$\sum_{k\in\mathbb{Z}^1} \Phi(\xi + 2k\pi) = \sum_{n\in\mathbb{Z}^1} \mu_1(n)e^{in\xi}, \ \xi \in \mathbb{R}^1. \tag{3.2}$$

Since Φ is supported by the interval $[-\epsilon, \epsilon] \subset [-\pi, \pi]$, the equation (3.2) implies

$$\Phi(\xi) = \sum_{n\in\mathbb{Z}^1} \mu_1(n)e^{in\xi}. \tag{3.3}$$

In particular, for $\xi = 0$ we get

$$1 = \Phi(0) = \sum_{n\in\mathbb{Z}^1} \mu_1(n). \tag{3.4}$$

This shows that the distribution μ on $\mathbb{Z}^1$ defined as $\mu(\{n\}) = \mu_1(n)$ is a probability distribution. Its characteristic function Φ coincides with $\Phi_0 = e^{-g}$ around zero.

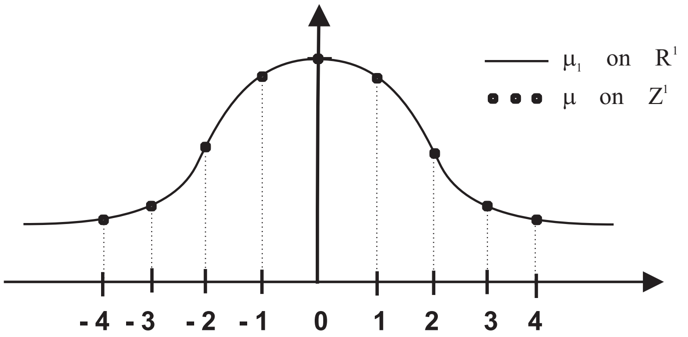

FIG. 7. *Construction of the probability measure μ on $\mathbb{Z}^1$.*

Hence we have:

$$\mathbb{P}(S_{2n} \in I) \asymp \mu^{*2n}(0) = \frac{1}{\pi} \int_0^{\epsilon} [\Phi(x)]^{2n} = \frac{1}{\pi} \int_0^{\epsilon'} e^{-2ng(x)} dx + O(e^{-\lambda n}) \sim$$

$$\sim \frac{1}{\pi} \int_0^{\epsilon'} e^{-2ng(x)} dx = \frac{1}{\pi} \int_0^{A} e^{-2ns} f(s) ds.$$

Then we can proceed as on the group $\mathbb{R}^1$. Finally we get the following result

Theorem 3.1. *For any non-decreasing function $G : \mathbb{R}^1_+ \to \mathbb{R}^1_+$, which is $o(t)$ at ∞ there exists a symmetric probability measure μ on $\mathbb{Z}^1$ such that*

$$\mu^{*n}(e) \preceq e^{-G(n)} \qquad at \quad \infty.$$

In fact, μ can be chosen such that

$$-\log \mu^{*n}(e)/G(n) \to \infty \qquad at \quad \infty.$$

4. Symmetric random walks on infinite periodic groups.

Let $\mathbb{G}$ be an infinite discrete periodic (torsion) group, that is each element $g \in \mathbb{G}$ has a finite order. Then $\mathbb{G}$ can be represented as union of increasing sequence of finite periodic (torsion) groups $\mathbb{G}_k$, $k = 0, 1, \ldots,$

$$\mathbb{G} = \bigcup_{k=0}^{\infty} \mathbb{G}_k, \ \mathbb{G}_0 = \{e\}, \ \mathbb{G}_k \subset \mathbb{G}_{k+1} \text{ and } |\mathbb{G}_k| < \infty.$$

The group $\mathbb{G}_k$ can be constructed as follows: Let $\{\overline{a}_0, \overline{a}_1, \overline{a}_2, ...\} \subset \mathbb{G}$ be the set of independent elements of the group $\mathbb{G}$. Let $\mathbb{G}_0 = \{e\}$ and $\mathbb{G}_k =< \overline{a}_0, \overline{a}_1, \overline{a}_2, ..., \overline{a}_k >$ be the group generated by $\{\overline{a}_i\}_{i \leq k}$. By this construction every $a \in \mathbb{G}_k$ is of the form $\overline{a}_1^{m_1} \cdot \overline{a}_2^{m_2} \cdot ... \cdot \overline{a}_k^{m_k}$ where $m_i \leq \max\{\text{order } a_i\}$ and we see that for $k = 0, 1, 2, \ldots$

$$\mathbb{G}_k \subset \mathbb{G}_{k+1} \subset \mathbb{G}.$$

Clearly $|\mathbb{G}_k| < \infty$. In fact, by the structure theory each $\mathbb{G}_k$ is a finite product of cyclic groups $\mathbb{Z}(n_i)$.

Example. Let $\mathbb{Z}(2)^\infty = \mathbb{Z}(2) \times \mathbb{Z}(2) \times ...$, where $\mathbb{Z}(2) = \{-1, 1\}$, then all elements $\xi = (\xi_0, \xi_1, ...) \in \mathbb{Z}(2)^\infty$ have order 1 or 2. Now define infinite discrete periodic group $\mathbb{G} = \mathbb{Z}(2)^{(\infty)} \subset \mathbb{Z}(2)^\infty$ as follows

$$\mathbb{G} = \{\xi \in \mathbb{Z}(2)^\infty : \text{ for some } k, \ \xi_{k+1} = \xi_{k+2} = ... = 1\}.$$

Put $\overline{\xi}_i = \{\xi : \xi_i = -1 \text{ and for any } k \neq i \ \xi_k = 1\}$, then

$$\mathbb{G} = \bigcup_{k=0}^{\infty} \mathbb{G}_k,$$

where

$$\mathbb{G}_k =< \overline{\xi}_0, \overline{\xi}_1, ..., \overline{\xi}_k >= \{\xi \in \mathbb{Z}(2)^{(\infty)} : \xi_i = 1 \text{ for all } i > k\} = \mathbb{Z}(2)^k.$$

Very generally, we have the following statement.

Proposition 4.1. *Let d_k be an arbitrary sequence of natural numbers such that the quotient $\frac{d_{k+1}}{d_k}$ is an integer equal or great then 2. Then there exists an infinite periodic discrete group $\mathbb{G}$ and an increasing sequence of groups $\mathbb{G}_k \subset \mathbb{G}$ such that $\mathbb{G} = \bigcup_{k=0}^\infty \mathbb{G}_k$ and d_k is the cardinality of $\mathbb{G}_k$.*

Proof. Let $\mathbb{Z}(n) = \{\xi \in \mathbb{C} : |\xi| = 1 \text{ and } \xi^n = 1\}$. Now define $c_k := \frac{d_{k+1}}{d_k}$, $k = 0, 1, 2, \ldots$. Then $d_n = d_0 \cdot c_0 \cdot \cdots \cdot c_{n-1}$, $n = 0, 1, 2, \ldots$. Put $\tilde{\mathbb{G}}_0 = \mathbb{Z}(d_0)$, $\tilde{\mathbb{G}}_n = \mathbb{Z}(d_0) \times \mathbb{Z}(c_0) \times \cdots \times \mathbb{Z}(c_{n-1})$, $n = 1, 2, 3, \ldots$. We have $|\tilde{\mathbb{G}}_n| = d_0 \cdot c_0 \cdot \cdots \cdot c_{n-1} = d_n$. Let now $\mathbb{G}_0 = \{(e_0, 1, 1, \ldots) : e_0 \in \tilde{\mathbb{G}}_0\}, \ldots, \mathbb{G}_n = \{(e_0, e_1, \ldots, e_n, 1, 1, \ldots) : (e_0, e_1, \ldots, e_n) \in \tilde{\mathbb{G}}_n\}$. We have $\mathbb{G}_0 \subset \mathbb{G}_1 \subset \cdots \subset \mathbb{G}$, where $\mathbb{G} = \bigcup_{k=0}^\infty \mathbb{G}_k$. Also $|\mathbb{G}_k| = |\tilde{\mathbb{G}}_k| = d_k$. The group $\mathbb{G}$ is infinite periodic discrete group.

□

Let now $H = \hat{\mathbb{G}}$ be the dual of the group $\mathbb{G}$. According to the structure theory of Abelian groups, H is a compact totally disconnected group. Some examples which are basic for our purpose are given below.

Examples.

- $\mathbb{G} \cong \mathbb{Z}(p^\infty)$, $H \cong \Delta_p$ - the group of p-adic integers,
- $\mathbb{G} \cong \mathbb{Z}(l)^{(\infty)}$, $H \cong \mathbb{Z}(l)^\infty$, $l \geq 2$.

More generally,

- $\mathbb{G} \cong (\prod_{k=0}^\infty)^* \mathbb{Z}(l_k)$, $H \cong \prod_{k=0}^\infty \mathbb{Z}(l_k)^{(*)}$.

* $\prod^* X_k$ - the weak product of groups X_k, that is the set of all sequences $x = (x_i) \in \prod X_k$ which are eventually identities.

Let m_k be the uniform distribution on $\mathbb{G}_k$, that is for $A \subset \mathbb{G}_k$:

$$m_k(A) = \frac{\#\{a \in A\}}{\#\{a \in \mathbb{G}_k\}} = \frac{|A|}{|\mathbb{G}_k|}.$$

Let $\{c_k\} \subset \mathbb{R}^1_+$ be a sequence of positive reals such that $\sum_{k=0}^{\infty} c_k = 1$. Define a probability measure μ on $\mathbb{G}$ as follows:

$$\mu = \mu(c) = c_0 m_0 + c_1 m_1 + \dots . \tag{4.1}$$

Evidently the measure μ is symmetric. We want to find the Fourier transform $\hat{\mu}$ of the measure μ, that is

$$\hat{\mu}(y) = \int_{\mathbb{G}} < y, x > d\mu(x), \quad y \in H.$$

Let $H_k = A(H, \mathbb{G}_k) = \{y \in H :< y, x >= 1, \forall x \in \mathbb{G}_k\}$ be the annihilator of the group $\mathbb{G}_k$ in the group $H = \hat{\mathbb{G}}$.

Proposition 4.2. *The Fourier transform $\hat{\mu}$ of the measure μ has the following form: For $k = 0, 1, 2, \dots$,*

$$\hat{\mu}(y) = c_0 + c_1 + \dots + c_k, \quad y \in A(H, \mathbb{G}_k) \backslash A(H, \mathbb{G}_{k+1}). \tag{4.2}$$

Proof. First we compute the Fourier transform $\hat{m}_k$ of the measure m_k

$$\hat{m}_k(y) = \int_{\mathbb{G}_k} < y, x > dm_k(x).$$

Because m_k is the uniform distribution, we have $m_k * m_k = m_k$. Then $\widehat{m_k * m_k} = \hat{m}_k \cdot \hat{m}_k = \hat{m}_k$, so $\hat{m}_k(y) \in \{0, 1\}$. Let $H_k = A(H, \mathbb{G}_k)$, in particular $H_0 = H$. For example, if $\mathbb{G} = (\prod_{i=1}^{\infty})^* \mathbb{Z}(p_i)$, $p_i \equiv p$, then

$$H_0 = H = \prod_{i=1}^{\infty} \mathbb{Z}(p_i),$$

$$\mathbb{G}_k = \prod_{i=1}^{k} \mathbb{Z}(p_i) \times \{0\} \subset \mathbb{G},$$

$$H_k = \prod_{i=k+1}^{\infty} \mathbb{Z}(p_i) \subset \mathbb{H}.$$

Consider two cases:

1) If $y \in H \backslash H_k$ then there exists $x \in \mathbb{G}_k$ such that $< y, x > \neq 1$, hence $\hat{m}_k(y) \neq 1$. Since

$$\begin{aligned} \hat{m}_k(y) &= \int_{\mathbb{G}_k} < y, x > dm_k(x) = \int_{\mathbb{G}_k} Re < y, x > dm_k(x) = \\ &= \frac{\sum_{\{x \in \mathbb{G}_k\}} Re < y, x >}{|\mathbb{G}_k|} < 1, \end{aligned}$$

we must have $\hat{m}_k(y) = 0$.

2) If $y \in H_k$ then for every $x \in \mathbb{G}_k$ we have $< y, x >= 1$. Therefore we have $\hat{m}_k(y) = 1$.
Finally we obtain the equality

$$\hat{m}_k(y) = \begin{cases} 1, & y \in H_k \\ 0, & y \in H \backslash H_k \end{cases} \tag{4.3}$$

Now we can find the Fourier transform $\hat{\mu}$ of the measure μ. We have

$$\hat{\mu}(y) = \sum_{k=0}^{\infty} c_k \hat{m}_k(y).$$

Since $\mathbb{G}_k$ increase, $H_k = A(H, \mathbb{G}_k)$ decrease. We can write the following decomposition of the group H

$$H = (H_0 \backslash H_1) \bigcup (H_1 \backslash H_2) \bigcup \dots . \tag{4.4}$$

Because of the structure of the measure μ, equation (4.3), we have

$$\hat{\mu}(y) = \begin{cases} c_0, & y \in H_0 \backslash H_1 \\ c_0 + c_1, & y \in H_1 \backslash H_2 \\ \vdots & \\ c_0 + c_1 + \dots + c_k, & y \in H_k \backslash H_{k+1} \end{cases}$$

This finishes the proof of the proposition.

□

Proposition 4.3. *The Fourier transform $\hat{\mu}^{*n}(y)$ of the measure μ^{*n} has the following form: For $k = 0, 1, 2, \dots,$*

$$\hat{\mu}^{*n}(y) = (c_0 + \dots + c_k)^n, \quad y \in A(H, \mathbb{G}_k) \backslash A(H, \mathbb{G}_{k+1}). \tag{4.5}$$

In particular,

$$\mu^{*n} \equiv \sum_{k \geq 0} \left(\sigma_k^n - \sigma_{k-1}^n \right) m_k. \tag{4.6}$$

Proof. Let $\sigma_k := c_0 + c_1 + \dots + c_k$, $\sigma_{-1} := 0$, then $c_k = \sigma_k - \sigma_{k-1}$. Since $\hat{\mu}^{*n} = (\hat{\mu})^n$ we have

$$\hat{\mu}^{*n}(y) = \begin{cases} \sigma_0^n, & y \in H_0 \backslash H_1 \\ \sigma_1^n, & y \in H_1 \backslash H_2 \\ \vdots & \\ \sigma_k^n, & y \in H_k \backslash H_{k+1} \end{cases}$$

It is easy to see that the measure μ^{*n} has the same structure as μ, that is $\mu^{*n} = \sum a_k m_k$. Hence by the proposition 4.2 we have

$$\hat{\mu}^{*n}(y) = \sum_{k=0}^{\infty} a_k \hat{m}_k(y) = \begin{cases} a_0, & y \in H_0 \backslash H_1 \\ a_0 + a_1, & y \in H_1 \backslash H_2 \\ \vdots & \\ a_0 + a_1 + \cdots + a_k, & y \in H_k \backslash H_{k+1} \end{cases}$$

It follows that $a_k := \sigma_k^n - \sigma_{k-1}^n$ for $k = 0, 1, 2, \ldots$. The proof is finished.
□

Corollary 4.1. *The measure $\mu = \mu(c)$ defined on the group $\mathbb{G}$ is infinitely divisible. More precisely, for any $n = 2, 3, \ldots$ $\mu = \mu^{*n}(a)$, where $a = (a_k)$ and $a_k = \sigma_k^{1/n} - \sigma_{k-1}^{1/n}$.*

Proof. For any $a = (a_i)$ we have

$$\hat{\mu}^{*n}(a) = \begin{cases} a_0^n, & y \in H_0 \backslash H_1 \\ (a_0 + a_1)^n, & y \in H_1 \backslash H_2 \\ \vdots & \\ (a_0 + a_1 + \cdots + a_k)^n, & y \in H_k \backslash H_{k+1} \end{cases}$$

We want to find $a = (a_i)$ such that $\mu(c) = \mu^{*n}(a)$. This evidently gives an infinite system of algebraic equations:

$$\begin{cases} c_0 \\ c_0 + c_1 \\ \vdots \\ c_0 + c_1 + \cdots + c_k \end{cases} = \begin{cases} a_0^n \\ (a_0 + a_1)^n \\ \vdots \\ (a_0 + a_1 + \cdots + a_k)^n \end{cases}$$

Equivalently

$$\begin{cases} a_0 = c_0^{1/n} \\ a_0 + a_1 = (c_0 + c_1)^{1/n} \\ \vdots \\ a_0 + a_1 + \cdots + a_k = (c_0 + c_1 + \cdots + c_k)^{1/n} \end{cases}$$

Hence, we obtain the desired equalities.

$$\begin{cases} a_0 = c_0^{1/n} \\ a_1 = (c_0 + c_1)^{1/n} - c_0^{1/n} \\ \vdots \\ a_k = (c_0 + c_1 + \cdots + c_k)^{1/n} - (c_0 + c_1 + \cdots + c_{k-1})^{1/n} \end{cases}$$

The proof is finished. □

Corollary 4.2. *(The Lévy-Khinchin formula). The Fourier transform $\hat{\mu}$ of the measure $\mu = \mu(c)$ can be represented in the form:*

$$\hat{\mu}(\theta) = \exp(-\Psi(\theta)), \quad \theta \in H,$$

where the negative-definite function Ψ has the following representation:

$$\Psi(\theta) = \int_{\mathbb{G}} [1 - \langle x, \theta \rangle] \, d\Pi(x).$$

The measure Π on $\mathbb{G}$ (the Lévy measure) can be written in the form $\Pi = \lambda \mathbb{P}$, where $\lambda > 0$ and $\mathbb{P}$ is a probability measure on $\mathbb{G}$ with the following representation

$$\mathbb{P} = \sum_{k \geq 0} p_k m_k, \quad p_k > 0, \ k = 0, 1, 2, \dots. \tag{4.7}$$

The coefficients p_k are given by the following equations

$$p_k = \begin{cases} 1 - \frac{1}{\lambda} \log \frac{1}{c_0}, & k = 0 \\ \frac{1}{\lambda} \log \left[1 + \frac{c_k}{c_0 + \cdots + c_{k-1}} \right], & k \geq 1 \end{cases} \tag{4.8}$$

In particular, $\lambda p_k \sim c_k$ as $k \to \infty$.

Proof. By the corollary 4.1, the measure μ is infinite divisible, hence by the Lévy-Khinchin formula valid for any locally compact Abelian group (see [10], [4]) its Fourier transform $\hat{\mu}$ has the following representation

$$\hat{\mu}(\theta) = \exp\{-\Psi(\theta)\} \quad \theta \in H,$$

where $\Psi : H \to \mathbb{C}$ is a negative-definite function on H. Since μ is symmetric Ψ is real-valued. Also $\mu(1) = 1$, hence the Lévy-Khinchin formula has the following form

$$\Psi(\theta) = \phi(\theta) + \int_{\mathbb{G} \setminus \{e\}} Re[1 - \langle x, \theta \rangle] d\Pi(x),$$

where ϕ is a non-negative definite quadratic form on H and Π is a symmetric measure on $\mathbb{G} \setminus \{e\}$. Since the group $H = \hat{\mathbb{G}}$ is totally disconnected, $\phi \equiv 0$. Since $\mathbb{G}$ is discrete, Π is a finite symmetric measure on $\mathbb{G} \setminus \{e\}$. Extend the measure Π to the whole group $\mathbb{G}$ putting $\Pi(\{e\}) = \pi_0 > 0$. Evidently it does not change the value of the function $\Psi(\theta)$, $\theta \in H$. After these preparations we can write the following equality

$$\Psi(\theta) = \int_{\mathbb{G}} [1 - \langle x, \theta \rangle] d\Pi(x) = \Pi(\mathbb{G}) - \hat{\Pi}(\theta). \tag{4.9}$$

On the other hand we have:

$$\Psi(\theta) = -\log \hat{\mu}(\theta) = \begin{cases} \log \frac{1}{c_0}, & \theta \in H_0 \backslash H_1 \\ \vdots & \\ \log \frac{1}{c_0+c_1+\cdots+c_k}, & \theta \in H_k \backslash H_{k+1} \\ \vdots & \end{cases} \tag{4.10}$$

Put $\lambda = \Pi(G)$, then we will have $\hat{\Pi}(\theta) = \lambda - \Psi(\theta)$ which implies

$$\hat{\Pi}(\theta) = \begin{cases} \lambda - \log \frac{1}{c_0}, & \theta \in H_0 \backslash H_1 \\ \vdots & \\ \lambda - \log \frac{1}{c_0+c_1+\cdots+c_k}, & \theta \in H_k \backslash H_{k+1} \\ \vdots & \end{cases} \tag{4.11}$$

It is easy to see that we can choose the value $\pi_0 = \Pi(\{e\})$ big enough so that $\lambda = \Pi(\mathbb{G}) > \log \frac{1}{c_0}$. Put $\mathbb{P} = \frac{1}{\lambda}\Pi$, then $\mathbb{P}$ is a probability on $\mathbb{G}$ having the same structure as μ. Hence it admits the following representation

$$\mathbb{P} = \sum_{k \geq 0} p_k m_k.$$

To find $\{p_k\}$ we solve the following system of equations:

$$\begin{cases} 1 - \frac{1}{\lambda} \log \frac{1}{c_0} = p_0 \\ \vdots \\ 1 - \frac{1}{\lambda} \log \frac{1}{c_0+c_1+\cdots+c_{k-1}+c_k} = p_0 + p_1 + \cdots + p_{k-1} + p_k \\ \vdots \end{cases} \tag{4.12}$$

From these equations we find coefficients p_k, for $k = 0, 1, 2, \ldots$.

$$\begin{aligned} p_0 &= 1 - \frac{1}{\lambda} \log \frac{1}{c_0}, \\ p_k &= \frac{1}{\lambda} \log \frac{1}{c_0 + c_1 + \cdots + c_{k-1}} - \frac{1}{\lambda} \log \frac{1}{c_0 + c_1 + \cdots + c_{k-1} + c_k} = \\ &= \frac{1}{\lambda} \log \frac{c_0 + c_1 + \cdots + c_{k-1} + c_k}{c_0 + c_1 + \cdots + c_{k-1}} = \\ &= \log \left[1 + \frac{c_k}{c_0 + c_1 + \cdots + c_{k-1}} \right]. \end{aligned}$$

Finally, since $c_k \to 0$ as $k \to \infty$ we get the last assertion of Corollary 4.2

$$\lambda p_k = \log \left[1 + \frac{c_k}{c_0 + c_1 + \cdots + c_{k-1}} \right] \sim c_k, \text{ as } k \to \infty.$$

The proof is finished.

□

Notation. For any finite measure $\mathbb{P}$ on $\mathbb{G}$ we define the following probability measure on $\mathbb{G}$

$$e(\mathbb{P}) := e^{-\mathbb{P}(\mathbb{G})} \sum_{k\geq 0} \frac{1}{k!} \mathbb{P}^{*k}$$

and call this measure the compound Poisson measure.

Corollary 4.3. *(The compound Poisson semigroup.) The measure $\mu = \mu(c)$ can be embedded in a weakly continuous convolution semigroup $(\mu_t)_{t>0}$ of symmetric infinite divisible measures on $\mathbb{G}$. Moreover the following properties hold:*
1) Each measure μ_t has the following representation

$$\mu_t = e(\lambda t \mathbb{P}), \quad t > 0,$$

where $\lambda > 0$, the probability measure $\mathbb{P}$ has the following structure

$$\mathbb{P} = \sum_{k\geq 0} p_k m_k, \quad p_k > 0,$$

and the coefficients p_k are given by the equations

$$p_k = \begin{cases} 1 - \frac{1}{\lambda} \log \frac{1}{c_0}, & k = 0 \\ \frac{1}{\lambda} \log \left[1 + \frac{c_k}{c_0 + \cdots + c_{k-1}} \right], & k \geq 1 \end{cases}$$

2) In particular, the measure μ_t has the same structure as μ, that is

$$\mu_t = \sum_{k\geq 0} c_k(t) m_k,$$

where

$$c_k(t) = \begin{cases} c_0^t, & k = 0 \\ (c_0 + c_1 + \cdots + c_k)^t - (c_0 + c_1 + \cdots + c_{k-1})^t, & k \geq 1 \end{cases}$$

Proof. Let Ψ be the negative definite function defined by μ. For each $t > 0$ we define the probability measure μ_t by its Fourier transform

$$\hat{\mu}_t(\theta) = \exp\{-t\Psi(\theta)\}, \quad \theta \in H.$$

That this equation defines μ_t as a probability on $\mathbb{G}$ follows from the celebrated theorem of Bochner valid on any locally compact Abelian group. Evidently $\mu = \mu_t$ for $t = 1$. The first statement follows from Corollary 4.2. The second statement for rational $t = m/n$ is a consequence of Proposition 4.2 and Corollary 4.1. Then, for any real $t > 0$ it follows by continuity.

□

Corollary 4.4. *(The heat kernel.) Let U be the Haar measure on $\mathbb{G}$ chosen such that $U(\{x\}) = 1$ for any $x \in \mathbb{G}$. For any $t > 0$ the measure μ_t is absolutely continuous with respect to the measure U and has density $x \to \mu_t(x)$ given by the following equation*

$$\mu_t(x) = \sum_{k\geq 0} \frac{c_k(t)}{|\mathbb{G}_k|} 1_{\mathbb{G}_k}(x), \quad x \in \mathbb{G}.$$

In particular, we have:

1) The return probability to the origin $\mu_t(\{e\})$ has the following form

$$\mu_t(\{e\}) = \mu_t(e) = \sum_{k\geq 0} \frac{c_k(t)}{|\mathbb{G}_k|}, \quad t > 0.$$

2) For any finite subset $\mathbb{F} \subset \mathbb{G}$

$$\mu_t(x) \asymp \mu_t(e) \qquad as\ t \to \infty \quad uniformly\ in\ x \in \mathbb{F}.$$

Proof. For any $x \in \mathbb{G}_n \setminus \mathbb{G}_{n-1}$ we have

$$\mu_t(x) = \sum_{k\geq n} \frac{\sigma_k^t - \sigma_{k-1}^t}{|\mathbb{G}_k|} = \sum_{k\geq n} \frac{\sigma_k^t}{|\mathbb{G}_k|} - \sum_{k\geq n-1} \frac{\sigma_k^t}{|\mathbb{G}_{k+1}|}.$$

From this equation we obtain

$$\mu_t(x) \leq \sum_{k\geq n} \frac{\sigma_k^t}{|\mathbb{G}_k|} = \mu_t(e) - \sum_{k<n} \frac{\sigma_k^t}{|\mathbb{G}_k|} \tag{4.13}$$

For the low bound we note that since each $\mathbb{G}_k$ is a finite product of cyclic groups $|\mathbb{G}_k|/|\mathbb{G}_{k+1}| \leq 1/2$. Therefore we have

$$\begin{aligned} \mu_t(x) &= \sum_{k\geq n} \frac{\sigma_k^t}{|\mathbb{G}_k|} - \sum_{k\geq n-1} \frac{|\mathbb{G}_k|}{|\mathbb{G}_{k+1}|} \frac{\sigma_k^t}{|\mathbb{G}_k|} \geq \sum_{k\geq n} \frac{\sigma_k^t}{|\mathbb{G}_k|} - \frac{1}{2} \sum_{k\geq n-1} \frac{\sigma_k^t}{|\mathbb{G}_k|} = \\ &= \frac{1}{2} \sum_{k\geq n} \frac{\sigma_k^t}{|\mathbb{G}_k|} - \frac{1}{2} \frac{\sigma_{n-1}^t}{|\mathbb{G}_{n-1}|} = \frac{1}{2}\mu_t(e) - \frac{1}{2} \sum_{k<n} \frac{\sigma_k^t}{|\mathbb{G}_k|} - \frac{1}{2} \frac{\sigma_{n-1}^t}{|\mathbb{G}_{n-1}|}. \end{aligned}$$

Since each term $\sigma_k^t/|\mathbb{G}_k|$ as a function of $t > 0$ has exponential decay at infinity and the function $t \to \mu_t(e)$ has subexponential decay at infinity (this property holds for any amenable group!) we must have

$$\mu_t(x) \asymp \mu_t(e) \qquad \text{at infinity}$$

which is true for any $x \in \mathbb{F} \cap (\mathbb{G}_n \setminus \mathbb{G}_{n-1})$. Since we assume that $\mathbb{F}$ is finite this gives the result.

□

We want to investigate asymptotic properties of the function $n \to \mu^{*n}(e)$ at infinity. Let $d_k := |\mathbb{G}_k|$ be the cardinality of the group $\mathbb{G}_k$. We have

$$\mu^{*n}(e) = \sum_{k=0}^{\infty} \frac{\sigma_k^n - \sigma_{k-1}^n}{d_k} = \sum_{k=0}^{\infty} \frac{\sigma_k^n}{d_k} - \sum_{k=0}^{\infty} \frac{\sigma_{k-1}^n}{d_k}.$$

This implies the upper bound

$$\mu^{*n}(e) = \sum_{k=0}^{\infty} \frac{\sigma_k^n}{d_k} - \sum_{k=0}^{\infty} \frac{\sigma_{k-1}^n}{d_k} \le \sum_{k=0}^{\infty} \frac{\sigma_k^n}{d_k}.$$

Since $d_k \ge 2d_{k-1}$ we also have

$$\mu^{*n}(e) \ge \sum_{k=0}^{\infty} \frac{\sigma_k^n}{d_k} - \frac{1}{2}\sum_{k=1}^{\infty} \frac{\sigma_{k-1}^n}{d_{k-1}} = \frac{1}{2}\sum_{k=0}^{\infty} \frac{\sigma_k^n}{d_k}.$$

Since for any finite $k \ge 0$, $\sum_{i=0}^{k} \sigma_i^n / d_i = O(e^{-\alpha n})$ we finally obtain a simple comparison formula for $\mu^{*n}(e)$,

$$\mu^{*n}(e) \asymp \sum_{k>>1} \frac{\sigma_k^n}{d_k} \qquad \text{as} \qquad n \to \infty. \tag{4.14}$$

Denote $\sigma(k) := \sum_{i>k} c_k = 1 - \sigma_k$. Then we have

$$\sigma_k^n = (1 - \sigma(k))^n = e^{n \cdot \log(1-\sigma(k))}.$$

Since $0 < \sigma(k) < 1$ for $k \ge 0$, we obtain

$$-\sigma(k) \le \log(1 - \sigma(k)) \le -\sigma(k) + \frac{\sigma^2(k)}{2} \le -\frac{\sigma(k)}{2}. \tag{4.15}$$

The inequality (4.15) implies that for $n >> 1$,

$$e^{-n\cdot\sigma(k)} \le \sigma_k^n \le e^{-\frac{1}{2} n \cdot \sigma(k)}. \tag{4.16}$$

Finally, we obtain the following inequality

$$b_1 \sum_{k>>1} \frac{1}{d_k} e^{-n\cdot\sigma(k)} \le \mu^{*n}(e) \le b_2 \sum_{k>>1} \frac{1}{d_k} e^{-\frac{1}{2} n \cdot \sigma(k)}, \quad \text{for } n >> 1. \tag{4.17}$$

with some constants b_1, $b_2 > 0$. Define a function $x \to \mathbb{N}(x)$ as follows: It has jumps at the points $\lambda_k = \sigma(k)$ and the values of jumps are $1/d_k$.

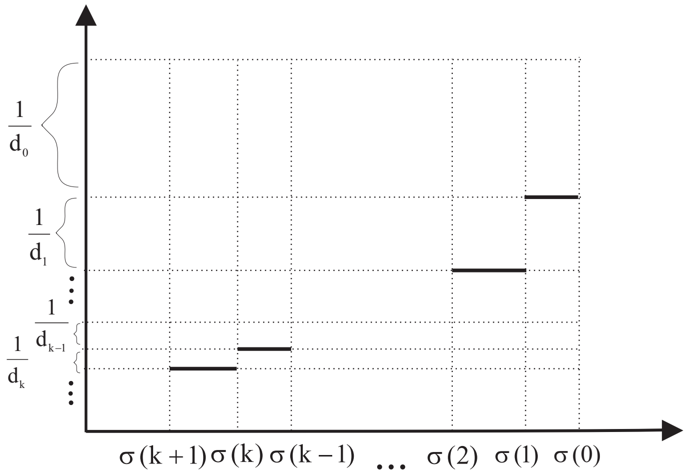

FIG. 8. *Construction of the function* $\mathbb{N}$.

Evidently we can write for any $c > 0$ and some $\varepsilon > 0$,

$$\sum_{k>>1} \frac{1}{d_k} e^{-c\cdot n\cdot\sigma(k)} = \int_0^{\varepsilon} e^{-c\cdot n\cdot\lambda} d\mathbb{N}(\lambda)$$

Hence, the bounds for the function $n \to \mu^{*n}(e)$ take the following form

$$(4.18) \quad b_1 \int_0^{\varepsilon} e^{-n\cdot\lambda} d\mathbb{N}(\lambda) \leq \mu^{*n}(e) \leq b_2 \int_0^{\varepsilon} e^{-\frac{1}{2}n\cdot\lambda} d\mathbb{N}(\lambda), \text{ as } n \to \infty.$$

Note, that for any non-decreasing function $f(t)$ such that $f(t) \downarrow 0$ as $t \downarrow 0$ one can construct a step function $\lambda \to \mathbb{N}(\lambda)$ which has jumps at the points $\sigma(k)$ and the values of jumps $1/d_k$ (see Figure 9) such that

$$\mathbb{N}(\lambda) \leq f(\lambda), \quad \lambda > 0.$$

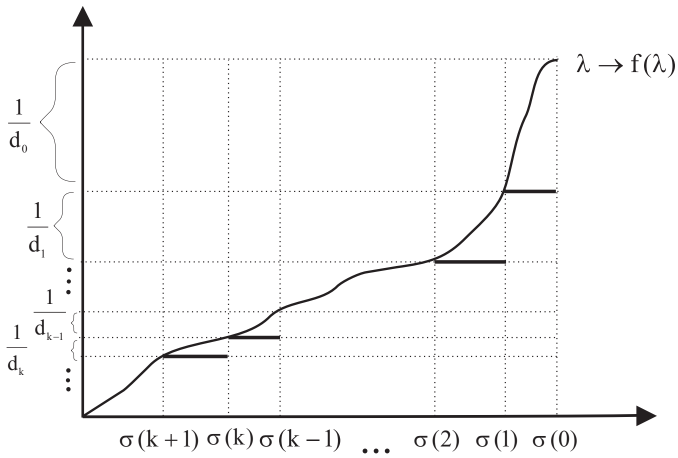

FIG. 9. *Construction of the step function* $\mathbb{N}$*, such that* $\mathbb{N}(x) \leq f(x)$.

Put $c_k = \sigma(k) - \sigma(k+1)$ and consider the following measure $\mu = \sum_{k=0}^{\infty} c_k m_k$. According to the consideration above we have

$$\mu^{*n}(e) \le b_2 \int_0^{\varepsilon} e^{-\frac{1}{2}n\cdot\lambda} f(\lambda) d\lambda \quad , \text{ as } \quad n \to \infty. \tag{4.19}$$

Now with this bound in mind we can apply asymptotic properties of the Laplace integral (see Theorem 2.1) to get the following statement.

Theorem 4.1. *Let $\mathbb{G}$ be infinite discrete periodic group. Then for any function $F(t)$ such that $F(t) = o(t)$ as $t \to \infty$, there exists a symmetric infinitely divisible probability measure μ on $\mathbb{G}$ such that*

$$\mu^{*n}(e) \preceq e^{-F(n)} \qquad as \quad n \to \infty.$$

In fact, the measure μ can be chosen in such a way that

$$-\log \mu^{*n}(e)/F(n) \to \infty \qquad at \quad \infty.$$

For any non-decreasing function $g(t)$ such that $\lim_{t\to 0} g(t) = 0$ one can construct a step function $\mathbb{N}(\lambda)$ as before (see Figure 10) such that

$$\mathbb{N}(\lambda) \ge g(\lambda), \quad \lambda > 0.$$

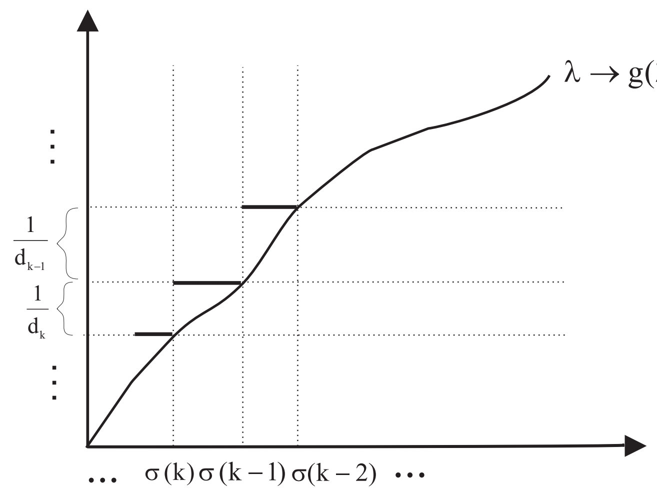

FIG. 10. *Construction of the step function $\mathbb{N}$, such that $\mathbb{N}(\lambda) \ge g(\lambda)$.*

Then we put $c_k = \sigma(k) - \sigma(k+1)$ and define the probability measure $\mu = \sum_{k=0}^{\infty} c_k m_k$. From the left hand side of the inequality (4.18) we obtain:

$$\mu^{*n}(e) \ge b_1 \int_0^{\varepsilon} e^{-n\cdot\lambda} d\mathbb{N}(\lambda) \ge b_1 \int_0^{\varepsilon} e^{-n\cdot\lambda} dg(\lambda) \quad \text{ as } n \to \infty.$$

Using this bound and applying asymptotic properties of the Laplace integral (see Theorem 2.1) we obtain the following statement.

Theorem 4.2. *Let $\mathbb{G}$ be infinite discrete periodic group. Then for any function $R(t)$ such that $R(t) = o(t)$ as $t \to \infty$ there exists a symmetric infinite divisible probability measure μ on $\mathbb{G}$ such that*

$$\mu^{*n}(e) \succeq e^{-R(n)} \quad \text{as } n \to \infty.$$

Proof. Choose a concave increasing non-negative function $\widetilde{R}$ such that $\widetilde{R}(t) \leq R(t)$ and $\widetilde{R}(t) = o(t)$ as $t \to \infty$. That such a choice is possible, follows from the simple geometric construction (see Figure 11).

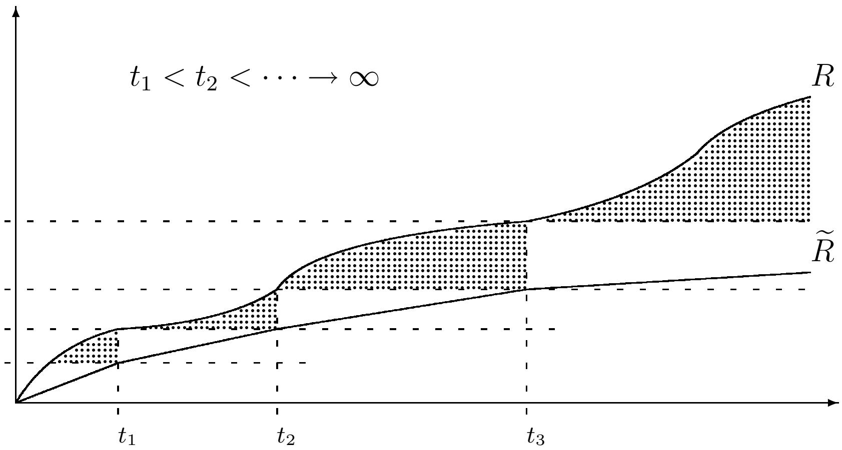

FIG. 11. *Construction of the function $\widetilde{R}$.*

Now define $g(x) = e^{-\mathcal{L}^*(\widetilde{R})(x)}$ and construct the step function $\mathbb{N}(\lambda)$ as on the Figure 10. By Theorem 2.2 we obtain the following inequality valid for $n >> 1$

$$\mu^{*n}(e) \succeq \int_0^\infty e^{-n\cdot\lambda} dg(\lambda) = \int_0^\infty e^{-n\cdot\lambda} de^{-\mathcal{L}^*(\widetilde{R})(\lambda)} \succeq e^{-\mathcal{L}(\mathcal{L}^*(\widetilde{R}))(n)}.$$

Since $\widetilde{R}$ is concave $\mathcal{L}(\mathcal{L}^*(\widetilde{R})) = \widetilde{R}$ and we get the desired result

$$\mu^{*n}(e) \succeq e^{-\widetilde{R}(n)} \geq e^{-R(n)}, \quad n \to \infty.$$

The proof of the theorem is finished.

□

Examples. Assume that $g(t)$ is a non-decreasing function such that $\lim_{t\to 0} g(t) = 0$ and let the function $\mathbb{N}(\lambda) \geq g(\lambda)$, $0 < \lambda < 1$ be chosen as on the Figure 10. Then as $t \to \infty$ we have:

$$\begin{aligned} \mu_t(0) &\sim \int_0^\varepsilon e^{-t\cdot\lambda} d\mathbb{N}(\lambda) \geq \int_0^\varepsilon e^{-t\cdot\lambda} dg(\lambda) \sim t \int_0^\infty e^{-t\cdot\lambda} g(\lambda) d\lambda = \\ &= \int_0^\infty e^{-s} g\left(\frac{s}{t}\right) ds = g\left(\frac{1}{t}\right) \int_0^\infty e^{-s} \left[g\left(\frac{s}{t}\right) / g\left(\frac{1}{t}\right) \right] e^{-s} ds. \end{aligned}$$

For some classes of functions $t \to g(t)$ the ratio $g(\lambda\tau)/g(\tau)$ has dominated convergence as $\tau \to \infty$ to some integrable function (in fact, always to the function $\lambda \to \lambda^\alpha$, $0 \le \alpha < \infty$, see [5]). This simple observation leads us to the examples presented in Table 2.

	$g(t) \asymp$ at zero	$\mu_t(0) \asymp$ at infinity
1	$t^\alpha, \quad \alpha > 0$	$\frac{1}{t^\alpha}$
2	$\exp\{-\left(\log \frac{1}{t}\right)^\alpha\}, \quad 0 < \alpha < 1$	$\exp\{-\left(\log t\right)^\alpha\}$
3	$\left(\log \frac{1}{t}\right)^{-\frac{1}{\alpha}}, \quad \alpha > 0$	$\left(\frac{1}{\log t}\right)^{\frac{1}{\alpha}}$
4	$\left[\log\left(\log \frac{1}{t}\right)\right]^{-\frac{1}{\alpha}}, \quad \alpha > 0$	$\left[\frac{1}{\log(\log t)}\right]^{\frac{1}{\alpha}}$
5	$\left[\log_{(k)} \frac{1}{t}\right]^{-\frac{1}{\alpha}}, \quad \alpha > 0$ (*)	$\left[\frac{1}{\log_{(k)} t}\right]^{\frac{1}{\alpha}}$

(*) $\log_{(k)}(t) = \log(\log(\ldots\log(t)))$ at ∞.

TABLE. 2. *Some examples of slow decaying function $t \to \mu_t(0)$.*

Remark 4.1. *It is well known fact that if the locally compact non-compact group $\mathbb{G}$ is compactly generated (in particular, for discrete $\mathbb{G}$ - finitely generated) the upper rate of decay of the return probability associated to any symmetric random walk on $\mathbb{G}$ exists and is a geometric invariant of the group $\mathbb{G}$. For example, if $\mathbb{G}$ is an Abelian, compactly generated group, then by structure theory*

$$\mathbb{G} \cong \mathbb{R}^l \times \mathbb{Z}^m \times H$$

where H is a compact group. Then, for any symmetric random walk on $\mathbb{G}$ we have $\mathbb{P}(S_{2n} \in U) \preceq n^{-(l+m)/2}$, $n \to \infty$. Theorem 4.2 shows that if $\mathbb{G}$ is not compactly generated then the upper bound may not exist in the sense explained above.

5. Symmetric random walks on general Abelian groups.

Let $\mathbb{G}$ be a locally compact non-compact second-countable Abelian group. The main result of this section is the following theorem, which is central in this paper.

Theorem 5.1. *For any non-decreasing function $F : \mathbb{R}^1_+ \to \mathbb{R}^1_+$ which is $o(t)$ at infinity there exists a symmetric probability measure μ on $\mathbb{G}$*

such that

$$\mu^{*n}(e) \preceq e^{-F(n)} \qquad at \quad \infty.$$

Moreover μ can be chosen in such a way that

$$-\log \mu^{*n}(e)/F(n) \to \infty \qquad at \quad \infty.$$

In other words the random walk on $\mathbb{G}$ generated by μ has probability of return to any relatively compact symmetric neighbourhood of the neutral element of $\mathbb{G}$ decaying faster than that of the function $n \to \exp(-F(n))$ at infinity.

Proof. According to the structure theory (see [8], [9]) we have

$$\mathbb{G} = \mathbb{R}^n \times \Gamma,$$

where $n \geq 0$ is an integer and Γ is an Abelian group which contains an open compact subgroup $\Gamma_0 \subset \Gamma$. So that, in particular Γ/Γ_0 is a discrete countable Abelian group. We shall consider the following two cases:

First case: Assume that $n > 0$, then we define a random walk S_m on $\mathbb{G}$ as follows:

$$S_m = S_m^{\mathbb{R}^n} + U_m^{\Gamma_0},$$

where $S_m^{\mathbb{R}^n}$ is an "optimal" random walk on $\mathbb{R}^n$ constructed via Theorem 2.2 and $U_m^{\Gamma_0}$ is the random walk on Γ_0 such that $\mathbb{P}_{U_1^{\Gamma_0}}$ is the uniform distribution on Γ_0. We also assume that $S_m^{\mathbb{R}^n}$ and $U_m^{\Gamma_0}$ are independent. Then we will have

$$\mathbb{P}(S_m \in I \times \Gamma_0) = \mathbb{P}(S_m^{\mathbb{R}^n} \in I), \quad m = 0, 1, 2, \ldots.$$

Hence in this case the theorem is proved thanks to Theorem 2.2.

Second case: Assume that $n = 0$, then $\mathbb{G} = \Gamma$, $\Gamma_0 \subset \Gamma$ and Γ/Γ_0 is discrete.

(a) Assume that Γ/Γ_0 contains a *non-compact* element a, that is

$$a^k \neq e, \qquad k = \pm 1, \pm 2, \ldots.$$

Let $< a >$ be a subgroup of the group Γ/Γ_0 generated by the element a, that is

$$< a >= \{e, a^{\pm 1}, a^{\pm 2}, ...\}$$

Clearly the mapping

$$\gamma : \mathbb{Z}^1 \to < a >, \quad n \to a^n$$

is an *isomorphism* of the group $\mathbb{Z}^1$ onto the group $< a >$. We choose an "optimal" random walk on $\mathbb{Z}^1$ as in the Theorem 3.1. Via the isomorphism γ this gives an "optimal" random walk on the group $< a >$. Let now

$$\pi : \Gamma \to \Gamma/\Gamma_0$$

be the canonical homomorphism of Γ onto Γ/Γ_0. Evidently $\pi^{-1}(< a >)$ is an open subgroup of Γ. If it is clear that the group $\pi^{-1}(< \ a >)$ is compactly generated: it is generated by the compact set $\Gamma_0 \bigcup \pi^{-1}(a)$. Hence by the structure theory ([8, Thm. 9.8])

$$\pi^{-1}(< a >) = \mathbb{R}^m \times \mathbb{Z}^l \times K,$$

where K is a compact Abelian group. Evidently $m = 0$. Also since

$$\pi^{-1}(< a >)/\Gamma_0 \cong \mathbb{Z}^1,$$

we must have $l = 1$. Hence we obtain the following identification

$$\pi^{-1}(< a >) \cong \mathbb{Z}^1 \times \Gamma_0.$$

Finally we construct an "optimal" random walk on $\Gamma \supset \pi^{-1}(< a >)$ as follows: We let $S_1^{\mathbb{Z}^1}$ and $U_1^{\Gamma_0}$ be independent random variables taking values in the group $\mathbb{Z}^1$ and Γ_0 respectively. Put $S_n := \gamma(S_n^{\mathbb{Z}^1}) + U_n^{\Gamma_0}$. If we choose the random variable $S_1^{\mathbb{Z}^1}$ as in Theorem 3.1, then the random walk (S_n) on the group Γ has desired property. This proves the theorem in the case (a).

(b) Assume now that all elements of the group Γ/Γ_0 have finite order, that is the group $\Gamma/\Gamma_0 = \{a_0 = e, a_1, a_2, ...\}$ is an infinite discrete periodic group. Then we construct an "optimal" random walk on the group Γ/Γ_0 and use this random walk in order to construct desired random walk on Γ. Indeed, let μ be a probability measure on $\Gamma/\Gamma_0 = \{a_i : i = 0, 1, 2, \ldots\}$ such that $\mu(\{a_i\}) = \mu_i$. We assume that μ is "optimal" in the sense of Theorem 4.1. Define a probability measure $\tilde{\mu}$ on Γ as follows: Write the decomposition $\Gamma = \bigcup \pi^{-1}(a_i)$ and define $\tilde{\mu}$ on the compact set $\pi^{-1}(a_i)$, $i = 0, 1, 2, \ldots$, as uniform distribution such that $\tilde{\mu}(\pi^{-1}(a_i)) = \mu_i$. Evidently this defines a probability distribution $\tilde{\mu}$ on the group Γ such that $\pi(\tilde{\mu}) = \mu$. It follows that

$$\mu^{*n} = (\pi(\tilde{\mu}))^{*n} = \pi\left(\tilde{\mu}^{*n}\right).$$

In particular, since $\pi^{-1}(e) = \Gamma_0$ we obtain

$$\tilde{\mu}^{*n}(\Gamma_0) = \mu^{*n}(e) \qquad \text{for all } n = 1, 2, \ldots.$$

This relation proves the theorem in the last case (b). Thus the proof of the theorem is finished. □

Remark 5.1. *Our construction of the measure μ with desired properties does not give in general an infinite divisible distribution on the group $\mathbb{G}$. This is because the construction on the group $\mathbb{Z}^1$ does not give such a distribution. We think that this construction can be improved so that we do obtain an infinite divisible distribution on $\mathbb{Z}^1$ with desired properties, and thus an infinite divisible distribution on the group $\mathbb{G}$.*

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