ON THE EXISTENCE OF U-POLYGONS OF CLASS $c \geq 4$ IN PLANAR POINT SETS

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Abstract. For a finite set U of directions in the Euclidean plane, a convex non-degenerate polygon P is called a U-polygon if every line parallel to a direction of U that meets a vertex of P also meets another vertex of P. We characterize the numbers of edges of U-polygons of class $c \geq 4$ with all their vertices in certain subsets of the plane and derive explicit results in the case of cyclotomic model sets.

1. Introduction

The (discrete parallel) X-ray of a finite subset F of the Euclidean plane in direction u is the corresponding line sum function that gives the numbers of points of F on each line parallel to u. It was shown in [18, Proposition 4.6] that the convex subsets of an *algebraic Delone set* Λ are determined by their discrete parallel X-rays in the directions of a set U of at least two pairwise non-parallel Λ-directions (i.e., directions parallel to non-zero interpoint vectors of Λ) if and only if there is no U-polygon with all its vertices in Λ. By [18, Lemma 4.5], there always exists a U-polygon with all its vertices in Λ if U is a set of at most three pairwise non-parallel Λ-directions. This leads to the question which U-polygons exist with all their vertices in Λ for sets U of four or more pairwise non-parallel Λ-directions. We refer the reader to [12, 15, 16, 17, 18, 19] for more on discrete tomography and [11] for the role of U-polygons in geometric tomography, where the X-ray of a compact convex set in a direction gives the lengths of all chords of the set in this direction. Dulio and Peri have introduced the notion of *class* of a U-polygon and demonstrated that for planar *lattices* L the numbers of edges of U-polygons of class $c \geq 4$ with all their vertices in L are precisely 8 and 12; cf. [10, Theorem 12]. As a first step beyond the case of planar lattices, this text provides a generalization of this result to planar sets that are non-degenerate in some sense and satisfy a certain affinity condition on finite scales (Theorem 3.1). It turns out that, for these sets Λ, the existence of U-polygons of class $c \geq 4$ with all their vertices in Λ is equivalent to the existence of certain *affinely regular polygons* with all their vertices in Λ, a problem that was addressed in [19]. The obtained characterization of numbers of vertices of U-polygons of class $c \geq 4$ with all their vertices in Λ can be expressed in terms of a simple inclusion of real field extensions of $\mathbb{Q}$ and particularly applies to *algebraic Delone sets*, thus including *cyclotomic model sets*, which form an important class of planar *mathematical quasicrystals*; cf. [2, 7]. For cyclotomic model

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sets $\varLambda$, the numbers of vertices of U-polygons of class $c \geq 4$ with all their vertices in $\varLambda$ can be expressed by a simple divisibility condition (Corollary 4.1). In particular, the above result on lattice U-polygons of class $c \geq 4$ by Dulio and Peri is contained as a special case (Corollary 4.3(a)).

2. Definitions and preliminaries

Natural numbers are always assumed to be positive and the set of rational primes is denoted by $\mathcal{P}$. Primes $p \in \mathcal{P}$ for which the number $2p+1$ is prime as well are called *Sophie Germain primes*. We denote by $\mathcal{P}_{\mathrm{SG}}$ the set of Sophie Germain primes. The first few ones are

$$2, 3, 5, 11, 23, 29, 41, 53, 83, 89, 113, 131, 173, 179, 191, 233, 239, \dots ;$$

see entry A005384 of [20] for further details. The group of units of a given ring R is denoted by $R^{\times}$. As usual, for a complex number $z \in \mathbb{C}$, $|z|$ denotes the complex absolute value $|z| = \sqrt{z\bar{z}}$, where $\bar{\cdot}$ denotes the complex conjugation. Occasionally, we identify $\mathbb{C}$ with $\mathbb{R}^2$. The unit circle $\{x \in \mathbb{R}^2 \mid |x| = 1\}$ in $\mathbb{R}^2$ is denoted by $\mathbb{S}^1$. Moreover, the elements of $\mathbb{S}^1$ are also called *directions*. For a direction $u \in \mathbb{S}^1$, the *angle between u and the positive real axis* is understood to be the unique angle $\theta \in [0, \pi)$ with the property that a rotation of $1 \in \mathbb{C}$ by θ in counter-clockwise order is a direction parallel to u. For $r > 0$ and $x \in \mathbb{R}^2$, $B_r(x)$ denotes the open ball of radius r about x. A subset $\varLambda$ of the plane is called *uniformly discrete* if there is a radius $r > 0$ such that every ball $B_r(x)$ with $x \in \mathbb{R}^2$ contains at most one point of $\varLambda$. Further, $\varLambda$ is called *relatively dense* if there is a radius $R > 0$ such that every ball $B_R(x)$ with $x \in \mathbb{R}^2$ contains at least one point of $\varLambda$. $\varLambda$ is called a *Delone set* if it is both uniformly discrete and relatively dense. A direction $u \in \mathbb{S}^1$ is called a *$\varLambda$-direction* if it is parallel to a non-zero element of the difference set $\varLambda - \varLambda$ of $\varLambda$. Further, a bounded subset C of $\varLambda$ is called a *convex subset of $\varLambda$* if its convex hull contains no new points of $\varLambda$. A *non-singular affine transformation* of the Euclidean plane is given by $z \mapsto Az + t$, where $A \in \mathrm{GL}(2, \mathbb{R})$ and $t \in \mathbb{R}^2$. Further, recall that a *homothety* of the Euclidean plane is given by $z \mapsto \lambda z + t$, where $\lambda \in \mathbb{R}$ is positive and $t \in \mathbb{R}^2$. A *convex polygon* is the convex hull of a finite set of points in $\mathbb{R}^2$. For a subset $S \subset \mathbb{R}^2$, a *polygon in S* is a convex polygon with all vertices in S. A *regular polygon* is always assumed to be planar, non-degenerate and convex. An *affinely regular polygon* is a non-singular affine image of a regular polygon. In particular, it must have at least 3 vertices. Let $U \subset \mathbb{S}^1$ be a finite set of pairwise non-parallel directions. A non-degenerate convex polygon P is called a *U-polygon* if it has the property that whenever v is a vertex of P and $u \in U$, the line ℓ_u^v in the plane in direction u which passes through v also meets another vertex v' of P. Then, every direction of U is parallel to one of the edges of P; cf. [10, Lemma 5(i)]. Further, one can easily see that a U-polygon has $2m$ edges, where $m \geq \mathrm{card}(U)$. For example, an affinely regular polygon with an even number of vertices is a U-polygon if and only if each direction of U is parallel to one of its edges. The following notion of *class* of a U-polygon was introduced by Dulio and Peri; cf. [10, Definition 1]. For $0 < c \leq \mathrm{card}(U)$, a U-polygon P is said to be of *class c* with respect to U if c is the maximal number of consecutive edges of P whose directions belong to U.

Definition 2.1. For a subset $\varLambda \subset \mathbb{C}$, we denote by $\mathbb{K}_\varLambda$ the intermediate field of $\mathbb{C}/\mathbb{Q}$ that is given by

$$\mathbb{K}_\varLambda := \mathbb{Q}\left((\varLambda - \varLambda) \cup (\overline{\varLambda - \varLambda})\right) .$$

Further, we set $\mathbb{k}_\varLambda := \mathbb{K}_\varLambda \cap \mathbb{R}$, the maximal real subfield of $\mathbb{K}_\varLambda$.

For $n \in \mathbb{N}$, we always let $\zeta_n := e^{2\pi i/n}$, as a specific choice for a primitive nth root of unity in $\mathbb{C}$. Denoting by ϕ Euler's totient function, one has the following standard result for the nth cyclotomic field $\mathbb{Q}(\zeta_n)$.

Fact 2.2 (Gauß)**.** [21, Theorem 2.5] $[\mathbb{Q}(\zeta_n) : \mathbb{Q}] = \phi(n)$. *The field extension $\mathbb{K}_n/\mathbb{Q}$ is a Galois extension with Abelian Galois group $G(\mathbb{Q}(\zeta_n)/\mathbb{Q}) \simeq (\mathbb{Z}/n\mathbb{Z})^\times$, where $a \,(\mathrm{mod}\, n)$ corresponds to the automorphism given by $\zeta_n \mapsto \zeta_n^a$.*

It is well known that $\mathbb{Q}(\zeta_n + \bar{\zeta}_n)$ is the maximal real subfield of $\mathbb{Q}(\zeta_n)$ and is of degree $\phi(n)/2$ over $\mathbb{Q}$; see [21]. Throughout this text, we shall use the notation

$$\mathbb{K}_n = \mathbb{Q}(\zeta_n),\ \mathbb{k}_n = \mathbb{Q}(\zeta_n + \bar{\zeta}_n),\ \mathcal{O}_n = \mathbb{Z}[\zeta_n],\ \scriptstyle{\mathcal{O}}\displaystyle_n = \mathbb{Z}[\zeta_n + \bar{\zeta}_n] .$$

Note that that $\mathcal{O}_n$ and $\scriptstyle{\mathcal{O}}\displaystyle_n$ are the rings of integers in $\mathbb{K}_n$ and $\mathbb{k}_n$, respectively; cf. [21, Theorem 2.6 and Proposition 2.16]. For n odd, one has $\phi(2n) = \phi(n)$ by the multiplicativity of the arithmetic function ϕ and thus $\mathbb{K}_n = \mathbb{K}_{2n}$; cf. Fact 2.2.

Definition 2.3. For a set $\varLambda \subset \mathbb{R}^2$, we define the following properties:

(Alg) $[\mathbb{K}_\varLambda : \mathbb{Q}] < \infty$.

(Aff) For all finite subsets F of $\mathbb{K}_\varLambda$, there is a non-singular affine transformation Ψ of the plane such that $\Psi(F) \subset \varLambda$.

(Hom) For all finite subsets F of $\mathbb{K}_\varLambda$, there is a homothety h of the plane such that $h(F) \subset \varLambda$.

Moreover, we call $\varLambda$ *degenerate* if and only if $\mathbb{K}_\varLambda$ is a subfield of $\mathbb{R}$.

Remark 2.4. For any non-degenerate $\varLambda \subset \mathbb{R}^2$, the field $\mathbb{K}_\varLambda$ is a complex extension of $\mathbb{Q}$. Trivially, property (Hom) implies property (Aff). If $\varLambda$ satisfies property (Alg), then one has $[\mathbb{k}_\varLambda : \mathbb{Q}] < \infty$, meaning that $\mathbb{k}_\varLambda$ is a real algebraic number field.

We need the following result of Darboux [9] on second mid-point polygons, where the *midpoint polygon* $M(P)$ of a convex polygon P is the convex polygon whose vertices are the midpoints of the edges of P; compare also [13, Lemma 5] or [11, Lemma 1.2.9].

Fact 2.5. *Let P_0 be a convex n-gon in $\mathbb{R}^2$ with centroid at the origin. For each $k \in \mathbb{N}$, define $P_k := \sec(\pi/n)M(P_{k-1})$. Then the sequence $(P_{2k})_{k=0}^\infty$ converges in the Hausdorff metric to an affinely regular polygon.*

If, in the situation of Fact 2.5, P_0 is a U-polygon of class c, then, for all k, P_{2k} is a U-polygon of class c, whence also $R := \lim_{k\to\infty} P_{2k}$ is a U-polygon of class c. This proves the next

Lemma 2.6. *Let $U \subset \mathbb{S}^1$ be a finite set of directions and let $0 < c \leq \operatorname{card}(U)$. Then, there exists a U-polygon of class c if and only if there is an affinely regular U-polygon of class c.*

Let (t_1, t_2, t_3, t_4) be an ordered tuple of four distinct elements of the set $\mathbb{R} \cup \{\infty\}$. Then, its *cross ratio* $\langle t_1, t_2, t_3, t_4 \rangle$ is defined by

$$\langle t_1, t_2, t_3, t_4 \rangle := \frac{(t_3 - t_1)(t_4 - t_2)}{(t_3 - t_2)(t_4 - t_1)}\,,$$

with the usual conventions if one of the t_i equals ∞, thus $\langle t_1, t_2, t_3, t_4 \rangle \in \mathbb{R}$. The following property of cross ratios of slopes s_z of elements $z \in \mathbb{R}^2 \setminus \{0\}$ is standard.

Fact 2.7. *Let $z_j \in \mathbb{R}^2 \setminus \{0\}$, $j \in \{1, \dots, 4\}$, be four pairwise non-parallel elements of the Euclidean plane and let $A \in \mathrm{GL}(2, \mathbb{R})$. Then, one has*

$$\langle s_{z_1}, s_{z_2}, s_{z_3}, s_{z_4} \rangle = \langle s_{Az_1}, s_{Az_2}, s_{Az_3}, s_{Az_4} \rangle\,.$$

Lemma 2.8. [18, Fact 4.7] *For a set $\varLambda \subset \mathbb{R}^2$, the cross ratio of slopes of four pairwise non-parallel $\varLambda$-directions is an element of the field $\mathbb{k}_\varLambda$.*

3. The characterization

Theorem 3.1. *For a non-degenerate subset $\varLambda$ of the plane with property* (Aff) *and an even number $m \geq 8$, the following statements are equivalent:*

(i) *There is a U-polygon of class $c \geq 4$ in $\varLambda$ with m edges.*
(ii) *There is an affinely regular U-polygon of class $c \geq 4$ with m edges for a set U of $\varLambda$-directions.*
(iii) $\mathbb{k}_{m/2} \subset \mathbb{k}_\varLambda$.
(iv) *There is an affinely regular polygon in $\varLambda$ with* $\operatorname{lcm}(m/2, 2)$ *edges.*

If $\varLambda$ additionally fulfils property (Alg)*, then the above assertions only hold for finitely many values of m.*

Proof. Direction (i) $\Rightarrow$ (ii) immediately follows from Lemma 2.6. For direction (ii) $\Rightarrow$ (iii), let P be an affinely regular U-polygon of class $c \geq 4$ with m edges for a set U of $\varLambda$-directions. There is then a non-singular affine transformation Ψ of the plane such that $R_m = \Psi(P)$ is a regular m-gon. Since P is a U-polygon of class $c \geq 4$ for a set U of $\varLambda$-directions and since, by Fact 2.7, the cross ratio of slopes of directions of edges is preserved by non-singular affine transformations, there are four consecutive edges of R_m whose cross ratio q of slopes of their directions, say arranged in order of increasing angle with the positive real axis, is an element of $\mathbb{k}_\varLambda$; cf. Lemma 2.8. Applying a suitable rotation, if necessary, we may assume that these directions are given in complex form by $1, \zeta_m, \zeta_m^2$ and ζ_m^3; cf. Fact 2.7 again. Using $\sin(\theta) = -e^{-i\theta}(1 - e^{2i\theta})/2i$, one easily calculates that

$$\begin{aligned} q &= \frac{(\tan(\frac{3\pi}{m/2}) - \tan(\frac{\pi}{m/2}))(\tan(\frac{2\pi}{m/2}) - \tan(\frac{0\pi}{m/2}))}{(\tan(\frac{3\pi}{m/2}) - \tan(\frac{0\pi}{m/2}))(\tan(\frac{2\pi}{m/2}) - \tan(\frac{\pi}{m/2}))} = \frac{\sin(\frac{2\pi}{m/2})\sin(\frac{2\pi}{m/2})}{\sin(\frac{\pi}{m/2})\sin(\frac{3\pi}{m/2})} \\ &= \frac{(1 - \zeta_{m/2}^2)(1 - \zeta_{m/2}^2)}{(1 - \zeta_{m/2})(1 - \zeta_{m/2}^3)} = \frac{2 + \zeta_{m/2} + \bar{\zeta}_{m/2}}{1 + \zeta_{m/2} + \bar{\zeta}_{m/2}} \in \mathbb{k}_\varLambda\,. \end{aligned}$$

This implies that
$$\frac{q}{q-1} - 2 = \zeta_{m/2} + \bar{\zeta}_{m/2} \in \Bbbk_{\Lambda}\,,$$
the latter being equivalent to (iii). Direction (iii) $\Rightarrow$ (iv) is an immediate consequence of [19, Theorem 3.3] in conjunction with the identity $\Bbbk_{m/2} = \Bbbk_{\mathrm{lcm}(m/2,2)}$. For direction (iv) $\Rightarrow$ (i), assume first that $m/2$ is odd. Here, we are done since every affinely regular polygon in Λ with $\mathrm{lcm}(m/2,2) = m$ edges is a U-polygon of class $c = m/2$ with respect to any set U of directions parallel to $m/2$ consecutive of its edges. If $m/2$ is even, there is an affinely regular polygon P in Λ with $\mathrm{lcm}(m/2,2) = m/2$ edges. Attach $m/2$ translates of P edge-to-edge to P in the obvious way and consider the convex hull P' of the resulting point set. Clearly, P' is a U'-polygon in $\mathbb{K}_{\Lambda}$ of class $c = \mathrm{card}(U')$ with m edges, where U' consists of the $m/2$ pairwise non-parallel Λ-directions given by the edges and diagonals of P. By property (Aff), there is a non-singular affine transformation Ψ of the plane such that $\Psi(P')$ is a polygon in Λ. Then, $\Psi(P')$ is a U-polygon of class $c = \mathrm{card}(U)$ in Λ with m edges, where U is a set of $m/2$ pairwise non-parallel Λ-directions parallel to the elements of $\Psi(U')$. Assertion (i) follows. If Λ additionally has property (Alg), then $\Bbbk_{\Lambda}$ is an algebraic number field by Remark 2.4. Thus, the field extension $\Bbbk_{\Lambda}/\mathbb{Q}$ has only finitely many intermediate fields and the assertion follows from condition (iii) in conjunction with [19, Corollary 2.7, Remark 2.8 and Lemma 2.9]. □

Corollary 3.2. *Let $\mathbb{L}$ be a complex algebraic number field with $\overline{\mathbb{L}} = \mathbb{L}$ and let $\mathcal{O}_{\mathbb{L}}$ be the ring of integers in $\mathbb{L}$. Let Λ be a translate of $\mathbb{L}$ or a translate of $\mathcal{O}_{\mathbb{L}}$. Further, let $m \geq 8$ be an even number. Denoting the maximal real subfield of $\mathbb{L}$ by $\mathbb{l}$, the following statements are equivalent:*

(i) *There is a U-polygon of class $c \geq 4$ in Λ with m edges.*
(ii) *There is an affinely regular U-polygon of class $c \geq 4$ with m edges for a set U of Λ-directions.*
(iii) $\Bbbk_{m/2} \subset \mathbb{l}$.
(iv) *There is an affinely regular polygon in Λ with* $\mathrm{lcm}(m/2,2)$ *edges.*

Additionally, the above assertions only hold for finitely many values of m.

Proof. This follows immediately from Theorem 3.1 in conjunction with the fact that Λ has properties (Aff) and (Alg) with $\mathbb{K}_{\Lambda} = \mathbb{L}$; cf. [19, Section 3]. □

Remark 3.3. In particular, Corollary 3.2 applies to translates of complex cyclotomic fields and their rings of integers, respectively, with $\mathbb{l} = \Bbbk_n$ for a suitable $n \geq 3$; cf. Fact 2.2 and also compare the equivalences of Corollary 4.1 below.

4. Application to cyclotomic model sets

Delone subsets of the plane satisfying properties (Alg) and (Hom) were introduced as *algebraic Delone sets* in [18, Definition 4.1]. Note that algebraic Delone sets are always non-degenerate, since this is true for all relatively dense subsets of the plane. Examples of algebraic Delone sets are the so-called *cyclotomic model sets* Λ; cf. [18, Proposition 4.31]. By definition, any cyclotomic model set Λ is contained in a translate of $\mathcal{O}_n$, where $n \geq 3$, in which case the $\mathbb{Z}$-module $\mathcal{O}_n$ is called the *underlying* $\mathbb{Z}$*-module* of $\varLambda$. More precisely, for $n \geq 3$, let $.^\star\colon \mathcal{O}_n \to (\mathbb{R}^2)^{\phi(n)/2-1}$ be any map of the form

$$z \mapsto (\sigma_2(z), \dots, \sigma_{\phi(n)/2}(z)),$$

where the set $\{\sigma_2, \dots, \sigma_{\phi(n)/2}\}$ arises from $G(\mathbb{K}_n/\mathbb{Q}) \setminus \{\mathrm{id}, \bar{.}\}$ by choosing exactly one automorphism from each pair of complex conjugate ones; cf. Fact 2.2. Then, for any such choice, each translate of the set $\{z \in \mathcal{O}_n \,|\, z^\star \in W\}$, where $W \subset (\mathbb{R}^2)^{\phi(n)/2-1}$ is a sufficiently 'nice' set with non-empty interior and compact closure, is a cyclotomic model set with underlying $\mathbb{Z}$-module $\mathcal{O}_n$; cf. [16, 17, 18, 19] for more details and properties of (cyclotomic) model sets. Since $\mathcal{O}_n = \mathcal{O}_{2n}$ for odd n, we might restrict ourselves to values $n \not\equiv 2 \pmod 4$ when dealing with cyclotomic model sets with underlying $\mathbb{Z}$-module $\mathcal{O}_n$. With the exception of the crystallographic cases of translates of the square lattice $\mathcal{O}_4$ and translates of the triangular lattice $\mathcal{O}_3$, cyclotomic model sets are aperiodic (they have no non-zero translational symmetries) and have long-range order; cf. [18, Remark 4.23]. Well-known examples of cyclotomic model sets with underlying $\mathbb{Z}$-module $\mathcal{O}_n$ are the vertex sets of aperiodic tilings of the plane like the Ammann-Beenker tiling [1, 4, 14] ($n = 8$), the Tübingen triangle tiling [5, 6] ($n = 5$) and the shield tiling [14] ($n = 12$); cf. Figure 1 for an illustration. For definitions of the above vertex sets of aperiodic tilings of the plane in algebraic terms, we refer the reader to [17, Section 1.2.3.2] or [16]. As an immediate consequence of Theorem 3.1 in conjunction with [19, Corollary 4.1] and the identity $\mathbb{k}_{m/2} = \mathbb{k}_{\mathrm{lcm}(m/2,2)}$, one obtains the following

Corollary 4.1. *Let $m, n \in \mathbb{N}$ with $m \geq 8$ an even number and $n \geq 3$. Further, let $\varLambda$ be a cyclotomic model set with underlying $\mathbb{Z}$-module $\mathcal{O}_n$. The following statements are equivalent:*

(i) *There is a U-polygon of class $c \geq 4$ in $\varLambda$ with m edges.*
(ii) *There is an affinely regular U-polygon of class $c \geq 4$ with m edges for a set U of $\varLambda$-directions.*
(iii) $\mathbb{k}_{m/2} \subset \mathbb{k}_{\varLambda}$.
(iv) *There is an affinely regular polygon in $\varLambda$ with* $\mathrm{lcm}(m/2, 2)$ *edges.*
(v) $\mathbb{k}_{m/2} \subset \mathbb{k}_n$.
(vi) $m \in \{8, 12\}$*, or* $\mathbb{K}_{m/2} \subset \mathbb{K}_n$.
(vii) $m \in \{8, 12\}$*, or* $m|2n$*, or* $m = 4d$ *with d an odd divisor of n.*
(viii) $m \in \{8, 12\}$*, or* $\mathcal{O}_{m/2} \subset \mathcal{O}_n$.
(ix) $\mathcal{o}_{m/2} \subset \mathcal{o}_n$.

Remark 4.2. Combining Corollary 4.1 and Fact 2.7, one sees that the cross ratios of slopes of directions of edges of U-polygons of class $c \geq 4$ in cyclotomic model sets $\varLambda$, say arranged in order of increasing angle with the positive real axis, easily follow from a direct computation with a finite number of regular polygons; cf. [12, 8] for deep insights into this in the case of planar lattices.

The following consequence follows immediately from Corollary 4.1 in conjunction with [19, Corollary 4.2]. Restricted to values $n \not\equiv 2 \pmod 4$, it deals with the two cases where the degree $\phi(n)/2$ of $\mathbb{k}_n$ over $\mathbb{Q}$ is either 1 or a prime number $p \in \mathcal{P}$; cf. [19, Lemma 2.10].

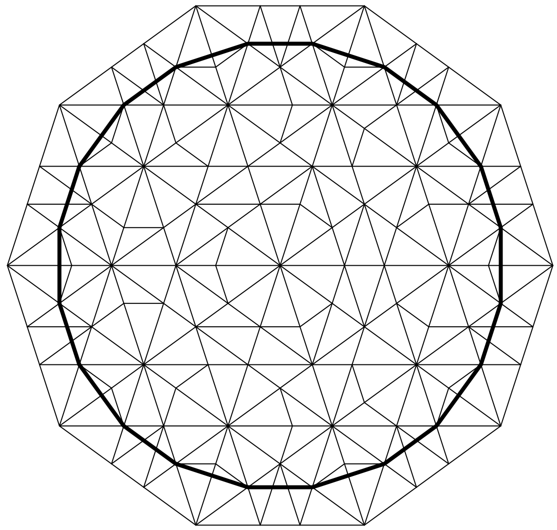

FIGURE 1. An U-icosagon of class $c = \mathrm{card}(U) = 10$ in the vertex set $\varLambda_{\mathrm{TTT}}$ of the Tübingen triangle tiling with respect to the set U of $\varLambda_{\mathrm{TTT}}$-directions given by the edges and diagonals of the central regular decagon.

Corollary 4.3. *Let $m, n \in \mathbb{N}$ with $m \geq 8$ an even number and $n \geq 3$. Further, let $\varLambda$ be a cyclotomic model set with underlying $\mathbb{Z}$-module $\mathcal{O}_n$. Then, one has:*

(a) *If $n \in \{3, 4\}$, there is a U-polygon of class $c \geq 4$ in $\varLambda$ with m edges if and only if $m \in \{8, 12\}$.*
(b) *If $n \in \{8, 9, 12\} \cup \{2p + 1 \,|\, p \in \mathcal{P}_{\mathrm{SG}}\}$, there is a U-polygon of class $c \geq 4$ in $\varLambda$ with m edges if and only if*

$$\begin{cases} m \in \{8, 12, 2n\}, & \text{if } n = 8 \text{ or } n = 12, \\ m \in \{8, 12, 2n, 4n\}, & \text{otherwise.} \end{cases}$$

Example 4.4. As mentioned above, the vertex set $\varLambda_{\mathrm{TTT}}$ of the Tübingen triangle tiling is a cyclotomic model set with underlying $\mathbb{Z}$-module $\mathcal{O}_5$. By Corollary 4.3 there is a U-polygon of class $c \geq 4$ in $\varLambda_{\mathrm{TTT}}$ with m edges if and only if $m \in \{8, 10, 12, 20\}$; see Figure 1 for an U-icosagon of class $c = 10$ in $\varLambda_{\mathrm{TTT}}$.

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