A NOTE ON COINCIDENCE ISOMETRIES OF MODULES IN EUCLIDEAN SPACE

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Abstract. It is shown that the coincidence isometries of certain modules in Euclidean n-space can be decomposed into a product of at most n coincidence reflections defined by their non-zero elements. This generalizes previous results obtained for lattices to situations that are relevant in quasicrystallography.

1. Introduction

The classification of grain boundaries in (quasi)crystals is closely related to the existence of coincidence submodules of their underlying $\mathbb{Z}$-modules; cf. [1], [3], [8], [12] and references therein. Here, two $\mathbb{Z}$-modules (additive subgroups) Γ, Γ' of Euclidean n-space are called *commensurate*, denoted by $\Gamma \sim \Gamma'$, if their intersection $\Gamma \cap \Gamma'$ has finite subgroup index both in Γ and Γ'. A (linear) *coincidence isometry* of Γ is an element f of the real orthogonal group $\mathrm{O}(n)$ such that $\Gamma \sim f(\Gamma)$. We denote by $\mathrm{OC}(\Gamma)$ the set of linear coincidence isometries of Γ. For $f \in \mathrm{OC}(\Gamma)$, the intersection $\Gamma \cap f(\Gamma)$ is called a *coincidence submodule* of Γ. In fact, for finitely generated $\mathbb{Z}$-modules Γ, the set $\mathrm{OC}(\Gamma)$ is a subgroup of $\mathrm{O}(n)$ (Corollary 2.3). For the main results, we consider free S-modules $\Gamma \subset \mathbb{R}^n$ of rank n that span $\mathbb{R}^n$, where S is a subring of $\mathbb{R}$ that is finitely generated as a $\mathbb{Z}$-module, and extend previous results obtained for the crystallographic case $S = \mathbb{Z}$, where Γ is a lattice, to the more general situation; cf. [15]. In particular, we consider *S-modules over K in $\mathbb{R}^n$* (Definition 3.4), where K is the field of fractions of S, and show that the coincidence isometries of these modules Γ can be decomposed into at most n coincidence reflections defined by non-zero elements of Γ (Corollary 3.5). This class of modules includes many examples that are relevant in (quasi)crystallography (Example 3.6).

2. Preliminaries and general results

Frequently, and without further mention, we use the consequence of Lagrange's theorem saying that, for any subgroup Γ' of an Abelian group Γ of finite subgroup index $m := [\Gamma : \Gamma']$, the group $m\Gamma$ is a subgroup of Γ'; compare [9, Ch. 1, Propositions 2.2 and 4.1].

Lemma 2.1. *Let $\Gamma, \Gamma' \subset \mathbb{R}^n$ be free $\mathbb{Z}$-modules of finite rank r. The following assertions are equivalent:*

(i) $\Gamma \sim \Gamma'$.
(ii) *$\Gamma \cap \Gamma'$ contains a free $\mathbb{Z}$-module of rank r.*
(iii) *$\Gamma \cap \Gamma'$ is a free $\mathbb{Z}$-module of rank r.*

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Proof. If $\Gamma \sim \Gamma'$, then the subgroup index $m = [\Gamma : (\Gamma \cap \Gamma')]$ is finite. Hence $\Gamma \cap \Gamma'$ contains the free $\mathbb{Z}$-module $m\Gamma$ of rank r. This proves direction (i) $\Rightarrow$ (ii). Since Γ is a free $\mathbb{Z}$-module of rank r and since $\mathbb{Z}$ is a principal ideal domain, direction (ii) $\Rightarrow$ (iii) follows from [9, Ch. 3, Theorem 7.1]. Direction (iii) $\Rightarrow$ (i) is an immediate consequence of [4, Ch. 2, Lemma 6.1.1]. $\square$

Lemma 2.2. *Commensurateness of free $\mathbb{Z}$-modules $\Gamma \subset \mathbb{R}^n$ of the same finite rank is an equivalence relation.*

Proof. Reflexivity and symmetry are clear by definition. For the transitivity, let $\Gamma_1 \sim \Gamma_2$ and $\Gamma_2 \sim \Gamma_3$. In particular, the indices $m_{12} = [\Gamma_2 : (\Gamma_1 \cap \Gamma_2)]$ and $m_{23} = [\Gamma_2 : (\Gamma_2 \cap \Gamma_3)]$ are finite. Since $m_{12}\Gamma_2 \subset (\Gamma_1 \cap \Gamma_2)$ and $m_{23}\Gamma_2 \subset (\Gamma_2 \cap \Gamma_3)$, one sees that $\Gamma_1 \cap \Gamma_3$ contains the free $\mathbb{Z}$-module $m_{12}m_{23}\Gamma_2$, which has the same finite rank as Γ_1 and Γ_3, whence $\Gamma_1 \sim \Gamma_3$ by Lemma 2.1. $\square$

Corollary 2.3. *For free $\mathbb{Z}$-modules $\Gamma \subset \mathbb{R}^n$ of finite rank,* $\mathrm{OC}(\Gamma)$ *is a group. In particular, it is a subgroup of* $\mathrm{O}(n)$*.*

Proof. Since commensurateness of free $\mathbb{Z}$-modules of the same finite rank is reflexive (Lemma 2.2), one has $\mathrm{id} \in \mathrm{OC}(\Gamma)$. Now let $f, g \in \mathrm{OC}(\Gamma)$, i.e., $\Gamma \sim f(\Gamma)$ and $\Gamma \sim g(\Gamma)$. By the symmetry of commensurateness of free $\mathbb{Z}$-modules of the same finite rank (Lemma 2.2), the latter implies $\Gamma \sim g^{-1}(\Gamma)$ and, further, that $f(\Gamma) \sim fg^{-1}(\Gamma)$. Finally, the transitivity property of commensurateness of free $\mathbb{Z}$-modules of the same finite rank gives $\Gamma \sim fg^{-1}(\Gamma)$ (Lemma 2.2). $\square$

We use the convention that subrings of $\mathbb{R}$ always contain the number 1. Recall that an algebraic number field K is a field extension of $\mathbb{Q}$ of finite degree $d := [K : \mathbb{Q}]$ over $\mathbb{Q}$. An order $\mathcal{O}$ of K is a subring of K that is free of rank d as a $\mathbb{Z}$-module. Among the various orders of K, there is one maximal order which contains all the other orders, namely the ring $\mathcal{O}_K$ of algebraic integers in K; cf. [4] for general background material on algebraic number fields. As a generalization of the ring $\mathbb{Z}$ of rational integers, we now work with subrings S of $\mathbb{R}$ that are finitely generated as a $\mathbb{Z}$-module and hence are free $\mathbb{Z}$-modules of finite rank s by [9, Ch. 3, Theorem 7.3]. These rings can be characterized as follows.

Proposition 2.4. *Let S be a subring of $\mathbb{R}$ that is finitely generated as a $\mathbb{Z}$-module. Then, its field K of fractions in $\mathbb{R}$ is a real algebraic number field and S is contained in $\mathcal{O}_K$, with equality if and only if S is a Dedekind domain. Vice versa, every subring S of the ring of algebraic integers in a real algebraic number field K is finitely generated as a $\mathbb{Z}$-module.*

Proof. Let S be a subring of $\mathbb{R}$ that is finitely generated as a $\mathbb{Z}$-module, say $S = \mathbb{Z}\alpha_1 + \cdots + \mathbb{Z}\alpha_m$. By the explanations given in §1 of [9, Ch. 7], every element of S is integral over $\mathbb{Z}$, i.e., $S \subset \mathcal{O}_K$. Since $S = \mathbb{Z}[\alpha_1, \ldots, \alpha_m]$, one has

$$K = \mathbb{Q}(\alpha_1, \ldots, \alpha_m)\,,$$

whence K is indeed a real algebraic number field; cf. [9, Ch. 5, Proposition 1.6]. Suppose that S is also a Dedekind domain and let $\alpha \in \mathcal{O}_K$. Trivially, this implies that α is integral over S. But this means that $\alpha \in S$ because, in particular, S is integrally closed. Hence, for Dedekind domains S, one has $S = \mathcal{O}_K$. On the other hand, rings $\mathcal{O}_K$ of algebraic integers in a real algebraic number field K are always Dedekind domains; cf. [10]. Now let S be a subring of the ring of algebraic integers $\mathcal{O}_K$ in a real algebraic number field K. Since $\mathcal{O}_K$ is free of rank $d := [K : \mathbb{Q}]$ as a $\mathbb{Z}$-module, S is finitely generated as a $\mathbb{Z}$-module; cf. [9, Ch. 3, Theorem 7.1]. □

Further, we consider free S-modules $\Gamma \subset \mathbb{R}^n$ of rank n that span $\mathbb{R}^n$. In other words, these modules Γ are the S-span of an $\mathbb{R}$-basis of $\mathbb{R}^n$. In particular, Γ is a free $\mathbb{Z}$-module of rank sn. In the simplest case $S = \mathbb{Z}$, Γ is a *lattice* in $\mathbb{R}^n$. Further, K will always denote the field of fractions of S in $\mathbb{R}$. There is the following characterization of commensurateness; compare [15, Theorem 2.1] for the lattice case. Originally, the proof in the case of lattices in Euclidean 3-space is due to Grimmer; cf. [8].

Theorem 2.5. *Let $\Gamma_1, \Gamma_2 \subset \mathbb{R}^n$ be free S-modules of rank n that span $\mathbb{R}^n$ and let $B_1, B_2 \in \mathrm{GL}(n, \mathbb{R})$ be basis matrices of the S-modules Γ_1 and Γ_2, respectively. The following assertions are equivalent:*

(i) $\Gamma_1 \sim \Gamma_2$.
(ii) *The intersection $\Gamma_1 \cap \Gamma_2$ contains a free S-module of rank n that spans $\mathbb{R}^n$.*
(iii) $B_2^{-1} B_1 \in \mathrm{GL}(n, K)$.

Proof. If $\Gamma_1 \sim \Gamma_2$, the subgroup index $m = [\Gamma : (\Gamma \cap \Gamma')]$ is finite. Hence, $\Gamma \cap \Gamma'$ contains the free S-module $m\Gamma$ of rank n that spans $\mathbb{R}^n$. This proves direction (i) ⇒ (ii). For direction (ii) ⇒ (iii), let $B \in \mathrm{GL}(n, \mathbb{R})$ be the basis matrix of an $\mathbb{R}$-basis contained in $\Gamma \cap \Gamma'$. This implies the existence of non-singular matrices $Z_1, Z_2 \in \mathrm{Mat}(n, S)$ such that

$$B_1 Z_1 = B = B_2 Z_2 \, ,$$

whence $B_2^{-1} B_1 = Z_2 Z_1^{-1} \in \mathrm{GL}(n, K)$ by the standard formula for the inverse of a matrix. Finally, for direction (iii) ⇒ (i), assume that $B_2^{-1} B_1 \in \mathrm{GL}(n, K)$. Then, there is a non-zero number $\alpha \in S$ such that $B := \alpha B_2^{-1} B_1 \in \mathrm{Mat}(n, S)$. The identity $\alpha B_1 = B_2 B$ now implies $\alpha \Gamma_1 \subset (\Gamma_1 \cap \Gamma_2)$. Since $\alpha \Gamma_1$ is a free $\mathbb{Z}$-module of rank sn, Lemma 2.1 implies $\Gamma_1 \sim \Gamma_2$. □

Corollary 2.6. *Let $\Gamma \subset \mathbb{R}^n$ be a free S-module of rank n that spans $\mathbb{R}^n$. Then, for all basis matrices $B_\Gamma \in \mathrm{GL}(n, \mathbb{R})$ of the S-module Γ, one has*

$$\mathrm{OC}(\Gamma) \simeq \mathrm{O}(n, \mathbb{R}) \cap (B_\Gamma \, \mathrm{GL}(n, K) B_\Gamma^{-1})$$

via the map that assigns to an element of $\mathrm{O}(n)$ its representing matrix with respect to the canonical basis of $\mathbb{R}^n$.

Proof. With $R \in \mathrm{O}(n, \mathbb{R})$, RB_Γ is a basis matrix of the S-module $R\Gamma$. The assertion now follows from Theorem 2.5. □

Example 2.7. For a free S-module Γ of rank n that spans $\mathbb{R}^n$ and is contained in K^n (and hence has a basis matrix $B_\Gamma \in \mathrm{GL}(n, K)$), Corollary 2.6 shows that

$$\mathrm{OC}(\Gamma) \simeq \mathrm{O}(n, K) := \mathrm{O}(n, \mathbb{R}) \cap \mathrm{GL}(n, K) \, .$$

In particular, this applies to the S-module S^n.

By definition, a *similarity isometry* of $\varGamma$ is an element $f \in \mathrm{O}(n)$ such that $\varGamma \sim \alpha f(\varGamma)$ for a suitable positive real number α; compare [2]. For a relation between the group $\mathrm{OC}(\varGamma)$ and its supergroup $\mathrm{OS}(\varGamma)$ of similarity isometries of $\varGamma$, we refer the reader to [7]. Our aim is now to gain some insight into the structure of the group $\mathrm{OC}(\varGamma)$ for free S-modules $\varGamma$ of rank n that span $\mathbb{R}^n$.

3. Decomposition of coincidence isometries into reflections

The proofs of this section are parallel to the corresponding ones of [15]. For convenience, we prefer to present the details. We start with the following relative of the well known Cartan-Dieudonné theorem on decomposing elements of $\mathrm{O}(n)$ into a product of at most n reflections; compare [11].

Theorem 3.1. *Let $\varGamma \subset \mathbb{R}^n$ be a free S-module of rank n that spans $\mathbb{R}^n$. If any reflection defined by a non-zero element of $\varGamma$ is a coincidence isometry of $\varGamma$, then, any coincidence isometry of $\varGamma$ can be decomposed as a product of at most n reflections defined by non-zero elements of $\varGamma$.*

Proof. Let $\{\gamma_1, \dots, \gamma_n\}$ be an S-basis of $\varGamma$ and let $f \in \mathrm{OC}(\varGamma)$. We argue by induction on n. Clearly, the assertion holds for $n = 1$. Assume that it holds for $n \geq 1$ and consider the case $n + 1$.

Assume first that $f(\gamma_1) = \gamma_1$ and let H be the hyperplane orthogonal to γ_1, i.e.,

$$H := \{x \in \mathbb{R}^{n+1} \mid \langle x, \gamma_1 \rangle = 0\}\,.$$

Then, H is an n-dimensional subspace of $\mathbb{R}^{n+1}$ and, due to $f \in \mathrm{O}(n+1)$, H is f-invariant. More precisely, one has $f(H) = H$. It follows that, via restriction, f induces an isometry of the n-dimensional Euclidean space H. Compare the orthogonal projection $\pi\colon \mathbb{R}^{n+1} \to H$ along γ_1, given by

$$\pi(x) = x - \frac{\langle x, \gamma_1 \rangle}{\langle \gamma_1, \gamma_1 \rangle}\gamma_1\,,$$

with the reflection ρ_{γ_1} of $\mathbb{R}^{n+1}$ along γ_1, given by

$$\rho_{\gamma_1}(x) = x - 2\frac{\langle x, \gamma_1 \rangle}{\langle \gamma_1, \gamma_1 \rangle}\gamma_1\,.$$

By assumption, one has $m := [\rho_{\gamma_1}(\varGamma) : (\varGamma \cap \rho_{\gamma_1}(\varGamma))] < \infty$, wherefore $m\rho_{\gamma_1}(\gamma_i) \in \varGamma$ for all $1 \leq i \leq n+1$. Hence, $2m\pi(\gamma_i) = m\gamma_i + m\rho_{\gamma_1}(\gamma_i) \in \varGamma$ for all $1 \leq i \leq n+1$. In other words,

$$2m\pi(\varGamma) \subset \varGamma\,. \tag{1}$$

Clearly, $\pi(\varGamma)$ is a free S-module of rank n with S-basis $\{\pi(\gamma_2), \dots, \pi(\gamma_{n+1})\}$ and, further, $\pi(\varGamma)$ spans the n-dimensional Euclidean space H. Secondly, we claim that the (co-)restriction $f|_H^H$ is a coincidence isometry of $\pi(\varGamma)$. In order to see this, note that, since $f \in \mathrm{OC}(\varGamma)$, the subgroup index $p := [\varGamma : (\varGamma \cap f(\varGamma))]$ is finite, whence $p\varGamma \subset (\varGamma \cap f(\varGamma))$. Further, since $f|_H^H\pi = \pi f$, one obtains that

$$\begin{aligned} p\pi(\varGamma) \;\subset\; \pi(\varGamma \cap f(\varGamma)) \;&\subset\; \pi(\varGamma) \cap f(\pi(\varGamma)) \\ &=\; \pi(\varGamma) \cap f|_H^H(\pi(\varGamma))\,, \end{aligned}$$

thereby proving the claim by virtue of Theorem 2.5. Finally, any reflection of $\pi(\Gamma)$ defined by a non-zero element of $\pi(\Gamma)$ is a coincidence isometry of $\pi(\Gamma)$. To see this, consider the reflection $\rho_{\pi(\gamma)}|_H^H$ of H along a non-zero element $\pi(\gamma)$ of $\pi(\Gamma)$. By Relation (1), one has $\rho_{\pi(\gamma)} = \rho_{2m\pi(\gamma)} = \rho_\lambda$, where $\lambda := 2m\pi(\gamma)$ is a non-zero element of Γ. Since $\rho_\lambda \in \mathrm{OC}(\Gamma)$ by assumption, the subgroup index $q := [\Gamma : (\Gamma \cap \rho_\lambda(\Gamma))]$ is finite, whence $q\Gamma \subset (\Gamma \cap \rho_\lambda(\Gamma))$. Further, since $\rho_\lambda|_H^H \pi = \pi\rho_\lambda$, one obtains that

$$\begin{aligned} q\pi(\Gamma) \ \subset \ \pi(\Gamma \cap \rho_\lambda(\Gamma)) \ &\subset \ \pi(\Gamma) \cap \rho_\lambda(\pi(\Gamma)) \\ &= \ \pi(\Gamma) \cap \rho_\lambda|_H^H(\pi(\Gamma))\,, \end{aligned}$$

thereby proving the claim by virtue of Theorem 2.5. By the induction hypothesis, $f|_H^H$ is a product of at most n reflections defined by non-zero elements of $\pi(\Gamma)$, say

$$f|_H^H = \rho_{\pi(\lambda_1)}|_H^H \cdots \rho_{\pi(\lambda_j)}|_H^H\,.$$

By Relation (1) together with $\rho_{\pi(\lambda_i)} = \rho_{2m\pi(\lambda_i)}$ for all $1 \le i \le j$, one obtains that

$$f = \rho_{2m\pi(\lambda_1)} \cdots \rho_{2m\pi(\lambda_j)}\,.$$

This completes the proof in this case.

Secondly, assume that $f(\gamma_1) \neq \gamma_1$, whence $z := f(\gamma_1) - \gamma_1 \neq 0$. Since $f \in \mathrm{OC}(\Gamma)$, the subgroup index $m := [f(\Gamma) : (\Gamma \cap f(\Gamma))]$ is finite, wherefore $mz \in \Gamma$. Since $\rho_z = \rho_{mz}$, one sees that $\rho(z)$ is a reflection defined by a non-zero element of Γ. It follows that $\rho_z f \in \mathrm{OC}(\Gamma)$. One can easily verify that $\rho_z f(\gamma_1) = \gamma_1$. Hence, by the first case, $\rho_z f$ is the product of at most n reflections defined by non-zero vectors of Γ, say $\rho_z f = \rho_{\lambda_1} \dots \rho_{\lambda_j}$. Since $\rho_z^2 = \mathrm{id}$, one has

$$f = \rho_z \rho_{\lambda_1} \dots \rho_{\lambda_j} = \rho_{mz} \rho_{\lambda_1} \dots \rho_{\lambda_j}$$

is a product of at most $n+1$ reflections defined by non-zero elements of Γ. This completes the proof in the second case. □

Theorem 3.2. *Let $\Gamma \subset \mathbb{R}^n$ be a free S-module of rank n that spans $\mathbb{R}^n$ and let $\{\gamma_1, \dots, \gamma_n\}$ be an S-basis of Γ. Then, any reflection defined by a non-zero element of Γ is a coincidence isometry of Γ if and only if, for all $1 \le i, j, k \le n$, one has*

$$\frac{\langle \gamma_i, \gamma_j \rangle}{\langle \gamma_k, \gamma_k \rangle} \in K\,.$$

Proof. Assume first that any reflection defined by a non-zero element of Γ is a coincidence isometry of Γ. In particular, the reflections ρ_{γ_i}, where $1 \le i \le n$, are coincidence isometries of Γ, whence $m_i := [\rho_{\gamma_i}(\Gamma) : (\Gamma \cap \rho_{\gamma_i}(\Gamma))] < \infty$ and, further, $m_i \rho_{\gamma_i}(\Gamma) \subset \Gamma$ for all $1 \le i \le n$. Since, for all $1 \le i, j \le n$, one has

$$\rho_{\gamma_i}(\gamma_j) = \gamma_j - 2\frac{\langle \gamma_j, \gamma_i \rangle}{\langle \gamma_i, \gamma_i \rangle}\gamma_i\,,$$

the $\mathbb{R}$-linear independence of the γ_i shows that, for all $1 \le i, j \le n$, one has

$$\frac{\langle \gamma_i, \gamma_j \rangle}{\langle \gamma_i, \gamma_i \rangle} \in K\,. \tag{2}$$

If $\langle\gamma_i,\gamma_k\rangle \neq 0$, the last observation shows that

$$\frac{\langle\gamma_i,\gamma_i\rangle}{\langle\gamma_k,\gamma_k\rangle} = \frac{\langle\gamma_i,\gamma_i\rangle}{\langle\gamma_i,\gamma_k\rangle}\frac{\langle\gamma_i,\gamma_k\rangle}{\langle\gamma_k,\gamma_k\rangle} \in K\,.$$

If $\langle\gamma_i,\gamma_k\rangle = 0$, consider the reflection ρ_λ, where $\lambda := \gamma_i - \gamma_k \in \Gamma\setminus\{0\}$. By assumption, $\rho_\lambda \in \mathrm{OC}(\Gamma)$ and similar to the argumentation above, this gives

$$\frac{\langle\gamma_i,\lambda\rangle}{\langle\lambda,\lambda\rangle} = \frac{1}{1+\frac{\langle\gamma_k,\gamma_k\rangle}{\langle\gamma_i,\gamma_i\rangle}} \in K\,,$$

wherefore

$$\frac{\langle\gamma_i,\gamma_i\rangle}{\langle\gamma_k,\gamma_k\rangle} \in K\,.$$

Employing Equation (2), this shows that, for all for all $1 \leq i,j,k \leq n$, one has

$$\frac{\langle\gamma_i,\gamma_j\rangle}{\langle\gamma_k,\gamma_k\rangle} = \frac{\langle\gamma_i,\gamma_j\rangle}{\langle\gamma_i,\gamma_i\rangle}\frac{\langle\gamma_i,\gamma_i\rangle}{\langle\gamma_k,\gamma_k\rangle} \in K\,.$$

For the other direction, let $\gamma \in \Gamma\setminus\{0\}$, say $\gamma = \sum_{j=1}^n \alpha_j\gamma_j$ for uniquely determined $\alpha_j \in S$, $1 \leq j \leq n$. Then, by assumption, for any $1 \leq i \leq n$,

$$\frac{\langle\gamma_i,\gamma\rangle}{\langle\gamma,\gamma\rangle} = \frac{\sum_{j=1}^n \alpha_j\langle\gamma_i,\gamma_j\rangle}{\sum_{k,l=1}^n \alpha_k\alpha_l\langle\gamma_k,\gamma_l\rangle} = \frac{\sum_{j=1}^n \alpha_j\frac{\langle\gamma_i,\gamma_j\rangle}{\langle\gamma_i,\gamma_i\rangle}}{\sum_{k,l=1}^n \alpha_k\alpha_l\frac{\langle\gamma_k,\gamma_l\rangle}{\langle\gamma_i,\gamma_i\rangle}}$$

is an element of K. The reflection ρ_γ of $\mathbb{R}^n$ along γ satisfies

$$\rho_\gamma(x) = x - 2\frac{\langle x,\gamma\rangle}{\langle\gamma,\gamma\rangle}\gamma\,.$$

Since every element of K is of the form α/β, where $\alpha \in S$ and $\beta \in S\setminus\{0\}$, one sees that, for any $1 \leq i \leq n$, there is an element $\alpha_i \in S\setminus\{0\}$ such that $\alpha_i\rho_\gamma(\gamma_i) \in \Gamma$. Setting $\alpha := \prod_{i=1}^n \alpha_i$, one obtains $\alpha\rho_\gamma(\Gamma) \subset (\Gamma\cap\rho_\gamma(\Gamma))$, which implies that $\rho_\gamma \in \mathrm{OC}(\Gamma)$ by virtue of Theorem 2.5. This completes the proof. □

The following consequence of Theorems 3.1 and 3.2 is immediate.

Corollary 3.3. *Let $\Gamma \subset \mathbb{R}^n$ be a free S-module of rank n that spans $\mathbb{R}^n$. If there is a basis matrix $B_\Gamma \in \mathrm{GL}(n,\mathbb{R})$ of Γ such that $B_\Gamma^t B_\Gamma$ has only entries in K, then every non-zero vector of Γ defines a coincidence reflection of Γ and every coincidence isometry of Γ can be decomposed into a product of at most n reflections defined by non-zero elements of Γ.* □

Definition 3.4. We call a subset Γ of $\mathbb{R}^n$ an *S-module over K in $\mathbb{R}^n$* if Γ is a free S-module of rank n that spans $\mathbb{R}^n$ and satisfies $\langle\gamma,\gamma\rangle \in K$ for all $\gamma \in \Gamma$.

Due to the polarization identity, the relation $\langle\gamma,\gamma\rangle \in K$ holds for all $\gamma \in \Gamma$ if and only if one has $\langle\gamma,\gamma'\rangle \in K$ for all $\gamma,\gamma' \in \Gamma$. As an immediate consequence of Corollary 3.3, one obtains the following

Corollary 3.5. *Let $\Gamma \subset \mathbb{R}^n$ be an S-module over K in $\mathbb{R}^n$. Then, every non-zero vector of Γ defines a coincidence reflection of Γ and every coincidence isometry of Γ can be decomposed into a product of at most n reflections defined by non-zero elements of Γ.* □

Example 3.6. The $\mathbb{Z}$-modules over $\mathbb{Q}$ in $\mathbb{R}^n$ are precisely the *rational lattices* in $\mathbb{R}^n$; cf. [5, 6] for specific examples. Interesting examples of S-modules over K in $\mathbb{R}^4$ are given by the *icosian ring* and the *octahedral* ring. In three dimensions, there are the body and face centred *icosahedral modules* of quasicrystallography; cf. [1, 3] and references therein for details. An important class of planar examples is formed by the rings of *cyclotomic integers* in complex cyclotomic fields.; cf. [12, 14] and also see [7]. These rings appear in the description of planar mathematical quasicrystals with n-fold cyclic symmetry; see [13] for genuine quasicrystals with cyclic symmetries of orders $5, 8, 10$ and 12, respectively.

4. Outlook

In addition to the structure of the group of coincidence isometries, one is naturally interested in the characterization of the occuring coincidence submodules and subgroup indices, respectively. We hope to report on progress in this direction for the above setting in the near future.

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References

[1] Baake, M.: Solution of the coincidence problem in dimensions $d \leq 4$. In: R. V. Moody (Ed.): The Mathematics of Long-Range Aperiodic Order. NATO-ASI Series C **489**, Kluwer, Dordrecht (1997), pp. 9–44; revised version `arXiv:math/0605222v1 [math.MG]`
[2] Baake, M.; Grimm, U.; Heuer, M.; Zeiner, P.: Coincidence rotations of the root lattice A_4. European J. Combin., in press; `arXiv:0709.1341v1 [math.MG]`
[3] Baake, M.; Pleasants, P. A. B.; Rehmann, U.: Coincidence site modules in 3-space. Discrete Comput. Geom. **38** (2007) 111–138; `arXiv:math/0609793v1 [math.MG]`.
[4] Borevich, Z. I.; Shafarevich, I. R.: Number Theory. Academic Press, New York (1966).
[5] Conway, J. H.; Rains, E. M.; Sloane, N. J. A.: On the existence of similar sublattices. Can. J. Math. **51** (1999) 1300–1306.
[6] Conway, J. H.; Sloane, N. J. A.: Sphere packings, Lattices and Groups. 3rd ed., Springer, New York (1999).
[7] Glied, S.: Similarity and coincidence isometries for modules. Submitted.
[8] Grimmer, H.: Coincidence-site lattices. Acta Cryst. A**32** (1976) 783–785
[9] Lang, S.: Algebra. 3rd ed., Addison-Wesley, Reading, MA (1993).
[10] Neukirch, J.: Algebraic Number Theory. Springer, Berlin (1999).
[11] O'Meara, O. T.: Inroduction to Quadratic Forms. 3rd corr. printing, Springer, Berlin (1973).
[12] Pleasants, P. A. B.; Baake, M.; Roth, J.: Planar coincidences for N-fold symmetry. J. Math. Phys. **37** (1996) 1029–1058; corr. version, `arXiv:math/0511147v1 [math.MG]`.
[13] Steurer, W.: Twenty years of structure research on quasicrystals. Part I. Pentagonal, octagonal, decagonal and dodecagonal quasicrystals. Z. Kristallogr. **219** (2004) 391–446.
[14] Washington, L. C.: Introduction to Cyclotomic Fields. 2nd ed., Springer, New York (1997).
[15] Zou, Y. M.: Structures of coincidence symmetry groups. Acta Cryst. A**62** (2006) 109–114.

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