

# Word metrics on Lie groups and asymptotic cones of 2-step nilpotent groups

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## Introduction

The two inexhaustible sources of invariant distance functions on real Lie groups are Riemannian metrics and the "word norms". The relation between them is a subject of many papers but still many questions remained to be solved. The word norm on a connected topological group

$G$  is defined in the spirit of geometric group theory. Namely, if  $\Omega$  is a compact symmetric neighbourhood of identity such that  $G = \cup_{n \geq 1} \Omega^n$ , then a  $\Omega$ -word norm  $|g|_\Omega$  of an element  $g \in G$  is a minimal nonnegative  $n$  such that  $g \in \Omega^n$ . In other words,

$$|g|_\Omega = \min \{n : g = a_1 \cdots a_n, \quad a_i \in \Omega\}. \quad (1)$$

In a Lie group the exponential elements admit any real powers and this gives rise to some variation of a word norm. We say that a set  $A \subseteq \mathfrak{g}$  in a Lie algebra of  $G$  is an  $\mathbb{R}$ -generating set for a connected real Lie groups  $G$  if any  $g \in G$  can be represented as a product

$$g = \exp(t_1 a_1) \cdots \exp(t_k a_k)$$

for some  $a_i \in A, t_i \in \mathbb{R}, k \in \mathbb{N}$ . The Lie  $A$ -word norm  $|g|_A$  of  $g \in G$  (or simply a Lie word norm when  $A$  understood) is the infimum of the numbers  $\sum_1^k |t_i|$  over all possible such representations of  $g$ . Although this type of norm looks similar to a word norm, it can not be reduced to that one, even if the extended "weighted word norm" is being used. It turns out, rather unexpectedly, that Lie word norms are much closer to the norms coming from generalized Riemannian metric, namely

**Theorem A.** *For any connected real Lie group  $G$  the class of all Lie word norms on  $G$  coincides with the class of norms coming from Carnot-Caratheodory-Finsler metrics on  $G$ .*

In fact we will prove a more detailed version of this result, which describes very explicitly the correspondence between this two classes in one direction, see Theorem 17.

The Theorem A is related to the result of V.N.Berestovskii [Ber88], which describes in the same terms the structure of "inner norms" on connected real Lie groups.

We apply the Theorem A to the study of asymptotic cones of 2-step nilpotent groups. Our main result here is

**Theorem B.** *Let  $\Gamma$  be a finitely generated torsionfree 2-step nilpotent group with a horizontal word norm  $d$ . Then the asymptotic cone  $\mathcal{C}\Gamma$  is within a finite Hausdorff distance from  $\Gamma$ .*

The theorem was known in the case of 3-dimensional Heisenberg group [Kra99], [Kra01]. For abelian groups the result was known for much more wider class of norms, see [AM04], [Bur92].

**The plan of the paper.** In section 1 we introduce and study the notion of Lie word norm on a connected real Lie group and prove the Theorem A. In Section 2 we consider 2-step nilpotent groups and prove M.Stoll's result on approximation of Lie word norm by "ordinary" word norms. In Section 3 we apply this result to the description of the asymptotic cone of a 2-step nilpotent Lie group and we prove the Theorem B.

## 1 Word norm on a real Lie group

By a norm on a group  $G$  we mean a function  $g \mapsto |g| : G \rightarrow \mathbb{R}$ , satisfying the axioms:

**N1**  $|g| \geq 0$  for all  $g \in G$  and  $|g| = 0$  if and only if  $g = 1$  (Positivity);

**N2**  $|g| = |g^{-1}|$  for all  $g \in G$  (Symmetry);

**N3**  $|gh| \leq |g| + |h|$  for all  $g, h \in G$  (Triangle inequality).

Any norm gives rise to a left invariant distance  $d$  on  $G$  as usual:  $d(g, h) \stackrel{def}{=} |g^{-1}h|$ . And conversely, any left invariant distance defines a norm by setting  $|g| = d(1, g)$  for  $g \in G$ . Thus we may frequently and interchangeably use both notions.

### 1.1 Preliminaries on Lie groups

In this section we will freely use some basic facts from Lie group theory, see for example [GOV97]. Let  $G$  be a connected Lie group,  $\mathfrak{g} = T_e G$  be its Lie algebra. Let  $l_g$  denote the left translation by  $g \in G$ . For any  $\xi \in T_h G$  denote

$$g\xi = dl_g(\xi) \in T_{gh}G.$$

For any differentiable path  $g(t)$  in  $G$  we denote by  $\dot{g}(t) \in T_{g(t)}G$  its derivative. There is a unique path  $g'(t)$  in  $\mathfrak{g}$  such that

$$\dot{g}(t) = g(t)g'(t)$$

It is called velocity vector<sup>1</sup> and can be described directly as  $\xi(t) = dl_{g^{-1}(t)}\dot{g}(t)$ . Note that the velocity vector does not change under left

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<sup>1</sup>or "angular velocity in the body" in mechanics, see [Arn90]

translation of the curve by group element. For any  $\xi \in \mathfrak{g}$  there is a unique 1-parameter subgroup  $g_\xi(t), t \in \mathbb{R}$  in  $G$  with a constant velocity vector  $\dot{g}_\xi(t) = \xi$ . Set

$$\exp(\xi) \stackrel{def}{=} g_\xi(1), \quad (\xi \in \mathfrak{g}).$$

Such a defined map is the exponential map  $\exp : \mathfrak{g} \rightarrow G$ . By a linear transformation  $\xi \mapsto t\xi$  we obtain that  $\exp(t\xi) = g_\xi(t)$ , ( $t \in \mathbb{R}, \xi \in \mathfrak{g}$ ). The exponential map is a diffeomorphism on some neighbourhood of 0. For a basis  $\{e_1, \dots, e_n\}$  of  $\mathfrak{g}$  each of the following two maps

$$\begin{aligned} (t_1, \dots, t_n) &\mapsto \exp(t_1 e_1 + \dots + t_n e_n), \\ (t_1, \dots, t_n) &\mapsto \exp(t_1 e_1) \cdots \exp(t_n e_n) \end{aligned}$$

is a diffeomorphism of some neighbourhood of origin in  $\mathbb{R}^n$  into a neighbourhood of  $1 \in G$ . Such defined coordinate systems are called the coordinate systems of the first and second kind respectively in a neighborhood of the identity  $1 \in G$ .

## 1.2 $\mathbb{R}$ -generating sets and Lie word norms on a Lie group

Recall the standard way of geometrization of an abstract group. Let  $G$  be a finitely generated group and  $A$  a finite set and  $a \mapsto \bar{a}$  a map of  $A$  to a monoid generating set  $\bar{A} \subseteq G$ . As is usual,  $A^*$  denotes the free monoid on  $A$  and the natural projection  $A^* \rightarrow G$  is denoted  $w \mapsto \bar{w}$ . The Cayley graph  $C_A(G)$  is the directed graph with vertex set  $G$  and a directed edge from  $g$  to  $g\bar{a}$  for each  $g \in G$  and  $a \in A$ ; we give this edge a label  $a$ . We require that  $\bar{A} = \overline{A^{-1}}$ . The “distance function”  $d(g, h)$  is defined as the length of a shortest directed path from  $g$  to  $h \in C_A(G)$ . Each word  $w \in A^*$  defines a path  $[0, \infty) \rightarrow C_A(G)$  in the Cayley graph  $C_A(G)$  as follows (we denote this path also by  $w$ ):  $w(t)$  is the value of the  $t$ -th initial segment of  $w$  for  $t = 0, \dots, \ell(w)$ , is on the edge from  $w(s)$  to  $w(s+1)$  for  $s < t < s+1 \leq \ell(w)$  and equals  $\bar{w}$  for  $t \geq \ell(w)$ . We refer to the translate by  $g \in G$  of a path  $w$  by  $gw$ .

Passing to Lie groups it is natural to consider not only integral powers of group elements but also any real powers of them. This is possible to do for exponentials of Lie algebra elements. Thus we naturally come to

the following definitions. We say that a precompact set <sup>2</sup>  $A \subseteq \mathfrak{g}$  is an  $\mathbb{R}$ -generating set for a connected real Lie groups  $G$  if any  $g \in G$  can be represented as a product

$$g = \exp(t_1 a_1) \cdots \exp(t_k a_k)$$

for some  $a_i \in A, t_i \in \mathbb{R}, k \in \mathbb{N}$ . The expression  $w = a_1^{t_1} a_2^{t_2} \cdots a_k^{t_k}$  is called the Lie  $A$ -word representing  $g$  or the Lie  $A$ -name of  $g$ . Formally speaking  $w$  is an element of a free product  $\star_{a \in A} a^{\mathbb{R}}$  of exemplars of additive groups of reals. The number  $\sum_1^k |t_i| = |w|_A$  is called the  $A$ -norm of  $w$  and  $n$  is called the width of  $w$ . We also write  $\bar{w} = g$  and call  $w \mapsto \bar{w}$  the evaluation map. Associated to  $w$  is an  $\mathbb{R}$ -word path  $p_w : [0, T] \rightarrow G, T = \sum |t_i|$ , which on each interval

$$|t_1| + \cdots + |t_i| \leq t \leq |t_1| + \cdots + |t_i| + |t_{i+1}|, i = 1, \dots, k - 1.$$

is given by

$$p_w(t) = \exp(t_1 a_1) \cdots \exp(t_i a_i) \exp(t a_{i+1}).$$

This path starts at the unit element and ends at  $g$ . In fact  $p_w(t)$  is smooth on each subinterval since it is a translated segment of a 1-parameter subgroup. The Lie  $A$ -norm  $|g|_A$  of  $g \in G$  (or just a norm when  $A$  is understood) is the infimum of the norms of names of  $g$ . It is easy to see that Lie norm is non-negative and satisfies the triangle inequality. The point which is not entirely trivial is to check that the Lie norm is positive for nontrivial elements and we shall prove this later on in the case of precompact generating sets, see Lemma 5. For this reason we always assume our  $\mathbb{R}$ -generating sets to be precompact.

**Remark 1** *Note that our "Lie word norm" in general is different from the notion of word norm (possibly weighted) on a topological group. First, small difference, consists in the fact that if  $A \leq \mathfrak{g}$  is an  $\mathbb{R}$ -generating set in our sense, we can not formally put the weight  $t$  onto  $\exp(ta)$  since  $ta$  is not necessarily in  $A$ . Of course we can extend  $A$  by adding segments from 0 to  $a$ . But another essential difference is that we do not ask  $A$  to contain the neighbourhood of unity and this difference can not be repaired.*

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<sup>2</sup>A subset in a topological space is precompact if its closure is compact. Some authors (notably Bourbaki) use precompact differently - as a synonym for totally bounded. "Relatively compact" is then used to mean "precompact" as it is defined here.

### 1.3 Generating sets in Lie algebra and in Lie group

The following result surely is folklore but I was not able to find out the reference.

**Theorem 2** *Let  $G$  be a connected real Lie group and let  $\mathfrak{g}$  be its Lie algebra. The set  $A \subseteq \mathfrak{g}$  generates  $\mathfrak{g}$  as a Lie algebra if and only if  $A$  is an  $\mathbb{R}$ -generating set for  $G$ . Moreover, if  $A$  is an  $\mathbb{R}$ -generating set for  $G$  then there is a neighbourhood  $U$  of unity in  $G$  which has a bounded width relative to  $A$ . If  $G$  is a connected nilpotent Lie group and the set  $A \subseteq \mathfrak{g}$  generates  $\mathfrak{g}$  modulo  $[\mathfrak{g}, \mathfrak{g}]$ , then  $A$  is an  $\mathbb{R}$ -generating set for  $G$ .*

**Proof.** It is easy to prove the "only if" part. Suppose to the contrary that  $A$  generates a proper Lie subalgebra  $\mathfrak{h}$  in  $\mathfrak{g}$ . Then any 1-parameter subgroup  $\exp(ta)$ ,  $a \in A$  lies in a proper Lie subgroup  $H$  of  $G$ , corresponding to  $\mathfrak{h}$ , contradiction. Suppose now, that  $A$  generates  $\mathfrak{g}$ . We use the notation  $A_0 = \mathbb{R}A$ ,  $A_{i+1} = [A_0, A_i]$ . Then condition that the Lie algebra  $\mathfrak{g}$  be generated by the vector space  $A_0$  means that for some number  $m$  we have  $A_m = \mathfrak{g}$ . We take an arbitrary basis  $\{a_1, \dots, a_l\}$  in  $A_0$  with all  $a_i \in A$ . As in [Mit85], see also [Ber88] for a natural number  $k$  and a multiindex  $I = i_1 i_2 \cdots i_k$ , where  $1 \leq i_1, \dots, i_k \leq l$ , we denote by  $a_I$  the  $k$ -fold commutator

$$a_I = [a_{i_1}, [a_{i_2}, [\dots, [a_{i_{k-1}}, a_{i_k}] \dots]]].$$

Choose multiindices  $I_1, \dots, I_n$ , (assuming  $I_1 = 1, \dots, I_l = l$ ) of the length not longer than  $m$  such that the vectors  $a_{I_1}, \dots, a_{I_n}$  form a basis in  $\mathfrak{g}$ . For a multiindex of the form indicated, define a curve  $\alpha_I$  in  $G$  as follows. If  $I = i$ ,  $1 \leq i \leq l$ , then we set  $\alpha_i(t) = \exp(ta_i)$ . If  $I = iJ$ , then for  $t \geq 0$

$$\alpha_I(t) = \alpha_i(-\sqrt{t})\alpha_J(-\sqrt{t})\alpha_i(\sqrt{t})\alpha_J(\sqrt{t}).$$

and  $\alpha_I(t) = \alpha_I(-t)^{-1}$  for  $t \leq 0$ . Next we define the map  $\alpha : \mathbb{R}^n \rightarrow G$ , setting

$$\alpha(t_1, \dots, t_n) = \alpha_{I_1}(t_1)\alpha_{I_2}(t_2) \cdots \alpha_{I_n}(t_n), \quad (t_1, \dots, t_n) \in \mathbb{R}^n.$$

It is easy to see that  $\alpha$  is a  $C^1$ -map and

$$d\alpha \left( \frac{\partial}{\partial t_j} \right) (0) = a_{I_j}, \quad j = 1, \dots, n.$$

By the inverse function theorem  $\alpha$  is a  $C^1$ -diffeomorphism in some neighborhood of unity. On the other hand, for each  $(t_1, \dots, t_n) \in \mathbb{R}^n$  the element  $\alpha(t_1, \dots, t_n) \in G$  is a value of a certain  $\mathbb{R}$ -word

$$w(t_1, \dots, t_n) = w_{I_1}(t_1)w_{I_2}(t_2) \cdots w_{I_n}(t_n),$$

where the words  $w_I(t)$  are defined inductively similarly to  $\alpha_I(t)$  : if  $I = i$ ,  $1 \leq i \leq l$ , then we set  $w_i(t) = a_i^t$ . If  $I = iJ$ , then for  $t \geq 0$

$$w_I(t) = w_i(-\sqrt{t})w_J(-\sqrt{t})w_i(\sqrt{t})w_J(\sqrt{t}).$$

and  $w_I(t) = w_I(-t)^{-1}$  for  $t \leq 0$ . We infer that some neighbourhood  $U$  of unity is  $\mathbb{R}$ -generated by  $A$  and moreover by construction  $U$  has a bounded width relative to  $A$ . Next  $\cup_{n \geq 1} U^n$  is an open subgroup of  $G$  and thus in view of connectedness coincides with  $G$ . This means that  $A$  is an  $\mathbb{R}$ -generating set of  $G$ . For the nilpotent case, it is well known that  $A$  generates  $\mathfrak{g}$  as a Lie algebra if and only if  $A$  generates  $\mathfrak{g}/\mathfrak{g}_{ab}$  and similar statement is true in a Lie group, see f.e. [KM79], Theorem 6.2.5.

■

**Remark 3** *It is interesting to compare Theorem 2 with the following more delicate result from [BG03]: Let  $G$  be a connected real Lie group with Lie algebra  $\mathfrak{g}$  and suppose that the commutator subgroup  $[G, G]$  is dense in  $G$ . Then there is an identity neighborhood  $U$ , on which  $\log = \exp^{-1}$  is a well defined diffeomorphism, such that  $g_1, \dots, g_m \in U$  generate a dense subgroup whenever  $\log(g_1), \dots, \log(g_m)$  generate  $\mathfrak{g}$ .*

**Remark 4** *The theorem 2 raises a natural question: Is it true that a connected Lie group  $G$  is boundedly generated by any its finite  $\mathbb{R}$ -generating set  $A$ , that is, for  $n$  sufficiently large, the map  $(\mathbb{R}A)^n \rightarrow G$  given by  $t_1 a_1 + \dots + t_k a_k \mapsto \exp(t_1 a_1) \cdots \exp(t_k a_k)$  is onto? It seems it is easily true for a specific choice of generators in a connected simply-connected complex semisimple algebraic group via Bruhat-Tits decomposition, see [Hel01], Theorem 1.4, pg. 403. Besides, the answer is positive in the case of connected nilpotent Lie groups via Malcev's coordinates, see f.e. [KM79].*

#### 1.4 Equivalences of Riemannian and Lie word norms

Let  $G$  be a connected real Lie group. Recall that any Euclidean norm  $|\cdot|$  on a Lie algebra  $\mathfrak{g}$  of  $G$  gives rise to a left invariant Riemannian metric on  $G$ , namely the norm  $|\cdot|_g$  on the tangent space  $T_g$  at  $g \in G$  is given by  $|v|_g = |g^{-1}v|$ ,  $v \in T_g$ . In its turn, this Riemannian metric gives rise to a Riemannian norm on  $G$  as follows. Define the length of a continuous piecewise smooth path  $\alpha : [a, b] \rightarrow G$  as  $\int_a^b |\dot{\alpha}(t)| dt$ . The norm  $|g|$  of  $g \in G$  (with a slight ambiguity we retain the notation) is defined as the infimum of lengths of all paths between  $e$  and  $g$ . When proving that the function  $g \mapsto |g|$  is a norm, the only point which is not entirely trivial to check is positivity.

**Lemma 5** *For any connected Lie group  $G$  and any Riemannian metric on it the corresponding Riemannian norm  $|\cdot|$  is positive and its topology is compatible with the manifold topology on  $G$ . Moreover, the norm  $|\cdot|$  is proper and complete.*

**Proof.** Positivity is a standard fact of Riemannian geometry, see f.e. [CE75]. Namely, let  $(x_1, \dots, x_n)$  be a local coordinate system in the neighbourhood  $U$  about  $1 \in G$ , and let  $B(r)$  denote the closed Euclidean ball of small radius  $r$  about  $0$  in  $\mathbb{R}^n$ . In coordinates the Riemannian metric is a field  $P_x$  of positively definite symmetric matrices whose smallest eigenvalue  $\lambda(x)$  is positive and continuous. It follows easily that any path from  $0$  to the point of  $r$ -sphere has length at least  $r/\lambda_0$ , where  $\lambda_0$  is the infimum of  $\lambda(x)$  on the ball  $B(r)$ . Then any path from  $0$  to  $g$  must contain an initial subpath from  $0$  to the point of  $r$ -sphere hence its length is at least  $r/\lambda_0$  and thus the norm is at least  $r/\lambda_0$ . Similar argument shows that the norm topology is compatible with the manifold topology of  $G$ .

To prove completeness note that the previous argument shows that the small  $|\cdot|$ -ball  $B_{|\cdot|}(\varepsilon, 1)$  about  $1 \in G$  is compact. Then, by easy induction and compactness argument, a ball  $B_{|\cdot|}(\varepsilon, R)$  is compact for every  $R > 0$ , thus the norm is proper. Now any Cauchy sequence is contained in some ball, hence is convergent, thus the completeness. ■

### 1.4.1 Local equivalence

We are going to compare a Lie word norm with the Riemannian one. The Lemma below gives the global lower estimation of a Riemannian norm.

**Lemma 6** *Let  $G$  be a connected real Lie group and let  $\mathfrak{g}$  be its Lie algebra. For any Lie word norm  $|\cdot|_A$  corresponding to a  $\mathbb{R}$ -generating set  $A \subseteq \mathfrak{g}$  and any Riemannian norm  $|\cdot|$  on  $G$  the following estimation takes place for any  $g \in G$  :*

$$|g| \leq |A| |g|_A. \quad (2)$$

**Proof.** For a word  $w = a_1^{t_1} \cdots a_n^{t_n}$  representing  $g$ , the velocity vector of the path  $p_w(t)$  on the time interval  $I : |t_1| + \cdots + |t_i| \leq t \leq |t_1| + \cdots + |t_i| + |t_{i+1}|$  is equal to  $a_{i+1}$ . Indeed,  $p_w(t) = h \exp(ta_{i+1})$  for a constant  $h \in G$  on  $I$ . Note that the velocity vector does not change under left translation of the curve by group element. To get the Riemannian length of the path  $p_w(t)$  we integrate the speed (= the norm of the velocity vector) over the time interval. Hence the Riemannian length of  $p_w(t)$  does not exceed  $|A| (\sum |t_i|) = |A| \cdot |w|_A$ . Infimizing, we get the necessary estimation. ■

**Lemma 7** *Let  $G$  be a connected real Lie group and let  $\mathfrak{g}$  be its Lie algebra. For any Lie word norm  $|\cdot|_A$  and any Riemannian norm  $|\cdot|$  on  $G$  there is a neighbourhood  $U$  of unity in  $G$  and a continuous real function  $\psi$ , defined in the neighbourhood of zero in  $\mathbb{R}$  and vanishing at 0, such that for any  $g \in U$*

$$|g|_A \leq \psi(|g|). \quad (3)$$

.

**Proof.** For the second one let us estimate the norm of the word  $w_I(t)$ . To get the idea, consider examples:

$$w_i = a_i^t, w_{ij}(t) = a_i^{-\sqrt{|t|}} a_j^{-\sqrt{|t|}} a_i^{\sqrt{|t|}} a_j^{\sqrt{|t|}},$$

$$w_{ijk} = a_i^{-\sqrt[4]{t}} (a_j^{-\sqrt[4]{t}} a_k^{-\sqrt[4]{t}} a_j^{\sqrt[4]{t}} a_k^{\sqrt[4]{t}})^{-1} a_i^{\sqrt[4]{t}} (a_j^{-\sqrt[4]{t}} a_k^{-\sqrt[4]{t}} a_j^{\sqrt[4]{t}} a_k^{\sqrt[4]{t}}), 1 \leq i, j, k \leq l.$$

The norms clearly are  $|t|, 4\sqrt{|t|}, 2\sqrt{|t|} + 4\sqrt[4]{|t|}$  respectively. In general case define inductively the sequence of functions:  $\phi_1(t) = |t|, \phi_{i+1}(t) = 2\left(\phi_i\left(\sqrt{|t|}\right) + \sqrt{|t|}\right)$ . Then

$$|w(t_1, \dots, t_n)|_A = \sum_{i=1}^n \phi_{|I_i|}(t_i) \stackrel{def}{=} \phi(t_1, \dots, t_n).$$

Since  $w(t_1, \dots, t_n)$  evaluates onto the element  $\alpha(t_1, \dots, t_n)$ , we obtain

$$|\alpha(t_1, \dots, t_n)|_A \leq \sum_{i=1}^n \phi_{|I_i|}(t_i).$$

Denoting by  $B_r$  the riemannian ball about 1 in  $G$  the following function  $\psi(r)$  is well defined for sufficiently small  $r$

$$\psi(r) = \max_{\alpha^{-1}B_r} \phi(t_1, \dots, t_n).$$

Moreover,  $\psi(0) = 0$ ,  $\psi$  is continuous and positive outside of 0. Then for  $g \in B_{|g|}$  sufficiently closed to 1, there is  $(t_1, \dots, t_n) \in \alpha^{-1}B_{|g|}$  such that  $g = \alpha(t_1, \dots, t_n)$ . Then

$$|g|_A = |\alpha(t_1, \dots, t_n)|_A \leq \phi(t_1, \dots, t_n) \leq \psi(|g|).$$

■

**Corollary 8** *A Lie word norm is compatible with the standard topology on  $G$  and the metric space  $(G, |\cdot|_A)$  is proper and complete.*

**Proof.** Compatibility follows from the Lemmas above. To prove completeness note that the previous argument shows that the small  $|\cdot|_A$ -ball  $B_{|\cdot|_A}(\varepsilon, 1)$  about  $1 \in G$  is compact. Then, by easy induction and compactness argument, a ball  $B_{|\cdot|_A}(R, 1)$  is compact for every  $R > 0$ , thus the norm is proper. Now any Cauchy sequence is contained in some ball, hence is convergent, thus the completeness. ■

#### 1.4.2 Global equivalence

In the previous section we have established a local equivalence of Riemannian and Lie word norms which is weaker than Lipschitz equivalence.

However, this distortion takes place only locally. We are going to show that our norms are "large-scale" Lipschitz equivalent. First we recall basic notions of metric geometry. For definitions, following below, we refer to [BH99]. For us, a curve or a path in a metric space  $(X, d)$  is a continuous map  $\alpha$  from a compact interval  $[a, b] \subset \mathbb{R}$  to  $X$ . The length  $l(\alpha)$  of a path  $\alpha : [a, b] \rightarrow X$  is

$$l(\alpha) = \sup \left\{ \sum d(\alpha(t_i), \alpha(t_{i+1})) : a \leq t_0 \leq t_1 \leq \dots \leq t_n = b \right\},$$

that is supremum is taken over all possible partitions (no bound on  $n$ ) with  $a \leq t_0 \leq t_1 \leq \dots \leq t_n = b$ . The length of  $\alpha$  is either a non-negative number or it is infinite. The path  $\alpha$  is said to be rectifiable if its length is finite. Any rectifiable path admits parameterization by arc length, which we will instantly use, see ([BH99], Ch. I, Pr. 1.20). A metric space  $(X, d)$  is said to be a length space if the distance between every pair of points  $x, y$  of  $X$  is equal to the infimum of the lengths of rectifiable curves joining them. (If there are no such curves then  $d(x, y) = \infty$ .) If  $(X, d)$  is a length space, then  $d$  is called a length distance (otherwise known as an interior distance). We need a simple criterion, [BH99].

**Lemma 9** *A complete metric space  $X$  is a length space if and only if it has approximate midpoints in the sense that for every  $x, y \in X$  and every  $\varepsilon > 0$  there exists  $z \in X$  such that  $\max\{d(x, z), d(z, y)\} \leq \varepsilon + d(x, y)/2$ .*

As an application we prove the following

**Lemma 10** *Both Riemannian and Lie word norms on a connected Lie group are the length norms.*

**Proof.** We provide the argument only for Lie word norm, since for Riemannian norm it is literally the same. Let  $A \subset \mathfrak{g}$  be an  $\mathbb{R}$ -generating set for a Lie group  $G$ . Recall, that by the Corollary8, the norm  $|\cdot|_A$  is complete. Hence we are allowed to apply the above Lemma. For this purpose consider for  $g \in G$  and  $\varepsilon > 0$  a word path  $w : [0, T] \rightarrow G$  from  $e$  to  $g$  such that  $T \leq \varepsilon + |x|_A$ . Then  $w : [0, \frac{T}{2}] \rightarrow G$  is also a word path and we assert that  $w(\frac{T}{2})$  is an  $\varepsilon$ -midpoint for the pair  $e, x$ . Indeed,

$|w\left(\frac{T}{2}\right)|_A \leq \frac{T}{2} \leq \frac{\varepsilon}{2} + \frac{|x|_A}{2}$ . The other inequality is obtained by translating  $w\left(\frac{T}{2}\right)$  to  $e$ . It follows that  $|\cdot|_A$  is a length norm. ■

Recall that, according to [Gro93], a normed group  $(G, |\cdot|)$  is quasi-geodesic if there exist  $\delta > 0$  and  $\lambda \geq 1$  such that, for all  $g \in G$ , there exists a sequence  $g_1, \dots, g_n$  of elements of  $G$  such that  $g = g_1 \cdots g_n$  and

$$\sum_{i=1}^n |g_i| \leq \lambda |g| \quad \text{and} \quad |g_i| \leq \delta \quad \text{for all } 1 \leq i \leq n. \quad (4)$$

**Lemma 11** *Any length norm on  $G$  is a quasi-geodesic norm.*

**Proof.** We assert that  $\delta = 1$  and  $\lambda = 2$  do the job. Indeed, inductively applying Lemma 9, we can prove that for every  $g \in G$ , every  $\varepsilon > 0$  and every integral  $k \geq 1$  there exist  $g_1, \dots, g_{2^k} \in G$  such that  $|g_i| \leq \frac{|g|}{2^k} + \varepsilon$  and  $g = g_1 \cdots g_{2^k}$ . For given  $g \in G$  choose  $k$  so that  $\frac{|g|}{2^k} \leq \frac{1}{2}$  and take  $\varepsilon = \frac{1}{2^k}$ , then there are  $g_1, \dots, g_{2^k} \in G$  such that  $|g_i| \leq \frac{|g|}{2^k} + \frac{1}{2^k} \leq 1$  ( $1 \leq i \leq 2^k$ ) and  $g = g_1 \cdots g_{2^k}$ . By triangle inequality  $\sum_{i=1}^{2^k} |g_i| \leq |g| + 1$ . This implies that if  $|g| \geq 1$  then  $\sum_{i=1}^{2^k} |g_i| \leq 2|g|$  as we need. In the remaining case, when  $|g| < 1$ , we just take  $g_1 = g$ , then inequality (4) clearly holds. ■

The norms  $|\cdot|_1, |\cdot|_2$  on a group  $G$  are weakly equivalent if each norm is bounded on each ball defined by other norm. In other words, there is a real function  $\psi$  of positive variable such that  $|\cdot|_2 \leq \psi(|\cdot|_1)$  and  $|\cdot|_1 \leq \psi(|\cdot|_2)$ . The norms  $|\cdot|_1, |\cdot|_2$  are large-scale Lipschitz equivalent if there exist  $C, D > 0$ , such that for all  $g \in G$

$$C|g|_1 - D \leq |g|_2 \leq C|g|_1 + D. \quad (5)$$

The seeming asymmetry of the definition is easily clarified. The following proposition is a variation of the one, formulated in [Gro93], p. 7 and proved in [GK04].

**Proposition 12** *Any two weakly equivalent quasi-geodesic norms on a group  $G$  are large-scale Lipschitz equivalent. In particular any Lie word norm on a connected Lie group  $G$  is large-scale Lipschitz equivalent to any Riemannian norm.*

**Proof.** By definition there exist  $\delta > 0$  and  $\lambda \geq 1$  such that, for all  $g \in G$ , there exists a sequence  $g_1, \dots, g_n$  of elements of  $G$  such that  $g = g_1 \cdots g_n$  and

$$\sum_{i=1}^n |g_i|_1 \leq \lambda |g|_1 \quad \text{and} \quad |g_i|_1 \leq \delta \quad \text{for all } 1 \leq i \leq n. \quad (6)$$

Subdivide the word  $g_1 \cdots g_n$  into subsequent subwords  $h_1, \dots, h_m, s$  so that for each  $i = 1, \dots, m$  we have  $|h_i|_1 \geq \delta/2$  but the initial subwords  $h_i$  have norm  $\leq \delta/2$ , and the word  $s$  has a norm  $\leq \delta/2$  as well as each of its initial subwords. Then  $\delta/2 \leq |h_i|_1 \leq 3\delta/2$  for each  $i = 1, \dots, m$ . By triangle inequality and weak equivalence we have

$$|g|_2 \leq \sum_1^m |h_i|_2 + |s|_2 \leq \sum_1^m \psi(|h_i|_1) + \psi(s) \leq m\psi\left(\frac{3\delta}{2}\right) + \psi\left(\frac{\delta}{2}\right). \quad (7)$$

On the other hand

$$m\frac{\delta}{2} \leq \sum_1^m |h_i|_1 \leq \sum_1^n |g_i|_1 \leq \lambda |g|_1 \quad (8)$$

From the second estimate, we conclude that  $m \leq 2\delta^{-1}\lambda |g|_1$ , which, combined with the first estimate, yields

$$|g|_2 \leq \frac{2}{\delta}\lambda\psi\left(\frac{3\delta}{2}\right) |g|_1 + \psi\left(\frac{\delta}{2}\right). \quad (9)$$

Exchanging the norms, we obtain the converse inequality. ■

**Remark 13** *In fact, it will be shown in the next section that if the linear span of an  $\mathbb{R}$ -generating set  $A$  coincides with a Lie algebra  $\mathfrak{g}$  of a Lie group  $G$  then the Lie word norm  $|\cdot|_A$  turns out to be a Finslerian norm and consequently it is Lipschitz equivalent to (any) Finslerian norm on  $G$ , in particular to any Riemannian norm. Lipschitz equivalence is claimed in ([Osi01], Lemma 3.1) for the case of an  $\mathbb{R}$ -generating set which is a basis of a Lie algebra. But the proof is not convincing to the present author. Particularly weak is the claim that any 1-parameter subgroup in a Lie group is a local geodesic. The simplest counterexample is a 2-dimensional solvable group  $\mathbb{R}_+ \rtimes \mathbb{R}$  with multiplication  $(a, b)(a', b') = (aa', ab' + b)$ . Such examples also exist in the class of 2-step nilpotent connected simply connected Lee groups [Ebe04].*

**Remark 14** *The Lipschitz equivalence is not true in general. Consider a subbundle  $H \subset T(V)$  of the tangent bundle of a smooth manifold  $V$ . Suppose that  $H$  is spanned by the vector fields  $X_i$  and  $T(V)$  is spanned by the commutators of degree  $\leq 2$  of  $X_i$ . It is shown in [Gro96], Section 1.1.B. that the CC-distance along the curve, transversal to  $H$ , is (infinitesimally)  $\geq \sqrt{\text{Riemannian distance}}$ .*

### 1.5 Lie word norm versus CCF-norm. The proof of Theorem A

In this section we prove that a Lie word norm on a connected real Lie group is of generalized Riemannian type, more precisely, it is a Carnot-Caratheodory-Finsler norm.<sup>3</sup>

**Nonholonomic distributions.** Let  $M$  be a smooth manifold endowed with a smooth distribution of  $k$ -planes. Such a distribution  $\Delta$  associates with each point  $m \in M$  a  $k$ -dimensional subspace of the tangent space  $T_m M$ . A piecewise continuously differentiable curve  $\alpha$  on  $M$  is said to be horizontal, if it is tangent to the distribution  $\Delta$  everywhere. The class of such curves will be denoted by  $\mathcal{PC}^1 = \mathcal{PC}^1(\Delta)$ . Similarly a horizontal vector field on  $X$  is defined. Following Rashevskii [Ras38], we shall call the  $k$ -distribution  $\Delta$  completely nonholonomic, if the smallest Lie algebra of smooth vector fields on  $M$  over the ring of smooth real functions on  $M$  containing the submodule of smooth horizontal fields on  $M$  (i.e., tangent to the distribution  $\Delta$ ), is the Lie algebra  $X(M)$  of all smooth vector fields on  $M$ . A continuous almost everywhere differentiable curve is called horizontal if its derivative is tangent to  $\Delta$  almost everywhere. It is proved in [Ras38] that any two points of a connected manifold  $M$  endowed with a completely nonholonomic distribution can be joined by a piecewise continuously differentiable horizontal curve.

**Sub-Finslerian geometry.** Suppose, that for each  $p \in M$  a norm  $F_p$  on  $\Delta_p$  is given, such that for any continuous horizontal vector field  $X$  on  $M$  the function  $p \mapsto F_p(X(p))$  is continuous. This field of norms we call by the sub-Finslerian metric (=Carnot -Caratheodory-Finsler metric=CCF-metric). If all norms are Euclidean ones then we speak about

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<sup>3</sup>Bewaring of reader's (and ours) confusion by two-faced understanding of the word "metric" in the riemannian geometry (as a distance and as a field of Euclidean norms), we retain the term "metric" only in riemannian context and use the term "distance (function)" when considering metric spaces.

sub-Riemannian metric or CC-metric. Suppose next that we distinguish a set  $\mathcal{H}$  of horizontal curves in  $M$ , which is closed under concatenations and such that the subset  $\mathcal{H}_{pq}$  of horizontal curves joining  $p$  and  $q$  is nonempty for every  $p, q \in M$ . By CCF- distance between the points  $p, q \in M$  is meant

$$d_{\mathcal{H}}(p, q) = \inf_{\omega \in \mathcal{H}_{pq}} \{\text{length}(\omega)\}, \quad (10)$$

It is understood that the length of a curve  $\omega$  is measured with respect to the norm  $F_p$ , defined on the  $k$ -dimensional subbundle of  $TM$  equipped with the distribution  $\Delta$ . By this definition CCF-distance a priori depends on the choice of  $\mathcal{H}$ .

Further we consider only the case when  $M$  is a Lie group  $G$ , thus we can speak about norms on  $G$  rather than about distances. We are going to prove that the class of CCF-distances on  $G$  coincides with the class of Lie word norms. The following is proven in [Ber88], in particular see Lemma 8 therein:

**Theorem 15** *Any CCF-norm  $|\cdot|$  on a Lie group  $G$  is a length norm, and the norm topology coincides with the manifold topology. Moreover, for any Riemannian norm  $|\cdot|_r$  on  $G$  there is a continuous real function  $\psi$ , defined in the neighbourhood of zero in  $\mathbb{R}$  and vanishing at 0 and such that  $|g| \leq \psi(|g|_r)$  for  $g$  sufficiently closed to 1.*

The main result of [Ber88] is the following

**Theorem 16** *Let  $G$  be a connected Lie group with a length norm  $\nu$ . Then there exists a vector subspace  $\mathfrak{g}_0$  of the Lie algebra  $\mathfrak{g}$  of  $G$  which generates  $\mathfrak{g}$  as a Lie algebra, and a norm  $|\cdot|$  on  $\mathfrak{g}$  such that  $\nu$  coincides with the CCF-norm corresponding to  $(\mathfrak{g}_0, |\cdot|)$ .*

The main result of this section is

**Theorem 17** *A norm on a connected real Lie group  $G$  is a Lie word norm if and only if it is a CCF-norm.*

In one direction the Theorem is easy to prove. By Lemma 10 a Lie word norm is a length norm, hence by the Theorem 16 it is a CCF-norm. This proof does not give any construction of a CCF-metric

corresponding to a given  $\mathbb{R}$ -generating system  $A$ . Rather lucid picture is given as follows. Associated to  $A$  is a linear span  $\mathbb{R}A \subset \mathfrak{g}$  and a left invariant distribution  $\{g(\mathbb{R}A)\}_{g \in G}$ . Then the CCF-metric in question is associated to the norm  $F$  on a subspace  $\mathbb{R}A$  given by

$$F(v) = \lim_{t \rightarrow 0} \frac{|\exp(tv)|_A}{t}, v \in \mathbb{R}A, \quad (11)$$

see Lemma 9 from [Ber88]. It follows from this formula that the unit  $F$ -ball  $B_F$  contains  $A$  and thus contains a closed convex hull  $\overline{ch(A)}$  of  $A$ . We do not know whether it is true that  $B_F = \overline{ch(A)}$ .

For the converse we claim that a CCF-norm associated to a pair  $(V, |\cdot|)$  is a Lie word norm corresponding to  $\mathbb{R}$ -generating system  $A = \{v : |v| \leq 1\}$ . As a preliminary step we formulate and prove now the "fellow traveller property" for paths in Lie groups, which means that the paths of speed bounded by  $C_0 > 0$  and whose velocity vectors differ by  $\varepsilon > 0$ , diverge distance at most  $C\sqrt{\varepsilon}$  with a constant  $C$  depending only on  $C_0$  and the choice of (Riemannian) measurement (compare [Pan83], Section (43), [Bre07], Lemma 6.8).

**Lemma 18** *Let  $G$  be a connected Lie group and let  $|\cdot|$  denote a Euclidean norm on the Lie algebra  $\mathfrak{g}$  as well as a corresponding Riemannian norm on  $G$ . For every  $C_0 > 0$  there is a constant  $C = C(|\cdot|, C_0) > 0$  with the following property. If  $x, y : [0, 1] \rightarrow G$  are continuous piecewise smooth paths in  $G$  satisfying conditions  $x(0) = y(0) = e$ ,  $|\dot{x}(t)|, |\dot{y}(t)| \leq C_0, \forall t \in [0, 1]$  then for sufficiently small  $\varepsilon > 0$  the following implication holds:*

$$|x'(t) - y'(t)| \leq \varepsilon \quad (\forall t \in [0, 1]) \implies |x^{-1}(t)y(t)| \leq C\sqrt{\varepsilon} \quad (\forall t \in [0, 1]). \quad (12)$$

**Proof.** In the course of the proof we will introduce step by step several constants  $C_1, C_2, \dots$ , each depending only on  $|\cdot|$  and

$C_0$ . In all cases the existence will be clear. Also, in subsequent calculations we sometimes skip parameter  $t$  in notations and we mean all estimates to be hold for every value of  $t$  in  $[0, 1]$ .

For the case when  $G$  is a linear Lie group we give the proof due to V. M. Gichev (private communication). In fact we will prove one more

estimate :  $|\dot{z}(t)|_{z(t)} \leq C\sqrt{\varepsilon}$  for each  $t \in [0, 1]$ . Suppose  $G \leq GL_n(\mathbb{R})$ . It follows from assumptions that the lengths of curves  $x, y$  are bounded from above by  $C_0$ , so they, as well as the curve  $z = x^{-1}y$  are contained in a closed ball  $B_{|\cdot|}(C_0)$  about  $1 \in G$ , which is contained in a Euclidean ball  $B_{euc}(C_1)$  in a space of  $(n \times n)$ -matrices  $Mat_n = \mathbb{R}^{n^2}$  (relative to the standard Euclidean norm). Let us estimate the length of the curve  $z$ . Differentiating, we obtain

$$\dot{z} = -x^{-1}\dot{x}x^{-1}y + x^{-1}\dot{y} = -x'z + zy' = (y' - x')z + [z, y']. \quad (13)$$

Denote  $f(t) = z(t) \cdot z(t)$  (the scalar product on matrices is given by  $(a_{ij}) \cdot (b_{ij}) = \sum a_{ij}b_{ij}$ ). Then

$$\dot{f} = 2\dot{z} \cdot z = 2(y' - x')z \cdot z + 2[z, y'] \cdot z \leq C_2\varepsilon + C_2f, \quad (14)$$

Now applying the Gronwall's inequality <sup>4</sup> we obtain  $f \leq C_3\varepsilon$  or equivalently  $|z| \leq C_3\sqrt{\varepsilon}$ . Further, we need the estimation  $|\dot{z}|_z \leq C_4\sqrt{\varepsilon}$  which now immediately follows from (13) ( $|\dot{z}|_z \in T_z$  means of course the derivative of  $z(t)$  at the point  $z(t)$ ).

Our next step is to consider the case when  $G$  is a connected virtual Lie subgroup in a linear Lie group [GOV97]. By definition this means that  $G$  is a Lie group and at the same time it is a subgroup of  $G_n = GL_n(\mathbb{R})$  for some  $n \geq 1$  and the natural embedding  $\iota$  of  $G$  into  $G_n$  is a homomorphism of Lie groups. The Euclidean norm  $|\cdot|$  can be extended onto Lie algebra  $\mathfrak{g}_n$  and thus to a left-invariant Riemannian metric  $|\cdot|_g$  on  $G_n$  such that  $d\iota$  is an isometric embedding at any point of  $G$ . Let  $x, y : [0, 1] \rightarrow G$  be continuous piecewise smooth paths satisfying conditions  $x(0) = y(0) = e$ ,  $|\dot{x}|_x, |\dot{y}|_y \leq C_0$ . If  $|x' - y'| \leq \varepsilon$  then we considering  $x, y$  as paths in a linear group  $G_n$ , hence we conclude from the above argument for linear Lie groups gives, for suitable  $C_1 > 0$ , the estimates

$$|z|_{G_n} \leq C_1\sqrt{\varepsilon}, \quad |\dot{z}|_z \leq C_1\sqrt{\varepsilon} \text{ for all } \varepsilon \text{ smaller than some } \varepsilon_0 > 0. \quad (15)$$

(We again skip parameter  $t$  in inequalities, assuming always  $t \in [0, 1]$ ). The second inequality is valid in  $G$  too, because  $\dot{z} \in T_zG$  and the

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<sup>4</sup>The Gronwall's inequality asserts that if  $\dot{f} \leq \varepsilon + Kf$  and  $f(0) = 0$ , then  $f(1) \leq \varepsilon \frac{e^K}{K}$ . Proof: denoting  $g = \varepsilon + Kf$  we have  $\dot{g} = K\dot{f} \leq Kg$  and then  $\dot{g}/g \leq K$ . Integrating, obtain  $\ln g(t) - \ln g(0) \leq K(t - a)$  thus  $\ln g(1) \leq K + \ln g(0)$ , thus  $g(1) \leq \varepsilon e^K$  and  $f(1) \leq \varepsilon \frac{e^K}{K}$ .

Riemannian metrics agree on  $G$  and  $G_n$ . And the first inequality is obtained by integration:

$$|z(t)|_{G_n} \leq \int_0^t |\dot{z}(s)|_{z(s)} ds \leq C_1 \sqrt{\varepsilon}. \quad (16)$$

For general connected Lie group  $G$  we proceed as follows. Consider the adjoint representation  $\pi : g \mapsto \bar{g}$  from  $G$  onto the linear group  $\bar{G} = \text{Ad}(G) \leq GL_n(\mathbb{R})$ . By the result of Yamabe [Yam50] any arcwise connected subgroup in a Lie group is a virtual Lie subgroup, hence  $\bar{G}$  is a virtual Lie subgroup in  $GL_n(\mathbb{R})$ . The kernel  $Z = \ker \pi$  is a central subgroup of  $G$ . The projection  $\pi$  is a submersion, which means that  $d\pi_g : T_g G \rightarrow T_{\pi g} \bar{G}$  is surjective for all  $g \in G$ . The implicit function theorem then guarantees that  $Z$  is a submanifold of  $G$  and thus  $Z$  is a Lie subgroup of  $G$ . The cosets  $gZ$  are called the fibers of the submersion. By the implicit function theorem, the tangent space  $V_g G$  to the fiber through  $g \in G$  is just the kernel of the differential of the projection  $\pi$ . This space is called the space of vertical vectors, and a vector field on  $G$  whose values at every point are vertical will be called a vertical vector field.

Now suppose that  $G$  is given with a left-invariant Riemannian metric  $|\cdot|_g$ . We supply  $\bar{G}$  with a left-invariant Riemannian metric  $|\cdot|_{\bar{g}}$  in such a way that the projection  $\pi$  became a Riemannian submersion in a sense that the horizontal space  $H_g G = V_g G^\perp$  at each  $g \in G$  is mapped by  $d\pi$  isometrically onto  $T_{\pi g} \bar{G}$ , i.e. is bijective and preserves the scalar product of tangent vectors.

We assert now that  $\pi$  is complete in the Ehresmann sense which means that any path  $\gamma$  in  $\bar{G}$  starting at  $h \in \bar{G}$  has a horizontal lift in  $G$  starting at any given point in the fiber  $\pi^{-1}h$  over  $h$ . Indeed, this follows from the completeness of Riemannian metric  $(G, |\cdot|_g)$ , see ([Bes87], Theorem 9.42 (due to R. Herman) and the last one is equivalent to the completeness of the norm  $|\cdot|$  by the Hopf-Rinow theorem.

Now suppose that  $x, y : [0, 1] \rightarrow G$  are continuous piecewise smooth paths in  $G$  satisfying conditions  $x(0) = y(0) = e$ ,  $|x'(t)|, |y'(t)| \leq C_0, \forall t \in [0, 1]$ . Since  $\pi$  is a Riemannian submersion, for descended

curves  $\bar{x}, \bar{y}$  we have  $|\bar{x}' - \bar{y}'| \leq \varepsilon$  and since  $\bar{G}$  is a virtual Lie subgroup of the linear Lie group  $GL_n(\mathbb{R})$ , by the previous argument we have  $\bar{y} = \bar{x}\bar{\alpha}$ , where the curve  $\bar{\alpha}$  satisfies  $|\bar{\alpha}|_{\bar{G}} \leq C_1\sqrt{\varepsilon}, \left| \dot{\bar{\alpha}} \right|_{\bar{\alpha}} \leq C_1\sqrt{\varepsilon}$ . Making use of completeness of  $\pi$  we lift  $\bar{\alpha}$  to the horizontal curve  $\alpha$  in  $G$ , such that  $\alpha(0) = e$  and  $|\alpha|_G \leq C_1\sqrt{\varepsilon}, |\dot{\alpha}|_\alpha \leq C_1\varepsilon$ . The images of  $y, x\alpha$  in  $\bar{G}$  coincide so they differ by a curve  $\beta$  lying in  $ZG = \ker \pi$ , that is,  $y = x\alpha\beta, \beta \in Z$ . Differentiating, we obtain

$$y' = x'^{\alpha} + \alpha' + \beta', \quad (17)$$

( $\alpha$ -exponent means the adjoint action) which we rewrite in the form

$$\beta' = -(Ad_\alpha - Id)x' + (y' - x') - \alpha'. \quad (18)$$

By assumption  $|y' - x'| \leq \varepsilon$ , thus we need only estimate the norm of operator  $Ad_\alpha - Id = Ad_{\bar{\alpha}} - Id$  for  $\bar{\alpha}$  close to  $e$ . Write  $\bar{\alpha} = \exp a, a \in \mathfrak{g}$ , then  $Ad_{\bar{\alpha}} = \exp(\text{ad}(a))$  and  $|a| \leq C_2\sqrt{\varepsilon}$ . Furthermore

$$|Ad_{\bar{\alpha}} - Id| = |\exp(\text{ad}(a)) - Id| \leq \exp|\text{ad}(a)| - 1 \leq \exp C_3|a| - 1 \quad (20)$$

$$\leq C_4|a| \leq C_5\sqrt{\varepsilon} \quad (21)$$

for  $|a|$  small enough.  $|\beta'| \leq C_6\sqrt{\varepsilon}$  that is  $|\beta| \leq C_6\sqrt{\varepsilon}$ . Thus we get that  $|x^{-1}y| = |\alpha\beta| \leq C\sqrt{\varepsilon}$  for suitable  $C = C(|\cdot|, C_0)$ . ■

**The end of the proof of the Theorem 17.** We have to prove that the CCF-norm associated to a pair  $(V, |\cdot|)$  is a Lie word norm corresponding to  $\mathbb{R}$ -generating system  $A = \{v : |v| \leq 1\}$ . We first fix some auxiliary Riemannian norm  $\|\cdot\|$  on  $G$ . By definition the CCF-norm  $|g|$  of the element  $g \in G$  can be approximated by the length  $T$  of the  $\mathcal{PC}^1$ -path  $\alpha(t) : [0, T] \rightarrow G$  of unit speed:  $|\alpha'(t)| = 1, t \in [0, T]$ . For a given  $\varepsilon > 0$  take a subdivision  $0 = t_0 < t_1 < \dots < t_n = T$  so that the velocity vector  $\alpha'$  is continuous on each segment of this subdivision and moreover the  $\|\alpha'(s) - \alpha'(t)\| < \varepsilon$  for  $s, t$  from one segment. Now approximate  $\alpha$  inductively:  $\beta(t) = \exp(t\alpha'(0))$  on  $[0, t_1]$ ,  $\beta(t) = \beta(t_1) \exp((t - t_1)\alpha'(t_1))$  on  $[t_1, t_2]$  and in general  $\beta(t) = \beta(t_i) \exp((t - t_i)\alpha'(t_i))$  on  $[t_i, t_{i+1}]$  for  $1 \leq i \leq n - 1$ . The

path  $\beta(t)$  is a word path corresponding to the word

$$w = \alpha'(0)^{t_1} \alpha'(t_1)^{t_2-t_1} \dots \alpha'(t_{n-1})^{t_n-t_{n-1}} \quad (22)$$

whose norm equals to  $T$ . By Lemma 18  $|\beta^{-1}(T)\alpha(T)|_r \leq C\varepsilon$ . Both CCF-norm and Lie word norm of  $\beta(T)$  are equal to  $T$ . The error  $\beta(T)^{-1}\alpha(T)$  can be made arbitrarily small in both metrics due to Lemma 15. Thus both the Lie word norm  $|g|_A$  and CCF-norm  $|g|$  coincide. ■

## 2 Lie word norm versus word norm. Two-step nilpotent Lie groups

Suppose  $G$  is a Lie group with the Lie algebra  $\mathfrak{g}$  and  $\Gamma$  is a finitely generated subgroup in  $G$ . Given an  $\mathbb{R}$ -generating set  $A \leq \mathfrak{g}$  of  $G$  it may happen that  $\exp A$  is contained in  $\Gamma$  and, moreover, generates it. Then we have two norms on  $\Gamma$ . The first is the word norm  $|\cdot|_{\exp A}$  corresponding to  $\exp A$  and the second is the restriction to  $\Gamma$  of the Lie word norm  $|\cdot|_A$  on  $G$ . Clearly the first one is greater or equals to the second one. The problem of further comparison of these two norms on  $\Gamma$  seems to be rather delicate. In this section we will solve this problem in the case when  $G$  is a connected simply connected 2-step nilpotent Lie group and  $\Gamma$  is a lattice in  $G$ . Namely, we will prove, following to [Sto98], that for a finite  $A = \exp A$  the two norms under consideration are coarsely equal, i.e. differ by a bounded amount.

### 2.1 2-step nilpotent Lie groups and algebras

By a free 2-step nilpotent real Lie algebra on  $n$ -generators we mean the space

$$\mathfrak{g}_n = \mathbb{R}^n \oplus \wedge^2 \mathbb{R}^n,$$

with a Lie bracket

$$[(x, u), (y, v)] = (0, x \wedge y).$$

The set  $A_n = \{(e_1, 0), (e_2, 0), \dots, (e_n, 0)\}$  generates  $\mathfrak{g}_n$  as a Lie algebra and  $\mathfrak{g}_n$  is "free" in a sense that any map from  $A_n$  into any 2-step nilpotent Lie algebra  $\mathfrak{g}$  can be extended to a Lie algebra homomorphism

from  $\mathfrak{g}_n$  to  $\mathfrak{g}$ . In particular, any 2-step nilpotent  $n$ -generated Lie algebra is a homomorphic image of  $\mathfrak{g}_n$ . The corresponding simply connected Lie group  $G_n$  is called the free 2-step nilpotent Lie group on  $n$  generators. The underlying set for  $G_n$  is  $\mathfrak{g}_n$  with a group multiplication given by

$$(x, u) \cdot (y, v) = (x + y, u + v + \frac{1}{2}(x \wedge y)).$$

The exponential map  $\exp : \mathfrak{g}_n \rightarrow G_n$  is identical. It is easy to see that the set  $A_n$  is an  $\mathbb{R}$ -generating set of  $G_n$ .<sup>5</sup> If  $G$  is any simply connected 2-step nilpotent Lie group with an  $n$ -element Lie generating set  $S = \{s_1, \dots, s_n\}$ , there is an epimorphism of Lie groups  $G_n \rightarrow G$  sending  $e_i$  to  $s_i$  for  $i = 1, \dots, n$ . Sometimes we find it convenient to denote the elements  $(x, 0)$  as  $x$  and  $(\lambda x, 0)$  as  $x^\lambda$ ,  $\lambda \in \mathbb{R}$ .

## 2.2 Bounded geodesic width

Recall that the norm of a Lie word  $w = a_1^{t_1} a_2^{t_2} \cdots a_n^{t_n}$  is  $|t_1| + |t_2| + \cdots + |t_n|$  and the width is  $n$ . A word  $w$  is geodesic if its norm equals to  $|\bar{w}|_{\mathbb{R}A}$ , that is  $\mathbb{R}$ -length of the element  $\bar{w}$ . We say that a Lie group  $G$  has a bounded geodesic width relative to an  $\mathbb{R}$ -generating set  $A$  if there is  $n_0$  such that any  $g \in G$  has a geodesic name of width  $\leq n_0$ . The following result is a slight generalization of the Lemma 3.3 in [Sto98].

**Theorem 19** (*Bounded geodesic width*) *Let  $A$  be an  $\mathbb{R}$ -generating set for the simply connected 2-step nilpotent Lie group  $G$ . Then  $G$  has a bounded geodesic width relative to  $A$ .*

**Proof.** It suffices to prove the theorem for  $G = G_n$  and  $A = A_n$ . We show first that any  $g \in G$  can be represented by a word of width  $\leq n^2$ . Take a word  $w(a_1, \dots, a_n)$  of the shortest width, representing  $g$  and suppose that this width is  $> n^2$ . Then at least one generator, say  $a_n$ , has  $> n$  occurrences in  $w$ . Write  $w = w_0 a_n^{t_1} w_1 a_n^{t_2} w_2 a_n^{t_3} \cdots w_{k-1} a_n^{t_k} w_k$ , where  $k > n$ , all  $w_i$  do not depend on  $a_n$  and  $t = (t_1, \dots, t_k)$  has nonzero coordinates. Write  $\overline{w_0 w_1 \cdots w_j} = (x_j, u_j)$  with  $x_j \in \sum_1^{n-1} \mathbb{R} e_i \subset \mathbb{R}^n$

<sup>5</sup>In fact, any basis of  $\mathbb{R}^n \subset \mathfrak{g}_n$ , serves as well as  $A_n$

and  $u_j \in \sum_{1 \leq i, k \leq n-1} \mathbb{R} e_i \wedge e_k \subset \wedge^2 \mathbb{R}^n$ . Then

$$\bar{w} = \left( x_k + \left( \sum_1^k t_i \right) e_n, u_k + \frac{1}{2} \left( \sum_1^k t_i x_{i-1} \right) e_n \right).$$

We wish to change from  $t$  to some other  $t'$  which: 1) does not affect  $\bar{w} = g$ , i.e. satisfies  $n$  linear equations  $\sum_1^k t_i = \sum_1^k t'_i$ ,  $\sum_1^k t_i x_{i-1} = \sum_1^k t'_i x_{i-1}$  and 2) does not increase the contribution to the Lie word norm, i.e.  $\sum |t'_i| \leq |t| = \sum |t_i|$ . We now show that conditions 1), 2) are satisfied by some vector  $t'$  at least one of whose coordinates is zero. This will contradict the minimality of width of  $w$ . Since  $k > n$ , the affine subspace  $T$  of  $\mathbb{R}^k$  given by the linear equations in 1), is at least one-dimensional. Since all  $t_i$  are nonzero, the set of  $t'$  satisfying  $t'_i t_i \geq 0$ ,  $|t'| \leq |t|$ ,  $i = 1, \dots, k$  is a  $k$ -dimensional simplex  $\Delta$ . The point  $t \in T \cap \Delta$  belongs to a face of  $\Delta$  given by inequality  $|t'| \leq |t|$ . Hence  $T$  has to meet one of the other faces of  $\Delta$  as well. The corresponding point  $t'$  has at least one nonzero coordinate and satisfies 1) and 2), thus we get the desired contradiction.

It remains to prove the existence of geodesic Lie word representing an arbitrary element  $g \in G$ . Consider the set of words  $W_g$  of width  $\leq n^2$  and such that each letter occurs with an exponent  $\leq |g|_A + 1$ . This set can be parameterized by a compact set in the space  $\mathbb{R}^{n^2}$ . The Lie word norm is a continuous function on  $W_g$  by the Corollary 8 and thus it achieves its infimum on some  $w \in W_g$ . This  $w$  is a geodesic Lie word representing  $g$ . ■

### 2.3 Approximation of $\mathbb{R}$ -paths by integral paths

In this section we prove the result of M. Stoll about comparison of Lie word norm and word norm in 2-step nilpotent groups.

**Theorem 20** ([Sto98]) *Let  $G$  be a simply connected 2-step nilpotent Lie group, and let  $\Gamma$  be a discrete cocompact subgroup in  $G$  with a finite generating set  $A$ . Then the Lie word norm  $|\cdot|_{\mathbb{R}A}$  and the word norm  $|\cdot|_A$  coarsely equal on  $\Gamma$  in a sense that they differ by a bounded amount.*

**Proof.** We first consider the case when  $G = G_n$  is the free 2-step nilpotent Lie group,  $n \geq 2$  and  $\Gamma = \Gamma_n$  is a cocompact discrete subgroup generated by the set  $A = A_n = (e_1, 0), \dots, (e_n, 0)$ . The elements of  $\Gamma_n$  are said to be integral (although it is not literally true: in fact  $\Gamma_n = (\sum \mathbb{Z}e_i, \frac{1}{2} \sum \mathbb{Z}(e_i \wedge e_j))$ ). We will use one more norm  $|\cdot|$  on  $G_n = (\mathbb{R}^n \oplus \wedge^2 \mathbb{R}^n)$ , namely sup-norm, corresponding to the basis  $\{e_i, e_i \wedge e_j\}, 1 \leq i, j \leq n$ . The expression  $a \stackrel{C}{=} b$  will denote the  $C$ -coarse equality, i.e. an equality up to an error not exceeding  $C$ . We will use the notation  ${}^g h = ghg^{-1}$  for a conjugation in a group.

We start from the problem of approximation of elements of  $G_n$  of the form  $(me_i, v \wedge me_i), m \in \mathbb{Z}, v \in \wedge^2 \mathbb{R}^n$  by elements of  $\Gamma_n$  of the form  $(me_i, w), w \in \sum \mathbb{Z}(e_i \wedge e_j)$  in such a way that  $(me_i, v \wedge me_i), (me_i, w)$  differ by a bounded amount and the  $A$ -length  $(me_i, w)$  is bounded from above by  $|m| + 2n(n-1)|v|$ . ■

**Lemma 21** (Cf. [Sto98], Lemma 4.2.) *Let  $G_n = \mathbb{R}^n \oplus \wedge^2 \mathbb{R}^n$  be a free 2-step nilpotent  $n$ -generated Lie group with a standard generating set  $A_n$ . Then  $\forall a \in \mathbb{N}$  and  $\forall v \in \mathbb{R}^n, s.t. |v| \leq a$  and  $\forall m \in \mathbb{Z}$  there is an element  $w \in \sum \mathbb{Z}(e_i \wedge e_j)$  such that  $w \stackrel{an}{=} v \wedge me_i$  (relative to the sup-norm on  $G_n$ ), and*

$$|(me_i, w)|_A \leq |m| + 2n(n-1)a.$$

**Proof of the Lemma.** For illustration consider first the case  $n = 2$ . For an element  $v_1 e_1 \wedge me_2, 0 \leq v_1 \leq a$  denote by  $m_1$  the integer next to  $\frac{mv_1}{a}$  and of modulus not greater than  $|\frac{mv_1}{a}|$ . We assert that one can take  $w = ae_1 \wedge m_1 e_2$ . Indeed  $ae_1 \wedge m_1 e_2 \stackrel{a}{=} ae_1 \wedge \frac{mv_1}{a} e_2 = v_1 e_1 \wedge me_2$ . To compute the word length we note that

$$(me_2, ae_1 \wedge m_1 e_2) = {}^{ae_1} (m_1 e_2, 0) \cdot (m - m_1) e_2, \quad (23)$$

from which it follows that  $|(me_i, w)|_A \leq |m| + 2a$ .

In general case, without loss of generality, assume that  $j = n, m > 0$  and  $0 \leq v_1 \leq v_2 \leq \dots \leq v_{n-1} \leq a, v_n = 0$  (re-order the basis and multiply by suitable signs to achieve this). Decomposition

$$v = \sum_1^{n-1} v_i e_i = v_1 e'_1 + (v_2 - v_1) e'_2 + \dots + (v_{n-1} - v_{n-2}) e'_{n-1},$$

where

$$e'_1 = \sum_1^{n-1} e_i, e'_2 = \sum_2^{n-1} e_i, \dots, e'_{n-1} = e_{n-1}, \quad (24)$$

allows to represent the element  $g = v \wedge m e_n$  in the form

$$a e'_1 \wedge m \frac{v_1}{a} e_n + a e'_2 \wedge m \frac{v_2 - v_1}{a} e_n + \dots + a e'_{n-1} \wedge m \frac{v_{n-1} - v_{n-2}}{a} e_n.$$

Similarly to 2-dimensional case we  $m_1 = \lceil m \frac{v_1}{a} \rceil, m_2 = \lceil m \frac{v_2 - v_1}{a} \rceil, \dots, m_{n-1} = \lceil m \frac{v_{n-1} - v_{n-2}}{a} \rceil$ . We assert that one can take

$$w = a e'_1 \wedge m_1 e_n + a e'_2 \wedge m_2 e_n + \dots + a e'_{n-1} \wedge m_{n-1} e_n. \quad (25)$$

Similarly to 2-dimensional case we show that  $w \stackrel{an}{=} v \wedge m e_n$ . Writing down each multiple as a conjugator allows to get the following presentation for  $g = (m e_n, w)$  :

$$g = a e'_1 (m_1 e_n) \cdot a e'_2 (m_2 e_n) \cdot \dots \cdot a e'_{n-1} (m_{n-1} e_n) \cdot ((m - \sum m_i) e_n).$$

It follows that the  $g$  is represented by a word of the length at most  $(2na + m_1) a + \dots + (2na + m_{n-1}) a + (m - \sum m_i) = m + 2n(n-1)a$ . (26)

Lemma is proved.  $\blacksquare$

Returning to the proof of the Theorem 20, we note that by bounded generation width  $\exists l$ , s.t.  $\forall g \in G_n$

$$g = (r_1 e_{i_1}, 0) \cdots (r_l e_{i_l}, 0) = \left( \sum_i r_i e_{k_i}, \frac{1}{2} \sum_{i < j} r_i e_{k_i} \wedge r_j e_{k_j} \right),$$

$$|g|_{\mathbb{R}A} = \sum |r_i|.$$

We have to show that for any  $g = (x, v) \in \Gamma$  the difference  $|g|_A - |g|_{\mathbb{R}A}$  is bounded independently of  $g$ . Fix a geodesic Lie word for  $g$  as above and denote  $g_j = (r_1 e_{k_1}, 0) \cdots (r_j e_{k_j}, 0) = (x_j, v_j)$ . Choose  $m_i = \lceil r_i \rceil$  or  $\lceil r_i \rceil + 1$  so that for each  $g'_j = (m_1 e_{k_1}, 0) \cdots (m_j e_{k_j}, 0) = (x'_j, v'_j), j = 1, \dots, c$  we have  $|x_j - x'_j| \leq \frac{1}{2}$ . Then  $x_l = x'_l = x$  and  $r_i = m_i + \varepsilon_i, |\varepsilon_i| \leq 1$ . Denote  $g' = (m_1 e_{k_1}, 0) \cdots (m_l e_{k_l}, 0)$ . We have

$$g = \prod ((m_i + \varepsilon_i) e_{k_i}, 0) = \left( \sum_i r_i e_{k_i}, v \right), \quad (27)$$

where

$$2v = \sum_{i<j} m_i e_{k_i} \wedge m_j e_{k_j} + \sum_{i<j} m_i e_{k_i} \wedge \varepsilon_j e_{k_j} + \sum_{i<j} \varepsilon_i e_{k_i} \wedge m_j e_{k_j} + \sum_{i<j} \varepsilon_i e_{k_i} \wedge \varepsilon_j e_{k_j}.$$

Collecting members, rewrite  $\sum_{i<j} m_i e_{k_i} \wedge \varepsilon_j e_{k_j} + \sum_{i<j} \varepsilon_i e_{k_i} \wedge m_j e_{k_j}$  in the form  $\sum_{j=1}^l u_j \wedge m_j e_{k_j}$  with  $|u_j| \leq l$  ( $j = 1, \dots, l$ ). Applying the above Lemma to each  $u_j \wedge m_j e_{k_j}$  we find  $u'_j \in \sum \mathbb{Z}(e_i \wedge e_j)$  such that each  $(m_j e_{k_j}, u'_j)$  coarsely  $g' = \prod (m_j e_{k_j}, u'_j) \in \Gamma_n$  coarsely coincides with  $g$  and  $|g'|_A \leq \sum m_j + 2n(n-1)l^2 \leq |g|_{\mathbb{R}A}$  coarsely coincides with that of  $g$ .

general case.

Let  $G$  be a simply connected 2-step nilpotent Lie group, and let  $\Gamma$  be a discrete cocompact subgroup with finite generating set  $S$ . Let  $S = \{s_1, \dots, s_n\}$ . There is a unique continuous group epimorphism  $\pi : G_n \rightarrow G$ , taking  $g_i$  to  $s_i, i = 1, \dots, n$ . Then  $\pi$  also gives an epimorphism  $\pi_\Gamma : \Gamma_n \rightarrow \Gamma$ . Note first that  $\Gamma_n \cap H$  is cocompact in  $H = \ker \pi$ . Indeed, if  $G = F\Gamma$  for a compact  $F$ , then  $\pi F \cap \Gamma$  is finite, say  $\pi F \cap \Gamma = \{\pi\gamma_1, \dots, \pi\gamma_k\}$ . Then  $\cup_i (H \cap F\gamma_i) = F'$  is a fundamental domain for  $\Gamma_n \cap H$  in  $H$ . To compare the metrics, note that for  $\gamma \in \Gamma$  and  $\gamma' \in \Gamma_n$  such that  $\pi\gamma' = \gamma$

$$|\gamma|_S = \inf |\gamma'(H \cap \Gamma_n)|_{S_n}, \quad |\gamma|_{\mathbb{R}S_n} = \inf |\gamma'H|_{\mathbb{R}S_n} = \inf |F'\gamma'(H \cap \Gamma_n)|_{\mathbb{R}S_n}.$$

The boundedness of the difference now follows.

### 3 Asymptotic cones

In this section we show that the asymptotic cone of a finitely generated torsionfree 2-step nilpotent group  $\Gamma$  with a horizontal word norm is within a finite Hausdorff distance from  $\Gamma$ . The theorem is known in the case of 3-dimensional Heisenberg group [Kra99], [Kra01].

### 3.1 Preliminaries

A non-principal ultrafilter on  $\mathbb{N}$  is a finitely additive measure  $\omega$  such that all subsets  $S \subset \mathbb{N}$  are measurable with  $\omega(S) \in \{0, 1\}$  and  $\omega(S) = 0$  for finite  $S$ . Let  $X$  be a metrics space. We say that  $x \in X$  is the ultralimit of a sequence  $(x_n)$  in  $X$  relative to  $\omega$ , and write  $\lim_{\omega} x_n = x$ , if for any positive  $\varepsilon$  the inequality  $d(x_n, x) < \varepsilon$  holds on a set of  $n$ -s of full measure. A nice feature of this notion is given by

**Lemma 22** *Any bounded sequence in any proper metric space  $X$  has an ultralimit relative to any ultrafilter on  $\mathbb{N}$ . For a fixed ultrafilter the ultralimit is unique.*

**Proof.** Recall that properness means that all closed balls in  $X$  are compact. Any bounded sequence  $(x_n)$  in  $X$  entirely lies in some closed, hence compact, ball  $K$ . If no point of  $K$  were a limit of  $(x_n)$ , then each  $q \in K$  would have a neighborhood  $V_q$  such that the inclusion  $x_n \in V_q$  holds almost anywhere. Then for ant finite subset  $F$  of  $K$  the inclusion  $x_n \in \cup_{q \in F} V_q$  also holds almost anywhere. It is clear now that no finite subcollection of  $(V_q)$  can cover  $K$  contradicting the compactness of  $K$ . The uniqueness is easy. ■

For a metric space  $X = (X, d)$  and positive number  $\varepsilon$  we denote by  $\varepsilon X$  the metric space  $(X, \varepsilon d)$ . Given a metric space  $X$  with a basepoint  $p$ , ultrafilter  $\omega$ , and a sequence of positive scaling factors  $\varepsilon_n \rightarrow 0$ , the associated asymptotic cone  $\mathcal{C}X = \mathcal{C}(X, p, (\varepsilon_n), \omega)$  is, roughly, a metric space which includes information on finite configurations of points which occur in almost all (measured with  $\omega$ ) of the metric spaces  $\varepsilon_n X$ . Precisely,  $\mathcal{C}X$  is the space of sequences  $(x_n)$  in  $X$  of growth rate at most  $\frac{1}{\varepsilon_n}$  (relative to  $p$ ), endowed with the pseudometric  $\lim_{\omega} \varepsilon_n d(x_n, y_n)$ , which made a metric by identifying the sequences zero pseudodistance apart. That is, we consider the sequences  $(x_n)$ , such that  $\varepsilon_n d(x_n, p) \leq C, \forall n \in \mathbb{N}$  for some constant  $C$ , depending on a sequence but, not on  $n$ . The equivalence  $(x_n) \sim (y_n)$  is given by condition  $\lim_{\omega} \varepsilon_n d(x_n, y_n) = 0$  and the equivalence class is denoted by  $[x_n]$ . Remarkable feature of the correspondence  $X \mapsto \mathcal{C}X$  is that it converts a coarsely isometric embedding  $X \rightarrow Y$  into isometric embedding  $\mathcal{C}X \rightarrow \mathcal{C}Y$ .

**Examples.** The map  $[x_n] \mapsto \lim_{\omega} \varepsilon_n x_n$  gives rise to the isometry  $\mathcal{C}\mathbb{R}^m = \mathbb{R}^m, m \in \mathbb{N}$ , where  $\mathbb{R}^m$  is considered with a metric induced by any norm. Next, consider  $\mathbb{Z}^n$  with a word norm  $|\cdot|_A$ , corresponding to a finite (symmetric) generating set  $A$ . Then  $\mathcal{C}\mathbb{Z}^m = \mathbb{R}^m$  with the same isometry, and  $\mathbb{R}^m$  is considered with a norm whose unit ball is a convex hull of the set  $A$ . The essential property, had been used in this example, is the presence of dilations on  $\mathbb{R}^m$ . We can generalize this as follows. Let  $(\delta_t)_{t>0}$  be a 1-parameter group of dilations of a metric space  $X$  with a basepoint  $p$ , that is each  $\delta_t$  is a bijection fixing  $p$ ,  $\delta_s \delta_t = \delta_{st}$ ,  $\forall s, t$ ,  $\delta_1 = id$ , and  $d(\delta_t(x), \delta_t(y)) = td(x, y)$ ,  $x, y \in X, \forall t$ . Then, if  $X$  is complete and proper, the isometry  $\mathcal{C}X = X$  takes place and it is given by the map  $\iota : (x_n) \mapsto \lim_{\omega} \delta_{\varepsilon_n}(x_n)$ . In the Proposition below we generalize this property even further.

**Proposition 23** *Suppose that a proper metric space  $(X, d)$  admits a 1-parameter group of dilations  $(\delta_t)_{t>0}$  with a basepoint  $p$ . Suppose that  $X_0$  is a coarsely dense subset of  $X$  (which means that some  $d$ -neighbourhood of  $X_0$  coincides with  $X$ ) and  $X_0$  is given with a metric  $d_0$  which is coarsely equivalent to  $d|_{X_0}$ , that is  $|d - d_0|$  is bounded on  $X_0$ . Then  $\mathcal{C}X_0$  is isometric to  $X$  via the map  $\iota : (x_n) \mapsto \lim_{\omega} \delta_{\varepsilon_n}(x_n)$ . Furthermore,  $(X_0, d_0)$  is within finite Hausdorff distance from  $(X, d)$ .*

**Proof.** First verify that the map  $\iota$  is well defined. Suppose  $[x_n] \in \mathcal{C}X_0$ , that is  $\varepsilon_n d_0(x_n, p) \leq C, n \in \mathbb{N}$  for some constant  $C$ . By assumption  $|d - d_0|$  is bounded by some constant  $C_1$ , hence  $d(\delta_{\varepsilon_n}(x_n), p) = \varepsilon_n d(x_n, p) \leq \varepsilon_n (d_0(x_n, p) + C_1) \leq C + C_1$  for  $n \gg 0$ , that is the sequence  $\delta_{\varepsilon_n}(x_n)$  is  $d$ -bounded. In view of properness  $\lim_{\omega} \delta_{\varepsilon_n}(x_n)$  exists.

Next, the map  $\iota$  is onto: for  $y \in X$  consider the sequence  $\delta_{\frac{1}{\varepsilon_n}}(y) \in X$ . By coarse density there is a sequence  $(x_n)$  in  $X_0$  such that  $d\left(x_n, \delta_{\frac{1}{\varepsilon_n}}(y)\right)$  is bounded. We assert that  $\iota[x_n] = y$ . Indeed  $\lim_{\omega} d(\delta_{\varepsilon_n}(x_n), y) = \lim_{\omega} \varepsilon_n d\left(x_n, \delta_{\frac{1}{\varepsilon_n}}(y)\right) = 0$ . The map  $\iota$  is isometric. Indeed, denoting  $d_{\omega}^0, d_{\omega}$  the limiting metrics in  $\mathcal{C}X_0, \mathcal{C}X$  respectively, for  $[x_n], [y_n] \in \mathcal{C}X_0$  we have  $d_{\omega}^0([x_n], [y_n]) = \lim_{\omega} \varepsilon_n d_0(x_n, y_n) = \lim_{\omega} \varepsilon_n d(x_n, y_n) = d_{\omega}([x_n], [y_n])$ . (In the last expression we keep in mind that  $[x_n], [y_n] \in \mathcal{C}X$ ).

To prove the final statement, we first recall some definitions. For a map of metric spaces  $f : X \rightarrow Y$  define the distortion  $\delta(f) = \sup_{x,y \in X} |d(x,y) - d(fx, fy)|$ . Setting  $\delta(X, Y) = \inf\{\delta(f) \mid f : X \rightarrow Y\}$  we define the Hausdorff distance between  $X$  and  $Y$  as  $d_H(X, Y) = \max\{\delta(X, Y), \delta(Y, X)\}$ . Thus to prove the finiteness of the Hausdorff distance  $d_H(X_0, X)$  it is sufficient (and necessary) to find maps  $X_0 \xrightarrow{f} X \xrightarrow{g} X_0$  of finite distortion. As  $f$  we take a natural embedding map, then its distortion is finite, since by assumption the function  $|d - d_0|$  is bounded on  $X_0$  (from above) by some constant  $C_1$ . Now, by assumption there is an  $r$  such that  $N_r(X_0) = X$  (neighbourhood is taken in metric  $d$  of course). Thus for any point  $x \in X$  there is a point  $g(x) \in X_0$  distance at most  $r$  away from  $x$ . The resulting map  $g$  has a distortion at most  $2r + 2C_1$ . ■

We say that the pseudodistances  $d, d'$  on a set  $X$  are asymptotically equivalent if  $\frac{d(x,y)}{d'(x,y)}$  tends to 1 when both  $d(x,y), d'(x,y)$  tend to infinity.

**Proposition 24** *Let  $d, d'$  be two asymptotically equivalent proper pseudodistances on a set  $X$ . Let  $(\delta_t)_{t>0}$  be a 1-parameter group of dilations of  $(X, d')$  with a basepoint  $p$ . Then for any nonprincipal ultrafilter  $\omega$  and any scaling sequence  $(\varepsilon_n)$  the asymptotic cone  $\mathcal{C}(X, d)$  is isometric to  $(X, d')$  via the map  $\iota : (x_n) \mapsto \lim_{\omega} \delta_{\varepsilon_n}(x_n)$ .*

**Proof.** First verify that the map  $\iota$  is well defined. Suppose  $[x_n] \in \mathcal{C}(X, d)$ , that is  $\varepsilon_n d(x_n, p) \leq C, n \in \mathbb{N}$  for some constant  $C$ . Then, by asymptotic equivalence,  $\varepsilon_n d'(x_n, p) \leq C + 1$  for  $n \gg 0$ . By dilation property  $d'(\delta_{\varepsilon_n}(x_n), p) = \varepsilon_n d'(x_n, p) \leq C + 1$  for  $n \gg 0$ , hence  $\delta_{\varepsilon_n}(x_n)$  is bounded and thus in view of properness of  $(X, d')$  it has a limit. If  $(y_n) \sim (x_n)$ , that is  $\lim_{\omega} \varepsilon_n d(x_n, y_n) = 0$ , then by asymptotic equivalence and dilation property  $\lim_{\omega} \varepsilon_n d'(x_n, y_n) = 0 = \lim_{\omega} d'(\delta_{\varepsilon_n}(x_n), \delta_{\varepsilon_n}(y_n))$ , that is  $\lim_{\omega} \delta_{\varepsilon_n}(x_n) = \lim_{\omega} \delta_{\varepsilon_n}(y_n)$ .

Let us verify that  $\iota$  is surjective. For  $x \in X$  consider the sequence  $(x_n)$  with  $x_n = \delta_{\frac{1}{\varepsilon_n}}(x)$ . We have  $\varepsilon_n d'(x_n, p) = d'(\delta_{\varepsilon_n}(x_n), p) = d'(x, p)$  is bounded. Hence  $\varepsilon_n d(x_n, p)$  is bounded too, that is  $[x_n]$  is defined. The image of  $[x_n]$  is precisely  $\lim_{\omega} \delta_{\varepsilon_n}(x_n) = x$ . Finally, putting  $\lim \delta_{\varepsilon_n} x_n = x$  and  $\lim \delta_{\varepsilon_n} y_n = y$ , isometricity follows from the following calculation:

$$d_\infty([x_n], [y_n]) = \lim \varepsilon_n d(x_n, y_n) = \lim \varepsilon_n d'(x_n, y_n) = \lim d'(\delta_{\varepsilon_n} x_n, \delta_{\varepsilon_n} y_n) = d'(x, y). \quad \blacksquare$$

## 4 Asymptotic cones of nilpotent groups

### 4.1 Nilpotent Lie algebras, gradings and automorphisms $\delta_t$ .

For a fuller treatment we refer the reader to [Bre07]. Let  $\mathfrak{g}$  be a finite dimensional nilpotent Lie algebra over  $\mathbb{R}$ . Define the lower central filtration by

$$\mathfrak{g}^1 = \mathfrak{g}, \quad \mathfrak{g}^{k+1} = [\mathfrak{g}, \mathfrak{g}^k],$$

then  $\mathfrak{g}^1 \supset \mathfrak{g}^2 \supset \dots \supset \mathfrak{g}^d \supset \mathfrak{g}^{d+1} = 0$ , where  $\mathfrak{g}^d \neq \{0\}$  and  $d$  is the length of filtration. For each  $k$  choose in  $\mathfrak{g}^k$  a complementary to  $\mathfrak{g}^{k+1}$  subspace  $V_k$  - we call such a choice by polarization of  $\mathfrak{g}$ . Then  $\mathfrak{g}$  is the linear direct sum of these subspaces

$$\mathfrak{g} = V_1 \oplus \dots \oplus V_d \tag{28}$$

and  $V_1$  generates  $\mathfrak{g}$  as a Lie algebra. We identify polarization  $\pi$  with a tuple  $(\pi_1, \dots, \pi_d)$  of natural projections  $\pi_i : \mathfrak{g} \rightarrow V_i, i = 1, \dots, d$  according to decomposition (28). Associated to each

polarization  $\pi$  is a 1-parameter group of linear automorphisms  $(\delta_t)_{t>0}$  of the vector space  $\mathfrak{g}$ , where  $\delta_t = \delta_t^\pi$  acts on  $V_k$  by multiplication by  $t^k, k = 1, \dots, d$ . In general  $(\delta_t)_{t>0}$  does not preserve the Lie bracket on  $\mathfrak{g}$ . This happens if and only if

$$[V_i, V_j] \subseteq V_{i+j} \text{ for every } i, j = 1, \dots, d \tag{29}$$

(recall that  $[V_i, V_j]$  is the subspace generated by all commutators  $[a, b], a \in V_i, b \in V_j$ ). If (29) holds, we say that  $\pi$  is a grading of the Lie algebra  $\mathfrak{g}$ , and that  $\mathfrak{g}$  is a graded Lie algebra. If (29) does not hold, we can however consider a new Lie algebra structure on the vector space  $\mathfrak{g}$  by defining the new Lie bracket as  $[x, y]_\pi = \pi_{i+j}([x, y])$  if  $x \in V_i$  and  $y \in V_j$ . For this new Lie algebra  $\mathfrak{g}_\pi$  polarization  $\pi$  turns out to be a grading, thus  $\mathfrak{g}_\pi$  is graded and  $\delta_t, t > 0$ , are all automorphisms of the Lie algebra  $\mathfrak{g}_\pi$ . The construction of graded Lie algebra  $\mathfrak{g}_\pi$  depends on the choice of  $\pi$  but any such  $\mathfrak{g}_\pi$  is isomorphic to the canonically defined

graded Lie algebra  $gr(\mathfrak{g}) = \bigoplus_i (\mathfrak{g}^i / \mathfrak{g}^{i+1})$  with the commutation, induced by those in  $\mathfrak{g}$ . Namely, the natural map  $V_i \rightarrow \mathfrak{g}^i \rightarrow \mathfrak{g}^i / \mathfrak{g}^{i+1}, i \geq 1$  gives rise to an isomorphism of vector spaces  $\mathfrak{g} \simeq gr(\mathfrak{g})$ , which at the same time is a Lie algebra isomorphism between the new Lie algebra structure  $\mathfrak{g}_\pi$  and  $gr(\mathfrak{g})$ .

## 4.2 Nilpotent Lie groups

Let  $\mathfrak{g}$  be a finite dimensional nilpotent Lie algebra over  $\mathbb{R}$ . Introduce a Lie group structure on  $\mathfrak{g}$  via Campbell–Hausdorff–Dynkin formula:

$$xy = \log(e^x e^y) = x + y + \frac{1}{2}[x, y] + P_3(x, y) + \dots + P_d(x, y), x, y \in \mathfrak{g}, \quad (30)$$

where  $P_k(x, y), k = 3, \dots, d$  is the sum of Lie products of the length  $k$ . Then the so defined Lie group  $G$  is a connected simply connected nilpotent real Lie group with a Lie algebra  $\mathfrak{g}$ .

Let  $\pi$  be a polarization of  $\mathfrak{g}$ . The associated  $(\delta_t^\pi)_{t>0}$  is a 1-parameter group of automorphisms both of the Lie algebra  $\mathfrak{g}_\pi$  and of the Lie group  $G_\pi$ . In particular, if  $\pi$  is a grading, then  $(\delta_t^\pi)_{t>0}$  is a 1-parameter group of Lie algebra and of Lie group automorphisms of  $\mathfrak{g}, G$  respectively. <sup>6</sup>

## 4.3 Asymptotic equivalence of a norm on a discrete nilpotent group to a norm on a corresponding graded Lie group

Let  $\Gamma$  be a finitely generated torsionfree nilpotent group. We fix an asymptotically geodesic norm  $|\cdot|$  on  $\Gamma$ . This means that for every  $\varepsilon > 0$  there exists  $s > 0$  such that for any  $g \in \Gamma$  one can find a sequence of elements  $g_1, g_2, \dots, g_n$  in  $G$ , each of the norm  $\leq s$ , such that  $g_1 g_2 \dots g_n = g$  and  $\sum_{i=1}^{n-1} |g_i| \leq (1 + \varepsilon) |g|$ . Moreover, we require  $|\cdot|$  to be locally bounded (= bounded in some neighbourhood of any point) and proper, that is all the balls are assumed to be finite.<sup>7</sup> Embed  $\Gamma$  into a connected simply connected nilpotent Lie group  $G$  as a cocompact discrete group

<sup>6</sup>In fact the original Lie bracket  $[x, y]$  on Lie group on  $\mathfrak{g}$  can be deformed continuously to  $[x, y]_\pi$  through a continuous family of Lie algebra structures by setting

$$[x, y]_t = \delta_{\frac{1}{t}} [\delta_t x, \delta_t y],$$

letting  $t \rightarrow \infty$ .

<sup>7</sup>Two main sources of such norms are word norms and displacement norms  $|g| = d(gx_0, x_0), g \in \Gamma$  for the isometric proper action  $\Gamma : X$ .

(Malcev's hull [Mal49]). Fix a relatively compact fundamental domain  $K$  for  $\Gamma$  in  $G$ , so that  $1 \in K$ ,  $G = \Gamma K$  and the product map  $\Gamma \times K \rightarrow X$  is bijective. Extend the norm to  $\Gamma$ -invariant pseudodistance  $d$  on  $G$  by setting  $d(x, y) = |\gamma_x^{-1}\gamma_y|$ , where  $x \in \gamma_x K, y \in \gamma_y K$  (preposition "pseudo" means that  $|g|$  might be zero for nonidentical  $g$ ). A routine verification shows that the asymptotic cones of  $\Gamma, G$  with reference point 1, are naturally isometric.

We identify, as before,  $G$  with its Lie algebra  $\mathfrak{g}$  and we fix some polarization  $\pi$  of  $\mathfrak{g}$ . We wish now to descend  $d$  to the norm onto subspace  $V_1 = \pi_1 \mathfrak{g}$  as follows: Let  $A$  be a topological closure of a convex hull of the set

$$A_0 = \left\{ \frac{\pi_1(x)}{d(1, x)} : x \in \mathfrak{g} \right\}. \quad (31)$$

Thus we have a Lie word norm  $|\cdot|_A$  on  $G$ , which, taking into account our Theorem A, can be described as a certain CCF-norm. One of the main results of E. Breuillard [Bre07], which generalizes the results of [Pan83], can be read off as follows:

**Theorem 25** ([Bre07], Thm. 6.2) *For a locally bounded proper asymptotically geodesic norm  $|\cdot|$  on a finitely generated torsionfree nilpotent group  $\Gamma$  the corresponding pseudodistance  $d$  on a Malcev hull  $G$  is asymptotically equivalent to the Lie word norm  $|\cdot|_A$  in the following sense:*

$$\lim_{|x^{-1}y|_A \rightarrow \infty} \frac{d(x, y)}{|x^{-1}y|_A} = 1. \quad (32)$$

What is the advantage of Lie word norm  $|\cdot|_A$  compare to the starting norm  $|\cdot|$ ? First it is (left)  $G_\pi$ -invariant, so it admits a big connected Lie group of isometries (although this Lie group is not isomorphic to  $G$  in general !) compare to a smaller group  $\Gamma$  of isometries of  $d$ . Second a 1-parameter group of automorphisms  $(\delta_t^\pi)_{t>0}$  of the Lie group  $G_\pi$  is a dilation group for  $|\cdot|_A$ .

The above Theorem and the presence of dilations together with Proposition 24 yields

**Theorem 26** *For a locally bounded proper asymptotically geodesic norm  $|\cdot|$  on a finitely generated torsionfree nilpotent group  $\Gamma$  the asymptotic*

cone  $\mathcal{C}(\Gamma, |\cdot|)$  is isometric to  $(G_\pi, |\cdot|_A)$  and isometry is given by  $[x_n] \mapsto \lim_\omega (\delta_{\varepsilon_n} x_n)$  all classes  $[x_n]$  in  $\mathcal{C}(\Gamma, |\cdot|)$ .

**Remark 27** If  $G$  is a simply connected nilpotent Lie group, not necessarily graded, then the group of dilations  $(\delta_t)_{t>0}$  associated to a choice of polarization  $\pi$  will not consist of automorphisms of  $\mathfrak{g}$ . Note also that if we are given two different choices of polarizations  $\pi, \pi'$ , then the Carnot-Carathéodory metrics on the corresponding graded Lie groups are isometric if and only if the normed spaces  $(V_1, |\cdot|)$  and  $(V'_1, |\cdot|')$  are isometric. It turns out that in general there a lot of graded group structures relative to which  $d'$  is left invariant.

#### 4.4 Proof of the Theorem B

Let  $\mathfrak{g}$  be a finite dimensional nilpotent Lie algebra over  $\mathbb{R}$  and let  $G = \mathfrak{g}$  be the corresponding Campbell-Hausdorff group. Let  $\Gamma < G$  be a cocompact lattice, that is a discrete subgroup with cocompact quotient  $\Gamma \backslash G$ . For a finite generating set  $A$  of  $\Gamma$  we have  $\mathfrak{g} = \mathbb{R}A + [\mathfrak{g}, \mathfrak{g}]$ , see for example [KM79], Theorem 6.2.5. We say that a finite generating set  $A$  of  $\Gamma$  is horizontal if  $\mathbb{R}A \cap [\mathfrak{g}, \mathfrak{g}] = 0$ . We fix a horizontal generating set  $A$  of  $\Gamma$  and we fix a polarization  $\pi$  of  $\mathfrak{g}$ , such that  $\mathfrak{g} = V_1 \oplus \dots \oplus V_d$  and  $V_1 = \mathbb{R}A$ . Note that  $A$  is an  $\mathbb{R}$ -generating set for  $G$ , since  $\mathfrak{g} = \mathbb{R}A + [\mathfrak{g}, \mathfrak{g}]$ , see again loc. cit.

**Lemma 28** *The group  $(\delta_t^\pi)_{t>0}$  is a 1-parameter group of dilations of the Lie word norm  $|\cdot|_A$ , namely  $|\delta_t^\pi(g)|_A = t |g|_A$  for  $g \in G, t > 0$ .*

**Proof.** For a  $\mathbb{R}$ -word  $w = a_1^{s_1} \dots a_m^{s_m}, a_1, \dots, a_m \in A$  define  $\delta_t(w) = a_1^{ts_1} \dots a_m^{ts_m}$  for  $t > 0$ , then

$$|g|_A = \inf\{|w|_A : \bar{w} = g\} = \inf\{\frac{1}{t} |\delta_t(w)|_A : \bar{w} = g\} = \frac{1}{t} |\delta_t(g)|_A.$$

■

From now on we assume that  $\mathfrak{g}$  is 2-step nilpotent. Then for any choice of horizontal generating system  $A$  of  $\Gamma$  the algebra  $\mathfrak{g}$  admits a grading  $\pi$  with  $V_1 = \mathbb{R}A, V_2 = [\mathfrak{g}, \mathfrak{g}]$ . The following result is a detailed version of the Theorem B.

**Theorem 29** *Let  $\Gamma$  be a finitely generated torsionfree 2-step nilpotent group with a horizontal generating set  $A$  and let  $G$  be the Malcev hull of  $\Gamma$ . Then the asymptotic cone  $\mathcal{C}(\Gamma, |\cdot|_A)$  is isometric to  $(G, |\cdot|_A)$  and it is within a finite Hausdorff distance from  $(\Gamma, |\cdot|_A)$ .*

**Proof.** We wish to use the Proposition 23 to prove the first statement. Since  $\Gamma$  is cocompact in  $G$ , we have  $G = \Gamma K$  for some compact set  $K$ , which thus has a bounded norm  $|K|_A$ . Then for any  $g \in G$  we can write  $g = \gamma k$  for some  $\gamma \in \Gamma, k \in K$  and hence  $d_A(g, \gamma) = |g^{-1}\gamma|_A \leq |k^{-1}|_A \leq |K^{-1}|_A$  which means that the set  $\Gamma$  is coarsely dense in  $(G, |\cdot|_A^G)$ . Next, the restriction of  $|\cdot|_A^G$  onto  $\Gamma$  is coarsely equivalent to  $|\cdot|_A^\Gamma$  by Theorem 20. Finally, by Lemma 28,  $(G, |\cdot|_A^G)$  admits 1-parameter group of dilations  $(\delta_t)_{t>0}$  with a basepoint  $1 \in G$ . Thus, by Proposition 23  $\mathcal{C}(\Gamma, |\cdot|_A^\Gamma)$  is isometric to  $(G, |\cdot|_A^G)$  via the isometry  $\iota : [x_n] \mapsto \lim_\omega \delta_{\varepsilon_n}(x_n)$ ,  $[x_n] \in \mathcal{C}(\Gamma, |\cdot|_A^\Gamma)$  and moreover  $(G, |\cdot|_A^G)$  is within a finite Hausdorff distance from  $(\Gamma, |\cdot|_A)$ . ■

#### 4.5 Examples

##### **A group which is not quasi-isometric to its asymptotic cone.**

For a finitely generated nilpotent torsionfree group  $\Gamma$  we denote by  $G$  its Malcev hull and by  $\mathfrak{g}$  the Lie algebra on  $G$ . If we consider several such  $\Gamma$ -s, say  $\Gamma_i$ , then the corresponding Lie groups and Lie algebras will be denoted by  $G_i, \mathfrak{g}_i$  respectively. There exist two finitely generated nilpotent groups  $\Gamma_1, \Gamma_2$  satisfying the following properties:

1. The graded Lie algebras  $gr(\mathfrak{g}_1), gr(\mathfrak{g}_2)$  are isomorphic,
2. The second Betti numbers of  $\mathfrak{g}_1, \mathfrak{g}_2$  are different.

(the examples were built up by Y. Benoist, exposition in [Sha04]). Moreover,  $\dim \mathfrak{g}_1 = \dim \mathfrak{g}_2$  can be an arbitrary integer  $\geq 7$ . It follows from the Theorem [Bre07] that for any choice of locally bounded proper asymptotically geodesic metrics  $d_1, d_2$  on  $G_1, G_2$  respectively, their asymptotic cones  $(\mathcal{C}G_1, d_{1\infty}), (\mathcal{C}G_2, d_{2\infty})$  are Campbell-Hausdorff groups built up on  $gr(\mathfrak{g}_1) = gr(\mathfrak{g}_2)$ , respectively, hence they are isomorphic

as Lie groups. Moreover they are quasi-isometric as metric spaces with their limit metrics  $d_{1\infty}, d_{2\infty}$ . Nevertheless, by the result of Y. Shalom [Sha04],  $\Gamma_1, \Gamma_2$  are not quasi-isometric because they have distinct Betti numbers. If both  $\Gamma_1$  and  $\Gamma_2$  were quasi-isometric to  $\mathcal{C}G_1$  and  $\mathcal{C}G_2$  respectively, then since  $\mathcal{C}G_1$  and  $\mathcal{C}G_2$  are quasi-isometric, they would be quasi-isometric to each other, contradicting to the previous claim. It follows that at least one of the groups  $\Gamma_1, \Gamma_2$  is not quasi-isometric to its asymptotic cone. The evident reason for this phenomena is that  $\Gamma_1, \Gamma_2$  are not graded. <sup>8</sup>

**Example of a 2-step nilpotent group which is infinite distance from its asymptotic cone.** The following example is due to E. Breuillard [Bre07] Consider the real Lie algebra  $\mathfrak{g}$  with a basis  $X, Y, Z, V$  such that  $Z, V$  are central elements and  $[X, Y] = Z$ , then the lower central filtration reduces to  $\mathfrak{g}^1 = \mathfrak{g}, \mathfrak{g}^2 = [\mathfrak{g}, \mathfrak{g}] = \mathbb{R}Z, \mathfrak{g}^3 = 0$ . Let  $G$  be a Lie group structure on  $\mathfrak{g}$  given by Campbell-Hausdorff formula, then  $G$  is a direct product  $H \times \mathbb{R}T$ , where  $H = \mathbb{R}X + \mathbb{R}Y + \mathbb{R}Z$ . The subgroup  $\Gamma = \mathbb{Z}X + \mathbb{Z}Y + \mathbb{Z}Z + \frac{1}{2}\mathbb{Z}T$  is a discrete cocompact subgroup in  $G$ . The set  $A = \{X, Y, TZ, TZ^{-1}\}$  generates  $\Gamma$  but is not a horizontal one. It turns out that any CCF-metric on  $G$  (and moreover any homogeneous quasi-norm) is unbounded distance away from  $|\cdot|_A$ .

**Questions and Problems.** Is it true that if  $G$  is a nilpotent non-graded Lie group with some nice metric then the asymptotic cone of  $G$  is not quasi-isometric to corresponding graded lie group each other? Is it true that for any polyhedral CCF-metric for any two points there is a geodesic between them which is piecewise  $C^1$ ? Is Stoll's result true for "nonword" norms too? It seems plausible that a connected simply connected nilpotent Lie group is of finite Hausdorff distance from its asymptotic cone if and only if it is graded.

## References

- [AM04] HERBERT ABELS AND GREGORY MARGULIS. Coarsely geodesic metrics on reductive groups. In "Modern dynamical systems and applications", pages 163–183. Cambridge Univ. Press, Cambridge (2004).
- [Arn90] V. I. ARNOL'D. "Mathematical methods of classical mechanics", volume 60 of "Graduate Texts in Mathematics". Springer-Verlag, New York (1990). Translated from the 1974

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<sup>8</sup>I am grateful to A. Karlsson who brought my attention to the paper [Sha04] as a possible source of example.

- Russian original by K. Vogtmann and A. Weinstein, Corrected reprint of the second (1989) edition.
- [Ber88] V. N. BERESTOVSKIĪ. Homogeneous manifolds with an intrinsic metric. I. *Sibirsk. Mat. Zh.* **29**(6), 17–29 (1988).
  - [Bes87] ARTHUR L. BESSE. “Einstein manifolds”, volume 10 of “Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]”. Springer-Verlag, Berlin (1987).
  - [BG03] E. BREUILLARD AND T. GELANDER. On dense free subgroups of Lie groups. *J. Algebra* **261**(2), 448–467 (2003).
  - [BH99] MARTIN BRIDSON AND ANDRE HAEFLIGER. “Metric Spaces of Non-Positive Curvature”, volume 319 of “Grundlehren der Mathematischen Wissenschaften”. Springer, Berlin, Heidelberg, first edition (1999).
  - [Bre07] EMMANUEL BREUILLARD. Geometry of locally compact groups and polynomial growth and shape of large balls. Preprint (2007).
  - [Bur92] D. YU. BURAGO. Periodic metrics. In “Representation theory and dynamical systems”, volume 9 of “Adv. Soviet Math.”, pages 205–210. Amer. Math. Soc., Providence, RI (1992).
  - [CE75] JEFF CHEEGER AND DAVID G. EBIN. “Comparison theorems in Riemannian geometry”. North-Holland Publishing Co., Amsterdam (1975). North-Holland Mathematical Library, Vol. 9.
  - [Ebe04] PATRICK EBERLEIN. Geometry of 2-step nilpotent Lie groups. In “Modern dynamical systems and applications”, pages 67–101. Cambridge Univ. Press, Cambridge (2004).
  - [GK04] ERIK GUENTNER AND JEROME KAMINKER. Exactness and uniform embeddability of discrete groups. *J. London Math. Soc. (2)* **70**, 703–718 (2004).
  - [GOV97] V. V. GORBATSEVICH, A. L. ONISHCHIK, AND E. B. VINBERG. “Foundations of Lie theory and Lie transformation groups”. Springer-Verlag, Berlin (1997). Translated from the Russian by A. Kozłowski, Reprint of the 1993 translation [*Lie groups and Lie algebras. I*, Encyclopaedia Math. Sci., 20, Springer, Berlin, 1993; MR1306737 (95f:22001)].
  - [Gro93] M. GROMOV. *Asymptotic invariants of infinite groups*. In “Geometric group theory, Vol. 2 (Sussex, 1991)”, volume 182 of “London Math. Soc. Lecture Note Ser.”, pages 1–295. Cambridge Univ. Press, Cambridge (1993).
  - [Gro96] MIKHAEL GROMOV. Carnot-Carathéodory spaces seen from within. In “Sub-Riemannian geometry”, volume 144 of “Progr. Math.”, pages 79–323. Birkhäuser, Basel (1996).
  - [Hel01] SIGURDUR HELGASON. “Differential geometry, Lie groups, and symmetric spaces”, volume 34 of “Graduate Studies in Mathematics”. American Mathematical Society, Providence, RI (2001). Corrected reprint of the 1978 original.
  - [KM79] M. I. KARGAPOLOV AND JU. I. MERZLJAKOV. “Fundamentals of the theory of groups”, volume 62 of “Graduate Texts in Mathematics”. Springer-Verlag, New York (1979). Translated from the second Russian edition by Robert G. Burns.
  - [Kra99] S. A. KRAT. Asymptotic properties of the Heisenberg group. *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI)* **261**(Geom. i Topol. 4), 125–154, 268 (1999).

- [Kra01] S. A. KRAT. On pairs of metrics invariant under a cocompact action of a group. *Electron. Res. Announc. Amer. Math. Soc.* **7**, 79–86 (electronic) (2001).
- [Mal49] A. I. MAL'CEV. On a class of homogeneous spaces. *Izvestiya Akad. Nauk. SSSR. Ser. Mat.* **13**, 9–32 (1949).
- [Mit85] JOHN MITCHELL. On Carnot-Carathéodory metrics. *J. Differential Geom.* **21**(1), 35–45 (1985).
- [Osi01] D. V. OSIN. Subgroup distortions in nilpotent groups. *Comm. Algebra* **29**(12), 5439–5463 (2001).
- [Pan83] PIERRE PANSU. Croissance des boules et des géodésiques fermées dans les nilvariétés. *Ergodic Theory Dynam. Systems* **3**(3), 415–445 (1983).
- [Ras38] P. K. RASHEVSKII. Joinability of any two points of a completely nonholonomic space by an admissible line. *Uch. Zap. Mosk. Gos. Pedagog. Inst. im. K. Libknekhta. Ser. Fiz.-Mat.* (2), 83–94 (1938).
- [Sha04] YEHUDA SHALOM. Harmonic analysis, cohomology, and the large-scale geometry of amenable groups. *Acta Math.* **192**(2), 119–185 (2004).
- [Sto98] MICHAEL STOLL. On the asymptotics of the growth of 2-step nilpotent groups. *J. London Math. Soc.* (2) **58**(1), 38–48 (1998).
- [Yam50] H. YAMABE. On an arcwise connected subgroup of a Lie group. *Osaka Math. J.* **2**, 13–14 (1950).