

A WEYL TYPE FORMULA RELATING SPECTRAL DISTRIBUTION TO L^2 -ISOPERIMETRIC PROFILE

ALEXANDER BENDIKOV, CHRISTOPHE PITTET, AND ROMAN SAUER

To the memory of Andrzej Hulanicki

ABSTRACT. We give a formula relating the L^2 -isoperimetric profile to the spectral distribution of the Laplace operator associated to a finitely generated group Γ or a Riemannian manifold with a cocompact, isometric Γ -action. As a consequence, we can apply techniques from geometric group theory to estimate the spectral distribution of the Laplace operator in terms of the growth and the Følner's function of the group, generalizing previous estimates by Gromov and Shubin. This leads, in particular, to sharp estimates of the spectral distributions for several classes of solvable groups. Furthermore, we prove the asymptotic invariance of the spectral distribution under changes of measures with finite second moment.

1. INTRODUCTION

What is the relation between the asymptotic behavior of the *return probability* $p(t)$ of the random walk of a probability measure μ on a finitely generated group Γ , the L^2 -*isoperimetric profile* $\Lambda(v)$ of the Laplace operator Δ associated to μ , and the *spectral distribution* $N(\lambda)$ of Δ ?

The asymptotic relation between these three functions is fully understood for non-amenable and for virtually nilpotent groups due to work of Kesten, Varopoulos and Gromov-Shubin, respectively. The investigation of relations between $p(t)$ and $\Lambda(v)$, and also Følner's function, is a field of active research (see [9, 10, 12–14, 21] – just to name a few).

Our main contribution is to establish a surprisingly simple relation between $N(\lambda)$ and $\Lambda(v)$ that holds under a mild regularity assumption.

1.1. Basic notions. Some definitions are in order to state the precise result. Let Γ be a finitely generated group. A probability measure μ on Γ is called *symmetric* if $\mu(\{\gamma\}) = \mu(\{\gamma^{-1}\})$ for every $\gamma \in \Gamma$. It is said to have *finite second moment* if $\int_{\Gamma} l(\gamma)^2 d\mu(\gamma) < \infty$, where l denotes the length function associated to some word metric on Γ . Further, we say

2000 *Mathematics Subject Classification.* Primary: 58J35, 20F63; Secondary: 60G50.

Key words and phrases. L^2 -isoperimetric profile, Random walk, Spectral distribution of Laplace-operators.

A. Bendikov was supported by the University of Aix-Marseille I as an invited Professor and by the Polish Government Scientific Research Fund, Grant 1PO3A03129. Ch. Pittet is currently supported by the CNRS. He was supported by the Marie Curie Transfer of Knowledge Fellowship of the European Community's Sixth Framework Program under contract number MTKD-CT-2004-013389 with the University of Wrocław. A. Bendikov and Ch. Pittet are grateful to Prof. E. Damek who managed the ToK contract, and to Prof. A. Grigor'yan for an invitation at the University of Bielefeld. They are also grateful to the Erwin Schrödinger Institute for several invitations. R. Sauer was supported by DFG grant SA 1661/1-2. All authors are grateful for the financial support from Prof. W. Lück's Leibniz award for a meeting at the WWU Münster.

that μ is *admissible* if it is symmetric, has finite second moment, and its support contains a finite generating set. Let $l^2(\Gamma)$ be the Hilbert space of square-integrable, complex-valued functions on Γ . Right convolution with μ defines a self-adjoint operator (*Markov-operator*)

$$R_\mu : l^2(\Gamma) \rightarrow l^2(\Gamma), \quad R_\mu(f)(x) = \sum_{\gamma \in \Gamma} f(x\gamma^{-1})\mu(\gamma).$$

with operator norm bounded by 1. The *Laplace operator* Δ of μ is the positive operator defined as $\Delta = \text{id} - R_\mu$. Both Δ and R_μ lie in the *von Neumann algebra* $\mathcal{N}(\Gamma)$ of Γ which is defined as the algebra of bounded operators on $l^2(\Gamma)$ that are equivariant with respect to the obvious isometric left Γ -action on $l^2(\Gamma)$. The *von Neumann trace* $\text{tr}_\Gamma : \mathcal{N}(\Gamma) \rightarrow \mathbb{C}$ is defined as $\text{tr}_\Gamma(A) = \langle A(\delta_e), \delta_e \rangle_{l^2(\Gamma)}$. The *return probability* $p(n)$ of μ can be expressed as $p(n) = \text{tr}_\Gamma(R_\mu^n)$. For a self-adjoint operator $A \in \mathcal{N}(\Gamma)$ the spectral projection $E_\lambda^A = \chi_{(-\infty, \lambda]}(A)$ of A (see e.g. [17, p. 56]) lies in $\mathcal{N}(\Gamma)$. The *spectral distribution* of Δ is the right-continuous function $N : [0, \infty) \rightarrow [0, \infty)$ with

$$N(\lambda) = \text{tr}_\Gamma(E_\lambda^\Delta).$$

The *L^2 -isoperimetric profile* of Δ is the function $\Lambda : [1, \infty) \rightarrow (0, \infty)$ such that $\Lambda(v)$ is, by definition, the smallest eigenvalue of Δ restricted to a set $\Omega \subset \Gamma$ of cardinality $\leq v$:

$$\Lambda(v) = \inf_{1 \leq |\Omega| \leq v} \lambda_1(\Omega) \quad \text{where} \quad \lambda_1(\Omega) = \inf_{\emptyset \neq \text{supp}(f) \subset \Omega} \frac{\langle \Delta(f), f \rangle}{\|f\|_2^2}.$$

The function Λ is a decreasing right-continuous step function. For finitely generated groups, Cheeger's inequality (Theorem 4.1) and Følner's characterization of amenability imply that $\Lambda(v) \rightarrow 0$ for $v \rightarrow \infty$ if and only if the group is amenable.

If the dependence of N , Λ , Δ , or p on μ needs to be stressed, we write N_μ , Λ_μ , Δ_μ , and p_μ . We say that $f \preceq g$ holds *near zero*, for functions $f, g : [0, \infty) \rightarrow [0, \infty)$, if there are constants $C, D, \epsilon > 0$ such that $f(\lambda) \leq Cg(D\lambda)$ for all $\lambda \in [0, \epsilon)$. We say that $f \preceq g$ holds *near infinity* if there are constants $C, D, x_0 > 0$ such that $f(x) \leq Cg(Dx)$ for all $x \in [x_0, \infty)$. Further we write $f \simeq g$ *near zero* or *near infinity* if $f \preceq g$ and $g \preceq f$ hold near zero or near infinity, respectively. We say that $f \simeq g$ *holds in the dilatational sense* or $f \simeq g$ *are dilatationally equivalent* if the outer constant C can be taken to be $C = 1$ in the above definition. Similarly, one defines \preceq in the dilatational sense.

1.2. A Weyl type formula. The computations of the spectral distribution and the L^2 -isoperimetric profile of virtually nilpotent groups are classic results (see the first row of Table 1.3) due to the work of Gromov-Shubin and Varopoulos [6, 15].

Theorem 1.1 (Gromov-Shubin-Varopoulos – reformulated). *Let Γ be an infinite finitely generated group, and let μ be an admissible probability measure on Γ . Assume that Λ_μ is equivalent near infinity to a regularly varying function of index $-\infty < \alpha < 0$ [4, p. 18]. Then, near zero,*

$$N_\mu(\lambda) \simeq \frac{1}{\Lambda_\mu^{-1}(\lambda)}.$$

One can prove this using Varopoulos' results on the return probability and classical Karata theory [4]. We emphasize that a group satisfying the above hypothesis is virtually nilpotent; hence we actually have $\Lambda_\mu(v) \simeq v^\alpha$ near infinity with $\alpha = -2/d$, where d is the degree of growth of Γ . The reason we interpret Gromov-Shubin's computation of $N(\lambda)$ via

the inverse of the L^2 -isoperimetric profile in the above theorem is that it is, as turns out through the present work, suited for generalization.

The following theorem is our main result (proved in Section 3). Notice that we prove the same conclusion as in the above theorem under a hypothesis that works for functions in the boundary case $\alpha = 0$, like, for example, $\Lambda(v) = \log(v)^{-\gamma}$ with $\gamma > 0$. This opens the way for many computations beyond nilpotent groups.

Theorem 1.2. *Let Γ be an infinite finitely generated amenable group, and let μ be an admissible probability measure on Γ . Assume that the function $\Lambda_\mu \circ \exp$ is doubling near infinity (Definition 3.3). Then we have the following dilatational equivalence near zero between N_μ and the reciprocal of the generalized inverse (Definition 3.3) of Λ_μ :*

$$N_\mu(\lambda) \simeq \frac{1}{\Lambda_\mu^{-1}(\lambda)}.$$

We do not know an example of Γ and μ such that Λ_μ is neither regularly varying of index $-\infty < \alpha < 0$ nor $\Lambda_\mu \circ \exp$ is doubling near infinity.

There is an analogous version in the Riemannian setting:

Corollary 1.3. *Let M be a connected complete non-compact Riemannian manifold and Γ be an amenable group. Let Γ act freely, properly discontinuously and cocompactly on M by isometries. Let Λ be the L^2 -isoperimetric profile of M and $N(\lambda)$ be the spectral distribution of the Laplace operator of M on functions. Let Λ^{-1} be the generalized inverse of Λ . If $\Lambda \circ \exp$ is doubling near infinity, then, near zero,*

$$N(\lambda) \simeq \frac{1}{\Lambda^{-1}(\lambda)}.$$

Indeed, Efremov proved that the spectral distributions of the Riemannian Laplace operator on M and the combinatorial Laplace operator Δ_μ on Γ for the probability measure

$$(1.1) \quad \mu = \frac{1}{|S|} \sum_{s \in S} \delta_s$$

of the simple random walk associated to a finite, symmetric, generating set S are equivalent near zero [17, Section 2.4]. The corresponding statement for the L^2 -isoperimetric profile can be deduced from [8, 10].

In the case of $\Gamma = \mathbb{Z}^n$ acting on $M = \mathbb{R}^n$, one is tempted to invoke *Weyl's law* for the asymptotics of eigenvalues of the Laplacian of compact domains $\Omega_i \subset \mathbb{R}^n$; if one assumes that the discrete spectra of Ω_i suitably approximate the spectrum on M when $\Omega_i \rightarrow M$, one can heuristically deduce the correct asymptotics of $N(\lambda)$ in this case.

1.3. Exponential and subexponential growth, Følner's function, and almost flat spectra. If not specified otherwise below, we consider the probability measure (1.1) associated to a finite, symmetric generating set of the group in consideration. According to Theorem 1.7 we may as well take any other admissible probability measure as long as we are only interested in the asymptotic properties of N_μ .

Geometric methods allow to compute $\Lambda(v)$ and verify the doubling assumption in many cases – the computation usually uses information about *Følner's function* $F\text{øl} : (0, \infty) \rightarrow \mathbb{N}$ defined for finitely generated amenable groups by

$$(1.2) \quad F\text{øl}(r) = \min \left\{ |\Omega|; \Omega \subset \Gamma : \frac{|\partial_S \Omega|}{|\Omega|} < \frac{1}{r} \right\}.$$

Here the *boundary* $\partial_S \Omega$ of Ω is, by definition, $\partial_S \Omega = \{x \in \Omega; \exists s \in S : xs \in \Gamma \setminus \Omega\}$. The function $F\phi(r)$ is an increasing, right-continuous step function. It satisfies $F\phi(r) \rightarrow \infty$ as $r \rightarrow \infty$, if and only if the group Γ is infinite. If the Følner's function grows sufficiently fast, we can deduce an upper bound on $N(\lambda)$:

Proposition 1.4. *Let Γ be an infinite, finitely generated amenable group. Let $F : (x_0, \infty) \rightarrow (0, \infty)$ be a continuous, strictly increasing function such that*

- (1) $F(r) \rightarrow \infty$ as $r \rightarrow \infty$,
- (2) there is $C > 1$ such that $F(r)^2 \leq F(Cr)$ for large $r > 0$,
- (3) $F(r) \leq F\phi(r)$ near infinity.

Then, near zero,

$$N(\lambda) \leq \frac{1}{F(\lambda^{-1/2})}$$

in the dilatational sense.

The proof of the above proposition is based on Theorem 1.2 and Cheeger's inequality (see Section 4). We refer the reader to [7, 12, 14] for lower bounds on Følner's function. For lower bounds on $N(\lambda)$ we refer to Proposition 4.3 in Section 4.

Corollary 1.5. *Let Γ be a finitely generated amenable group. Let $0 < \alpha \leq 1$. Assume there exists $\epsilon > 0$ such that the cardinality of a ball of radius r in Γ is bounded below by $\exp(\epsilon r^\alpha)$ for large r . Then, near zero,*

$$N(\lambda) \leq \exp(-\lambda^{-\alpha/2}).$$

In particular, if Γ has exponential growth, then $N(\lambda) \leq \exp(-\lambda^{-1/2})$ near zero.

This follows from Proposition 1.4 and an inequality of Coulhon and Saloff-Coste [7], which implies that, under the above hypothesis, $F\phi(r) \geq \exp(r^\alpha)$ near infinity.

Given any locally bounded function $f : (x_0, \infty) \rightarrow (0, \infty)$, we can find a continuous function $F : (x_0, \infty) \rightarrow (0, \infty)$ with $F(r) \geq f(r^2)$ for every $r \in (x_0, \infty)$. By making F even bigger, we may also assume that $F(r)^2 \leq F(2r)$ and $F(r) \rightarrow \infty$ as $r \rightarrow \infty$. By a result of Erschler [13, Theorem 1] there exists a finitely generated amenable group whose Følner's function exceeds F . Thus we obtain the following corollary:

Corollary 1.6. *For any positive, locally bounded function f defined in a neighborhood of ∞ there exists a finitely generated amenable group Γ such that, near zero,*

$$N(\lambda) \leq \frac{1}{f(\lambda^{-1})}.$$

This means that $N(\lambda)$ can be as flat as desired for amenable groups; on the other hand, $N(\lambda)$ is identically zero near zero if and only if the group is not amenable and infinite due to Kesten's spectral gap characterization.

Table 1.3 gives some samples of explicit computations obtained from Proposition 1.4, Proposition 4.3, and Theorem 1.2. Previous to the present work, estimates for $N(\lambda)$ were known for virtually nilpotent groups [15] (first row in Table 1.3) and for rank 1 lamplighter groups with special generating sets [1] (third row in Table 1.3 with $d = 1$).

TABLE 1. Computations for several classes of solvable groups

Group	$p(2t)$ as $t \rightarrow \infty$	$N(\lambda)$ as $\lambda \rightarrow 0$	$\Lambda(v)$ as $v \rightarrow \infty$	$F(r)$ as $r \rightarrow \infty$
Virtually nilpotent of polynomial growth d	$t^{-d/2}$ (a)	$\lambda^{d/2}$ (b)	$v^{-2/d}$ (e)	r^d (f)
Virtually torsion-free solvable of exponential growth and finite Prüfer rank (c)	$\exp(-t^{1/3})$ (d)	$\exp(-\lambda^{-1/2})$	$\log(v)^{-2}$ (g)	$\exp(r)$ (h)
$F \wr N$, F finite and N of polynomial growth d	$\exp(-t^{\frac{d}{d+2}})$ (i)	$\exp(-\lambda^{-d/2})$	$\log(v)^{-2/d}$ (j)	$\exp(r^d)$ (l)
$\Lambda \wr N$, Λ infinite and of polynomial growth, N of polynomial growth d	$\exp(-t^{\frac{d}{d+2}}(\log(t))^{\frac{2}{d+2}})$ (k)	$\exp(-\lambda^{-\frac{d}{2}} \log(\frac{1}{\lambda}))$ (n)	$(\frac{\log(v)}{\log \log(v)})^{-2/d}$ (l)	$\exp(r^d \log(r))$ (o)
$F \wr (\dots (F \wr (\mathbb{Z} \wr \mathbb{Z})) \dots)$, F finite, k times iterated wreath product, $k \geq 2$	$\exp\left(-\frac{t}{(\log_{(k-1)}(t))^2}\right)$ (m)	$\exp(-\exp_{(k-1)}(\lambda^{-1/2}))$	$\log_{(k)}(v)^{-2}$ (t)	$\exp_{(k)}(r)$ (u)
$\mathbb{Z} \wr (\dots (\mathbb{Z} \wr (\mathbb{Z} \wr \mathbb{Z})) \dots)$, k times iterated wreath product, $k \geq 2$	$\exp\left(-t\left(\frac{\log_{(k)}(t)}{\log_{(k-1)}(t)}\right)^2\right)$ (m)	$\exp(-\exp_{(k-1)}(\lambda^{-1/2} \log(\frac{1}{\lambda})))$ (n)	$(\frac{\log_{(k)}(v)}{\log_{(k+1)}(v)})^{-2}$ (t)	$\exp_{(k)}(r \log(r))$ (o)

(a) This is a well known result of Varopoulos [6]. See [9] for a short proof. (b) This follows from [6, 15]. See also [17, pp. 94-95].

(c) Every polycyclic group of exponential growth is in this class. (d) See [21]. (e) See [9, 10]. (f) See [8, 16].

(g) See [10, 21]. (h) See [20]. See also [9] for an easier proof. (i) See [19] for the upper bound and [12] for the lower bound.

(j) See [8, 21]. (k) See [20]. See also [9] for an easier proof. (l) Proved in [12]. See [14] for an alternative proof. The notation \exp_k stands

for the k -times iterated exponential function. Similarly for $\log_{(k)}$. (m) This expression is different from but equivalent to the

functional inverse of the reciprocal of the one right next to it.

1.4. Stability. The following theorem (proved in Section 5) justifies that we call N_μ an invariant of the group.

Theorem 1.7. *Let Γ be a finitely generated group. Let μ_1 and μ_2 be admissible probability measures on Γ . Then we have the dilatational equivalence near zero*

$$N_{\mu_1} \simeq N_{\mu_2}.$$

Building on this, we prove in a forthcoming paper [2] the invariance under quasi-isometry.

Theorem 1.8. *Let Γ and Λ be finitely generated amenable groups. Let μ and ν be admissible probability measures on Γ and Λ , respectively. If Γ and Λ are quasi-isometric, then, near zero,*

$$N_\mu(\lambda) \simeq N_\nu(\lambda).$$

This result is an instance of a more general invariance result for arbitrary groups (i.e. not necessarily amenable) with respect to uniform measure equivalence that also holds in any degree, not only for the Laplace operator on functions. In [2] we also discuss how the stability of the return probability due to Pittet and Saloff-Coste [22] (actually, a slightly stronger version thereof) can be deduced from Theorem 1.8.

2. PROPERTIES OF THE LEGENDRE TRANSFORM

We collect some elementary properties of the *Legendre transform* for later reference.

Definition 2.1 (see [24, Section 26]).

(1) Let $M : (0, \infty) \rightarrow (0, \infty)$ be decreasing such that $\lim_{x \rightarrow 0} M(x) = \infty$. For $t > 0$, we define the Legendre transform $\mathfrak{L}\mathfrak{e}_M(t)$ of M as

$$\mathfrak{L}\mathfrak{e}_M(t) = \inf\{tx + M(x); x > 0\}.$$

(2) Let $G : [0, \infty) \rightarrow [0, \infty)$ be increasing such that $\lim_{x \rightarrow \infty} G(x)/x = 0$. For $t > 0$, we define the Legendre conjugate transform $\mathfrak{L}\mathfrak{e}_G^*(t)$ of G as

$$\mathfrak{L}\mathfrak{e}_G^*(t) = \sup\{-tx + G(x); x \geq 0\}.$$

Lemma 2.2. *Let M and G be as in Definition 2.1. Let $t > 0$.*

(1) *If M is right-continuous, then the infimum $\inf\{tx + M(x); x > 0\}$ is a minimum.*

(2) *If G is right-continuous, then the supremum $\sup\{-tx + G(x); x \geq 0\}$ is a maximum.*

Proof. The proofs of both assertions are very similar, and we only prove the first one as a sample. Since $f(x) = tx + M(x) \rightarrow \infty$ for both $x \rightarrow 0$ and $x \rightarrow \infty$, there exist $0 < a \leq b < \infty$ and a sequence $(x_n)_{n \in \mathbb{N}}$ in $[a, b]$ such that $f(x_n) \rightarrow \inf\{f(x); x > 0\}$. By compactness of $[a, b]$, we may assume that $x_n \rightarrow y \in [a, b]$. We have to show that y realizes the minimum. We may assume that either $x_n \geq y$ for every $n \in \mathbb{N}$ or $x_n \leq y$ for every $n \in \mathbb{N}$. In the first case, $f(x_n) \rightarrow f(y)$ follows from right-continuity. In the second case, we have $tx_n + M(y) \leq tx_n + M(x_n)$ since M is decreasing. Taking limits, we obtain that $f(y) \leq \inf\{f(x); x > 0\}$. \square

We will need the following reformulation of [3, Lemma 3.2].

Lemma 2.3. *Let F be an increasing positive right-continuous function defined on $[0, \infty)$ which is bounded by 1. Assume moreover that $F(0) = 0$ and that $F(\lambda) > 0$ if $\lambda > 0$. Let M be the decreasing positive function defined on $(0, \infty)$ as $M(x) = -\log(F(x))$. Then, for all $t > 0$, we have*

$$(2.1) \quad \exp(-\mathfrak{L}\mathfrak{e}_M(t)) \leq \int_0^\infty \exp(-t\lambda) dF(\lambda) \leq (1 + \mathfrak{L}\mathfrak{e}_M(t)) \exp(-\mathfrak{L}\mathfrak{e}_M(t)).$$

Remark 2.4. *The hypothesis of right-continuity should be added in [3, Lemma 3.2] and the convexity assumption on the function $x \mapsto tx + M(x)$ should be removed. (The convexity assumption is used only to ensure that the infimum $\inf_{x>0} \{tx + M(x)\}$ is a minimum and the convexity assumption in the context of [3, Lemma 3.2] implies anyway that M is right-continuous.)*

Proof. To prove Lemma 2.3, apply Lemma 2.2; then follow the proof of [3, Lemma 3.2] without using convexity (see the remark above). \square

Proposition 2.5. *Let $M : (0, \infty) \rightarrow (0, \infty)$ be right-continuous and decreasing such that $\lim_{\lambda \rightarrow 0} M(\lambda) = \infty$.*

(1) *Let $t_0 > 0$, and let $G : [t_0, \infty) \rightarrow (0, \infty)$ be a function such that, for every $t \geq t_0$,*

$$\mathfrak{L}\mathfrak{e}_M(t) \leq G(t).$$

If $G/\text{id} : [t_0, \infty) \rightarrow (0, \infty)$, defined as $(G/\text{id})(t) = G(t)/t$, has an inverse $(G/\text{id})^{-1}$, then

$$M(\lambda) \leq G \circ (G/\text{id})^{-1}(\lambda)$$

holds for all $\lambda \in (G/\text{id})([t_0, \infty))$.

(2) *Let $G : [0, \infty) \rightarrow [0, \infty)$ be right-continuous and increasing such that $\lim_{t \rightarrow \infty} G(t) = \infty$ and $\lim_{t \rightarrow \infty} G(t)/t = 0$. Assume that*

$$\mathfrak{L}\mathfrak{e}_M(t) \geq G(t)$$

near infinity. Then, near zero,

$$M(\lambda) \geq \mathfrak{L}\mathfrak{e}_G^*(\lambda).$$

(3) *Let $G : [0, \infty) \rightarrow [0, \infty)$ be continuous such that $\lim_{t \rightarrow \infty} G(t)/t = 0$. Assume that $t \mapsto G(t)/t$ is strictly decreasing near infinity. Let $\epsilon \in (0, 1)$. Then, near zero,*

$$\mathfrak{L}\mathfrak{e}_G^*(\lambda) \geq (1 - \epsilon)G \circ (G/\text{id})^{-1}(\lambda/\epsilon).$$

Proof. (1) By Lemma 2.2, for each $t \geq t_0$, there exists $y = y(t) > 0$ such that $\mathfrak{L}\mathfrak{e}_M(t) = ty + M(y)$. Hence $y \leq G(t)/t$. As M is decreasing, $M(G(t)/t) \leq M(y) \leq \mathfrak{L}\mathfrak{e}_M(t) \leq G(t)$.

(2) This follows from the fact that for every $t_0 > 0$ the function $t \mapsto -t\lambda + G(t)$ attains its maximum in (t_0, ∞) provided $\lambda > 0$ is sufficiently small: Let $T > t_0$ be such that $G(T) > 1 + G(t_0)$. Then for $\lambda \in (0, 1/T)$ we have $-t\lambda + G(t) < -T\lambda + G(T)$ for every $t \in (0, t_0)$ because G is increasing.

(3) By hypothesis, the inverse $(G/\text{id})^{-1}$ is well defined in a neighborhood of zero. For sufficiently small $\lambda > 0$, there exists $s = s(\lambda)$, such that $\lambda s = \epsilon G(s)$, and thus $(G/\text{id})^{-1}(\lambda/\epsilon) = s$. Hence, $\mathfrak{L}\mathfrak{e}_G^*(\lambda) \geq -\lambda s + G(s) = (1 - \epsilon)G(s) = (1 - \epsilon)G \circ (G/\text{id})^{-1}(\lambda/\epsilon)$. \square

3. PROOF OF THE MAIN THEOREM

Section 3 is devoted to the proof of Theorem 3.1, which implies Theorem 1.2.

Theorem 3.1. *Retain the setting of Theorem 1.2. Let $L : (0, \infty) \rightarrow (0, \infty)$ be a decreasing function such that $\lim_{x \rightarrow \infty} L(x) = 0$. Assume that $L \circ \exp$ is doubling near infinity.*

- (1) *If $L(v) \preceq \Lambda_\mu(v)$ near infinity, then $N_\mu(\lambda) \preceq \frac{1}{L^{-1}(\lambda)}$ near zero in the dilatational sense.*
(2) *If $L(v) \succeq \Lambda_\mu(v)$ near infinity, then $N_\mu(\lambda) \succeq \frac{1}{L^{-1}(\lambda)}$ near zero in the dilatational sense.*

3.1. The relation between $N_\mu(\lambda)$ and $p_\mu(t)$ via the Laplace transform. We recall the following equivalence near infinity between the continuous Poisson semi-group of Δ_μ and its discrete Markov semi-group.

Proposition 3.2. *Let Γ be a finitely generated infinite amenable group and let μ be an admissible probability measure on Γ . Then, near infinity,*

$$(3.1) \quad p_\mu(2t) \simeq \int_0^\infty \exp(-\lambda t) dN_\mu(\lambda).$$

This is well known – and rather easy to verify, if the Markov operator R_μ is positive (but it is often not, like in the case of the simple random walk $\mu = (\delta_1 + \delta_{-1})/2$ on $\Gamma = \mathbb{Z}$). In practice, it is often sufficient to apply the equivalence to the positive operator R_μ^2 , but we do need the general case here.

The proof relies on our stability result (Theorem 1.7). Since we could not locate a proof of the general case in the literature, we provide some details.

Sketch of proof. We find a sufficiently small $\lambda_0 > 0$ such that $(1 - \lambda)^{2t} \leq \exp(-2t\lambda)$ for every $t \geq 1$ and every $\lambda \in [0, \lambda_0]$. Thus,

$$(3.2) \quad \begin{aligned} p_\mu(2t) &= \int_{[0, \lambda_0]} (1 - \lambda)^{2t} dN_\mu(\lambda) + \int_{(\lambda_0, 2 - \lambda_0]} (1 - \lambda)^{2t} dN_\mu(\lambda) + \underbrace{\int_{(2 - \lambda_0, 2]} (1 - \lambda)^{2t} dN_\mu(\lambda)}_{=R(\lambda_0)} \\ &\leq \int_{[0, \lambda_0]} \exp(-2t\lambda) dN_\mu(\lambda) + \exp(-2t\lambda_0) + R(\lambda_0) \\ &\leq (1 + N_\mu(\lambda_0)^{-1}) \int_{[0, \lambda_0]} \exp(-2t\lambda) dN_\mu(\lambda) + R(\lambda_0). \end{aligned}$$

Note that there is no spectral gap, i.e. $N_\mu(\lambda_0) > 0$, since Γ is infinite and amenable. The estimate of $R(\lambda_0)$ relies on two observations: Firstly, for a self-adjoint operator A with $\|A\| \leq 1$ and $0 \leq \lambda \leq 1$ we have

$$E_\lambda^{1-A^2} = E_{1-\sqrt{1-\lambda}}^{1-A} + 1 - E_{1+\sqrt{1-\lambda}}^{1-A}.$$

When applied to the Markov-operator $A = R_\mu$, this implies that

$$\begin{aligned} 1 - N_\mu(2 - \lambda/2) &\leq 1 - N_\mu(2 - \lambda/2) + N_\mu(\lambda/2) \\ &\leq 1 - N_\mu\left(2 - \frac{\lambda}{1 + \sqrt{1 - \lambda}}\right) + N_\mu\left(\frac{\lambda}{1 + \sqrt{1 - \lambda}}\right) \\ &= 1 - N_\mu(1 + \sqrt{1 - \lambda}) + N_\mu(1 - \sqrt{1 - \lambda}) \\ &= N_{\mu^{(2)}}(\lambda). \end{aligned}$$

Secondly, the support of $\mu^{(2)}$ (convolution of μ with itself) is either $\Gamma = \text{supp}(\mu)$ or a subgroup in Γ of index 2. In the first case, Theorem 1.7 immediately yields that $N_{\mu^{(2)}} \simeq N_\mu$ in the dilatational sense. In the latter case, one uses Theorem 1.7 and [17, Theorem 2.55 (6) on p. 98] to see that $N_{\mu^{(2)}} \simeq 2N_\mu$ holds dilatationally. In particular, upon decreasing λ_0 , we may assume that there is a constant $D > 0$ such that for every $\lambda \in [0, \lambda_0]$

$$(3.3) \quad N_{\mu^{(2)}}(\lambda) \leq 2N_\mu(D\lambda).$$

From this one can conclude by a change of variable and an application of partial integration for Lebesgue-Stieltjes integrals [23, Chapitre II. 54] that

$$R(\lambda_0) = \int_{(0, \lambda_0]} (1 - \lambda)^{2t} d(1 - N_\mu(2 - \lambda)) \leq 2 \int_{(0, 2D\lambda_0]} \left(1 - \frac{\lambda}{2D}\right)^{2t} dN_\mu(\lambda).$$

Estimating the latter term proceeds as in (3.2); therefore, we have proved the \preceq -assertion in the theorem. The proof of the easier \succeq -assertion is omitted. \square

3.2. General properties of doubling functions and generalized inverses.

Definition 3.3. *Let $L : (0, \infty) \rightarrow (0, \infty)$ be a decreasing function.*

- (1) *The generalized inverse of L is the function $L^{-1} : (\inf L, \infty) \rightarrow (0, \infty)$ defined as $L^{-1}(x) = \inf\{v > 0; L(v) \leq x\}$.*
- (2) *We say that L is doubling (near infinity) if there exists a constant $c > 0$ such that $L(2x) \geq cL(x)$ for all $x > 0$ (sufficiently large $x > 0$).*

Lemma 3.4. *Let $L : (0, \infty) \rightarrow (0, \infty)$ be a decreasing function. Assume that L is doubling. Then there is a continuous, decreasing function $L_{ct} : (0, \infty) \rightarrow (0, \infty)$ such that $L_{ct} \simeq L$. If, in addition, $L \circ \exp$ is doubling, then $L_{ct} \circ \exp$ is doubling.*

Proof. The function

$$L_{ct}(x) = \frac{2}{x} \int_{x/2}^x L(s) ds$$

is continuous and differentiable almost everywhere by the Lebesgue differentiation theorem. It is decreasing because of

$$\begin{aligned} \frac{xL'_{ct}(x)}{2} &= L(x) - \frac{L(x/2)}{2} - \frac{1}{x} \int_{x/2}^x L(s) ds \\ &\leq L(x) - \frac{L(x/2)}{2} - \frac{L(x)}{2} = \frac{L(x) - L(x/2)}{2} \leq 0. \end{aligned}$$

It is equivalent to L because of $L_{ct}(x) \geq L(x)$ and $L_{ct}(x) \leq L(x/2) \leq c^{-1}L(x)$. The last assertion also follows from this. \square

Lemma 3.5. *Let $f, g : (0, \infty) \rightarrow (0, \infty)$ be decreasing functions such that $f(x) \rightarrow 0$ and $g(x) \rightarrow 0$ for $x \rightarrow \infty$. So the generalized inverses f^{-1} and g^{-1} are defined on $(0, \infty)$. Then the following holds:*

- (1) *If $f \circ \exp$ is doubling near infinity, then f is doubling near infinity.*
- (2) *Let $l : (0, \infty) \rightarrow (0, \infty)$ be a strictly increasing, continuous function (i.e. l^{-1} in the usual sense exists) such that $l(x) \rightarrow \infty$ as $x \rightarrow \infty$. Then $(f \circ l)^{-1} = l^{-1} \circ f^{-1}$.*
- (3) *If $f \simeq g$ and f, g are doubling near infinity, then there is a constant $D > 0$ such that $D^{-1}g(x) \leq f(x) \leq Dg(x)$ for large $x > 0$.*
- (4) *If $f \preceq g$ near infinity, then $f^{-1} \preceq g^{-1}$ near zero.*

- (5) If there is $D > 0$ with $f(x) \leq Dg(x)$ near infinity, then $f^{-1}(\lambda) \leq g^{-1}(D^{-1}\lambda)$ near zero.
(6) If $f \simeq g$ and f and g are doubling near infinity, then $f^{-1} \simeq g^{-1}$ holds near zero in the dilatational sense.

Proof. (1) If $x \geq 2$, then

$$f(2x) = f(\exp(\log(2x))) \geq f(\exp(2 \log(x))) \geq cf(\exp(\log(x))) = cf(x).$$

(2) By continuity, there is $x_0 > 0$ with $l(0, \infty) = (x_0, \infty)$. Hence, if $\lambda > 0$ is small enough, we have

$$\inf\{y > 0; f(y) \leq \lambda\} = \inf\{l(x); f(l(x)) \leq \lambda, x > 0\}.$$

Thus, as l^{-1} is increasing and continuous,

$$(l^{-1} \circ f^{-1})(\lambda) = l^{-1}(\inf\{l(x); f(l(x)) \leq \lambda, x > 0\}) = \inf\{x > 0; f(l(x)) \leq \lambda\} = (f \circ l)^{-1}(\lambda).$$

(3)–(5) are easy; the proof is omitted.

(6) follows from (3) and (5). \square

3.3. Coulhon's and Grigoryan's functional equation. In the proof of Theorem 3.1 we relate the L^2 -isoperimetric profile to the return probability. This relies on work of Coulhon and Grigor'yan [10].

Proposition 3.6. *Let $L : (0, \infty) \rightarrow (0, \infty)$ be decreasing and continuous. Assume that $\lim_{x \rightarrow \infty} L(x) = 0$. Then the functional equation*

$$(3.4) \quad t = \int_0^{v(t)} \frac{ds}{(L \circ \exp)(s)}$$

uniquely defines a strictly increasing C^1 -function $v : [0, \infty) \rightarrow [0, \infty)$. Further, the following properties hold:

- (1) $v(2t) \leq 2v(t)$ for $t \geq 0$.
(2) If $L \circ \exp$ is doubling near infinity, then there is $c > 0$ with $v'(2t) \geq cv'(t)$ near infinity.
(3) For $t > 0$, the function $t \mapsto v(t)t^{-1}$ is decreasing and strictly decreasing near infinity.

Proof. (1) Since the derivative $v'(t) = (L \circ \exp)(v(t))$ is decreasing,

$$\begin{aligned} v(2t) &= \int_0^{2t} v'(s) ds + v(0) \leq \int_0^{2t} v'(s/2) ds + v(0) \leq 2 \int_0^t v'(s) ds + 2v(0) \\ &= 2v(t). \end{aligned}$$

(2) We have

$$v'(2t) = (L \circ \exp)(v(2t)) \geq (L \circ \exp)(2v(t)) \geq c(L \circ \exp)(v(t)) = cv'(t).$$

(3) Differentiating yields

$$\begin{aligned} \frac{d}{dt} \frac{v(t)}{t} &= \frac{v(t)}{t^2} \left(\frac{v'(t)t}{v(t)} - 1 \right) = \frac{v(t)}{t^2} \left(\frac{(L \circ \exp)(v(t))t}{v(t)} - 1 \right) \\ &\leq \frac{v(t)}{t^2} \left(\frac{t}{\int_0^{v(t)} \frac{ds}{(L \circ \exp)(s)}} - 1 \right) = \frac{v(t)}{t^2} \left(\frac{t}{t} - 1 \right) = 0. \end{aligned}$$

Let us show that the inequality

$$\int_0^{v(t)} \frac{ds}{(L \circ \exp)(s)} \leq \frac{v(t)}{(L \circ \exp)(v(t))}$$

we just used is strict provided t is large enough. Assume the equality case for some $t > 0$. Then, for every $x \in [0, t]$, we have $(L \circ \exp)(v(x)) = (L \circ \exp)(v(t))$ since $L \circ \exp \circ v$ is decreasing. This is a contradiction for large $t > 0$ because $L(x) \rightarrow 0$ as $x \rightarrow \infty$. \square

Theorem 3.7 (Coulhon-Grigor'yan). *Let Γ be an infinite, finitely generated, amenable group, and let μ be an admissible measure on Γ . Let $L : (0, \infty) \rightarrow (0, \infty)$ be a decreasing, continuous function, and let $v(t)$ be defined by (3.4).*

- (1) *If $L(v) \preceq \Lambda_\mu(v)$ near infinity, then $p_\mu(t) \preceq \exp(-v(t))$ near infinity.*
(2) *Assume, in addition, that $L \circ \exp$ is doubling. If $L(v) \succeq \Lambda_\mu(v)$ near infinity, then $p_\mu(2t) \succeq \exp(-v(t))$.*

Proof. This is a straightforward application of [10, Section 3] (see also [9, Propositions 2.1 and 4.1]); the technical condition (D) of [10, Section 3], needed for the lower bounded on $p_\mu(t)$, is satisfied because of Proposition 3.6 (1) and (2). \square

Lemma 3.8. *Retain the setting of Proposition 3.6. Assume $L \circ \exp$ is doubling near infinity. Then there is a constant $D > 0$ such that near infinity*

$$(L \circ \exp \circ v)(t) \leq \frac{v(t)}{t} \leq D \cdot (L \circ \exp \circ v)(t).$$

In particular, we have

$$\lim_{t \rightarrow \infty} \frac{v(t)}{t} = 0.$$

Proof. Since L is decreasing, we obtain that

$$\frac{v(t)}{t} = v(t) \left(\int_0^{v(t)} \frac{ds}{(L \circ \exp)(s)} \right)^{-1} \geq (L \circ \exp)(v(t)).$$

If $C > 0$ is the doubling constant, then we obtain for every $v > 0$ that

$$\int_0^v \frac{ds}{(L \circ \exp)(s)} \geq \int_{v/2}^v \frac{ds}{(L \circ \exp)(s)} \geq \frac{v}{2(L \circ \exp)(v/2)} \geq \frac{v}{2C(L \circ \exp)(v)}.$$

Thus,

$$\begin{aligned} \frac{v(t)}{t} &= v(t) \left(\int_0^{v(t)} \frac{ds}{(L \circ \exp)(s)} \right)^{-1} \leq v(t) \left(\frac{v(t)}{2C(L \circ \exp \circ v)(t)} \right)^{-1} \\ &= 2C(L \circ \exp \circ v)(t). \end{aligned} \quad \square$$

3.4. Conclusion of the proof of Theorem 3.1. By assumption, $L \circ \exp$, thus, by Lemma 3.5 (1), L itself, are doubling near infinity. Let $L_1 = L_{ct}$ be the continuous, decreasing function obtained from Lemma 3.4; $L_1 \circ \exp$ is also doubling near infinity and $L_1 \simeq L$. Let v_1 be the continuous, increasing function defined by the functional equation (3.4) where we replace L with L_1 . The proof now splits into two parts.

Proof of the upper bound on $N_\mu(\lambda)$ in Theorem 3.1. Due to Theorem 3.7, we have $p_\mu(t) \preceq \exp(-v_1(t))$ near infinity since $L_1 \simeq L \preceq \Lambda_\mu$. Combining this with Proposition 3.2, we obtain that

$$-\log \left(\int_0^\infty \exp(-\lambda t) dN_\mu(\lambda) \right) \succeq v_1(t)$$

near infinity. Since Γ is infinite and amenable, $N_\mu(0) = 0$ and $N_\mu(\lambda) > 0$ for every $\lambda > 0$. Hence, Lemma 2.3 applies, and we deduce that near infinity,

$$\mathfrak{L}\mathfrak{e}_M(t) \succeq v_1(t),$$

where, by definition, $M(x) = -\log(N_\mu(x))$. In other words, there exists $\alpha, \beta > 0$, such that near infinity,

$$\mathfrak{L}\mathfrak{e}_M(t) \geq v_2(t),$$

where $v_2(t) = \alpha v_1(\beta t)$. Define the function L_2 by

$$(3.5) \quad L_2 \circ \exp(x) = \alpha\beta(L_1 \circ \exp)(\alpha^{-1}x).$$

One verifies that v_2 and L_2 satisfy the assumptions in Proposition 3.6 and, in particular, the functional equation (3.4). Furthermore, $L_2 \circ \exp$ is doubling near infinity because $L_1 \circ \exp$ is so. By Lemma 3.8, $v_2(t)/t \rightarrow 0$. Hence Proposition 2.5 (2) implies that

$$M(\lambda) \geq \mathfrak{L}\mathfrak{e}_{v_2}^*(\lambda)$$

near zero. By Proposition 3.6 and Lemma 3.8, the function $v_2(t)/t$ is strictly decreasing near infinity and converges to 0. Thus its inverse $(v_2/\text{id})^{-1}$ is well defined near zero. Proposition 2.5 (3) applies, and we deduce that

$$\mathfrak{L}\mathfrak{e}_{v_2}^*(\lambda) \geq \frac{1}{2}v_2 \circ (v_2/\text{id})^{-1}(2\lambda)$$

near zero. By Lemma 3.8, $v_2/\text{id} \geq L_2 \circ \exp \circ v_2$. Applying, in this order, Lemma 3.5 (5) and (2), yields

$$(v_2/\text{id})^{-1} \geq (L_2 \circ \exp \circ v_2)^{-1} = v_2^{-1} \circ (L_2 \circ \exp)^{-1}$$

near zero. Let $L_3 : (0, \infty) \rightarrow (0, \infty)$ be the function defined by $L_3(\exp(x)) = L_2(\exp(2x))$. Putting all together and using Lemma 3.5 (2) for the last two equalities below, we obtain that

$$\begin{aligned} -\log(N_\mu(\lambda/2)) &= M(\lambda/2) \geq \mathfrak{L}\mathfrak{e}_{v_2}^*(\lambda/2) \geq \frac{1}{2}(L_2 \circ \exp)^{-1}(\lambda) \\ &= (L_3 \circ \exp)^{-1}(\lambda) = \log \circ L_3^{-1}(\lambda) \end{aligned}$$

near zero. Thus, near zero,

$$N_\mu(\lambda) \preceq \frac{1}{L_3^{-1}(\lambda)}$$

holds in the dilatational sense. It remains to see that $L_3^{-1} \simeq L^{-1}$ near zero in the dilatational sense. Since near infinity $L_3 \simeq L_2 \simeq L_1 \simeq L$ and all these functions are doubling, this follows from Lemma 3.5 (6). \square

Proof of the lower bound on $N_\mu(\lambda)$ in Theorem 3.1. Since $L_1 \simeq L \succeq \Lambda_\mu$ and $L_1 \circ \exp$ is doubling at infinity, Theorem 3.7 can be applied and yields that $p_\mu(2t) \succeq \exp(-v_1(t))$ near infinity. We proceed similarly as for the upper bound and apply Proposition 3.2 and Lemma 2.3. This implies

$$v_1(t) \succeq -\log\left(\int_0^\infty \exp(-t\lambda) dN_\mu(\lambda)\right) \geq \mathfrak{L}\mathfrak{e}_M(t) - \log(1 + \mathfrak{L}\mathfrak{e}_M(t)) \geq \frac{1}{2}\mathfrak{L}\mathfrak{e}_M(t)$$

for large $t > 0$. Thus there are constants $\alpha, \beta > 0$ such that for $v_2(t) = \alpha v_1(\beta t)$ we have

$$v_2(t) \geq \mathfrak{L}\mathfrak{e}_M(t)$$

for large $t > 0$. Let L_2 be defined as in (3.5). With the same argument as for the upper bound, the function v_2/id has a well defined inverse near zero. Hence Proposition 2.5 (1) can be applied, implying that near zero:

$$M(\lambda) \leq v_2 \circ (v_2/\text{id})^{-1}(\lambda).$$

By Lemmas 3.8 and 3.5 (5) and (2),

$$(v_2/\text{id})^{-1}(\lambda) \leq v_2^{-1} \circ (L_2 \circ \exp)^{-1}(\lambda/D).$$

We conclude that

$$M(D\lambda) \leq \log \circ L_2^{-1}(\lambda).$$

near zero. Now we proceed exactly as for the upper bound, but with reversed inequalities. \square

4. FÖLNER'S FUNCTIONS AND CHEEGER'S INEQUALITY

In this section, we combine Theorem 3.1 with some geometric tools and prove Proposition 1.4. This leads to estimates of the spectral distribution in terms of the Følner's function (1.2), the growth function, and Følner couples.

Throughout this section, Γ denotes a finitely generated group and S denotes a finite symmetric generating set of Γ . The Laplace operator Δ , the spectral distribution N , and the L^2 -isoperimetric profile Λ are taken with respect to the probability measure (1.1). For statements up to equivalence, the specific choice of an admissible probability measure does not matter (Theorem 1.7).

We refer to [11, Theorem 2.3] for a proof of the combinatorial version of *Cheeger's inequality*:

Theorem 4.1 (Cheeger's inequality).

$$\lambda_1(\Omega) \geq \frac{1}{2|S|^2} \left(\inf_{\omega \subset \Omega} \frac{|\partial_S \omega|}{|\omega|} \right)^2.$$

Lemma 4.2. *Let F be a continuous strictly increasing positive function defined on a neighborhood of infinity with $\lim_{r \rightarrow \infty} F(r) = \infty$. Let $\alpha > 0$ and $\beta \geq 0$. For large $v > 0$ we define*

$$L(v) = \frac{\alpha}{(F^{-1}(v) - \beta)^2}.$$

Assume that there exists a constant $C > 1$ such that for large $r > 0$

$$F(Cr) \geq F(r)^2.$$

Then the function $L \circ \exp$ is doubling near infinity.

Proof. In a neighborhood of zero, we have:

$$L^{-1}(\lambda) = F(\alpha^{1/2} \lambda^{-1/2} + \beta).$$

This function is dilatationally equivalent to $F(\lambda^{-1/2})$. Hence we have the dilatational equivalence

$$\log \circ L^{-1}(\lambda) \simeq \log \circ F(\lambda^{-1/2}).$$

But,

$$2 \log(F(\lambda^{-1/2})) = \log(F(\lambda^{-1/2})^2) \leq \log(F(C\lambda^{-1/2})) = \log(F((C^{-2}\lambda)^{-1/2})).$$

Hence there is a constant $\delta > 0$ such that, near zero,

$$2 \log \circ L^{-1}(\lambda) \leq \log \circ L^{-1}(\alpha\lambda).$$

From this we deduce that $L \circ \exp$ is doubling near infinity. \square

Proof of Proposition 1.4. We will apply Theorem 3.1 (1) with

$$L(v) = \frac{1}{F^{-1}(v)^2}.$$

The doubling of $L \circ \exp$ follows from Lemma 4.2. Hence the corollary will be proved if we show that

$$\Lambda(v) \succeq L(v),$$

for large $v > 0$. Let $\alpha \geq 1, \beta \geq 1$ such that $F(r) \leq \alpha F\phi(\beta r)$ near infinity. As $F\phi$ increases and $\lim_{r \rightarrow \infty} F\phi(r) = \infty$, for a given large $v > 0$, there exists $x > 0$ such that

$$F\phi(x-1) < v \leq F\phi(x).$$

Hence, if $|\Omega| < v$, then

$$\frac{|\partial_S \Omega|}{|\Omega|} \geq \frac{1}{x}.$$

On the other hand, as F^{-1} is strictly increasing, we obtain

$$(x-1)/\beta = F^{-1}(F((x-1)/\beta)) \leq F^{-1}(\alpha F\phi(x-1)) < F^{-1}(\alpha v).$$

Cheeger's inequality implies that

$$\Lambda(v-1) \geq \frac{1}{2|S|^2} \min_{|\Omega| \leq v-1} \left(\frac{|\partial_S \Omega|}{|\Omega|} \right)^2.$$

We deduce that

$$\Lambda(v/2) \geq \frac{1}{2|S|^2(\beta F^{-1}(\alpha v) + 1)^2},$$

which yields $\Lambda \succeq L$. \square

Proposition 4.3. *Let Γ be a finitely generated group. Assume there exists a sequence of finite subsets $(\Omega_n)_{n \in \mathbb{N}}$ of Γ with the following properties:*

- (1) *There exists a constant $\alpha \geq 1$, such that $\lambda_1(\Omega_n) \leq \alpha/n^2$.*
- (2) *$|\Omega_n| < |\Omega_{n+1}|$.*
- (3) *There exists $C > 1$ such that the piecewise linear extension $F : [1, \infty) \rightarrow \mathbb{R}$ of the function $n \mapsto |\Omega_n|$ satisfies $F(Cr) \geq F(r)^2$ for large $r > 0$.*

Then, near zero,

$$N(\lambda) \succeq \frac{1}{F(\lambda^{-1/2})}.$$

Proof. Define

$$L(v) = \frac{\alpha}{(F^{-1}(v) - 1)^2}.$$

The inverse function $L^{-1}(\lambda) = F(\alpha^{1/2}\lambda^{-1/2} + 1)$ is dilatationally equivalent to $F(\lambda^{-1/2})$ near zero. By Lemma 4.2 the function $L \circ \exp$ is doubling near infinity. Hence the assertion will follow from Theorem 3.1 (2) once it has been shown that $\Lambda(v) \leq L(v)$ near infinity. Let

$v > 1$. Let $n \in \mathbb{N}$ be determined by $F^{-1}(v) - 1 < n \leq F^{-1}(v)$. From the definition of F we get

$$|\Omega_n| = F(n) \leq F(F^{-1}(v)) = v.$$

But, by hypothesis,

$$\lambda_1(\Omega_n) \leq \frac{\alpha}{n^2} \leq \frac{\alpha}{(1 - F^{-1}(v))^2} = L(v). \quad \square$$

The above proposition is inspired from [12, Proposition 2] and [9, Theorem 4.7]. As mentioned in [12, Proposition 2], the required upper bound $\lambda_1(\Omega_n) \leq \frac{\alpha}{n^2}$ holds in the case there exists $\omega_n \subset \Omega_n$ such that $d_S(\omega_n, \Gamma \setminus \Omega_n) > \epsilon n$, and such that $|\Omega_n| \leq C|\omega_n|$, where $\epsilon > 0, C \geq 1$ are constants independent of n . The pairs (Ω_n, ω_n) are called *Følner couples*. In several examples, this approach leads to lower bounds on $N(\lambda)$ that match the upper bounds deduced from Proposition 1.4. In particular, for all the examples listed in Table 1.3 it leads to matching bounds. We refer the reader to [9, 12, 18, 21] for the construction of Følner couples.

5. STABILITY

This section is devoted to the proof of Theorem 1.7. It is possible to deduce this theorem from [15, Proposition 4.1] if both measures are assumed to have finite support. The general case requires a direct proof:

Proof of Theorem 1.7. For a probability measure μ on Γ , consider the following bounded, Γ -equivariant operator

$$C_\mu : \bigoplus_{s \in \text{supp}(\mu)} l^2(\Gamma) \xrightarrow{\sum_{s \in \text{supp}(\mu)} (\mu(s)/2)^{1/2} R_{s-1}} l^2(\Gamma),$$

where $R_{s-1} : l^2(\Gamma) \rightarrow l^2(\Gamma)$ is right convolution by $\delta_s - \delta_e$ (or in other words, right multiplication by $s - 1$). If μ is symmetric, we compute

$$\begin{aligned} C_\mu C_\mu^* &= \sum_{s \in \text{supp}(\mu)} \frac{\mu(s)}{2} R_{2-s-s^{-1}} = \sum_s \mu(s) \text{id} - \frac{1}{2} \sum_s \mu(s) R_s - \frac{1}{2} \sum_s \mu(s) R_{s^{-1}} \\ (5.1) \quad &= 1 - \sum_s \mu(s) R_s \\ &= \Delta_\mu. \end{aligned}$$

So we have to show that the spectral distributions of $C_{\mu_1} C_{\mu_1}^*$ and $C_{\mu_2} C_{\mu_2}^*$ are equivalent near zero. Without loss of generality, we may and will assume that the density of μ_2 is $\sum_{s \in S} \delta_s / |S| \in l^1(\Gamma)$ for a finite, symmetric generating set S . Throughout, direct sums of Hilbert spaces are understood to be completed direct sums.

Firstly, we show that there is a bounded, Γ -equivariant operator F that makes the following square commutative.

$$\begin{array}{ccc} \bigoplus_{h \in \text{supp}(\mu_1)} l^2(\Gamma) & \xrightarrow{C_{\mu_1}} & l^2(\Gamma) \\ \downarrow F & & \downarrow \text{id} \\ \bigoplus_{s \in S} l^2(\Gamma) & \xrightarrow{C_{\mu_2}} & l^2(\Gamma) \end{array}$$

To this end, choose for every $\gamma \in \Gamma$ a path w^γ in the Cayley graph $\text{Cayl}(\Gamma, S)$ from e to γ of length $n = l(\gamma)$. Further, let $(v_0^\gamma, \dots, v_n^\gamma)$ denote the successive vertices of w^γ , and $(w_1^\gamma, \dots, w_n^\gamma)$ the successive oriented edges of w^γ . Note that $v_0^\gamma = e$ and w_i^γ has endpoints v_{i-1}^γ and v_i^γ . The matrix representation $F = (F_{s,h})_{h \in \text{supp}(\mu_1), s \in S}$ of F is given by

$$F_{s,h} = (|S|\mu_1(h))^{1/2} \sum_{1 \leq i \leq l(h), w_i^h = s} R_{v_{i-1}^h}.$$

The map F is well defined on the dense subset $\bigoplus_{s \in \text{supp}(\mu_1)} \mathbb{C}\Gamma$ and makes the square commute. It is straightforward to verify that F is bounded with operator norm

$$\|F\|^2 \leq \sum_{h \in \text{supp}(\mu_1)} |S|^2 \mu_1(h) l(h)^2 < \infty.$$

Let $\alpha : \mathbb{C}\Gamma \rightarrow \mathbb{C}$ be the linear map uniquely defined by $\alpha(\gamma) = 1$ for every $\gamma \in \Gamma$. For a finite generating set $T \subset \Gamma$, consider the map

$$\phi_T : \bigoplus_{t \in T} \mathbb{C}\Gamma \xrightarrow{\oplus R_{t-1}} \mathbb{C}\Gamma.$$

It satisfies $\text{im}(\phi_T) = \ker(\alpha)$. This is either proved directly or by noting that ϕ_T is the first differential in a free $\mathbb{C}\Gamma$ -resolution of \mathbb{C} (or, topologically, in the cellular chain complex of a model of the universal space $E\Gamma$ that has $\text{Cayl}(\Gamma, T)$ as its 1-skeleton [5, (4.3) Example on p. 16]). By assumption there is a finite generating set $T \subset \text{supp}(\mu_1)$. So we have

$$\text{im}(C_{\mu_1}|_{\bigoplus_{t \in T} \mathbb{C}\Gamma}) = \text{im}(\phi_T) = \ker(\alpha).$$

Similarly, we have

$$\text{im}(C_{\mu_2}|_{\bigoplus_{s \in S} \mathbb{C}\Gamma}) = \ker(\alpha).$$

For every $s \in S$, pick $x_s \in \bigoplus_{t \in T} \mathbb{C}\Gamma$ with $C_{\mu_1}(x_s) = C_{\mu_2}(1_s) \in \ker(\alpha)$ where $1_s \in \Gamma \subset \mathbb{C}\Gamma$ is the unit element in the s -th component of $\bigoplus_{s \in S} \mathbb{C}\Gamma$. Define $G : \bigoplus_{s \in S} \mathbb{C}\Gamma \rightarrow \bigoplus_{t \in T} \mathbb{C}\Gamma$ to be the unique Γ -equivariant, linear map with $G(1_s) = x_s$ for every $s \in S$. Then G is bounded. We obtain a commutative square:

$$\begin{array}{ccc} \bigoplus_{h \in \text{supp}(\mu_1)} l^2(\Gamma) & \xrightarrow{C_{\mu_1}} & l^2(\Gamma) \\ G \uparrow & & \uparrow \text{id} \\ \bigoplus_{s \in S} l^2(\Gamma) & \xrightarrow{C_{\mu_2}} & l^2(\Gamma). \end{array}$$

Let $\text{pr}_1 : \bigoplus_{h \in \text{supp}(\mu_1)} l^2(\Gamma) \circlearrowleft$ be the projection onto the orthogonal complement $\ker(C_{\mu_1})^\perp$ of $\ker(C_{\mu_1})$, and similarly, let pr_2 be the projection onto $\ker(C_{\mu_2})^\perp$. Then we obtain two commutative squares

$$\begin{array}{ccc} \ker(C_{\mu_1})^\perp & \xrightarrow{C_{\mu_1}} & l^2(\Gamma) \\ \downarrow \text{pr}_2 \circ F & & \downarrow \text{id} \\ \ker(C_{\mu_2})^\perp & \xrightarrow{C_{\mu_2}} & l^2(\Gamma) \end{array} \quad \begin{array}{ccc} \ker(C_{\mu_1})^\perp & \xrightarrow{C_{\mu_1}} & l^2(\Gamma) \\ \text{pr}_1 \circ G \uparrow & & \uparrow \text{id} \\ \ker(C_{\mu_2})^\perp & \xrightarrow{C_{\mu_2}} & l^2(\Gamma). \end{array}$$

The commutativity and the injectivity of C_{μ_1} and C_{μ_2} when restricted to $\ker(C_{\mu_1})^\perp$ and $\ker(C_{\mu_2})^\perp$, respectively, already imply that $\text{pr}_2 \circ F$ is an isomorphism with inverse $\text{pr}_1 \circ G$. One easily verifies that

$$(5.2) \quad (C_{\mu_i}|_{\ker^\perp}) \circ (C_{\mu_i}|_{\ker^\perp})^* = C_{\mu_i} \circ C_{\mu_i}^*.$$

We apply Efremov-Shubin's *min-max principle* [15, (1.3); 17, Definition 2.1 and Lemma 2.3 on pp. 73-74] which says that

$$\text{tr}_\Gamma(E_{\lambda^2}^{A^*A}) = \sup\{\text{tr}_\Gamma(\text{pr}_L); L \subset l^2(\Gamma) \text{ closed } \Gamma\text{-invariant, } \|Ax\| \leq \lambda\|x\| \forall x \in L\}.$$

to the operator $A = (C_{\mu_1}|_{\ker(C_{\mu_1})^\perp})^* = (\text{pr}_2 \circ F)^*(C_{\mu_2}|_{\ker(C_{\mu_2})^\perp})^*$. From this, combined with (5.1) and (5.2), we obtain, setting $B = (C_{\mu_2}|_{\ker(C_{\mu_2})^\perp})^*$ and $T = (\text{pr}_2 \circ F)^*$, that

$$\begin{aligned} N_{\mu_1}(\lambda^2) &= \sup\{\text{tr}_\Gamma(\text{pr}_L); L \subset l^2(\Gamma), \|Ax\| = \|TBx\| \leq \lambda\|x\| \forall x \in L\} \\ &\leq \sup\{\text{tr}_\Gamma(\text{pr}_L); L \subset l^2(\Gamma), \|Bx\| = \|T^{-1}TBx\| \leq \|T^{-1}\|\lambda\|x\| \forall x \in L\} \\ &= N_{\mu_2}(\|T^{-1}\|^2\lambda^2). \end{aligned}$$

A similar argument yields $N_{\mu_2}(\lambda^2) \leq N_{\mu_1}(\|T\|^2\lambda^2)$. \square

REFERENCES

- [1] Laurent Bartholdi and Wolfgang Woess, *private communication*.
- [2] Alexander Bendikov, Christophe Pittet, and Roman Sauer, *QI-invariance of the spectral distribution of Laplace operators and the return probability of random walks on groups*, preprint.
- [3] Alexander Bendikov, Thierry Coulhon, and Laurent Saloff-Coste, *Ultracontractivity and embedding into L^∞* , Math. Ann. **337** (2007), no. 4, 817–853.
- [4] N. H. Bingham, C. M. Goldie, and J. L. Teugels, *Regular variation*, Encyclopedia of Mathematics and its Applications, vol. 27, Cambridge University Press, Cambridge, 1989.
- [5] Kenneth S. Brown, *Cohomology of groups*, Graduate Texts in Mathematics, vol. 87, Springer-Verlag, 1982.
- [6] Nicolas Th. Varopoulos, Laurent Saloff-Coste, and Thierry Coulhon, *Analysis and geometry on groups*, Cambridge Tracts in Mathematics, vol. 100, Cambridge University Press, Cambridge, 1992.
- [7] Thierry Coulhon and Laurent Saloff-Coste, *Isopérimétrie pour les groupes et les variétés*, Rev. Mat. Iberoamericana **9** (1993), no. 2, 293–314 (French).
- [8] ———, *Variétés riemanniennes isométriques à l'infini*, Rev. Mat. Iberoamericana **11** (1995), no. 3, 687–726 (French).
- [9] Thierry Coulhon, Alexander Grigor'yan, and Christophe Pittet, *A geometric approach to on-diagonal heat kernel lower bounds on groups*, Ann. Inst. Fourier (Grenoble) **51** (2001), no. 6, 1763–1827 (English, with English and French summaries).
- [10] Thierry Coulhon and Alexander Grigor'yan, *On-diagonal lower bounds for heat kernels and Markov chains*, Duke Math. J. **89** (1997), no. 1, 133–199.
- [11] Jozef Dodziuk, *Difference equations, isoperimetric inequality and transience of certain random walks*, Trans. Amer. Math. Soc. **284** (1984), no. 2, 787–794.
- [12] Anna Erschler, *Isoperimetry for wreath products of Markov chains and multiplicity of selfintersections of random walks*, Probab. Theory Related Fields **136** (2006), no. 4, 560–586.
- [13] ———, *Piecewise automatic groups*, Duke Math. J. **134** (2006), no. 3, 591–613.
- [14] Mikhail Gromov, *Entropy and Isoperimetry for Linear and non-Linear Group Actions*, Groups, Geometry, and Dynamics **1** (2008), no. 4, 499–593.
- [15] Mikhail Gromov and Mikhail Shubin, *von Neumann spectra near zero*, Geom. Funct. Anal **1** (1991), no. 4, 375–404.
- [16] Vadim A. Kaĭmanovich and Anatoly M. Vershik, *Random walks on discrete groups: boundary and entropy*, Ann. Probab. **11** (1983), no. 3, 457–490.
- [17] Wolfgang Lück, *L^2 -invariants: theory and applications to geometry and K-theory*, Springer-Verlag, 2002.

- [18] Christophe Pittet, *The isoperimetric profile of homogeneous Riemannian manifolds*, J. Differential Geom. **54** (2000), no. 2, 255–302.
- [19] Christophe Pittet and Laurent Saloff-Coste, *Amenable groups, isoperimetric profiles and random walks*, Geometric group theory down under (Canberra, 1996), de Gruyter, Berlin, 1999, pp. 293–316.
- [20] ———, *On random walks on wreath products*, Ann. Probab. **30** (2002), no. 2, 948–977.
- [21] ———, *Random walks on finite rank solvable groups*, J. Eur. Math. Soc. (JEMS) **5** (2003), no. 4, 313–342.
- [22] ———, *On the stability of the behavior of random walks on groups*, J. Geom. Anal. **10** (2000), no. 4, 713–737.
- [23] Frédéric Riesz and Béla Sz.-Nagy, *Leçons d'analyse fonctionnelle*, Gauthier-Villars, Paris, 1972.
- [24] R. Tyrrell Rockafellar, *Convex analysis*, Princeton Landmarks in Mathematics, Princeton University Press, Princeton, NJ, 1997. Reprint of the 1970 original; Princeton Paperbacks.

INSTITUTE OF MATHEMATICS, WROCLAW UNIVERSITY

E-mail address: `bendikov@math.uni.wroc.pl`

CMI UNIVERSITÉ D'AIX-MARSEILLE I, MARSEILLE

Current address: Laboratoire Poncelet CNRS, Moscow

E-mail address: `pittet@cmi.univ-mrs.fr`

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CHICAGO

Current address: Mathematisches Institut, WWU Münster

E-mail address: `sauerr@uni-muenster.de`