On the Picard principle for negative perturbations

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Abstract

Given a local Kato measure $\mu$ on $\mathbb{R}^d \setminus \{0\}$, $d \geq 2$, let $\mathcal{H}_0^{\Delta + \mu}(U)$ be the convex cone of all continuous real solutions $u \geq 0$ to the equation $\Delta u + u\mu = 0$ on the punctured unit ball $U$ satisfying $\lim_{|x| \to 1} u(x) = 0$. It is shown that $\mathcal{H}_0^{\Delta + \mu}(U) \neq \{0\}$ if and only if the operator $f \mapsto \int_U G(\cdot, y) f(y) \, d\mu(y)$, where $G$ denotes the Green function on $U$, is a bounded operator on $L^2(U, \mu)$ having a norm which is at most one. Moreover, extremal rays in $\mathcal{H}_0^{\Delta + \mu}(U)$ are characterized and it is proven that the Picard principle holds for $\Delta + \mu$ on $U$, that is, that $\mathcal{H}_0^{\Delta + \mu}(U)$ consists of one ray, provided there exists a suitable sequence of shells in $U$ such that, on these shells, $\mu$ is either small or not too far from being radial. Finally, it is shown that the verification of the Picard principle can be localized.

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1 Introduction

For every relatively compact open set $V$ in $\mathbb{R}^d$, $d \geq 2$, let $G_V$ denote the (classical) Green function on $V$, normalized such that $\Delta G_V(\cdot, y) = -\varepsilon_y$, $y \in V$. Throughout this paper we fix a measure $\mu$ on $\mathbb{R}^d \setminus \{0\}$ which does not charge the origin and is a (local) Kato measure on $\mathbb{R}^d \setminus \{0\}$, that is, for some covering of $\mathbb{R}^d \setminus \{0\}$ by relatively compact open sets $V$, the potentials $x \mapsto \int_V G_V(x, y) \, d\mu(y)$ are continuous and real.

Moreover, we fix $R > 0$ and define

$$B := \{ x \in \mathbb{R} : |x| < R \}, \quad U := B \setminus \{0\}.$$ 

Let $\mathcal{H}_0^{\Delta + \mu}(U)$ be the convex cone of all continuous real solutions $u \geq 0$ to the Schrödinger equation

$$(1.1) \quad \Delta u + u\mu = 0$$

on $U$ which vanish at the boundary $\partial B$ of $B$. Here solution is meant in the sense of distributions, that is,

$$\int_U u\Delta \varphi \, d\lambda^d + \int_U u\varphi \, d\mu = 0$$

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1This is a natural assumption, since otherwise (1.1) would not admit continuous solutions $u \neq 0$. 
for all $C^\infty$-functions $\varphi$ with compact support in $U$, $\lambda^d$ being Lebesgue measure on $\mathbb{R}^d$.

By definition, $\dim_U(\Delta + \mu)$, the Picard dimension of $\Delta + \mu$ on $U$, is the number of extremal rays in $\mathcal{H}_0^{\Delta+\mu}(U)$. Of course, $\dim_U(\Delta + \mu) = 0$ if $\mathcal{H}_0^{\Delta+\mu}(U) = \{0\}$. If $\mathcal{H}_0^{\Delta+\mu}(U) \neq \{0\}$ and $x_0 := (R/2,0,\ldots,0)$, then $\{h \in \mathcal{H}_0^{\Delta+\mu}(U): h(x_0) = 1\}$ is a compact base of $\mathcal{H}_0^{\Delta+\mu}(U)$, $\dim_U(\Delta + \mu)$ is the number of extreme points of this set, and hence $\dim_U(\Delta + \mu) > 0$.

We say that $\Delta + \mu$ satisfies the Picard principle on $U$ provided

\begin{equation}
\dim_U(\Delta + \mu) \leq 1.
\end{equation}

In [NT97b] it is shown that (1.2) holds, if $d = 2$. Moreover, it is satisfied if $\mu$ has a locally Hölder continuous density with respect to $\lambda^d$ which is radial ([NT97a]). However, the problem seems to be open for $d \geq 3$ and general measures $\mu$.

In this paper, we prove that $\mathcal{H}_0^{\Delta+\mu}(U) \neq \{0\}$ if and only if the operator

$$K: f \mapsto \int_U G(\cdot, y)f(y)\,d\mu(y),$$

where $G := G_U = G_B|_{U \times U}$, is a bounded operator on $L^2(U,\mu)$ with $\|K\|_2 \leq 1$ (Corollary 4.6). In particular, (1.2) holds, unless $K$ is bounded on $L^2(U,\mu)$ and $\|K\|_2 \leq 1$.

Since $\mathcal{H}_0^{\Delta}(U)$ is the set of all positive multiples of $G_0 := G_B(\cdot,0)|_U$ (cf. (2.2)) and hence $\dim_U \Delta = 1$, it would be sufficient to consider the case $\mu(U) > 0$. We shall see first that, whatever $\mu$ is, every function in an extremal ray of $\mathcal{H}_0^{\Delta+\mu}(U)$ is either a multiple of $\sum_{n=0}^\infty K^nG_0$ or a continuous strictly positive $K$-invariant function $h$ (see Proposition 2.1). This implies that $\dim_U(\Delta + \mu) = 1$, if $K$ is a bounded operator on $L^2(U,\mu)$ such that $\|K\|_2 = 1$ and 1 is an eigenvalue of $K$ (Proposition 3.2 in connection with Corollary 4.6).

Further, we shall prove that the Picard principle holds for $\Delta + \mu$ on $U$ provided that there are arbitrarily small shells $A$ of constant relative thickness, where the potential $\int_AG(\cdot, y)\,d\mu(y)$ is small enough (Theorem 5.3) or the measure $1_A\mu$ is not too far from being invariant under rotations (Theorem 6.4).

Finally, we shall see that the verification of the Picard principle can be localized in different ways (Theorem 7.1).

## 2 Nature of extremal functions in $\mathcal{H}_0^{\Delta+\mu}(U)$

For every open set $W$ in $\mathbb{R}^d$, let $\mathcal{S}^+(W)$, $\mathcal{H}^+(W)$ denote the set of all positive functions on $W$ which are superharmonic, harmonic respectively. For every relatively compact open set $V$ in $\mathbb{R}^d$, let $K_V$ denote the mapping $f \mapsto \int_VG_V(\cdot, y)f(y)\,d\mu(y)$. If $V$ is regular and $0 \notin \partial V$, then $K_V$ maps the space $\mathcal{B}_0(V)$ of all bounded Borel measurable functions on $V$ into the subspace $\mathcal{C}_0(V)$ of all continuous functions in $\mathcal{B}_0(V)$ vanishing at the boundary $\partial V$. The harmonic kernel of $V$ will be denoted by $H_V$.

Let us note that every $u \in \mathcal{C}^+(U)$ satisfying (1.1) is superharmonic. Hence

$$\mathcal{H}_0^{\Delta+\mu}(U) \subset \mathcal{S}^+(U).$$

A function $h \in \mathcal{H}_0^{\Delta+\mu}(U) \setminus \{0\}$ is extremal, if it is contained in an extremal ray of $\mathcal{H}_0^{\Delta+\mu}(U)$, that is, if every $h \in \mathcal{H}_0^{\Delta+\mu}(U)$ such that $0 \leq \tilde{h} \leq h$ is a multiple of $h$. 

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PROPOSITION 2.1. Let $h$ be an extremal function in $H_0^{\Delta+\mu}(U) \setminus \{0\}$. Then $Kh = h$ or $h$ is a multiple of the series $\sum_{n=0}^{\infty} K^n G_0$.

Proof. For every $n \in \mathbb{N}$, let

$$A_n := \{ x \in \mathbb{R}^d : R/n < |x| < (1 - 1/n)R \}$$

and $g_n := H_{A_n} h \in H^+(A_n)$. Then

$$h - K_{A_n} h = g_n, \quad n \in \mathbb{N},$$

since $\Delta(h - K_{A_n} h) = \Delta h + h\mu = 0$ on $A_n$ and $K_{A_n} h \in C_0(A_n)$. The sequence $(K_{A_n} h)$ is increasing to $Kh$. So the sequence $(g_n)$ is decreasing to a function $g \in H^+(U)$, and we have

$$(2.1) \quad h = g + Kh.$$

To prove our result it remains to consider the case $g \neq 0$. Since $g \leq h$, we know that $\lim_{|x| \to R} g(x) = 0$. Therefore

$$(2.2) \quad g = \alpha G_0$$

for some $\alpha > 0$ (see [AG01, Exercise 2.11]). By an obvious induction, we see from (2.1) that, for every $m \in \mathbb{N},$

$$h = \sum_{n=0}^{m-1} K^n g + K^m h,$$

and therefore

$$\tilde{h} := \sum_{n=0}^{\infty} K^n g \leq h.$$

In particular, $\lim_{|x| \to R} \tilde{h}(x) = 0$. Since $K\tilde{h} + K(h - \tilde{h}) = Kh = h - g \in C^+(U)$, where both $K\tilde{h}$ and $K(h - \tilde{h})$ are lower semicontinuous, we obtain that $K\tilde{h} \in C^+(U)$. Moreover, clearly $\tilde{h} = g + K\tilde{h}$, hence $\tilde{h} \in C^+(U)$ and $\Delta \tilde{h} = \Delta K\tilde{h} = -\tilde{h}\mu$. Therefore $\tilde{h} \in H_0^{\Delta+\mu}(U)$. Since $h$ is extremal by our assumption and $h \geq \tilde{h} \geq g > 0$, we conclude that $\tilde{h} = \beta h$ for some $\beta > 0$. Thus we finally see that $h = (\alpha/\beta) \sum_{n=0}^{\infty} K^n G_0$ (and $\sum_{n=0}^{\infty} K^n G_0$ is an extremal function in $H_0^{\Delta+\mu}(U)$).

If $g_0 := \sum_{n=0}^{\infty} K^n G_0 < \infty$, then obviously

$$(2.3) \quad g_0 = Kg_0 + G_0 > Kg_0.$$ 

If $g_0$ is locally bounded on $U$ and $\lim_{|x| \to R} g_0(x) = 0$, then (2.3) implies that $g_0$ is continuous on $U$, $\Delta g_0 = -g_0\mu$, and hence $g_0 \in H_0^{\Delta+\mu}(U)$. Moreover, any locally bounded function $h$ on $U$ which vanishes at $\partial B$ and satisfies $Kh = h$ is contained in $H_0^{\Delta+\mu}(U)$.

Let us recall that, taking $x_0 := (R/2, 0, \ldots, 0)$, the convex set

$$(2.4) \quad H_0^{\Delta+\mu} := \{ h \in H_0^{\Delta+\mu}(U) : h(x_0) = 1 \}$$
is a compact base of $\mathcal{H}_0^{\Delta+\mu}(U)$, and hence, by Choquet’s theorem, for every function $h \in \mathcal{H}_0^{\Delta+\mu}(U) \setminus \{0\}$, there exists a probability measure $\chi$ on the set of extreme points of $H_0^{\Delta+\mu}$ such that

$$h(x) = \int \tilde{h} d\chi(\tilde{h}).$$

This leads to the following consequence of Proposition 2.1.

**COROLLARY 2.2.** Every function $h \in \mathcal{H}_0^{\Delta+\mu}(U)$ is $\mu$-integrable and satisfies $Kh \leq h$, where even $Kh = h$, if $g_0 \notin \mathcal{H}_0^{\Delta+\mu}(U)$.

If $g_0 \in \mathcal{H}_0^{\Delta+\mu}(U)$, then $g_0$ is extremal.

**Proof.** Let us fix $h \in \mathcal{H}_0^{\Delta+\mu}(U)$. Obviously, by Proposition 2.1, (2.3), and (2.5), $Kh \leq h$, and even $Kh = h$ if $g_0 \notin \mathcal{H}_0^{\Delta+\mu}(U)$. Of course, $h$ is bounded on the set $U_1 := \{x \in U : |x| > R/2\}$, and $\mu(U_1) < \infty$. Further, if $G(x, x_0) = 0$, and hence the inequality $\int G(x, x_0)h(x) d\mu(x) = Kh(x_0) \leq h(x_0) < \infty$ implies that $h$ is $\mu$-integrable on $U \setminus U_1$. Thus $h \in \mathcal{L}^1(U, \mu)$.

Finally, let us assume that $g_0 \in \mathcal{H}_0^{\Delta+\mu}(U)$. Then the measure $\chi$ associated with $g_0$ must charge $g_0/g(x_0)$, since otherwise we would obtain that $Kg_0 = g_0$, contradicting (2.3). So $g_0$ is extremal.

**3 Applications**

In the proof of the following result (and later on) we shall tacitly use the fact that, for every $s \in \mathcal{S}^+(U)$, there exists a unique extension to a function $\tilde{s} \in \mathcal{S}^+(B)$ (and $\tilde{s}(0) = \lim \inf_{y \to x} s(y)$; see [AG01, Corollary 5.2.2]).

**PROPOSITION 3.1.** Suppose that $g_0$ is bounded by a multiple of $G_0$. Then there exists $C > 0$ such that

$$\sum_{n=0}^\infty K^n s \leq C s \quad \text{for every } s \in \mathcal{S}^+(U).$$

In particular, $\Delta + \mu$ satisfies the Picard principle, $\mathcal{H}_0^{\Delta+\mu}(U) = \mathbb{R}^+ g_0$.

**Proof.** By Corollary 8.3, there exists $C > 0$ such that (3.1) holds. In particular, there is no $s \in \mathcal{S}^+(U)$ such that $Ks = s$. So there is no $h \in \mathcal{H}_0^{\Delta+\mu}(U)$ satisfying $Kh = h$. Thus, by Proposition 2.1, $\mathcal{H}_0^{\Delta+\mu}(U) = \mathbb{R}^+ g_0$.

In Section 4, we shall see that $\mathcal{H}_0^{\Delta+\mu}(U) \neq \{0\}$ if and only if $K$ is a bounded operator on $\mathcal{L}^2(U, \mu)$ with $\|K\|_2 \leq 1$ (Corollary 4.6). A first step in this direction is the following.

**PROPOSITION 3.2.** Let us suppose that $\mu(U) > 0$ and that $K$ is a bounded operator on $\mathcal{L}^2(U, \mu)$, where $\beta := \|K\|_2$ is an eigenvalue of $K$. Then there exists $u \in \mathcal{L}^2(U, \mu)$, $u > 0$, such that

$$\{v \in \mathcal{L}^2(U, \mu) : Kv = \beta v\} = \mathbb{R} u.$$
Moreover, for every \( v \in \mathcal{B}^+(U) \) satisfying \( K v \leq \beta v \), there exists \( c \geq 0 \) such that \( v = c u \) \( \mu \)-a.e. on \( U \).

In particular, \( \Delta + \mu \) satisfies the Picard principle on \( U \), if \( \beta \geq 1 \). More precisely, \( \mathcal{H}_0^{\Delta + \mu}(U) \subset \mathbb{R}^+ u \) if \( \beta = 1 \), \( \mathcal{H}_0^{\Delta + \mu}(U) = \{0\} \) if \( \beta > 1 \).

Before starting the proof let us note that the assumption of Proposition 3.2 implies that, in fact, \( \mathcal{H}_0^{\Delta + \mu}(U) = \mathbb{R}_+ u \) if \( \beta = 1 \) (see Corollary 4.6). Moreover, the first statement is more or less known (see e.g. [BAH01, Proposition 3.12]). However, since the first and the second statement can be obtained almost simultaneously, we shall present a complete proof.

Proof of Proposition 3.2. 1. By assumption, there exists a function \( u \in \mathcal{L}^2(U, \mu) \) such that \( \mu(\{u \neq 0\}) > 0 \) and \( Ku = \beta u \) \( \mu \)-a.e. Since we can replace \( u \) by \( \beta^{-1} Ku \) on the \( \mu \)-null set \( \{Ku \neq \beta u\} \), we may suppose without loss of generality that

\[
K u = \beta u \quad \text{(everywhere) on } U.
\]

Moreover, we may assume that \( \mu(\{u > 0\}) > 0 \) (if necessary, we replace \( u \) by \( -u \)).

2. Next let \( v \in \mathcal{B}(U) \) such that \( v \geq -|u| \) and \( Kv \leq \beta v \). We define

\[
w := (|u| - v)^+.
\]

Since \( K|u| \geq |Ku| = \beta|u| \), we know that \( Kw \geq K(|u| - v) \geq \beta(|u| - v) \). Hence

\[
K w \geq \beta w \geq 0.
\]

Moreover, \( w \in \mathcal{L}^2(U, \mu) \), since \( 0 \leq w \leq 2|u| \), and

\[
\int (Kw)^2 \, d\mu \leq \beta^2 \int w^2 \, d\mu,
\]

since \( \|K\|_2 = \beta \). Therefore

\[
K w = \beta w \quad \mu \text{-a.e. on } U.
\]

If \( \mu(\{w > 0\}) > 0 \), then \( Kw > 0 \), hence, by (3.6), \( w > 0 \) \( \mu \)-a.e on \( U \), that is,

\[
|u| - v > 0 \quad \mu \text{-a.e. on } U.
\]

If, however, \( w = 0 \) \( \mu \)-a.e., then \( Kw = 0 \), hence, by (3.5), \( w = 0 \), that is,

\[
v \geq |u|.
\]

3. If we take \( v := u \), then (3.7) does not hold, since \( \mu(\{u > 0\}) > 0 \) and \( |u| - u = 0 \) on \( \{u > 0\} \). So, by (3.8), \( u \geq |u| \geq 0 \). In fact, this shows that, for every function \( \tilde{u} \in \mathcal{L}^2(U, \mu) \) satisfying \( K\tilde{u} = \beta\tilde{u} \),

\[
\tilde{u} \geq 0 \quad \text{or} \quad -\tilde{u} \geq 0.
\]

So every \( v \in \mathcal{L}^2(U, \mu) \) satisfying \( Kv = \beta v \) is a multiple of \( u \), since otherwise there certainly is a linear combination \( \tilde{u} \) of \( u \) and \( v \) violating (3.9).
4. Let us next suppose that \( v \in \mathcal{B}^+(U) \) and \( Kv \leq \beta v \). To show that \( v \) is \( \mu \)-a.e. equal to a multiple of \( u \) we may suppose that \( \mu(\{u > v\}) > 0 \) (we replace \( v \) by \( \eta v \), where \( \eta > 0 \) is sufficiently small). Then \( \mu(\{w > 0\}) > 0 \) and therefore, by (3.7), \( v < u \) \( \mu \)-a.e. on \( U \). Since \( v \geq 0 \), we hence see that \( v \in \mathcal{L}^2(U, \mu) \). If \( \mu(\{Kv < \beta v\}) > 0 \), then, by the symmetry of \( G \),

\[
\beta \int uv \, d\mu > \int u(Kv) \, d\mu = \int (Ku)v \, d\mu = \beta \int uv \, d\mu,
\]

a contradiction. So \( Kv = \beta v \) \( \mu \)-a.e. Replacing \( v \) by \( \beta^{-1}Kv \) on the \( \mu \)-null set \( \{Kv < \beta v\}\) we obtain a function \( \tilde{v} \) satisfying \( K\tilde{v} = \beta\tilde{v} \). Then \( \tilde{v} = cu \) for some \( c \geq 0 \) and hence \( v = cu \) \( \mu \)-a.e.

5. If \( \beta = 1 \), then \( \mathcal{H}^{\Delta+\mu}_0(U) \subseteq \mathbb{R}u \) by Corollary 2.2 and the preceding considerations. Finally, let us suppose that \( \beta > 1 \) and let \( h \in \mathcal{H}^{\Delta+\mu}_0(U) \). Then \( Kh \leq h \leq \beta h \), hence \( h = cu \) \( \mu \)-a.e. for some \( c \geq 0 \). So \( h = Kh = \beta h \) and therefore \( h = 0 \).

We stress that our general assumption on \( \mu \) does not exclude the possibility that there is a function \( h \in \mathcal{H}^{\Delta+\mu}_0(U) \) which satisfies \( Kh = h \) but is not contained in \( \mathcal{L}^2(U, \mu) \).

**EXAMPLE 3.3.** For every \( n \in \mathbb{N} \), let \( B_n \) be the open ball of radius \( 2^{-(n+3)}R \) centered at \( x_n := (2^{-n}R, 0, \ldots, 0) \), let \( a_n \in (n, \infty) \) such that \( G(\cdot, x_n)/a_n < 2^{-n} \) on \( U \setminus B_n \), and

\[
p_n := \min\{a_n, G(\cdot, x_n)/a_n\}.
\]

Then \( p_n \in \mathcal{C}(U) \) and \( p_n = G^\nu_n \) for some measure \( \nu_n \) which has total mass \( 1/a_n \) and a support \( C_n \subseteq B_n \). Let \( \nu := \sum_{n=1}^\infty \nu_n \), \( p := G^\nu = \sum_{n=1}^\infty p_n \), and \( \mu := (1/p)\nu \). Then \( Kp = G^{\nu\mu} = G^\nu = p \). Since the balls \( B_n, n \in \mathbb{N} \), are pairwise disjoint and \( \sum_{n=1}^\infty 2^{-n} = 1 \), we have \( p \in \mathcal{C}(U) \) and

\[
(3.10) \quad a_n = p_n \leq p \leq a_n + 1 \quad \text{on supp}(\nu_n).
\]

Obviously, \( p \) vanishes at \( \partial B \). Hence \( p \in \mathcal{H}^{\Delta+\mu}_0(U) \). Moreover, for every \( n \in \mathbb{N} \),

\[
\int G^{\nu_n} \, d\nu_n = a_n/a_n = 1 \quad \text{and hence} \quad \int p^2 \, d\mu = \int G^{\nu} \, d\nu \geq \sum_{n=1}^\infty \int G^{\nu_n} \, d\nu_n = \infty.
\]

Corollary 4.6 will show that, nevertheless, \( K \) is a bounded operator on \( \mathcal{L}^2(U, \mu) \) and \( \|K\|_2 \leq 1 \). Further, (3.10) implies that, for every \( n \in \mathbb{N} \), \( K1_{C_n} = G(1/\nu)^{\nu_n} \geq a_n/(a_n + 1) \geq n/(n + 1) \) on \( C_n \), and hence \( \|K\|_2 \geq n/(n + 1) \). Therefore \( \|K\|_2 = 1 \). Finally, let us note that both Theorem 6.4 and Corollary 6.5 immediately imply that \( \dim_{\nu}(\Delta + \mu) = 1 \). We could even smear \( \nu \) a little, add Lebesgue measure on \( U \) (leading to a measure \( \mu \) having a strictly positive \( \mathcal{C}^\infty \)-density on \( U \)), and still have the same result.

### 4 Characterization of \( \mathcal{H}^{\Delta+\mu}_0(U) \neq \{0\} \)

In this section we shall see that \( \mathcal{H}^{\Delta+\mu}_0(U) \neq \{0\} \) if and only if the \( \mu \)-eigenvalues of \( \Delta \) on the shells \( \{x \in \mathbb{R}^d : R/(n+1) < |x| < R\}, n \in \mathbb{N} \), are at least 1 or – equivalently – if and only if \( K \) is a bounded operator on \( \mathcal{L}^2(U, \mu) \) with \( \|K\|_2 \leq 1 \) (Corollary 4.4 and Corollary 4.6). For a useful consequence of \( \|K\|_2 \leq 1 \) see Lemma 6.2.

Let us first recall the following from [HH88] (where the proof is so short that we may just as well include it).
LEMMA 4.1. Let $L$ be a bounded kernel on a measurable space $(E, \mathcal{E})$. Then the following statements are equivalent.

(i) The function $\sum_{n=0}^{\infty} L^n 1$ is bounded.

(ii) The operator $I - L$ on $(\mathcal{E}_b, \| \cdot \|_{\infty})$ is invertible, its inverse is the positive bounded operator $\sum_{n=0}^{\infty} L^n$.

(iii) The operator $I - L$ on $(\mathcal{E}_b, \| \cdot \|_{\infty})$ is invertible, its inverse is positive.

(iv) There exists $f \in \mathcal{E}_b^+$ such that $1 + Lf \leq f$.

Proof. (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii): Trivial.

(iii) $\Rightarrow$ (iv): The function $f := (I - L)^{-1} 1 \in \mathcal{E}_b^+$ satisfies $1 + Lf = f$.

(iv) $\Rightarrow$ (i): By induction, $\sum_{j=0}^{n-1} L^j 1 + L^n f \leq f$, and thus $\sum_{n=0}^{\infty} L^n 1 \leq f$.

For every $\gamma \geq 0$, let $\mathcal{H}^{\Delta+\gamma \mu}$ denote the sheaf of $(\Delta+\gamma \mu)$-harmonic functions: For every open set $W$ in $\mathbb{R}^d \setminus \{0\}$, $\mathcal{H}^{\Delta+\gamma \mu}(W)$ denotes the set of all $(\Delta+\gamma \mu)$-harmonic functions on $W$, that is, all continuous real solutions to the equation $\Delta u + \gamma \mu u = 0$ on $W$. Let us note that $(\mathbb{R}^d \setminus \{0\}, \mathcal{H}^{\Delta+\gamma \mu})$ is a Brelot space (see [BHH87, Theorem 7.7]). In particular, $(\Delta+\gamma \mu)$-harmonic functions satisfy Harnack’s inequalities. Of course, functions are called $(\Delta+\gamma \mu)$-superharmonic provided they are superharmonic with respect to the sheaf $\mathcal{H}^{\Delta+\gamma \mu}$.

Given an open set $V$ which is relatively compact in $\mathbb{R}^d \setminus \{0\}$, let $\Gamma_V$ denote the set of all $\gamma > 0$ such that $I - \gamma K_V : \mathcal{B}_b(V) \to \mathcal{B}_b(V)$ is invertible and the inverse is positive. In the next two propositions we collect results which are proven in [HH88] in a far more general setting. For the convenience of the reader we include a complete proof for our situation.

PROPOSITION 4.2. Let $V \neq \emptyset$ be a relatively compact regular open set in $\mathbb{R}^d \setminus \{0\}$ and $\gamma \in \Gamma_V$. Then the following holds.

1. For every $\gamma \in \Gamma_V$, the set $V$ is regular with respect to $\mathcal{H}^{\Delta+\gamma \mu}$ and the corresponding harmonic kernel is

$$
H_V^{\Delta+\gamma \mu} = (I - \gamma K_V)^{-1} H_V = \sum_{n=0}^{\infty} (\gamma K_V)^n H_V. 
$$

2. If $s$ is a bounded $(\Delta+\gamma \mu)$-superharmonic function on $V$ and $\liminf_{x \to z} s(x) \geq 0$ for every $z \in \partial V$, then $s \geq 0$.

Proof. 1. Let $L := \gamma K_V$ and $\varphi \in C(\partial V)$. Then the function $\psi := (I - L)^{-1} H_V \varphi$ satisfies $\psi - L \psi = H_V \varphi \in \mathcal{H}(V)$. Hence $\psi$ is $(\Delta+\gamma \mu)$-harmonic on $V$ and satisfies $\lim_{x \to z} \psi(x) = \varphi(z)$ for every $z \in \partial V$. Moreover, $\psi \geq 0$ if $\varphi \geq 0$.

If $\tilde{\psi}$ is any $(\Delta+\gamma \mu)$-harmonic function on $V$ such that $\tilde{\psi}$ tends to 0 at $\partial V$, then $g := \psi - \tilde{\psi}$ is harmonic on $V$ and vanishes at $\partial V$. So $g = 0$ and hence $\tilde{\psi} = 0$.

Therefore $V$ is regular with respect to $\mathcal{H}^{\Delta+\gamma \mu}$, and (4.1) follows from Lemma 4.1.

2. Next let $s$ be as indicated. Then, by [BHH87, Theorem 3.2], $t := s - Ls$ is a bounded superharmonic function on $V$. Moreover, $\liminf_{x \to z} t(x) \geq 0$ for every $z \in \partial V$. So $t \geq 0$, and therefore $s = (I - L)^{-1} t \geq 0$. □
Let

$$\alpha_V := \sup \Gamma_V.$$ 

If $$\mu(V) = 0$$, then obviously $$\Gamma_V = (0, \infty)$$ and hence $$\alpha_V = \infty$$.

**Proposition 4.3.** Let $$V$$ be a connected relatively compact regular open set in $$\mathbb{R}^d \setminus \{0\}$$ and $$\mu(V) > 0$$. Then the following holds.

1. $$0 < \alpha_V < \infty$$ and $$\Gamma_V = (0, \alpha_V)$$.

2. There exists a strictly positive $$(\Delta + \alpha_V \mu)$$-harmonic function $$h \in C_0(V)$$. Moreover, $$\ker(I - \alpha_V K_V) = \mathbb{R}h$$, and every positive bounded $$(\Delta + \alpha_V \mu)$$-superharmonic function on $$V$$ is a multiple of $$h$$.

3. For every $$\gamma > \alpha_V$$, the constant function $$0$$ is the only positive bounded function on $$V$$ which is $$(\Delta + \gamma \mu)$$-superharmonic.

In particular, $$\alpha_V$$ is the first $$\mu$$-eigenvalue of $$\Delta$$ on $$V$$ and the corresponding eigenfunctions are multiples of a strictly positive function.

**Proof.** Let $$A$$ be a compact set in $$V$$ such that $$\mu(A) > 0$$. Then $$K_V 1_A > 0$$ on $$V$$. So there exists $$\beta \in (0, \infty)$$ such that $$\beta K_V 1_A \geq 1_A$$. By induction, $$(\beta K_V)^n 1_A \geq 1_A$$ for every $$n \in \mathbb{N}$$, and hence $$\sum_{n=0}^{\infty} (\beta K_V)^n 1 = \infty$$ on $$A$$. Therefore, by Lemma 4.1, $$0 < \alpha_V \leq \beta < \infty$$ and $$\Gamma_V$$ is an interval from $$0$$ to $$\alpha_V$$. We still have to show that $$\Gamma_V$$ is open. To that end let us consider $$\gamma \in \Gamma_V$$ and $$0 < \varepsilon < \|(I + \gamma K_V)^{-1} K_V\|^{-1}$$.

Then

$$(I - (\gamma + \varepsilon) K_V)^{-1} 1 = \sum_{n=0}^{\infty} [\varepsilon (I + \gamma K_V)^{-1} K_V]^{-1} (I + \gamma K_V)^{-1} 1$$

is bounded, and hence $$\gamma + \varepsilon \in \Gamma_V$$. So $$\Gamma_V$$ is an open interval, $$\Gamma_V = (0, \alpha_V)$$. Let $$(\gamma_n)$$ be a sequence in $$\Gamma_V$$ which is increasing to $$\alpha_V$$. For every $$n \in \mathbb{N}$$, let

$$g_n := H_V^{\Delta + \gamma_n \mu} 1 \quad \text{and} \quad c_n := \|g_n\|_{\infty}.$$ 

By (4.1), for every $$n \in \mathbb{N}$$, $$1 \leq g_n \leq g_{n+1}$$ and

$$(4.2) \quad g_n - \gamma_n K_V g_n = 1.$$ 

If $$\sup c_n < \infty$$, then $$g := \lim_{n \to \infty} g_n$$ is bounded and $$g - \alpha_V K_V g = 1$$, and hence $$\alpha_V \in \Gamma_V$$ by Lemma 4.1, a contradiction. So $$\sup c_n = \infty$$.

Since $$K_V$$ is a compact operator on $$(B_0(V), \|\cdot\|_{\infty})$$ which maps $$B_0(V)$$ into $$C_0(V)$$, there exists a subsequence $$(h_n)$$ of $$(c_n^{-1} g_n)$$ such that the sequence $$(K_V h_n)$$ converges uniformly to a function $$h \in C_0^+(V)$$. By (4.2), the sequence $$(h_n)$$ itself converges uniformly to $$h$$ and $$h - \alpha_V K_V h = 0$$, that is, $$h \in \ker(I - \alpha_V K_V)$$. Of course, $$\|h\|_{\infty} = 1$$, since $$\|h_n\|_{\infty} = 1$$ for every $$n \in \mathbb{N}$$. Since $$h \geq 0$$, we hence see that $$h = \alpha_V K_V h > 0$$ on $$V$$. Finally, $$\Delta h = -\alpha_V h \mu$$. So $$h$$ is $$(\Delta + \alpha_V \mu)$$-harmonic.

Let $$\gamma \geq \alpha_V$$ and $$s$$ be a bounded $$(\Delta + \gamma \mu)$$-superharmonic function, $$s > 0$$. There exists a regular open set $$W$$ such that $$\overline{W} \subset V$$ and $$A := V \setminus W$$ satisfies $$\|\alpha_V K_V 1_W\|_{\infty} < 1$$. Then $$\alpha_V K_V 1 \leq \|\alpha_V K_V 1_W\|_{\infty} < 1$$ and hence $$\alpha_V \in \Gamma_W$$. Let

$$a := \sup \{\alpha \geq 0 : \alpha h_0 \leq s \text{ on } A\} \quad \text{and} \quad t := s - ah.$$
Then \( t \) is \((\Delta + \alpha_{V} \mu)\)-superharmonic on \( V \), \( t \geq 0 \) on \( A \), and there exists a point \( x \in A \) such that \( t(x) = 0 \). Clearly, \( \liminf_{y \to z} t(y) \geq 0 \) for every \( z \in \partial W \) (recall that \( h \to 0 \) at \( \partial V \)). Hence, applying (2) with \( W \) in place of \( V \), we obtain that \( t \geq 0 \) on \( W \), and therefore \( t \geq 0 \) on \( V \). Since \( V \) is connected and \( t(x) = 0 \) for some \( x \in A \), we conclude that \( t = 0 \), that is, \( s = ah \).

If \( \hat{W} \) is an open set such that \( \overline{W} \subset V \) and \( \|\gamma_{V} 1_{\hat{W}}\|_{\infty} < 1 \), then
\[
H_{\hat{W}}^{\Delta + \gamma_{V} \mu} h \geq H_{\hat{W}}^{\Delta + \alpha_{V} \mu} h = h.
\]
Since \( a > 0 \) and \( H_{\hat{W}}^{\Delta + \gamma_{V} \mu} s \leq s \), we conclude that \( H_{\hat{W}}^{\Delta + \gamma_{V} \mu} h = h \). Hence \( h \in \mathcal{H}_{\hat{W}}^{\Delta + \gamma_{V} \mu}(V) \), \( \Delta h + \gamma_{V} h \mu = 0 \). On the other hand \( h \in \mathcal{H}_{\hat{W}}^{\Delta + \alpha_{V} \mu}(V) \), \( \Delta h + \alpha_{V} h \mu = 0 \). Since \( \gamma \geq \alpha_{V} \), \( s > 0 \) on \( V \), and \( \mu(V) > 0 \), this implies that \( \gamma = \alpha_{V} \).

Finally, if \( g \in \ker(I - \alpha_{V} K_{V}) \), there exists \( b > 0 \) such that \( bh - g \geq 0 \) on \( A \) and hence \( bh - g \geq 0 \) on \( V \) by (2) (with \( W \) in place of \( V \)). By the preceding considerations, there exists \( c > 0 \) such that \( bh - g = ch \) and therefore \( g \in \mathbb{R}h \).

For every \( n \in \mathbb{N} \), let
\[
U_{n} := \{x \in \mathbb{R}^{d}: R/(n+1) < |x| < R \} \quad \text{and} \quad \alpha_{n} := \alpha_{U_{n}}.
\]

**COROLLARY 4.4.** The following statements are equivalent.

(i) \( \mathcal{H}_{0}^{\Delta + \mu}(U) \neq \{0\} \).

(ii) For every \( n \in \mathbb{N} \), \( \alpha_{n} > 1 \).

(iii) For every \( n \in \mathbb{N} \), \( \alpha_{n} \geq 1 \).

**Proof.** If \( \mu(U) = 0 \), then \( \mathcal{H}_{0}^{\Delta + \mu}(U) = \mathbb{R}^{+} G_{0} \) and \( \alpha_{n} = \infty, n \in \mathbb{N} \). So let us suppose that \( \mu(U) > 0 \).

(i) \( \Rightarrow \) (ii): Let \( g \in \mathcal{H}_{0}^{\Delta + \mu}(U) \setminus \{0\} \) and \( n \in \mathbb{N} \). Then \( s := g|_{U_{n}} \) is a positive bounded \((\Delta + \mu)\)-(super)harmonic function on \( U_{n} \), and \( s > 0 \) at \( \partial U_{n} \setminus \partial B \). By (3) in Proposition 4.3, \( 1 < \alpha_{n} \).

(ii) \( \Rightarrow \) (i): By Proposition 4.2, the sets \( U_{n}, n \in \mathbb{N} \), are regular with respect to \( \mathcal{H}_{\hat{W}}^{\Delta + \mu} \). Hence, for every \( n \in \mathbb{N} \), we have a function
\[
g_{n} := H_{U_{n}}^{\Delta + \mu} 1_{\partial U_{n} \setminus \partial B}
\]
which is strictly positive and \((\Delta + \mu)\)-harmonic on \( U_{n} \). Let \( x_{0} := (3R/4,0,\ldots,0) \) and \( \hat{g}_{n} := g_{n}/g_{n}(x_{0}), n \in \mathbb{N} \). Then there exists a subsequence of \( (\hat{g}_{n}) \) which is locally uniformly convergent to a positive \((\Delta + \mu)\)-harmonic function \( g \) on \( U \). The convergence is uniform on \( U_{1} \), since \( \hat{g}_{n} = H_{U_{1}}^{\Delta + \mu} \hat{g}_{n} \) and \( \hat{g}_{n} \) vanishes at \( \partial B, n \in \mathbb{N} \). Thus \( g \in \mathcal{H}_{0}^{\Delta + \mu}(U) \). Of course, \( g \neq 0 \) since \( g(x_{0}) = 1 \).

(ii) \( \Rightarrow \) (iii): Trivial.

(iii) \( \Rightarrow \) (ii): It suffices to show that \( \alpha_{n+1} < \alpha_{n} \), if \( n \) is sufficiently large. Since \( \mu(U) > 0 \), there exists \( n_{0} \in \mathbb{N} \) such that \( \mu(U_{n_{0}}) > 0 \). By Proposition 4.3, for every \( n \geq n_{0}, \alpha_{n} \in (0, \infty) \) and there exists a strictly positive function \( h_{n} \in \mathcal{C}_{0}(U_{n}) \cap \mathcal{H}_{\hat{W}}^{\Delta + \alpha_{n} \mu}(U_{n}) \). Let us now fix \( n \geq n_{0} \). Then \( s := h_{n+1}|_{U_{n}} \) is a strictly positive bounded \((\Delta + \alpha_{n+1} \mu)\)-(super)harmonic function on \( U_{n} \). Clearly, \( s \) is not a multiple of \( h_{n} \), since \( h_{n+1} > 0 \) on \( \partial U_{n} \setminus \partial B \). So \( \alpha_{n+1} < \alpha_{n} \), by Proposition 4.3.
PROPOSITION 4.5. If $\mathcal{H}_0^{\Delta+\mu}(U) \neq \{0\}$, then $K$ is a bounded operator on $L^2(U, \mu)$ and $\|K\|_2 = \sup \alpha_n^{-1} \leq 1$.

Proof. If $\mu(U) = 0$, then $K = 0$ and $\inf \alpha_n = \infty$. So we assume that $\mathcal{H}_0^{\Delta+\mu}(U) \neq \{0\}$ and $\mu(U) > 0$. Let $n \in \mathbb{N}$ such that $\mu(U_n) > 0$, and hence $\alpha_n = \alpha_{U_n} \in (0, \infty)$.

By Proposition 4.3, there exists $h_n \in \mathcal{H}^{\Delta+\alpha_n\mu}(U_n) \cap C_0(U_n)$, $h_n > 0$. We define $K_n := K_{U_n}$. Then $\alpha_n K_n h_n = h_n$, since $\alpha_n K_n h_n - h_n$ is harmonic on $U_n$ and vanishes at $\partial U_n$. Considering $h_n$ as a function on $U$ which vanishes on $U \setminus U_n$, we have $h_n \in L^2(U, \mu)$ and

$$\tag{4.3} \int_U (Kh_n)^2 d\mu \geq \int_{U_n} (K_n h_n)^2 d\mu = \alpha_n^{-2} \int_{U_n} h_n^2 d\mu = \alpha_n^{-2} \int_U h_n^2 d\mu.$$

By [BAH01, Theorem 2.5], $K_n := K_{U_n}$ is a compact operator on $L^2(U_n, \mu)$. In addition, $K_n$ is positive, self-adjoint, and $\alpha_n^{-1}$ is the first eigenvalue of $K_n$. Therefore $\|K_n\|_2 = \alpha_n^{-1}$ (see e.g. [HL99, Lemma 6.2.1 and Corollary 5.2.7]) and we conclude that, for every $f \in L^2(U, \mu)$, $f \geq 0$,

$$\tag{4.4} \int_U (Kf)^2 d\mu = \sup_n \int_{U_n} (K_n f|_{U_n})^2 d\mu \leq \sup_n \alpha_n^{-2} \int_U f^2 d\mu.$$

Combining (4.3) and (4.4), we see that $K$ is a bounded operator on $L^2(U, \mu)$ and $\|K\|_2 = \sup \alpha_n^{-1}$. By Corollary 4.4, $\sup \alpha_n^{-1} \leq 1$. \hfill \Box

COROLLARY 4.6. $\mathcal{H}_0^{\Delta+\mu}(U) \neq \{0\}$ if and only if $K$ is a bounded operator on $L^2(U, \mu)$ with $\|K\|_2 \leq 1$.

5 Smallness of $\mu$ on shells

For all $r > 0$ and $\delta \in (0, 1/4)$, let

$$B_r := \{x \in \mathbb{R}^d : |x| < r\}, \quad S_r := \{x \in \mathbb{R}^d : |x| = r\},$$

$$A_{r, \delta} := \{x \in \mathbb{R}^d : (1 - 2\delta)r < \|x\| < r\}.$$

For every measure $\nu$ on $U$, let

$$G\nu(x) := \int_U G(x, y) d\nu(y), \quad x \in U.$$

By [BHH87, Proposition 7.6], we know the following.

PROPOSITION 5.1. There exists $\eta > 0$ (which we shall fix once and for all) such that, for all $s \in \mathcal{S}^+(U)$ and measures $\nu$ on $U \cap B_{R/2}$,

$$G(s\nu) \leq s/2, \quad \text{whenever } G\nu \leq \eta.$$

PROPOSITION 5.2. For every $0 < \delta < 1/4$, there exists $c \geq 1$ such that, for all $g \in \mathcal{H}_0^{\Delta+\mu}(U)$ and all $r \in (0, R/2)$,

$$\sup g(S_{(1-\delta)r}) \leq c \inf g(S_{(1-\delta)r}), \quad \text{whenever } G(1_{A_{r, \delta}\mu}) \leq \eta.$$
Proof. Let \( \delta \in (0, 1/4) \). By scaling invariance and Harnack’s inequalities, there exists \( c > 1 \) such that, for every \( r > 0 \),

\[
(5.2) \quad \sup h(S_{(1-\delta)r}) \leq \frac{c}{2} \inf h(S_{(1-\delta)r}) \quad \text{for all } h \in \mathcal{H}^+(A_{r, \delta}).
\]

Let \( g \in \mathcal{H}^{\Delta + \mu}_+(U) \) and \( r \in (0, R/2) \) such that \( A := A_{r, \delta} \) satisfies \( G(1_A \mu) \leq \eta \). Since \( g \in \mathcal{S}^+(U) \cap \mathcal{C}(U) \), we see that \( s := H_A g \in \mathcal{S}^+(U) \cap \mathcal{C}(U) \), \( s \leq g \), and \( h := s|_A \in \mathcal{H}^+(A) \). Let \( S := S_{(1-\delta)r} \). By Proposition 5.1 and (5.2),

\[
G(1_A s \mu) \leq \frac{1}{2} s \quad \text{and} \quad \sup h(S) \leq \frac{c}{2} \inf h(S).
\]

By the first inequality, \( K_A h \leq G(1_A s \mu)|_A \leq (1/2) h \), hence

\[
g|_A = \sum_{n=0}^{\infty} (K_A)^n h \leq \sum_{n=0}^{\infty} 2^{-n} h = 2h
\]

and

\[
\sup g(S) \leq 2 \sup h(S) \leq c \inf h(S) \leq c \inf g(S).
\]

\[\square\]

**THEOREM 5.3.** Suppose that there exist \( \delta \in (0, 1/4) \) and \( r_n \in (0, R) \) such that \( r_n \downarrow 0 \) and, for every \( n \in \mathbb{N} \), the annulus \( A_n := A_{r_n, \delta} \) satisfies

\[
(5.3) \quad G(1_{A_n} \mu) \leq \eta.
\]

Then \( \Delta + \mu \) satisfies the Picard principle on \( U \).

Proof. Let us assume that we have extremal functions \( h_1, h_2 \in \mathcal{H}^{\Delta + \mu}_0 \setminus \{0\} \). We shall see that they are not linearly independent, and hence \( \dim_U (\Delta + \mu) \leq 1 \).

Let \( c \) denote the constant of Proposition 5.2 and let us fix \( C \geq 1 \) such that

\[
h_i \leq Ch_j \quad \text{on } U \setminus B_{R/2}.
\]

Assuming that already \( r_1 < R/2 \), we define \( x_n := ((1-\delta)r_n, 0, \ldots, 0) \in S_{(1-\delta)r_n}, \ n \in \mathbb{N} \). By Proposition 5.2, for all \( i \in \{1, 2\} \) and \( n \in \mathbb{N} \),

\[
c^{-1} h_i(x_n) \leq h_i \leq ch_i(x_n) \quad \text{on } S_{(1-\delta)r_n},
\]

and hence, for \( i, j \in \{1, 2\} \),

\[
h_i \leq c^2 \frac{h_i(x_n)}{h_j(x_n)h_j} \quad \text{on } S_{(1-\delta)r_n}.
\]

We may assume without loss of generality that there exist \( 1 \leq k_1 < k_2 < \ldots \) such that for some real \( a > 0 \) the sequence \( (h_1(x_{k_n})/h_2(x_{k_n})) \) is bounded by \( a \) (if \( \lim h_1(x_n)/h_2(x_n) = \infty \), we exchange the role of \( h_1 \) and \( h_2 \), and take \( a = 1 \)). Let \( \tilde{c} := \max\{ac^2, C\} \). Then, for every \( n \in \mathbb{N} \),

\[
h_1 \leq \tilde{c} h_2 \quad \text{on } S_{(1-\delta)r_n} \cup S_{R/2}. \]

By Proposition 4.3, (5.5) holds for all \( n \in \mathbb{N} \) and \( x \in U \) such that \( r_{k_n} \leq |x| \leq R/2 \). In view of (5.4), we obtain that (5.5) holds for all \( x \in U \). Since \( h_2 \) is an extremal function in \( \mathcal{H}^{\Delta + \mu}_0(U) \), we conclude that \( h_1 = \gamma h_2 \) for some \( \gamma > 0 \).
6 Almost radial measures

For $r > 0$, let $\sigma_r$ denote the normalized surface measure on $S_r$. Given $r \in (0, R/2)$, we define

\begin{equation}
(6.1) \quad a_r := \|G\sigma_r\|_{\infty}.
\end{equation}

To work with almost radial measures (see Definition 6.3) we shall need two simple properties of $\sigma_r$ (where it would be sufficient to have some constant $c_0$ instead of $2^{d-1}$ and $2^{d+1}$).

**Lemma 6.1.** Let $r \in (0, R/2)$. Then $G\sigma_r = G\sigma_r(0) = a_r$ on $B_r$. Moreover,

\begin{equation}
(6.2) \quad a_{r/2} \leq 2^{d-1}a_r \quad \text{and} \quad a_r \leq 2^{d+1}G(x, y) \quad \text{for all } x, y \in B_r.
\end{equation}

**Proof.** Let $z := (r, 0, \ldots, 0)$. Since the potential $G\sigma_r$ is a radial function which is harmonic on $B \setminus S_r$, we immediately see that

\begin{equation}
(6.3) \quad a_r = G(0, z) = G\sigma_r(0) = G\sigma_r \quad \text{on } B_r.
\end{equation}

To prove (6.2) we define

$$\phi(x, y) := \frac{(R^2 - |x|^2)(R^2 - |y|^2)}{R^2|x - y|^2} \quad (x, y \in B).$$

Clearly, $\phi(0, z) = (R/r)^2 - 1$ and, for all $x, y \in B_r$,

\begin{equation}
(6.4) \quad \frac{9}{16} |x - y|^{-2} \leq \phi(x, y) \leq |x - y|^{-2}.
\end{equation}

Let us first consider the case $d = 2$. Then, for all $x, y \in B$,

\begin{equation}
(6.5) \quad G(x, y) = (4\pi)^{-1} \ln(1 + \phi(x, y))
\end{equation}

(see [AG01, Theorem 4.1.5, Corollary 4.3.3]). Since $1 + \phi(0, z) = (R/r)^2$, we see that

$$a_r = (2\pi)^{-1} \ln(R/r).$$

Moreover, $\ln(2/r) = \ln 2 + \ln(1/r)$, where $\ln 2 \leq \ln(1/r)$. Therefore $a_{r/2} \leq 2a_r$ proving the first part of (6.2). If $x, y \in B_r$, then

$$\phi(x, y) \geq \frac{9}{16} \frac{R^2}{(2r)^2} \geq \frac{1}{8} \frac{R^2}{r^2}.$$ \[\text{Since } \ln 1 = 0 \text{ and the function } t \mapsto \ln(1 + t) \text{ is concave, we hence conclude that}

\begin{equation}
\ln(1 + \phi(x, y)) \geq \ln(1 + \frac{1}{8} \frac{R^2}{r^2}) \geq \frac{1}{8} \ln(1 + \frac{R^2}{r^2}) > \frac{1}{8} \ln(1 + \phi(0, z)).
\end{equation}

So $G(x, y) \geq (1/8)a_r$, by (6.5) and (6.3).

Now let $d \geq 3$. Then

\begin{equation}
(6.6) \quad G(x, y) = \kappa_d^{-1}(1 - (1 + \phi(x, y))^{(2-d)/2}) |x - y|^{2-d}
\end{equation}

\end{equation}
where \( \kappa_d \) is \( d - 2 \) times the surface of \( S_1 \); see [AG01, Theorem 4.1.5, Corollary 4.3.3] again. In particular (recall that \( 1 + \phi(0, z) = (R/r)^2 \)),

\[
\frac{r^{2-d}}{2} \leq r^{2-d} - R^{2-d} = \kappa_d a_r \leq r^{2-d}
\]

leading to the inequality \( a_r/2 \leq 2^{d-1} a_r \).

Finally, taking \( b := 2r/R/(R^2 + r^2) \in (0, 1) \), we have, for all \( x, y \in B_r \),

\[
1 + \phi(x, y) \geq 1 + \frac{(R^2 - r^2)^2}{(2rR)^2} = \frac{(R^2 + r^2)^2}{(2rR)^2} = b^{-2}
\]

and hence, by (6.6), (6.7), and since \( 1 - b = (R - r)^2/(R^2 + r^2) \geq 1/8 \),

\[
\kappa_d G(x, y) \geq (1 - b^{d-2})(2r)^{2-d} \geq (1-b)(2r)^{2-d} \geq 2^{-(d+1)} r^{2-d} \geq 2^{-(d+1)} \kappa_d a_r.
\]

\[
\square
\]

Let us recall that the mapping \( K \) is defined by

\[
Kf := G(f \mu)
\]

(whenever \( f: U \to \mathbb{R} \) is Borel measurable and \( G(f^+ \mu) - G(f^\mu) \) makes sense).

**Lemma 6.2.** Suppose that \( K \) is a bounded operator on \( \mathcal{L}^2(U, \mu) \) and \( \|K\|_2 \leq 1 \). Then

\[
\mu(B_r) \leq 2^{d+1}/a_r \quad \text{for every } r \in (0, R/2).
\]

**Proof.** Let \( r \in (0, R/2) \) and \( A := B_r \setminus \{0\} \). By Lemma 6.1, for every \( x \in A \),

\[
K1_A(x) = \int_A G(x, y) d\mu(y) \geq 2^{-(d+1)} a_r \mu(A)
\]

and therefore

\[
\int (K1_A)^2 d\mu \geq \int_A (K1_A)^2 d\mu \geq (2^{-(d+1)} a_r \mu(A))^2 \mu(A).
\]

On the other hand, since \( \|K\|_2 \leq 1 \), we know that \( \int (K1_A)^2 d\mu \leq \int (1_A)^2 d\mu = \mu(A) \). Since \( \mu(A) = \mu(B_r) \), (6.8) follows. \( \square \)

**Definition 6.3.** Given \( C > 0 \), \( r \in (0, R/2) \), and \( \delta \in (0, 1/4) \), we shall say that \( \mu \) is \( C \)-radial on \( A_{r, \delta} \), if for every annulus \( A := \{x \in \mathbb{R}^d : s = \|x\| < t\} \), where \( (1-2\delta)r \leq s < t \leq r \),

\[
G(1_A \mu) \leq Ca_r \mu(A).
\]

If \( \mu \) is radial on \( A := A_{r, \delta} \), then \( \mu \) is \( 2^{d-1} \)-radial on \( A \), by Lemma 6.1, since \( r/2 \leq (1-2\delta)r \). More generally, if \( C > 0 \) and

\[
1_A \mu = \int_{(1-2\delta)r}^r \mu_t dt,
\]

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such that
\[(6.9) \quad G\mu_t \leq Ca_t\mu_t(S_t) \quad \text{for every } (1 - 2\delta)r \leq t \leq r,\]
then \(\mu\) is \(2^{d-1}\)-radial on \(A\). Let us note that (6.9) holds if \(\mu_t = \varphi_t\sigma_t\) such that
\[(6.10) \quad \sup \varphi_t(S_t) \leq C \inf \varphi_t(S_t).\]
Indeed, (6.10) obviously implies that
\[G(\varphi_t\sigma_t) \leq \sup \varphi_t(S_t)a_t \leq C \inf \varphi_t(S_t)a_t \leq Ca_t\mu_t(S_t).\]

**THEOREM 6.4.** Suppose that there exist \(C > 0\) and \(\delta \in (0, 1/4)\) such that
\[
\inf \{r \in (0, R/2): \mu \text{ is } C\text{-radial on } A_{r, \delta}\} = 0.
\]
Then \(\Delta + \mu\) satisfies the Picard principle on \(U\).

**Proof.** By Corollary 4.6, it remains to consider the case, where \(K\) is a bounded operator on \(L^2(U, \mu)\) and \(\|K\|_2 \leq 1\). Then, by Lemma 6.2,
\[(6.11) \quad \mu(B_r) \leq 2^{d+1}/a_r \quad \text{for every } r \in (0, R/2).\]
We fix \(m \in \mathbb{N}\) and \(\tilde{\delta} \in (0, \delta)\) such that
\[(6.12) \quad 2^{2d-m}C \leq \eta \quad \text{and} \quad (1 - 2\tilde{\delta})2^m \geq 1 - 2\delta.
\]
Let \(r \in (0, R/2)\) such that \(\mu\) is \(C\)-radial on \(A := A_{r, \delta}\). By Theorem 5.3, it suffices to prove that there exists \(\tilde{r} \in (0, r)\) such that \(\tilde{A} := A_{\tilde{r}, \tilde{\delta}}\) satisfies
\[(6.13) \quad G(1_{\tilde{A}}\mu) \leq \eta.
\]
To that end we define, for every \(0 \leq j \leq 2^m - 1\),
\[r_j := (1 - 2\tilde{\delta})^j r \quad \text{and} \quad A_j := A_{r_j, \tilde{\delta}}.
\]
Let us note that the \(2^m\) annuli \(A_0, A_1, \ldots, A_{2^m-1}\) are pairwise disjoint sets in \(A \subset B_r\). So, by (6.11), there exists \(0 \leq j \leq 2^m - 1\) such that, defining \(\tilde{r} := r_j\) and \(\tilde{A} := A_j\), we have
\[\mu(\tilde{A}) \leq 2^{d+1-m}/a_r.
\]
Since \(r/2 \leq \tilde{r} \leq r\), we know, by Lemma 6.1, that
\[a_{\tilde{r}} \leq a_{r/2} \leq 2^{d-1}a_r.
\]
Since \(\mu\) is \(C\)-radial on \(A\), we finally conclude that
\[G(1_{\tilde{A}}\mu) \leq Ca_{\tilde{r}}\mu(\tilde{A}) \leq 2^{2d-m}C \leq \eta.
\]
This finishes the proof.

**COROLLARY 6.5.** If \(\mu\) is radial on \(U\) or, more generally, if there exist \(\delta \in (0, 1/4)\) and \(r_n \in (0, 1/2)\) such that \(r_n \downarrow 0\) and \(\mu\) is radial on every \(A_{r_n, \delta}\), \(n \in \mathbb{N}\), then \(\Delta + \mu\) satisfies the Picard principle on \(U\).
7 Localization

We recall that the sufficient conditions in Theorem 5.3, Theorem 6.4, and Corollary 6.5 depend only on the behavior of $\mu$ close to the origin. The following result shows that, even in the most general case, the verification of the Picard principle for $\Delta + \mu$ on $U$ can be localized at 0 in two different ways (which can be combined in an obvious manner; see the proof of Corollary 7.2). To that end let $r \in (0, R)$ and $V := \{x \in U : |x| < r\}$.

**THEOREM 7.1.** Let $\mu'$ be a measure on $\mathbb{R}^d \setminus \{0\}$ such that $1_V \mu \leq \mu' \leq \mu$. Then $\Delta + \mu$ satisfies the Picard principle on $U$, if $\Delta + \mu'$ satisfies the Picard principle on $U$ or if $\Delta + \mu$ satisfies the Picard principle on $V$.

Of course, there is no hope for reverse implications (unless we already know that $\dim_U(\Delta + \mu') \leq 1$ or $\dim_V(\Delta + \mu) \leq 1$), since, whatever $1_V \mu$ may be, we shall have $\mathcal{H}_0^{\Delta + \mu}(U) = \{0\}$, if $1_U \setminus V \mu$ is too large.

**Proof of Theorem 7.1.** 1. Given $h \in \mathcal{H}_0^{\Delta + \mu}(U) \setminus \{0\}$, we shall construct corresponding minorants in $\mathcal{H}_0^{\Delta + \mu}(V)$ and $\mathcal{H}_0^{\Delta + \mu'}(U)$. For $n \in \mathbb{N}$, let

$$U_n := \{x \in U : |x| > r/(n + 1)\} \quad \text{and} \quad V_n := V \cap U_n.$$

By Proposition 4.2 and Proposition 4.3, the open sets $U_n, V_n$ are regular with respect to $\Delta + \mu$ and $\Delta + \mu'$. We first define

$$v_n := H_{V_n}^{\Delta + \mu}(1_{U \setminus V} h) \quad (n \in \mathbb{N}).$$

The sequence $(v_n)$ is increasing to a function $\tilde{h} \leq h$ which is $(\Delta + \mu)$-harmonic on $V$ and equal to $h$ on $U \setminus V$. Since $v_1 \to h$ at $S := \partial V \setminus \{0\}$, we know that also $\tilde{h} \to h$ at $S$. Hence

$$g := h - \tilde{h} \in C^+(U) \quad \text{and} \quad g = 0 \quad \text{on} \quad U \setminus V$$

Moreover, $g$ is $(\Delta + \mu)$-harmonic on $V$, and hence $g|_V \in \mathcal{H}_0^{\Delta + \mu}(V)$ (we cannot and will not exclude the possibility that even $g|_V = 0$).

Obviously, $g$ is $(\Delta + \mu')$-subharmonic on $U$ and $h$ is a $(\Delta + \mu')$-superharmonic majorant of $g$. Therefore the functions

$$u_n := H_{U_n}^{\Delta + \mu'} g, \quad n \in \mathbb{N},$$

are increasing to a $(\Delta + \mu')$-harmonic function $h'$ which is the smallest $(\Delta + \mu')$-superharmonic majorant of $g$ on $U$. In particular, $h' \leq h$ and hence $h' \to 0$ at $\partial B$. So $h' \in \mathcal{H}_0^{\Delta + \mu'}(U)$.

There is a natural way to get $g$ back from $h'$: For $n \in \mathbb{N}$, let

$$(7.2) \quad v'_n := H_{V_n}^{\Delta + \mu'}(1_V h')$$

(where we could just as well write $\mu$ instead of $\mu'$, since $V_n \subset V$). We claim that

$$(7.3) \quad g = \lim_{n \to \infty} v'_n.$$
Indeed, for every \( n \in \mathbb{N} \), the function \( v'_n \) is \((\Delta + \mu')\)-harmonic on \( V_n \). It is equal to \( h' \) on \( V \setminus V_n \), it vanishes on \( U \setminus V \), and it is continuous on \( U \). Since \( g \leq 1_V h' \leq h' \), \( H_{V_n}^{\Delta + \mu'} g = g \), and \( H_{V_n}^{\Delta + \mu'} h' = h' \), we see that \( g \leq v'_n \leq h' \) and hence

\[
g \leq v'_n \leq 1_V h' \quad (n \in \mathbb{N}).
\]

In particular, for every \( n \in \mathbb{N} \), \( v'_{n+1} = H_{V_n}^{\Delta + \mu'} v'_{n+1} \leq H_{V_n}^{\Delta + \mu'} (1_V h') = v'_n \), that is, the sequence \((v'_n)\) is decreasing. Its limit \( f \) is \((\Delta + \mu')\)-harmonic on \( V \) and satisfies \( g \leq f \leq h' \), it vanishes on \( U \setminus V \), and is continuous on \( U \). Therefore the positive function \( f - g \) is \((\Delta + \mu')\)-subharmonic on \( U \), and we conclude that, for all \( n \in \mathbb{N} \),

\[
0 \leq f - g \leq H_{U_n}^{\Delta + \mu'} (f - g) \leq H_{U_n}^{\Delta + \mu'} h' - u_n \leq h' - u_n.
\]

Letting \( n \to \infty \) we see that \( f - g = 0 \) proving (7.3).

2. Now let \( h_1, h_2 \) be extremal functions in \( \mathcal{H}_0^{\Delta + \mu}(U) \setminus \{0\} \). Then we have corresponding functions \( \tilde{h}_1, \tilde{h}_2, g_1, g_2, \) and \( h'_1, h'_2 \). If \( \Delta + \mu' \) satisfies the Picard principle on \( U \), then \( h'_1, h'_2 \) are proportional and hence \( g_1, g_2 \) are proportional, by (7.2) and (7.3). If \( \Delta + \mu \) satisfies the Picard principle on \( V \), then we know immediately that \( g_1, g_2 \) are proportional.

3. So let us consider the case that \( g_1 = a g_2 \) for some \( a \geq 0 \). Of course, there exists \( b > 0 \) such that \( h_1 \leq bh_2 \) on \( S \) and hence \( h_1 \leq bh_2 \) on \( U \setminus V \). By (7.1), we see that \( \tilde{h}_1 \leq bh_2 \). Having \( h_j = g_j + \tilde{h}_j, j \in \{1,2\} \), we obtain that \( h_1 \leq (a + b)h_2 \). Since \( h_2 \) is extremal, we finally conclude that \( h_1 \) is a multiple of \( h_2 \).

A consequence of Theorem 7.1 is the following result (we note that, of course, (7.5) holds if \( \mu \) is a Kato measure on \( \mathbb{R}^d \)).

**COROLLARY 7.2.** Let us suppose that

\[
(7.5) \quad \lim \sup_{x \to 0} K_1(x) < \lim \inf_{x \to 0} K_1(x) + 1 < \infty
\]

or, more generally, that \( K_V \) is a bounded operator on \( (B_b(V), \| \cdot \|_\infty) \) having a spectral radius \( \rho(K_V) < 1 \). Then \( \Delta + \mu \) satisfies the Picard principle on \( U \).

*Proof.* If (7.5) holds, then \( K_1 \) is bounded, \( \sup K_1(\overline{V}) - \inf K_1(\overline{V}) < 1 \), if \( r \) is sufficiently small, and hence \( \| K_V 1 \|_\infty < 1 \). So let us assume that \( \rho(K_V) < 1 \) and let \( \mu' := 1_{B_{r/2}} \mu \). Of course, the spectral radius of the operator \( f \mapsto \int_V G(\cdot, y) f(y) \, d\mu'(y) \) is at most \( \rho(K_V) \).

By Proposition 3.1 and Corollary 8.3, \( \Delta + \mu' \) satisfies the Picard principle on \( V \). By Theorem 7.1, applied to \( V \) in place of \( U \), we obtain first that \( \Delta + \mu \) satisfies the Picard principle on \( V \). Using Theorem 7.1 again, we finally see that \( \Delta + \mu \) satisfies the Picard principle on \( U \). \( \square \)

### 8 Appendix: Triangle property on punctured sets

Let us recall the generalized triangle property. Given an arbitrary set \( X \) and functions \( w, w^* : X \to (0, \infty) \), a function \( F : X \times X \to [0, \infty] \) has the \((w, w^*)\)-triangle property, if there exists \( C > 0 \) such that, for all \( x, y, z \in X \),

\[
F(x, z) F(z, y) \leq C F(x, y) \max \left\{ \frac{w(z)}{w(x)} F(x, z), \frac{w^*(z)}{w^*(y)} F(z, y) \right\}
\]
or – equivalently – that the function \( F_{w,w^*} : (x, y) \mapsto F(x, y)/(w(x)w^*(y)) \) satisfies the *triangle property*, that is, for all \( x, y, z \in X \),

\[
F_{w,w^*}(x, z)F_{w,w^*}(z, y) \leq CF_{w,w^*}(x, y) \max\{F_{w,w^*}(x, z), F_{w,w^*}(z, y)\},
\]

which, in turn, can be rewritten as

\[
\min\{F_{w,w^*}(x, z), F_{w,w^*}(z, y)\} \leq CF_{w,w^*}(x, y).
\]

The following results are of independent interest.

**PROPOSITION 8.1.** Let \( X \) be an arbitrary set, \( a \in X \), \( X^a := X \setminus \{a\} \). Suppose that \( G : X \times X \to [0, \infty] \) is symmetric, \( 0 < G^a := G(\cdot, a)|_{X^a} < \infty \), and, for some \( w : X \to (0, \infty) \), \( G \) has the \((w, w)\)-triangle property.

Then \( G|_{X^a \times X^a} \) has the \((G^a, G^a)\)-triangle property.

**Proof.** 1. Let us suppose first that \( w = 1 \), that is, there exists \( C \geq 1 \) such that, for all \( x, y, z \in X \),

\[
\min\{G(x, z), G(z, y)\} \leq CG(x, y).
\]

We define \( \tilde{G} : X^a \times X^a \to [0, \infty] \) by

\[
\tilde{G}(x, y) := G_{G^a,G^a}(x, y) = \frac{G(x, y)}{G^a(x)G^a(y)}.
\]

Let us fix \( x, y, z \in X^a \). We claim that \( \min\{\tilde{G}(x, z), \tilde{G}(z, y)\} \leq C^2\tilde{G}(x, y) \), that is,

\[
\min\{G^a(y)G(x, z), G^a(x)G(z, y)\} \leq C^2G^a(z)G(x, y).
\]

By symmetry, we may assume that \( G^a(x) \leq G^a(y) \). If \( G^a(y) \leq CG^a(z) \), then

\[
\min\{G^a(y)G(x, z), G^a(x)G(z, y)\} \leq CG^a(z) \min\{G(x, z), G(z, y)\} \leq C^2G^a(z)G(x, y).
\]

So let us suppose \( CG^a(z) < G^a(y) \). Since \( \min\{G^a(y), G(y, z)\} \leq CG^a(z) \), we see that \( G(y, z) \leq CG^a(z) \). In addition, \( G^a(x) = \min\{G^a(x), G^a(y)\} \leq CG(x, y) \).

Therefore

\[
\min\{G^a(y)G(x, z), G^a(x)G(z, y)\} \leq G^a(x)G(y, z) \leq C^2G^a(z)G(x, y).
\]

Thus \( G|_{X^a \times X^a} \) has the \((G^a, G^a)\)-triangle property.

2. To reduce the general case to the special one, where \( w = 1 \), it suffices to note that \( G_{w,w} \) is symmetric and that, for all \( x, y \in X^a \),

\[
\frac{G_{w,w}(x, y)}{G_{w,w}(x, a)G_{w,w}(y, a)} = w(a)^2 \frac{G(x, y)}{G^a(x)G^a(y)}.
\]

\(\square\)
For a better understanding of the first corollary, let us recall that, given an inner product space \((V, \langle \cdot, \cdot \rangle)\),
\[
\rho : (x, y) \mapsto \frac{\|x - y\|}{\|x\| \|y\|}
\]
(where, of course, \(\|z\| := \langle x, x \rangle^{1/2}\)) is known to define a metric on \(V \setminus \{0\}\), since \(\rho(x, y) = \sqrt{\|x\|^{-2} - \|y\|^{-2}}\|y\|\) (see [Pin99, Lemma A.1]).

If \(X\) is an arbitrary set and \(\rho : X \times X \to \mathbb{R}^+\) is symmetric and vanishes on the diagonal, but nowhere else, then \(\rho\) is a quasi-metric if and only if \(\rho^{-1}\) has the triangle property (see e.g. [Han06, p. 646, Remark 2.1.2]). So Proposition 8.1 has an immediate consequence for quasi-metrics (and is more or less equivalent to it).

**COROLLARY 8.2.** Let \(\rho\) be a quasi-metric on a set \(X\). Let \(a \in X\), \(X^a := X \setminus \{a\}\), and
\[
\rho^a(x, y) := \frac{\rho(x, y)}{\rho(x, a) \rho(y, a)} \quad (x, y \in X^a).
\]
Then \(\rho^a\) is a quasi-metric on \(X^a\).

Let us note that the following corollary has obvious analogues in the more general situations considered in [Han06, Section 9] and [Han05].

**COROLLARY 8.3.** Let \(\nu\) be any measure on \(B\) and \(L f := \int G_B(\cdot, y) f(y) \, d\nu(y), f \in \mathcal{B}^+(B)\). Let us consider the following statements.

(i) There exists \(a \in B\) and \(c > 0\) such that \(\nu(\{a\}) = 0\) and
\[
(8.4) \quad \sum_{n=0}^{\infty} L^n G_B(\cdot, a) \leq c G_B(\cdot, a).
\]

(ii) There exists \(c > 0\) such that, for every \(s \in \mathcal{S}^+(B)\), \(\sum_{n=0}^{\infty} L^n s \leq cs\).

(iii) The function \(\sum_{n=0}^{\infty} L^n 1\) is bounded.

Then (i) \(\Leftrightarrow\) (ii) \(\Rightarrow\) (iii). If \(\nu\) has compact support in \(B\), then also (iii) \(\Rightarrow\) (i).

**Proof.** Let \(w := \min\{G_B(\cdot, 0), 1\}\). The function \(G_B\) has the \((w, w)\)-triangle property (see e.g. [Han06, Proposition 9.3]). So, by Proposition 8.1, \(G_B|_{B^a \times B^a}\) has the \((G^a, G^a)\)-triangle property.

(i) \(\Rightarrow\) (ii): We define \(L_a f(x) := \int_{B^a} G_B(x, y) f(y) \, d\nu(y), f \in \mathcal{B}^+(B^a), x \in B^a\). By (8.4), we know that \(\sum_{n=0}^{\infty} L_a^n G_B^a \leq c G_B^a\). Therefore, by [Han04, Proposition 7.4],
\[
L_a^n G_B^a \leq c \left(1 - \frac{1}{1/c}\right)^n G_B^a, \quad n \in \mathbb{N},
\]
where \(\lim_{n \to \infty} [c(1 - (1/c))^n]^{1/n} = 1 - (1/c) < 1\). Hence, by [Han06, Proposition 2.3 and Corollary 3.3], we infer that there exists \(C > 0\) such that, for every \(s \in \mathcal{S}^+(B^a)\), \(\sum_{n=0}^{\infty} L_a^n s \leq Cs\). Since \(\nu(\{x_0\}) = 0\), (iii) follows.

(ii) \(\Rightarrow\) (i),(iii): Trivial, since \(G_B(\cdot, a), 1 \in \mathcal{S}^+(B)\).

Finally, let us suppose that \(\nu\) is supported by a compact set \(A\) in \(B\) and that \(\sum_{n=0}^{\infty} L_a^n 1\) is bounded. Since \(\inf w(A) > 0\) and \(\sup w(A) < \infty\), we know that \(G_B|_{A \times A}\) has the triangle property. Thus, by [Han04, Proposition 3.10], (ii) follows. \(\square\)
Finally, let us note another consequence of Proposition 8.1 (where we shall not try to achieve the utmost generality).

**COROLLARY 8.4.** Let $G$ be a symmetric Green function for a connected Brelot space $(X, \mathcal{H})$ and $a \in X$ such that the following holds.

(i) $G(a, a) = \infty$ and $\limsup_{x \to \infty} G(x, a) < \infty$.

(ii) $G$ has the local triangle property.

(iii) There exist strictly positive real functions $w, w^*$ on $X$ such that $G$ has the $(w, w^*)$-triangle property.

Let $g := \min\{G^a, 1\}$. Then $G$ has the $(g, g)$-triangle property.

**Proof.** By (i), there exist a relatively compact open neighborhood $V$ of $a$ and $M \geq 1$ such that $G^a \geq 1$ on $\overline{V}$ and $G^a \leq M$ on $V^c$. Then $g = 1$ on $\overline{V}$ and $G^a \leq Mg$ on $V^c$. Let $L$ be a compact neighborhood of $\overline{V}$ and $W$ a relatively compact open neighborhood of $L$. We may assume without loss of generality that $g \geq 1/M$ on $W$.

By (ii), (iii), and Proposition 8.1, there exists $C \geq 1$ such that, for all $x, y, z \in W$,

\[(8.5) \quad \min\{G(x, z), G(z, y)\} \leq CG(x, y),\]

and, for all $x, y, z \in X \setminus \{a\}$,

\[(8.6) \quad \min\{G^a(y)G(x, z), G^a(x)G(z, y)\} \leq CG^a(z)G(x, y).\]

Moreover, there exists $c \geq 1$ such that

\[(8.7) \quad h(z) \leq ch(\tilde{z}),\]

whenever $h \in \mathcal{H}^+(W)$ and $z, \tilde{z} \in L$, or $h \in \mathcal{H}^0(L)$ and $z, \tilde{z} \in \overline{V}$.

We claim that, for all $x, y, z \in X$,

\[(8.8) \quad \min\{g(y)G(x, z), g(x)G(z, y)\} \leq \text{McC}g(z)G(x, y).\]

If $x, y, z \in W$, then (8.8) follows from (8.5), since $g \leq 1 \leq Mg(z)$. So we may assume that $W \neq X$.

Let us suppose next that $z \in V^c$. If $x, y \in X \setminus \{a\}$, then, by (8.6),

\[(8.9) \quad \min\{g(y)G(x, z), g(x)G(z, y)\} \leq MCg(z)G(x, y),\]

since $g \leq G^a$ and $G^a(z) \leq Mg(z)$. Then, by continuity, (8.9) holds as well, if $x \neq a$, but $y = a$. Analogously, if $y = a$ and $x \neq a$. If $x = y = a$, then (8.9) holds trivially, since $g(z) > 0$ and $G(a, a) = \infty$.

So we may and shall assume from now on that $z \in V$, whence $g(z) = 1$, and $(x, y) \notin W \times W$. If $x \in L$ and $y \notin W$, then we may apply (8.7) and obtain that $g(x)G(z, y) \leq G(z, y) \leq cG(x, y)$. Analogously, if $y \in L$ and $x \notin W$. Hence (8.8) holds in these two cases.

Therefore it remains to consider the case, where $x, y \notin L$. Since $X$ is connected and $W \neq X$, there exists a point $\tilde{z} \in \partial V$, and we know, by (8.9), that

\[\min\{g(y)G(x, \tilde{z}), g(x)G(\tilde{z}, y)\} \leq \text{MC}G(x, y).\]

By (8.7), $G(x, z) \leq cG(x, \tilde{z})$, $G(y, z) \leq cG(y, \tilde{z})$, and (8.8) follows. \qed
References


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