On the Picard principle for negative perturbations

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Abstract

Given a local Kato measure μ on $\mathbb{R}^d \setminus \{0\}$, $d \geq 2$, let $\mathcal{H}_0^{\Delta+\mu}(U)$ be the convex cone of all continuous real solutions $u \geq 0$ to the equation $\Delta u + u\mu = 0$ on the punctured unit ball U satisfying $\lim_{|x|\to 1} u(x) = 0$. It is shown that $\mathcal{H}_0^{\Delta+\mu}(U) \neq \{0\}$ if and only if the operator $f \mapsto \int_U G(\cdot, y) f(y)\, d\mu(y)$, where G denotes the Green function on U, is a bounded operator on $\mathcal{L}^2(U,\mu)$ having a norm which is at most one. Moreover, extremal rays in $\mathcal{H}_0^{\Delta+\mu}(U)$ are characterized and it is proven that the Picard principle holds for $\Delta + \mu$ on U, that is, that $\mathcal{H}_0^{\Delta+\mu}(U)$ consists of one ray, provided there exists a suitable sequence of shells in U such that, on these shells, μ is either small or not too far from being radial. Finally, it is shown that the verification of the Picard principle can be localized.

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1 Introduction

For every relatively compact open set V in $\mathbb{R}^d$, $d \geq 2$, let G_V denote the (classical) Green function on V, normalized such that $\Delta G_V(\cdot, y) = -\varepsilon_y$, $y \in V$. Throughout this paper we fix a measure μ on $\mathbb{R}^d$ which does not charge the origin and is a (local) Kato measure[1] on $\mathbb{R}^d \setminus \{0\}$, that is, for some covering of $\mathbb{R}^d \setminus \{0\}$ by relatively compact open sets V, the potentials $x \mapsto \int_V G_V(x,y)\, d\mu(y)$ are continuous and real.

Moreover, we fix $R > 0$ and define

$$B := \{x \in \mathbb{R}\colon |x| < R\}, \qquad U := B \setminus \{0\}.$$

Let $\mathcal{H}_0^{\Delta+\mu}(U)$ be the convex cone of all continuous real solutions $u \geq 0$ to the Schrödinger equation

$$\Delta u + u\mu = 0 \tag{1.1}$$

on U which vanish at the boundary ∂B of B. Here solution is meant in the sense of distributions, that is,

$$\int_U u \Delta\varphi \, d\lambda^d + \int_U u\varphi\, d\mu = 0$$

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[1]This is a natural assumption, since otherwise (1.1) would not admit *continuous* solutions $u \neq 0$.

for all $\mathcal{C}^\infty$-functions φ with compact support in U, λ^d being Lebesgue measure on $\mathbb{R}^d$.

By definition, $\dim_U(\Delta+\mu)$, the *Picard dimension of* $\Delta+\mu$ *on* U, is the number of extremal rays in $\mathcal{H}_0^{\Delta+\mu}(U)$. Of course, $\dim_U(\Delta+\mu)=0$ if $\mathcal{H}_0^{\Delta+\mu}(U)=\{0\}$. If $\mathcal{H}_0^{\Delta+\mu}(U)\neq\{0\}$ and $x_0:=(R/2,0,\dots,0)$, then $\{h\in\mathcal{H}_0^{\Delta+\mu}(U)\colon h(x_0)=1\}$ is a compact base of $\mathcal{H}_0^{\Delta+\mu}(U)$, $\dim_U(\Delta+\mu)$ is the number of extreme points of this set, and hence $\dim_U(\Delta+\mu)>0$.

We say that $\Delta+\mu$ satisfies the *Picard principle on* U provided

$$\dim_U(\Delta+\mu)\le 1. \tag{1.2}$$

In [NT97b] it is shown that (1.2) holds, if $d=2$. Moreover, it is satisfied if μ has a locally Hölder continuous density with respect to λ^d which is radial ([NT97a]). However, the problem seems to be open for $d\ge 3$ and general measures μ.

In this paper, we prove that $\mathcal{H}_0^{\Delta+\mu}(U)\neq\{0\}$ if and only if the operator

$$K\colon f\mapsto \int_U G(\cdot,y)f(y)\,d\mu(y),$$

where $G:=G_U=G_B|_{U\times U}$, is a bounded operator on $\mathcal{L}^2(U,\mu)$ with $\|K\|_2\le 1$ (Corollary 4.6). In particular, (1.2) holds, unless K is bounded on $\mathcal{L}^2(U,\mu)$ and $\|K\|_2\le 1$.

Since $\mathcal{H}_0^{\Delta}(U)$ is the set of all positive multiples of $G_0:=G_B(\cdot,0)|_U$ (cf. (2.2)) and hence $\dim_U\Delta=1$, it would be sufficient to consider the case $\mu(U)>0$. We shall see first that, whatever μ is, every function in an extremal ray of $\mathcal{H}_0^{\Delta+\mu}(U)$ is either a multiple of $\sum_{n=0}^\infty K^nG_0$ or a continuous strictly positive K-invariant function h (see Proposition 2.1). This implies that $\dim_U(\Delta+\mu)=1$, if K is a bounded operator on $\mathcal{L}^2(U,\mu)$ such that $\|K\|_2=1$ and 1 is an eigenvalue of K (Proposition 3.2 in connection with Corollary 4.6).

Further, we shall prove that the Picard principle holds for $\Delta+\mu$ on U provided that there are arbitrarily small shells A of constant relative thickness, where the potential $\int_A G(\cdot,y)\,d\mu(y)$ is small enough (Theorem 5.3) or the measure $1_A\mu$ is not too far from being invariant under rotations (Theorem 6.4).

Finally, we shall see that the verification of the Picard principle can be localized in different ways (Theorem 7.1).

2 Nature of extremal functions in $\mathcal{H}_0^{\Delta+\mu}$(U)

For every open set W in $\mathbb{R}^d$, let $\mathcal{S}^+(W)$, $\mathcal{H}^+(W)$ denote the set of all positive functions on W which are superharmonic, harmonic respectively. For every relatively compact open set V in $\mathbb{R}^d$, let K_V denote the mapping $f\mapsto\int_V G_V(\cdot,y)f(y)\,d\mu(y)$. If V is regular and $0\notin\overline{V}$, then K_V maps the space $\mathcal{B}_b(V)$ of all bounded Borel measurable functions on V into the subspace $\mathcal{C}_0(V)$ of all continuous functions in $\mathcal{B}_b(V)$ vanishing at the boundary ∂V. The harmonic kernel of V will be denoted by H_V.

Let us note that every $u\in\mathcal{C}^+(U)$ satisfying (1.1) is superharmonic. Hence

$$\mathcal{H}_0^{\Delta+\mu}(U)\subset\mathcal{S}^+(U).$$

A function $h\in\mathcal{H}_0^{\Delta+\mu}(U)\setminus\{0\}$ is *extremal*, if it is contained in an extremal ray of $\mathcal{H}_0^{\Delta+\mu}(U)$, that is, if every $\tilde h\in\mathcal{H}_0^{\Delta+\mu}(U)$ such that $0\le\tilde h\le h$ is a multiple of h.

PROPOSITION 2.1. *Let h be an extremal function in $\mathcal{H}_0^{\Delta+\mu}(U)\setminus\{0\}$. Then $Kh = h$ or h is a multiple of the series $\sum_{n=0}^\infty K^n G_0$.*

Proof. For every $n\in\mathbb{N}$, let

$$A_n := \{x\in\mathbb{R}^d\colon \frac{1}{n}R < |x| < \big(1-\frac{1}{n}\big)R\} \quad \text{and} \quad g_n := H_{A_n}h \in \mathcal{H}^+(A_n).$$

Then

$$h - K_{A_n}h = g_n, \qquad n\in\mathbb{N},$$

since $\Delta(h-K_{A_n}h) = \Delta h + h\mu = 0$ on A_n and $K_{A_n}h\in\mathcal{C}_0(A_n)$. The sequence $(K_{A_n}h)$ is increasing to Kh. So the sequence (g_n) is decreasing to a function $g\in\mathcal{H}^+(U)$, and we have

$$h = g + Kh. \tag{2.1}$$

To prove our result it remains to consider the case $g\neq 0$. Since $g\le h$, we know that $\lim_{|x|\to R} g(x) = 0$. Therefore

$$g = \alpha G_0 \tag{2.2}$$

for some $\alpha > 0$ (see [AG01, Exercise 2.11]). By an obvious induction, we see from (2.1) that, for every $m\in\mathbb{N}$,

$$h = \sum_{n=0}^{m-1} K^n g + K^m h,$$

and therefore

$$\tilde h := \sum_{n=0}^{\infty} K^n g \le h.$$

In particular, $\lim_{|x|\to R}\tilde h(x) = 0$. Since $K\tilde h + K(h-\tilde h) = Kh = h - g\in\mathcal{C}^+(U)$, where both $K\tilde h$ and $K(h-\tilde h)$ are lower semicontinuous, we obtain that $K\tilde h\in\mathcal{C}^+(U)$. Moreover, clearly $\tilde h = g + K\tilde h$, hence $\tilde h\in\mathcal{C}^+(U)$ and $\Delta\tilde h = \Delta K\tilde h = -\tilde h\mu$. Therefore $\tilde h\in\mathcal{H}_0^{\Delta+\mu}(U)$. Since h is extremal by our assumption and $h\ge\tilde h\ge g>0$, we conclude that $\tilde h = \beta h$ for some $\beta>0$. Thus we finally see that $h = (\alpha/\beta)\sum_{n=0}^\infty K^n G_0$ (and $\sum_{n=0}^\infty K^n G_0$ is an extremal function in $\mathcal{H}_0^{\Delta+\mu}(U)$). □

If $g_0 := \sum_{n=0}^\infty K^n G_0 < \infty$, then obviously

$$g_0 = Kg_0 + G_0 > Kg_0. \tag{2.3}$$

If g_0 is locally bounded on U and $\lim_{|x|\to R} g_0(x) = 0$, then (2.3) implies that g_0 is continuous on U, $\Delta g_0 = -g_0\mu$, and hence $g_0\in\mathcal{H}_0^{\Delta+\mu}(U)$. Moreover, any locally bounded function h on U which vanishes at ∂B and satisfies $Kh = h$ is contained in $\mathcal{H}_0^{\Delta+\mu}(U)$.

Let us recall that, taking $x_0 := (R/2, 0, \dots, 0)$, the convex set

$$H_0^{\Delta+\mu} := \{h\in\mathcal{H}_0^{\Delta+\mu}(U)\colon h(x_0) = 1\} \tag{2.4}$$

is a compact base of $\mathcal{H}_0^{\Delta+\mu}(U)$, and hence, by Choquet's theorem, for every function $h \in \mathcal{H}_0^{\Delta+\mu}(U)\setminus\{0\}$, there exists a probability measure χ on the set of extreme points of $H_0^{\Delta+\mu}$ such that

$$\frac{h}{h(x_0)} = \int \tilde{h}\, d\chi(\tilde{h}).^2 \tag{2.5}$$

This leads to the following consequence of Proposition 2.1.

COROLLARY 2.2. *Every function $h \in \mathcal{H}_0^{\Delta+\mu}(U)$ is μ-integrable and satisfies $Kh \le h$, where even $Kh = h$, if $g_0 \notin \mathcal{H}_0^{\Delta+\mu}(U)$.*

If $g_0 \in \mathcal{H}_0^{\Delta+\mu}(U)$, then g_0 is extremal.

Proof. Let us fix $h \in \mathcal{H}_0^{\Delta+\mu}(U)$. Obviously, by Proposition 2.1, (2.3), and (2.5), $Kh \le h$, and even $Kh = h$ if $g_0 \notin \mathcal{H}_0^{\Delta+\mu}(U)$. Of course, h is bounded on the set $U_1 := \{x \in U\colon |x| > R/2\}$, and $\mu(U_1) < \infty$. Further, $\inf\{G(x,x_0)\colon x \in U\setminus U_1\} > 0$, and hence the inequality $\int G(x,x_0)h(x)\,d\mu(x) = Kh(x_0) \le h(x_0) < \infty$ implies that h is μ-integrable on $U\setminus U_1$. Thus $h \in \mathcal{L}^1(U,\mu)$.

Finally, let us assume that $g_0 \in \mathcal{H}_0^{\Delta+\mu}(U)$. Then the measure χ associated with g_0 must charge $g_0/g(x_0)$, since otherwise we would obtain that $Kg_0 = g_0$, contradicting (2.3). So g_0 is extremal. □

3 Applications

In the proof of the following result (and later on) we shall tacitly use the fact that, for every $s \in \mathcal{S}^+(U)$, there exists a unique extension to a function $\tilde{s} \in \mathcal{S}^+(B)$ (and $\tilde{s}(0) = \liminf_{y\to x} s(y)$; see [AG01, Corollary 5.2.2]).

PROPOSITION 3.1. *Suppose that g_0 is bounded by a multiple of G_0. Then there exists $C > 0$ such that*

$$\sum_{n=0}^{\infty} K^n s \le Cs \qquad \text{for every } s \in \mathcal{S}^+(U). \tag{3.1}$$

In particular, $\Delta+\mu$ satisfies the Picard principle, $\mathcal{H}_0^{\Delta+\mu}(U) = \mathbb{R}^+ g_0$.

Proof. By Corollary 8.3, there exists $C > 0$ such that (3.1) holds. In particular, there is no $s \in \mathcal{S}^+(U)$ such that $Ks = s$. So there is no $h \in \mathcal{H}_0^{\Delta+\mu}(U)$ satisfying $Kh = h$. Thus, by Proposition 2.1, $\mathcal{H}_0^{\Delta+\mu}(U) = \mathbb{R}^+ g_0$. □

In Section 4, we shall see that $\mathcal{H}_0^{\Delta+\mu}(U) \neq \{0\}$ if and only if K is a bounded operator on $\mathcal{L}^2(U,\mu)$ with $\|K\|_2 \le 1$ (Corollary 4.6). A first step in this direction is the following.

PROPOSITION 3.2. *Let us suppose that $\mu(U) > 0$ and that K is a bounded operator on $\mathcal{L}^2(U,\mu)$, where $\beta := \|K\|_2$ is an eigenvalue of K.*[3] *Then there exists $u \in \mathcal{L}^2(U,\mu)$, $u > 0$, such that*

$$\{v \in \mathcal{L}^2(U,\mu)\colon Kv = \beta v\} = \mathbb{R}u. \tag{3.2}$$

[2] We even know that the measure χ is uniquely determined by h.

[3] Since G is symmetric, we know that $\|K\|_2$ is an eigenvalue of K, if K is a *compact* operator on $\mathcal{L}^2(U,\mu)$.

Moreover, for every $v \in \mathcal{B}^+(U)$ satisfying $Kv \le \beta v$, there exists $c \ge 0$ such that $v = cu$ μ-a.e. on U.

In particular, $\Delta + \mu$ satisfies the Picard principle on U, if $\beta \ge 1$. More precisely, $\mathcal{H}_0^{\Delta+\mu}(U) \subset \mathbb{R}u$ if $\beta = 1$, $\mathcal{H}_0^{\Delta+\mu}(U) = \{0\}$ if $\beta > 1$.

Before starting the proof let us note that the assumption of Proposition 3.2 implies that, in fact, $\mathcal{H}_0^{\Delta+\mu}(U) = \mathbb{R}^+ u$ if $\beta = 1$ (see Corollary 4.6). Moreover, the first statement is more or less known (see e.g. [BAH01, Proposition 3.12]). However, since the first and the second statement can be obtained almost simultaneously, we shall present a complete proof.

Proof of Proposition 3.2. 1. By assumption, there exists a function $u \in \mathcal{L}^2(U, \mu)$ such that $\mu(\{u \ne 0\}) > 0$ and $Ku = \beta u$ μ-a.e. Since we can replace u by $\beta^{-1} K u$ on the μ-null set $\{Ku \ne \beta u\}$, we may suppose without loss of generality that

$$Ku = \beta u \qquad \text{(everywhere) on } U. \tag{3.3}$$

Moreover, we may assume that $\mu(\{u > 0\}) > 0$ (if necessary, we replace u by $-u$).

2. Next let $v \in \mathcal{B}(U)$ such that $v \ge -|u|$ and $Kv \le \beta v$. We define

$$w := (|u| - v)^+. \tag{3.4}$$

Since $K|u| \ge |Ku| = \beta |u|$, we know that $Kw \ge K(|u| - v) \ge \beta(|u| - v)$. Hence

$$Kw \ge \beta w \ge 0. \tag{3.5}$$

Moreover, $w \in \mathcal{L}^2(U, \mu)$, since $0 \le w \le 2|u|$, and

$$\int (Kw)^2 \, d\mu \le \beta^2 \int w^2 \, d\mu,$$

since $\|K\|_2 = \beta$. Therefore

$$Kw = \beta w \qquad \mu\text{-a.e. on } U. \tag{3.6}$$

If $\mu(\{w > 0\}) > 0$, then $Kw > 0$, hence, by (3.6), $w > 0$ μ-a.e on U, that is,

$$|u| - v > 0 \qquad \mu\text{-a.e. on } U. \tag{3.7}$$

If, however, $w = 0$ μ-a.e., then $Kw = 0$, hence, by (3.5), $w = 0$, that is,

$$v \ge |u|. \tag{3.8}$$

3. If we take $v := u$, then (3.7) does not hold, since $\mu(\{u > 0\}) > 0$ and $|u| - u = 0$ on $\{u > 0\}$. So, by (3.8), $u \ge |u| \ge 0$. In fact, this shows that, for every function $\tilde{u} \in \mathcal{L}^2(U, \mu)$ satisfying $K\tilde{u} = \beta \tilde{u}$,

$$\tilde{u} \ge 0 \quad \text{or} \quad -\tilde{u} \ge 0. \tag{3.9}$$

So every $v \in \mathcal{L}^2(U, \mu)$ satisfying $Kv = \beta v$ is a multiple of u, since otherwise there certainly is a linear combination $\tilde{u}$ of u and v violating (3.9).

4. Let us next suppose that $v \in \mathcal{B}^+(U)$ and $Kv \le \beta v$. To show that v is μ-a.e. equal to a multiple of u we may suppose that $\mu(\{u > v\}) > 0$ (we replace v by ηv, where $\eta > 0$ is sufficiently small). Then $\mu(\{w > 0\}) > 0$ and therefore, by (3.7), $v < u$ μ-a.e. on U. Since $v \ge 0$, we hence see that $v \in \mathcal{L}^2(U, \mu)$. If $\mu(\{Kv < \beta v\}) > 0$, then, by the symmetry of G,

$$\beta \int uv\, d\mu > \int u(Kv)\, d\mu = \int (Ku)v\, d\mu = \beta \int uv\, d\mu,$$

a contradiction. So $Kv = \beta v$ μ-a.e. Replacing v by $\beta^{-1}Kv$ on the μ-null set $\{Kv < \beta v\}$ we obtain a function $\tilde{v}$ satisfying $K\tilde{v} = \beta\tilde{v}$. Then $\tilde{v} = cu$ for some $c \ge 0$ and hence $v = cu$ μ-a.e.

5. If $\beta = 1$, then $\mathcal{H}_0^{\Delta+\mu}(U) \subset \mathbb{R}u$ by Corollary 2.2 and the preceding considerations. Finally, let us suppose that $\beta > 1$ and let $h \in \mathcal{H}_0^{\Delta+\mu}(U)$. Then $Kh \le h \le \beta h$, hence $h = cu$ μ-a.e. for some $c \ge 0$. So $h = Kh = \beta h$ and therefore $h = 0$. □

We stress that our general assumption on μ does not exclude the possibility that there is a function $h \in \mathcal{H}_0^{\Delta+\mu}(U)$ which satisfies $Kh = h$ but is *not* contained in $\mathcal{L}^2(U, \mu)$.

EXAMPLE 3.3. For every $n \in \mathbb{N}$, let B_n be the open ball of radius $2^{-(n+3)}R$ centered at $x_n := (2^{-n}R, 0, \dots, 0)$, let $a_n \in (n, \infty)$ such that $G(\cdot, x_n)/a_n < 2^{-n}$ on $U \setminus B_n$, and

$$p_n := \min\{a_n, G(\cdot, x_n)/a_n\}.$$

Then $p_n \in \mathcal{C}(U)$ and $p_n = G^{\nu_n}$ for some measure ν_n which has total mass $1/a_n$ and a support $C_n \subset B_n$. Let $\nu := \sum_{n=1}^\infty \nu_n$, $p := G^\nu = \sum_{n=1}^\infty p_n$, and $\mu := (1/p)\nu$. Then $Kp = G^{p\mu} = G^\nu = p$. Since the balls B_n, $n \in \mathbb{N}$, are pairwise disjoint and $\sum_{n=1}^\infty 2^{-n} = 1$, we have $p \in \mathcal{C}(U)$ and

$$a_n = p_n \le p \le a_n + 1 \quad \text{on } \operatorname{supp}(\nu_n). \tag{3.10}$$

Obviously, p vanishes at ∂B. Hence $p \in \mathcal{H}_0^{\Delta+\mu}(U)$. Moreover, for every $n \in \mathbb{N}$, $\int G^{\nu_n}\, d\nu_n = a_n/a_n = 1$ and hence $\int p^2\, d\mu = \int G^\nu\, d\nu \ge \sum_{n=1}^\infty \int G^{\nu_n}\, d\nu_n = \infty$.

Corollary 4.6 will show that, nevertheless, K is a bounded operator on $\mathcal{L}^2(U, \mu)$ and $\|K\|_2 \le 1$. Further, (3.10) implies that, for every $n \in \mathbb{N}$, $K1_{C_n} = G^{(1/p)\nu_n} \ge a_n/(a_n+1) \ge n/(n+1)$ on C_n, and hence $\|K\|_2 \ge n/(n+1)$. Therefore $\|K\|_2 = 1$. Finally, let us note that both Theorem 6.4 and Corollary 6.5 immediately imply that $\dim_U(\Delta + \mu) = 1$. We could even smear ν a little, add Lebesgue measure on U (leading to a measure μ having a strictly positive $\mathcal{C}^\infty$-density on U), and still have the same result.

4 Characterization of $\mathcal{H}_0^{\Delta+\mu}(U) \neq \{0\}$

In this section we shall see that $\mathcal{H}_0^{\Delta+\mu}(U) \neq \{0\}$ if and only if the μ-eigenvalues of Δ on the shells $\{x \in \mathbb{R}^d : R/(n+1) < |x| < R\}$, $n \in \mathbb{N}$, are at least 1 or – equivalently – if and only if K is a bounded operator on $\mathcal{L}^2(U, \mu)$ with $\|K\|_2 \le 1$ (Corollary 4.4 and Corollary 4.6). For a useful consequence of $\|K\|_2 \le 1$ see Lemma 6.2.

Let us first recall the following from [HH88] (where the proof is so short that we may just as well include it).

LEMMA 4.1. *Let L be a bounded kernel on a measurable space $(E, \mathcal{E})$. Then the following statements are equivalent.*

(i) *The function $\sum_{n=0}^{\infty} L^n 1$ is bounded.*

(ii) *The operator $I-L$ on $(\mathcal{E}_b, \|\cdot\|_\infty)$ is invertible, its inverse is the positive bounded operator $\sum_{n=0}^{\infty} L^n$.*

(iii) *The operator $I-L$ on $(\mathcal{E}_b, \|\cdot\|_\infty)$ is invertible, its inverse is positive.*

(iv) *There exists $f \in \mathcal{E}_b^+$ such that $1 + Lf \le f$.*

Proof. (i) ⇒ (ii)⇒ (iii): Trivial.
(iii) ⇒ (iv): The function $f := (I-L)^{-1}1 \in \mathcal{E}_b^+$ satisfies $1 + Lf = f$.
(iv) ⇒ (i): By induction, $\sum_{j=0}^{n-1} L^j 1 + L^n f \le f$, and thus $\sum_{n=0}^{\infty} L^n 1 \le f$. □

For every $\gamma \ge 0$, let $\mathcal{H}^{\Delta+\gamma\mu}$ denote the sheaf of $(\Delta+\gamma\mu)$-harmonic functions: For every open set W in $\mathbb{R}^d \setminus \{0\}$, $\mathcal{H}^{\Delta+\gamma\mu}(W)$ denotes the set of all $(\Delta+\gamma\mu)$-*harmonic* functions on W, that is, of all continuous real solutions to the equation $\Delta u + \gamma u \mu = 0$ on W. Let us note that $(\mathbb{R}^d \setminus \{0\}, \mathcal{H}^{\Delta+\gamma\mu})$ is a Brelot space (see [BHH87, Theorem 7.7]). In particular, $(\Delta+\gamma\mu)$-harmonic functions satisfy Harnack's inequalities. Of course, functions are called $(\Delta+\gamma\mu)$-superharmonic provided they are superharmonic with respect to the sheaf $\mathcal{H}^{\Delta+\gamma\mu}$.

Given an open set V which is relatively compact in $\mathbb{R}^d \setminus \{0\}$, let Γ_V denote the set of all $\gamma > 0$ such that $I - \gamma K_V \colon \mathcal{B}_b(V) \to \mathcal{B}_b(V)$ is invertible and the inverse is positive. In the next two propositions we collect results which are proven in [HH88] in a far more general setting. For the convenience of the reader we include a complete proof for our situation.

PROPOSITION 4.2. *Let $V \ne \emptyset$ be a relatively compact regular open set in $\mathbb{R}^d \setminus \{0\}$ and $\gamma \in \Gamma_V$. Then the following holds.*

1. *For every $\gamma \in \Gamma_V$, the set V is regular with respect to $\mathcal{H}^{\Delta+\gamma\mu}$ and the corresponding harmonic kernel is*

$$H_V^{\Delta+\gamma\mu} = (I - \gamma K_V)^{-1} H_V = \sum_{n=0}^{\infty} (\gamma K_V)^n H_V. \tag{4.1}$$

2. *If s is a bounded $(\Delta+\gamma\mu)$-superharmonic function on V and $\liminf_{x\to z} s(x) \ge 0$ for every $z \in \partial V$, then $s \ge 0$.*

Proof. 1. Let $L := \gamma K_V$ and $\varphi \in \mathcal{C}(\partial V)$. Then the function $\psi := (I-L)^{-1} H_V \varphi$ satisfies $\psi - L\psi = H_V \varphi \in \mathcal{H}(V)$. Hence ψ is $(\Delta+\gamma\mu)$-harmonic on V and satisfies $\lim_{x\to z} \psi(x) = \varphi(z)$ for every $z \in \partial V$. Moreover, $\psi \ge 0$ if $\varphi \ge 0$.

If $\tilde{\psi}$ is any $(\Delta+\gamma\mu)$-harmonic function on V such that $\tilde{\psi}$ tends to 0 at ∂V, then $g := \tilde{\psi} - L\tilde{\psi}$ is harmonic on V and vanishes at ∂V. So $g = 0$ and hence $\tilde{\psi} = 0$.

Therefore V is regular with respect to $\mathcal{H}^{\Delta+\gamma\mu}$, and (4.1) follows from Lemma 4.1.

2. Next let s be as indicated. Then, by [BHH87, Theorem 3.2], $t := s - Ls$ is a bounded superharmonic function on V. Moreover, $\liminf_{x\to z} t(x) \ge 0$ for every $z \in \partial V$. So $t \ge 0$, and therefore $s = (I-L)^{-1} t \ge 0$. □

Let

$$\alpha_V := \sup \Gamma_V.$$

If $\mu(V) = 0$, then obviously $\Gamma_V = (0, \infty)$ and hence $\alpha_V = \infty$.

PROPOSITION 4.3. *Let V be a connected relatively compact regular open set in $\mathbb{R}^d \setminus \{0\}$ and $\mu(V) > 0$. Then the following holds.*

1. $0 < \alpha_V < \infty$ *and* $\Gamma_V = (0, \alpha_V)$.

2. *There exists a strictly positive $(\Delta + \alpha_V \mu)$-harmonic function $h \in \mathcal{C}_0(V)$. Moreover,* $\ker(I - \alpha_V K_V) = \mathbb{R}h$, *and every positive bounded $(\Delta + \alpha_V \mu)$-superharmonic function on V is a multiple of h.*

3. *For every $\gamma > \alpha_V$, the constant function 0 is the only positive bounded function on V which is $(\Delta + \gamma\mu)$-superharmonic.*

*In particular, α_V is the first μ-*eigenvalue *of Δ on V and the corresponding eigenfunctions are multiples of a strictly positive function.*

Proof. Let A be a compact set in V such that $\mu(A) > 0$. Then $K_V 1_A > 0$ on V. So there exists $\beta \in (0, \infty)$ such that $\beta K_V 1_A \geq 1_A$. By induction, $(\beta K_V)^n 1_A \geq 1_A$ for every $n \in \mathbb{N}$, and hence $\sum_{n=0}^{\infty} (\beta K_V)^n 1 = \infty$ on A. Therefore, by Lemma 4.1, $0 < \alpha_V \leq \beta < \infty$ and Γ_V is an interval from 0 to α_V. We still have to show that Γ_V is open. To that end let us consider $\gamma \in \Gamma_V$ and $0 < \varepsilon < \|(I + \gamma K_V)^{-1} K_V\|^{-1}$. Then

$$(I - (\gamma + \varepsilon) K_V)^{-1} 1 = \sum_{n=0}^{\infty} [\varepsilon (I + \gamma K_V)^{-1} K_V]^n (I + \gamma K_V)^{-1} 1$$

is bounded, and hence $\gamma + \varepsilon \in \Gamma_V$. So Γ_V is an open interval, $\Gamma_V = (0, \alpha_V)$.

Let (γ_n) be a sequence in Γ_V which is increasing to α_V. For every $n \in \mathbb{N}$, let

$$g_n := H_V^{\Delta + \gamma_n \mu} 1 \quad \text{and} \quad c_n := \|g_n\|_\infty.$$

By (4.1), for every $n \in \mathbb{N}$, $1 \leq g_n \leq g_{n+1}$ and

$$g_n - \gamma_n K_V g_n = 1. \tag{4.2}$$

If $\sup c_n < \infty$, then $g := \lim_{n \to \infty} g_n$ is bounded and $g - \alpha_V K_V g = 1$, and hence $\alpha_V \in \Gamma_V$ by Lemma 4.1, a contradiction. So $\sup c_n = \infty$.

Since K_V is a compact operator on $(\mathcal{B}_b(V), \|\cdot\|_\infty)$ which maps $\mathcal{B}_b(V)$ into $C_0(V)$, there exists a subsequence (h_n) of $(c_n^{-1} g_n)$ such that the sequence $(K_V h_n)$ converges uniformly to a function $h \in \mathcal{C}_0^+(V)$. By (4.2), the sequence (h_n) itself converges uniformly to h and $h - \alpha_V K_V h = 0$, that is, $h \in \ker(I - \alpha_V K_V)$. Of course, $\|h\|_\infty = 1$, since $\|h_n\|_\infty = 1$ for every $n \in \mathbb{N}$. Since $h \geq 0$, we hence see that $h = \alpha_V K_V h > 0$ on V. Finally, $\Delta h = -\alpha_V h \mu$. So h is $(\Delta + \alpha_V \mu)$-harmonic.

Let $\gamma \geq \alpha_V$ and s be a bounded $(\Delta + \gamma\mu)$-superharmonic function, $s > 0$. There exists a regular open set W such that $\overline{W} \subset V$ and $A := V \setminus W$ satisfies $\|\alpha_V K_V 1_W\|_\infty < 1$. Then $\alpha_V K_W 1 \leq \|\alpha_V K_V 1_W\|_\infty < 1$ and hence $\alpha_V \in \Gamma_W$. Let

$$a := \sup\{\alpha \geq 0 \colon \alpha h_0 \leq s \text{ on } A\} \quad \text{and} \quad t := s - ah.$$

Then t is $(\Delta + \alpha_V\mu)$-superharmonic on V, $t \geq 0$ on A, and there exists a point $x \in A$ such that $t(x) = 0$. Clearly, $\liminf_{y\to z} t(y) \geq 0$ for every $z \in \partial W$ (recall that $h \to 0$ at ∂V). Hence, applying (2) with W in place of V, we obtain that $t \geq 0$ on W, and therefore $t \geq 0$ on V. Since V is connected and $t(x) = 0$ for some $x \in A$, we conclude that $t = 0$, that is, $s = ah$.

If $\tilde{W}$ is an open set such that $\overline{\widetilde{W}} \subset V$ and $\|\gamma K_V 1_{\widetilde{W}}\|_\infty < 1$, then

$$H_{\widetilde{W}}^{\Delta+\gamma\mu} h \geq H_{\widetilde{W}}^{\Delta+\alpha\mu} h = h.$$

Since $a > 0$ and $H_{\widetilde{W}}^{\Delta+\gamma\mu} s \leq s$, we conclude that $H_{\widetilde{W}}^{\Delta+\gamma\mu} h = h$. Hence $h \in \mathcal{H}^{\Delta+\gamma\mu}(V)$, $\Delta h + \gamma h\mu = 0$. On the other hand $h \in \mathcal{H}^{\Delta+\alpha\mu}(V)$, $\Delta h + \alpha h\mu = 0$. Since $\gamma \geq \alpha_V$, $s > 0$ on V, and $\mu(V) > 0$, this implies that $\gamma = \alpha_V$.

Finally, if $g \in \ker(I - \alpha_V K_V)$, there exists $b > 0$ such that $bh - g \geq 0$ on A and hence $bh - g \geq 0$ on V by (2) (with W in place of V). By the preceding considerations, there exists $c > 0$ such that $bh - g = ch$ and therefore $g \in \mathbb{R}h$. □

For every $n \in \mathbb{N}$, let

$$U_n := \{x \in \mathbb{R}^d \colon R/(n+1) < |x| < R\} \quad \text{and} \quad \alpha_n := \alpha_{U_n}.$$

COROLLARY 4.4. *The following statements are equivalent.*

(i) $\mathcal{H}_0^{\Delta+\mu}(U) \neq \{0\}$.

(ii) *For every* $n \in \mathbb{N}$, $\alpha_n > 1$.

(iii) *For every* $n \in \mathbb{N}$, $\alpha_n \geq 1$.

Proof. If $\mu(U) = 0$, then $\mathcal{H}_0^{\Delta+\mu}(U) = \mathbb{R}^+ G_0$ and $\alpha_n = \infty$, $n \in \mathbb{N}$. So let us suppose that $\mu(U) > 0$.

(i) ⇒ (ii): Let $g \in \mathcal{H}_0^{\Delta+\mu}(U) \setminus \{0\}$ and $n \in \mathbb{N}$. Then $s := g|_{U_n}$ is a positive bounded $(\Delta + \mu)$-(super)harmonic function on U_n, and $s > 0$ at $\partial U_n \setminus \partial B$. By (3) in Proposition 4.3, $1 < \alpha_n$.

(ii) ⇒ (i): By Proposition 4.2, the sets U_n, $n \in \mathbb{N}$, are regular with respect to $\mathcal{H}^{\Delta+\mu}$. Hence, for every $n \in \mathbb{N}$, we have a function

$$g_n := H_{U_n}^{\Delta+\mu} 1_{\partial U_n \setminus \partial B}$$

which is strictly positive and $(\Delta + \mu)$-harmonic on U_n. Let $x_0 := (3R/4, 0, \ldots, 0)$ and $\tilde{g}_n := g_n/g_n(x_0)$, $n \in \mathbb{N}$. Then there exists a subsequence of $(\tilde{g}_n)$ which is locally uniformly convergent to a positive $(\Delta + \mu)$-harmonic function g on U. The convergence is uniform on U_1, since $\tilde{g}_n = H_{U_1}^{\Delta+\mu} \tilde{g}_n$ and $\tilde{g}_n$ vanishes at ∂B, $n \in \mathbb{N}$. Thus $g \in \mathcal{H}_0^{\Delta+\mu}(U)$. Of course, $g \neq 0$ since $g(x_0) = 1$.

(ii) ⇒ (iii): Trivial.

(iii) ⇒ (ii): It suffices to show that $\alpha_{n+1} < \alpha_n$, if n is sufficiently large. Since $\mu(U) > 0$, there exists $n_0 \in \mathbb{N}$ such that $\mu(U_{n_0}) > 0$. By Proposition 4.3, for every $n \geq n_0$, $\alpha_n \in (0, \infty)$ and there exists a strictly positive function $h_n \in \mathcal{C}_0(U_n) \cap \mathcal{H}^{\Delta+\alpha_n\mu}(U_n)$. Let us now fix $n \geq n_0$. Then $s := h_{n+1}|_{U_n}$ is a strictly positive bounded $(\Delta + \alpha_{n+1}\mu)$-(super)harmonic function on U_n. Clearly, s is not a multiple of h_n, since $h_{n+1} > 0$ on $\partial U_n \setminus \partial B$. So $\alpha_{n+1} < \alpha_n$, by Proposition 4.3. □

PROPOSITION 4.5. *If $\mathcal{H}_0^{\Delta+\mu}(U) \neq \{0\}$, then K is a bounded operator on $\mathcal{L}^2(U,\mu)$ and $\|K\|_2 = \sup \alpha_n^{-1} \le 1$.*

Proof. If $\mu(U)=0$, then $K=0$ and $\inf \alpha_n = \infty$. So we assume that $\mathcal{H}_0^{\Delta+\mu}(U) \neq \{0\}$ and $\mu(U)>0$. Let $n \in \mathbb{N}$ such that $\mu(U_n)>0$, and hence $\alpha_n = \alpha_{U_n} \in (0,\infty)$.

By Proposition 4.3, there exists $h_n \in \mathcal{H}^{\Delta+\alpha_n\mu}(U_n) \cap \mathcal{C}_0(U_n)$, $h_n > 0$. We define $K_n := K_{U_n}$. Then $\alpha_n K_n h_n = h_n$, since $\alpha_n K_n h_n - h_n$ is harmonic on U_n and vanishes at ∂U_n. Considering h_n as a function on U which vanishes on $U \setminus U_n$, we have $h_n \in \mathcal{L}^2(U,\mu)$ and

$$\int_U (Kh_n)^2\,d\mu \ge \int_{U_n} (K_n h_n)^2\,d\mu = \alpha_n^{-2} \int_{U_n} h_n^2\,d\mu = \alpha_n^{-2} \int_U h_n^2\,d\mu. \tag{4.3}$$

By [BAH01, Theorem 2.5], $K_n := K_{U_n}$ is a compact operator on $\mathcal{L}^2(U_n,\mu)$. In addition, K_n is positive, self-adjoint, and α_n^{-1} is the first eigenvalue of K_n. Therefore $\|K_n\|_2 = \alpha_n^{-1}$ (see e.g. [HL99, Lemma 6.2.1 and Corollary 5.2.7]) and we conclude that, for every $f \in \mathcal{L}^2(U,\mu)$, $f \ge 0$,

$$\int_U (Kf)^2\,d\mu = \sup_n \int_{U_n} (K_n f|_{U_n})^2\,d\mu \le \sup_n \alpha_n^{-2} \int_U f^2\,d\mu. \tag{4.4}$$

Combining (4.3) and (4.4), we see that K is a bounded operator on $\mathcal{L}^2(U,\mu)$ and $\|K\|_2 = \sup \alpha_n^{-1}$. By Corollary 4.4, $\sup \alpha_n^{-1} \le 1$, $\square$

COROLLARY 4.6. *$\mathcal{H}_0^{\Delta+\mu}(U) \neq \{0\}$ if and only if K is a bounded operator on $\mathcal{L}^2(U,\mu)$ with $\|K\|_2 \le 1$.*

5 Smallness of μ on shells

For all $r>0$ and $\delta \in (0,1/4)$, let

$$B_r := \{x \in \mathbb{R}^d \colon |x| < r\}, \qquad S_r := \{x \in \mathbb{R}^d \colon |x| = r\},$$

$$A_{r,\delta} := \{x \in \mathbb{R}^d \colon (1-2\delta)r < \|x\| < r\}.$$

For every measure ν on U, let

$$G\nu(x) := \int_U G(x,y)\,d\nu(y), \qquad x \in U.$$

By [BHH87, Proposition 7.6], we know the following.

PROPOSITION 5.1. *There exists $\eta > 0$ (which we shall fix once and for all) such that, for all $s \in \mathcal{S}^+(U)$ and measures ν on $U \cap B_{R/2}$,*

$$G(s\nu) \le s/2, \qquad \text{whenever } G\nu \le \eta.$$

PROPOSITION 5.2. *For every $0<\delta<1/4$, there exists $c \ge 1$ such that, for all $g \in \mathcal{H}_+^{\Delta+\mu}(U)$ and all $r \in (0,R/2)$,*

$$\sup g(S_{(1-\delta)r}) \le c \inf g(S_{(1-\delta)r}), \qquad \text{whenever } G(1_{A_{r,\delta}}\mu) \le \eta. \tag{5.1}$$

Proof. Let $\delta \in (0, 1/4)$. By scaling invariance and Harnack's inequalities, there exists $c > 1$ such that, for every $r > 0$,

$$\sup h(S_{(1-\delta)r}) \le \frac{c}{2} \inf h(S_{(1-\delta)r}) \quad \text{for all } h \in \mathcal{H}^+(A_{r,\delta}). \tag{5.2}$$

Let $g \in \mathcal{H}_+^{\Delta+\mu}(U)$ and $r \in (0, R/2)$ such that $A := A_{r,\delta}$ satisfies $G(1_A\mu) \le \eta$. Since $g \in \mathcal{S}^+(U) \cap \mathcal{C}(U)$, we see that $s := H_A g \in \mathcal{S}^+(U) \cap C(U)$, $s \le g$, and $h := s|_A \in \mathcal{H}^+(A)$. Let $S := S_{(1-\delta)r}$. By Proposition 5.1 and (5.2),

$$G(1_A s\mu) \le \frac{1}{2} s \quad \text{and} \quad \sup h(S) \le \frac{c}{2} \inf h(S).$$

By the first inequality, $K_A h \le G(1_A s\mu)|_A \le (1/2)\, h$, hence

$$g|_A = \sum_{n=0}^{\infty} (K_A)^n h \le \sum_{n=0}^{\infty} 2^{-n} h = 2h$$

and

$$\sup g(S) \le 2 \sup h(S) \le c \inf h(S) \le c \inf g(S).$$

□

THEOREM 5.3. *Suppose that there exist $\delta \in (0, 1/4)$ and $r_n \in (0, R)$ such that $r_n \downarrow 0$ and, for every $n \in \mathbb{N}$, the annulus $A_n := A_{r_n,\delta}$ satisfies*

$$G(1_{A_n}\mu) \le \eta. \tag{5.3}$$

Then $\Delta + \mu$ satisfies the Picard principle on U.

Proof. Let us assume that we have extremal functions $h_1, h_2 \in \mathcal{H}_0^{\Delta+\mu} \setminus \{0\}$. We shall see that they are not linearly independent, and hence $\dim_U(\Delta + \mu) \le 1$.

Let c denote the constant of Proposition 5.2 and let us fix $C \ge 1$ such that $h_i \le C h_j$ on $S_{R/2}$ for $i, j \in \{1, 2\}$. By Proposition 4.2 and Proposition 4.3,

$$h_i \le C h_j \quad \text{on } U \setminus B_{R/2}. \tag{5.4}$$

Assuming that already $r_1 < R/2$, we define $x_n := ((1-\delta)r_n, 0, \dots, 0) \in S_{(1-\delta)r_n}$, $n \in \mathbb{N}$. By Proposition 5.2, for all $i \in \{1, 2\}$ and $n \in \mathbb{N}$,

$$c^{-1} h_i(x_n) \le h_i \le c h_i(x_n) \quad \text{on } S_{(1-\delta)r_n},$$

and hence, for $i, j \in \{1, 2\}$,

$$h_i \le c^2 \frac{h_i(x_n)}{h_j(x_n)} h_j \quad \text{on } S_{(1-\delta)r_n}.$$

We may assume without loss of generality that there exist $1 \le k_1 < k_2 < \dots$ such that for some real $a > 0$ the sequence $(h_1(x_{k_n})/h_2(x_{k_n}))$ is bounded by a (if $\lim h_1(x_n)/h_2(x_n) = \infty$, we exchange the role of h_1 and h_2, and take $a = 1$). Let $\tilde{c} := \max\{ac^2, C\}$. Then, for every $n \in \mathbb{N}$,

$$h_1 \le \tilde{c}\, h_2 \tag{5.5}$$

on $S_{(1-\delta)r_n} \cup S_{R/2}$. By Proposition 4.3, (5.5) holds for all $n \in \mathbb{N}$ and $x \in U$ such that $r_{k_n} \le |x| \le R/2$. In view of (5.4), we obtain that (5.5) holds for all $x \in U$. Since h_2 is an extremal function in $\mathcal{H}_0^{\Delta+\mu}(U)$, we conclude that $h_1 = \gamma h_2$ for some $\gamma > 0$. □

6 Almost radial measures

For $r > 0$, let σ_r denote the normalized surface measure on S_r. Given $r \in (0, R/2)$, we define

$$a_r := \|G\sigma_r\|_\infty. \tag{6.1}$$

To work with almost radial measures (see Definition 6.3) we shall need two simple properties of σ_r (where it would be sufficient to have some constant c_0 instead of 2^{d-1} and 2^{d+1}).

LEMMA 6.1. *Let $r \in (0, R/2)$. Then $G\sigma_r = G\sigma_r(0) = a_r$ on $\overline{B}_r$. Moreover,*

$$a_{r/2} \le 2^{d-1} a_r \quad \text{and} \quad a_r \le 2^{d+1} G(x,y) \ \text{ for all } x, y \in B_r. \tag{6.2}$$

Proof. Let $z := (r, 0, \dots, 0)$. Since the potential $G\sigma_r$ is a radial function which is harmonic on $B \setminus S_r$, we immediately see that

$$a_r = G(0, z) = G\sigma_r(0) = G\sigma_r \quad \text{on } \overline{B}_r. \tag{6.3}$$

To prove (6.2) we define

$$\phi(x,y) := \frac{(R^2 - |x|^2)(R^2 - |y|^2)}{R^2 |x-y|^2} \qquad (x, y \in B).$$

Clearly, $\phi(0, z) = (R/r)^2 - 1$ and, for all $x, y \in B_r$,

$$\frac{9}{16} |x-y|^{-2} \le \phi(x,y) \le |x-y|^{-2}. \tag{6.4}$$

Let us first consider the case $d = 2$. Then, for all $x, y \in B$,

$$G(x,y) = (4\pi)^{-1} \ln(1 + \phi(x,y)) \tag{6.5}$$

(see [AG01, Theorem 4.1.5, Corollary 4.3.3]). Since $1 + \phi(0, z) = (R/r)^2$, we see that

$$a_r = (2\pi)^{-1} \ln(R/r).$$

Moreover, $\ln(2/r) = \ln 2 + \ln(1/r)$, where $\ln 2 \le \ln(1/r)$. Therefore $a_{r/2} \le 2a_r$ proving the first part of (6.2). If $x, y \in B_r$, then

$$\phi(x,y) \ge \frac{9}{16} \frac{R^2}{(2r)^2} \ge \frac{1}{8} \frac{R^2}{r^2}.$$

Since $\ln 1 = 0$ and the function $t \mapsto \ln(1+t)$ is concave, we hence conclude that

$$\ln(1 + \phi(x,y)) \ge \ln(1 + \frac{1}{8}\frac{R^2}{r^2}) \ge \frac{1}{8} \ln(1 + \frac{R^2}{r^2}) > \frac{1}{8} \ln(1 + \phi(0,z)).$$

So $G(x,y) \ge (1/8) a_r$, by (6.5) and (6.3).

Now let $d \ge 3$. Then

$$G(x,y) = \kappa_d^{-1} \bigl(1 - (1 + \phi(x,y))^{(2-d)/2}\bigr) \, |x-y|^{2-d} \tag{6.6}$$

(where κ_d is $d-2$ times the surface of S_1; see [AG01, Theorem 4.1.5, Corollary 4.3.3] again). In particular (recall that $1+\phi(0,z) = (R/r)^2$),

$$r^{2-d}/2 \le r^{2-d} - R^{2-d} = \kappa_d a_r \le r^{2-d} \tag{6.7}$$

leading to the inequality $a_{r/2} \le 2^{d-1} a_r$.

Finally, taking $b := 2rR/(R^2+r^2) \in (0,1)$, we have, for all $x, y \in B_r$,

$$1+\phi(x,y) \ge 1 + \frac{(R^2-r^2)^2}{(2rR)^2} = \frac{(R^2+r^2)^2}{(2rR)^2} = b^{-2}$$

and hence, by (6.6), (6.7), and since $1-b = (R-r)^2/(R^2+r^2) \ge 1/8$,

$$\kappa_d G(x,y) \ge (1-b^{d-2})(2r)^{2-d} \ge (1-b)(2r)^{2-d} \ge 2^{-(d+1)} r^{2-d} \ge 2^{-(d+1)} \kappa_d a_r.$$

□

Let us recall that the mapping K is defined by

$$Kf := G(f\mu)$$

(whenever $f\colon U \to \mathbb{R}$ is Borel measurable and $G(f^+\mu) - G(f^-\mu)$ makes sense).

LEMMA 6.2. *Suppose that K is a bounded operator on $\mathcal{L}^2(U,\mu)$ and $\|K\|_2 \le 1$. Then*

$$\mu(B_r) \le 2^{d+1}/a_r \qquad \text{for every } r \in (0, R/2). \tag{6.8}$$

Proof. Let $r \in (0,R/2)$ and $A := B_r \setminus \{0\}$. By Lemma 6.1, for every $x \in A$,

$$K1_A(x) = \int_A G(x,y)\, d\mu(y) \ge 2^{-(d+1)} a_r \mu(A)$$

and therefore

$$\int (K1_A)^2\, d\mu \ge \int_A (K1_A)^2\, d\mu \ge (2^{-(d+1)} a_r \mu(A))^2 \mu(A).$$

On the other hand, since $\|K\|_2 \le 1$, we know that $\int (K1_A)^2\, d\mu \le \int (1_A)^2\, d\mu = \mu(A)$. Since $\mu(A) = \mu(B_r)$, (6.8) follows. □

DEFINITION 6.3. *Given $C > 0$, $r \in (0,R/2)$, and $\delta \in (0,1/4)$, we shall say that μ is C-*radial *on $A_{r,\delta}$, if for every annulus $A := \{x \in \mathbb{R}^d\colon s < \|x\| < t\}$, where $(1-2\delta)r \le s < t \le r$,*

$$G(1_A\mu) \le C a_r \mu(A).$$

If μ is radial on $A := A_{r,\delta}$, then μ is 2^{d-1}-radial on A, by Lemma 6.1, since $r/2 \le (1-2\delta)r$. More generally, if $C > 0$ and

$$1_A\mu = \int_{(1-2\delta)r}^{r} \mu_t\, dt,$$

such that

$$G\mu_t \le C a_t \mu_t(S_t) \qquad \text{for every } (1-2\delta)r \le t \le r, \tag{6.9}$$

then μ is $2^{d-1}C$-radial on A. Let us note that (6.9) holds if $\mu_t = \varphi_t \sigma_t$ such that

$$\sup \varphi_t(S_t) \le C \inf \varphi_t(S_t). \tag{6.10}$$

Indeed, (6.10) obviously implies that

$$G(\varphi_t \sigma_t) \le \sup \varphi_t(S_t) a_t \le C \inf \varphi_t(S_t) a_t \le C a_t \mu_t(S_t).$$

THEOREM 6.4. *Suppose that there exist $C>0$ and $\delta \in (0,1/4)$ such that*

$$\inf\{r \in (0,R/2)\colon \mu \textit{ is } C\textit{-radial on } A_{r,\delta}\} = 0.$$

Then $\Delta + \mu$ satisfies the Picard principle on U.

Proof. By Corollary 4.6, it remains to consider the case, where K is a bounded operator on $\mathcal{L}^2(U,\mu)$ and $\|K\|_2 \le 1$. Then, by Lemma 6.2,

$$\mu(B_r) \le 2^{d+1}/a_r \qquad \text{for every } r \in (0,R/2). \tag{6.11}$$

We fix $m \in \mathbb{N}$ and $\tilde{\delta} \in (0,\delta)$ such that

$$2^{2d-m} C \le \eta \qquad \text{and} \qquad (1-2\tilde{\delta})^{2^m} \ge 1-2\delta. \tag{6.12}$$

Let $r \in (0,R/2)$ such that μ is C-radial on $A := A_{r,\delta}$. By Theorem 5.3, it suffices to prove that there exists $\tilde{r} \in (0,r)$ such that $\tilde{A} := A_{\tilde{r},\tilde{\delta}}$ satisfies

$$G(1_{\tilde{A}}\mu) \le \eta. \tag{6.13}$$

To that end we define, for every $0 \le j \le 2^m - 1$,

$$r_j := (1-2\tilde{\delta})^j r \qquad \text{and} \qquad A_j := A_{r_j,\tilde{\delta}}.$$

Let us note that the 2^m annuli $A_0, A_1, \dots, A_{2^m-1}$ are pairwise disjoint sets in $A \subset B_r$. So, by (6.11), there exists $0 \le j \le 2^m - 1$ such that, defining $\tilde{r} := r_j$ and $\tilde{A} := A_j$, we have

$$\mu(\tilde{A}) \le 2^{d+1-m}/a_r.$$

Since $r/2 \le \tilde{r} \le r$, we know, by Lemma 6.1, that

$$a_{\tilde{r}} \le a_{r/2} \le 2^{d-1} a_r.$$

Since μ is C-radial on A , we finally conclude that

$$G(1_{\tilde{A}}\mu) \le C a_{\tilde{r}} \mu(\tilde{A}) \le 2^{2d-m} C \le \eta.$$

This finishes the proof. □

COROLLARY 6.5. *If μ is radial on U or, more generally, if there exist $\delta \in (0,1/4)$ and $r_n \in (0,1/2)$ such that $r_n \downarrow 0$ and μ is radial on every $A_{r_n,\delta}$, $n \in \mathbb{N}$, then $\Delta + \mu$ satisfies the Picard principle on U.*

7 Localization

We recall that the sufficient conditions in Theorem 5.3, Theorem 6.4, and Corollary 6.5 depend only on the behavior of μ close to the origin. The following result shows that, even in the most general case, the verification of the Picard principle for $\Delta + \mu$ on U can be localized at 0 in two different ways (which can be combined in an obvious manner; see the proof of Corollary 7.2). To that end let $r \in (0, R)$ and $V := \{x \in U \colon |x| < r\}$.

THEOREM 7.1. *Let μ' be a measure on $\mathbb{R}^d \setminus \{0\}$ such that $1_V\mu \le \mu' \le \mu$. Then $\Delta + \mu$ satisfies the Picard principle on U, if $\Delta + \mu'$ satisfies the Picard principle on U or if $\Delta + \mu$ satisfies the Picard principle on V.*

Of course, there is no hope for reverse implications (unless we already know that $\dim_U(\Delta + \mu') \le 1$ or $\dim_V(\Delta + \mu) \le 1$), since, whatever $1_V\mu$ may be, we shall have $\mathcal{H}_0^{\Delta+\mu}(U) = \{0\}$, if $1_{U\setminus V}\mu$ is too large.

Proof of Theorem 7.1. 1. Given $h \in \mathcal{H}_0^{\Delta+\mu}(U) \setminus \{0\}$, we shall construct corresponding minorants in $\mathcal{H}_0^{\Delta+\mu}(V)$ and $\mathcal{H}_0^{\Delta+\mu'}(U)$. For $n \in \mathbb{N}$, let

$$U_n := \{x \in U \colon |x| > r/(n+1)\} \quad \text{and} \quad V_n := V \cap U_n.$$

By Proposition 4.2 and Proposition 4.3, the open sets U_n, V_n are regular with respect to $\Delta + \mu$ and $\Delta + \mu'$. We first define

$$v_n := H_{V_n}^{\Delta+\mu}(1_{U\setminus V} h) \qquad (n \in \mathbb{N}). \tag{7.1}$$

The sequence (v_n) is increasing to a function $\tilde{h} \le h$ which is $(\Delta+\mu)$-harmonic on V and equal to h on $U \setminus V$. Since $v_1 \to h$ at $S := \partial V \setminus \{0\}$, we know that also $\tilde{h} \to h$ at S. Hence

$$g := h - \tilde{h} \in \mathcal{C}^+(U) \quad \text{and} \quad g = 0 \quad \text{on } U \setminus V$$

Moreover, g is $(\Delta + \mu)$-harmonic on V, and hence $g|_V \in \mathcal{H}_0^{\Delta+\mu}(V)$ (we cannot and will not exclude the possibility that even $g|_V = 0$).

Obviously, g is $(\Delta + \mu')$-subharmonic on U and h is a $(\Delta + \mu')$-superharmonic majorant of g. Therefore the functions

$$u_n := H_{U_n}^{\Delta+\mu'} g, \qquad n \in \mathbb{N},$$

are increasing to a $(\Delta + \mu')$-harmonic function h' which is the smallest $(\Delta + \mu')$-superharmonic majorant of g on U. In particular, $h' \le h$ and hence $h' \to 0$ at ∂B. So $h' \in \mathcal{H}_0^{\Delta+\mu'}(U)$.

There is a natural way to get g back from h': For $n \in \mathbb{N}$, let

$$v_n' := H_{V_n}^{\Delta+\mu'}(1_V h') \tag{7.2}$$

(where we could just as well write μ instead of μ', since $V_n \subset V$). We claim that

$$g = \lim_{n\to\infty} v_n'. \tag{7.3}$$

Indeed, for every $n \in \mathbb{N}$, the function v_n' is $(\Delta+\mu')$-harmonic on V_n. It is equal to h' on $V \setminus V_n$, it vanishes on $U \setminus V$, and it is continuous on U. Since $g \le 1_V h' \le h'$, $H_{V_n}^{\Delta+\mu'} g = g$, and $H_{V_n}^{\Delta+\mu'} h' \le h'$, we see that $g \le v_n' \le h'$ and hence

$$g \le v_n' \le 1_V h' \qquad (n \in \mathbb{N}). \tag{7.4}$$

In particular, for every $n \in \mathbb{N}$, $v_{n+1}' = H_{V_n}^{\Delta+\mu'} v_{n+1}' \le H_{V_n}^{\Delta+\mu'}(1_V h') = v_n'$, that is, the sequence (v_n') is decreasing. Its limit f is $(\Delta+\mu')$-harmonic on V and satisfies $g \le f \le h'$, it vanishes on $U \setminus V$, and is continuous on U. Therefore the positive function $f - g$ is $(\Delta+\mu')$-subharmonic on U, and we conclude that, for all $n \in \mathbb{N}$,

$$0 \le f - g \le H_{U_n}^{\Delta+\mu'}(f-g) \le H_{U_n}^{\Delta+\mu'} h' - u_n \le h' - u_n.$$

Letting $n \to \infty$ we see that $f - g = 0$ proving (7.3).

2. Now let h_1, h_2 be extremal functions in $\mathcal{H}_0^{\Delta+\mu}(U) \setminus \{0\}$. Then we have corresponding functions $\tilde h_1, \tilde h_2$, g_1, g_2, and h_1', h_2'. If $\Delta + \mu'$ satisfies the Picard principle on U, then h_1', h_2' are proportional and hence g_1, g_2 are proportional, by (7.2) and (7.3). If $\Delta + \mu$ satisfies the Picard principle on V, then we know immediately that g_1, g_2 are proportional.

3. So let us consider the case that $g_1 = a g_2$ for some $a \ge 0$. Of course, there exists $b > 0$ such that $h_1 \le b h_2$ on S and hence $h_1 \le b h_2$ on $U \setminus V$. By (7.1), we see that $\tilde h_1 \le b \tilde h_2$. Having $h_j = g_j + \tilde h_j$, $j \in \{1,2\}$, we obtain that $h_1 \le (a+b) h_2$. Since h_2 is extremal, we finally conclude that h_1 is a multiple of h_2. □

A consequence of Theorem 7.1 is the following result (we note that, of course, (7.5) holds if μ is a Kato measure on $\mathbb{R}^d$).

COROLLARY 7.2. *Let us suppose that*

$$\limsup_{x\to 0} K1(x) < \liminf_{x \to 0} K1(x) + 1 < \infty \tag{7.5}$$

or, more generally, that K_V is a bounded operator on $(\mathcal{B}_b(V), \|\cdot\|_\infty)$ having a spectral radius $\rho(K_V) < 1$. Then $\Delta + \mu$ satisfies the Picard principle on U.

Proof. If (7.5) holds, then $K1$ is bounded, $\sup K1(\overline{V}) - \inf K1(\overline{V}) < 1$, if r is sufficiently small, and hence $\|K_V 1\|_\infty < 1$. So let us assume that $\rho(K_V)) < 1$ and let $\mu' := 1_{B_{r/2}} \mu$. Of course, the spectral radius of the operator $f \mapsto \int_V G(\cdot, y) f(y)\, d\mu'(y)$ is at most $\rho(K_V)$.

By Proposition 3.1 and Corollary 8.3, $\Delta + \mu'$ satisfies the Picard principle on V. By Theorem 7.1, applied to V in place of U, we obtain first that $\Delta + \mu$ satisfies the Picard principle on V. Using Theorem 7.1 again, we finally see that $\Delta + \mu$ satisfies the Picard principle on U. □

8 Appendix: Triangle property on punctured sets

Let us recall the generalized triangle property. Given an arbitrary set X and functions $w, w^*\colon X \to (0,\infty)$, a function $F\colon X \times X \to [0,\infty]$ has the (w,w^*)*-triangle property*, if there exists $C > 0$ such that, for all $x, y, z \in X$,

$$F(x,z)F(z,y) \le C F(x,y) \max\Big\{\frac{w(z)}{w(x)} F(x,z), \frac{w^*(z)}{w^*(y)} F(z,y)\Big\}$$

or – equivalently – that the function $F_{w,w^*}\colon (x,y) \mapsto F(x,y)/(w(x)w^*(y))$ satisfies the *triangle property*, that is, for all $x, y, z \in X$,

$$F_{w,w^*}(x,z)F_{w,w^*}(z,y) \le CF_{w,w^*}(x,y)\max\bigl\{F_{w,w^*}(x,z), F_{w,w^*}(z,y)\bigr\}, \tag{8.1}$$

which, in turn, can be rewritten as

$$\min\bigl\{F_{w,w^*}(x,z), F_{w,w^*}(z,y)\bigr\} \le CF_{w,w^*}(x,y). \tag{8.2}$$

The following results are of independent interest.

PROPOSITION 8.1. *Let X be an arbitrary set, $a \in X$, $X^a := X \setminus \{a\}$. Suppose that $G\colon X \times X \to [0,\infty]$ is symmetric, $0 < G^a := G(\cdot,a)|_{X^a} < \infty$, and, for some $w\colon X \to (0,\infty)$, G has the (w,w)-triangle property.*

Then $G|_{X^a\times X^a}$ has the (G^a,G^a)-triangle property.

Proof. 1. Let us suppose first that $w = 1$, that is, there exists $C \ge 1$ such that, for all $x, y, z \in X$,

$$\min\bigl\{G(x,z), G(z,y)\bigr\} \le CG(x,y).$$

We define $\tilde G\colon X^a \times X^a \to [0,\infty]$ by

$$\tilde G(x,y) := G_{G^a,G^a}(x,y) = \frac{G(x,y)}{G^a(x)G^a(y)}\,.$$

Let us fix $x, y, z \in X^a$. We claim that $\min\{\tilde G(x,z), \tilde G(z,y)\} \le C^2\tilde G(x,y)$, that is,

$$\min\bigl\{G^a(y)G(x,z), G^a(x)G(z,y)\bigr\} \le C^2G^a(z)G(x,y). \tag{8.3}$$

By symmetry, we may assume that $G^a(x) \le G^a(y)$. If $G^a(y) \le CG^a(z)$, then

$$\begin{aligned}\min\bigl\{&G^a(y)G(x,z), G^a(x)G(z,y)\bigr\}\\ &\le CG^a(z)\min\bigl\{G(x,z), G(z,y)\bigr\} \le C^2G^a(z)G(x,y).\end{aligned}$$

So let us suppose $CG^a(z) < G^a(y)$. Since $\min\bigl\{G^a(y), G(y,z)\bigr\} \le CG^a(z)$, we see that $G(y,z) \le CG^a(z)$. In addition, $G^a(x) = \min\bigl\{G^a(x), G^a(y)\bigr\} \le CG(x,y)$. Therefore

$$\min\bigl\{G^a(y)G(x,z), G^a(x)G(z,y)\bigr\} \le G^a(x)G(y,z) \le C^2G^a(z)G(x,y).$$

Thus $G|_{X^a\times X^a}$ has the (G^a,G^a)-triangle property.

2. To reduce the general case to the special one, where $w = 1$, it suffices to note that $G_{w,w}$ is symmetric and that, for all $x, y \in X^a$,

$$\frac{G_{w,w}(x,y)}{G_{w.w}(x,a)G_{w,w}(y,a)} = w(a)^2\frac{G(x,y)}{G^a(x)G^a(y)}.$$

□

For a better understanding of the first corollary, let us recall that, given an inner product space $(V, \langle\cdot,\cdot\rangle)$,

$$\rho\colon (x,y) \mapsto \frac{\|x-y\|}{\|x\|\|y\|}$$

(where, of course, $\|z\| := \langle x, x\rangle^{1/2}$) is known to define a metric on $V \setminus \{0\}$, since $\rho(x,y) = \|\|x\|^{-2}x - \|y\|^{-2}y\|$ (see [Pin99, Lemma A.1]).

If X is an arbitrary set and $\rho\colon X \times X \to \mathbb{R}^+$ is symmetric and vanishes on the diagonal, but nowhere else, then ρ is a quasi-metric if and only if ρ^{-1} has the triangle property (see e.g. [Han06, p. 646, Remark 2.1.2]). So Proposition 8.1 has an immediate consequence for quasi-metrics (and is more or less equivalent to it).

COROLLARY 8.2. *Let ρ be a quasi-metric on a set X. Let $a \in X$, $X^a := X\setminus\{a\}$, and*

$$\rho^a(x,y) := \frac{\rho(x,y)}{\rho(x,a)\rho(y,a)} \qquad (x,y \in X^a).$$

Then ρ^a is a quasi-metric on X^a.

Let us note that the following corollary has obvious analogues in the more general situations considered in [Han06, Section 9] and [Han05].

COROLLARY 8.3. *Let ν be any measure on B and $Lf := \int G_B(\cdot,y)f(y)\,d\nu(y)$, $f \in \mathcal{B}^+(B)$. Let us consider the following statements.*

(i) *There exists $a \in B$ and $c > 0$ such that $\nu(\{a\}) = 0$ and*

$$\sum_{n=0}^{\infty} L^n G_B(\cdot,a) \le c G_B(\cdot,a). \tag{8.4}$$

(ii) *There exists $c > 0$ such that, for every $s \in \mathcal{S}^+(B)$, $\sum_{n=0}^{\infty} L^n s \le cs$.*

(iii) *The function $\sum_{n=0}^{\infty} L^n 1$ is bounded.*

Then (i) $\Leftrightarrow$ (ii) $\Rightarrow$ (iii) *. If ν has compact support in B, then also* (iii) $\Rightarrow$ (ii).

Proof. Let $w := \min\{G_B(\cdot,0),1\}$. The function G_B has the (w,w)-triangle property (see e.g. [Han06, Proposition 9.3]). So, by Proposition 8.1, $G_B|_{B^a\times B^a}$ has the (G^a,G^a)-triangle property.

(i) $\Rightarrow$ (ii): We define $L_a f(x) := \int_{B^a} G_B(x,y)f(y)\,d\nu(y)$, $f \in \mathcal{B}^+(B^a)$, $x \in B^a$. By (8.4), we know that $\sum_{n=0}^{\infty} L_a^n G_B^a \le cG_B^a$. Therefore, by [Han04, Proposition 7.4],

$$L_a^n G_B^a \le c\big(1-(1/c)\big)^n G_B^a, \qquad n \in \mathbb{N},$$

where $\lim_{n\to\infty}[c(1-(1/c))^n]^{1/n} = 1-(1/c) < 1$. Hence, by [Han06, Proposition 2.3 and Corollary 3.3], we infer that there exists $C > 0$ such that, for every $s \in \mathcal{S}^+(B^a)$, $\sum_{n=0}^{\infty} L_a^n s \le Cs$. Since $\nu(\{x_0\}) = 0$, (iii) follows.

(ii) $\Rightarrow$ (i),(iii): Trivial, since $G_B(\cdot,a), 1 \in \mathcal{S}^+(B)$.

Finally, let us suppose that ν is supported by a compact set A in B and that $\sum_{n=0}^{\infty} L^n 1$ is bounded. Since $\inf w(A) > 0$ and $\sup w(A) < \infty$, we know that $G_B|_{A\times A}$ has the triangle property. Thus, by [Han04, Proposition 3.10], (ii) follows. ☐

Finally, let us note another consequence of Proposition 8.1 (where we shall not try to achieve the utmost generality).

COROLLARY 8.4. *Let G be a symmetric Green function for a connected Brelot space $(X,\mathcal{H})$ and $a \in X$ such that the following holds.*

(i) $G(a,a)=\infty$ *and* $\limsup_{x\to\infty} G(x,a)<\infty$.

(i) *G has the local triangle property.*

(ii) *There exist strictly positive real functions w, w^* on X such that G has the (w,w^*)-triangle property.*

Let $g := \min\{G^a, 1\}$. Then G has the (g,g)-triangle property.

Proof. By (i), there exist a relatively compact open neighborhood V of a and $M \ge 1$ such that $G^a \ge 1$ on $\overline{V}$ and $G^a \le M$ on V^c. Then $g=1$ on $\overline{V}$ and $G^a \le Mg$ on V^c. Let L be a compact neighborhood of $\overline{V}$ and W a relatively compact open neighborhood of L. We may assume without loss of generality that $g \ge 1/M$ on W. By (ii), (iii), and Proposition 8.1, there exists $C \ge 1$ such that, for all $x,y,z \in W$,

$$\min\{G(x,z), G(z,y)\} \le CG(x,y), \tag{8.5}$$

and, for all $x,y,z \in X\setminus\{a\}$,

$$\min\{G^a(y)G(x,z), G^a(x)G(z,y)\} \le CG^a(z)G(x,y). \tag{8.6}$$

Moreover, there exists $c \ge 1$ such that

$$h(z) \le ch(\tilde{z}), \tag{8.7}$$

whenever $h \in \mathcal{H}^+(W)$ and $z,\tilde{z} \in L$, or $h \in \mathcal{H}(\mathring{L})$ and $z,\tilde{z} \in \overline{V}$.

We claim that, for all $x,y,z \in X$,

$$\min\{g(y)G(x,z), g(x)G(z,y)\} \le McCg(z)G(x,y). \tag{8.8}$$

If $x,y,z \in W$, then (8.8) follows from (8.5), since $g \le 1 \le Mg(z)$. So we may assume that $W \ne X$.

Let us suppose next that $z \in V^c$. If $x,y \in X\setminus\{a\}$, then, by (8.6),

$$\min\{g(y)G(x,z), g(x)G(z,y)\} \le MCg(z)G(x,y), \tag{8.9}$$

since $g \le G^a$ and $G^a(z) \le Mg(z)$. Then, by continuity, (8.9) holds as well, if $x \ne a$, but $y=a$. Analogously, if $y=a$ and $x \ne a$. If $x=y=a$, then (8.9) holds trivially, since $g(z)>0$ and $G(a,a)=\infty$.

So we may and shall assume from now on that $z \in V$, whence $g(z)=1$, and $(x,y) \notin W\times W$. If $x \in L$ and $y \notin W$, then we may apply (8.7) and obtain that $g(x)G(z,y) \le G(z,y) \le cG(x,y)$. Analogously, if $y \in L$ and $x \notin W$. Hence (8.8) holds in these two cases.

Therefore it remains to consider the case, where $x,y \notin L$. Since X is connected and $W \ne X$, there exists a point $\tilde{z} \in \partial V$, and we know, by (8.9), that

$$\min\{g(y)G(x,\tilde{z}), g(x)G(\tilde{z},y)\} \le MCG(x,y).$$

By (8.7), $G(x,z) \le cG(x,\tilde{z})$, $G(y,z) \le cG(y,\tilde{z})$, and (8.8) follows. □

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